.

Weak Centers for Reversible Cubic Systems *

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ABSTRACT

Unlike Chicone and Jacobs' consideration in quadratic case and unlike Rousseau and Toni's in homogenuous cubic case, the problem of weak centers for general cubic differential systems involves too much complicated polynomials. In this paper we analyze reversible cubic systems using *Maple V.2* on Pentium/75 PC for its weak center and local bifurcation of critical period.

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1 Introduction

The differential system

$$\begin{cases} \dot{x} = & -y + \phi(x, y, \lambda) \\ \dot{y} = & x + \psi(x, y, \lambda), \end{cases} \tag{1.1}$$

where $\phi(0,0,\lambda) = \psi(0,0,\lambda) = 0, \forall\lambda \in R^n$, is a basic form to discuss the degenerate vector fields. In particular, for C^3 functions $\phi(x,y,\lambda), \psi(x,y,\lambda)$ we often locally consider a cubic system

$$\begin{cases} \dot{x} = & -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ \dot{y} = & x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{cases} \tag{1.2}$$

Concerning Eq.(1.2) one of important problems is the well-known *weak focus problem*, considering the effect of nonlinearity. As a question surrounding Hilbert's 16th problem it was given a set of analyzing methods[1]–[4]. Especially the method calculating the *Lyapunov value* with *succesion function* is often used[5]–[8]. Recently in aid of computer algebra systems and symbolic computation techniques this theory is developing rapidly [9]–[11].

Along with the deep research on weak focus, another important theme about *center* becomes more and more attractive. One concerns the monotonicity[12] of *period function* of the family of closed orbits surrounding the center, since monotonicity is a nondegeneracy condition for the bifurcation of subharmonic solutions of periodically forced Hamiltonian systems[13]. This makes it interesting to investigate critical points of the period function[14]. In the light of weak focus in 1989 C.Chicone and M.Jacobs[15] put forward the concept of so-called *weak centre*, answering how many critical periods bifurcating from the center. However, because of the complexity of computation only systems with quadratic and homogenuous cubic nonlinearities were discussed, e.g., in [15] and [16]. There were seldom found further results.

Unlike Chicone and Jacobs' consideration and unlike Rousseau and Toni's, In this paper we analyze reversible cubic systems using Maple V.2 software[18] on Pentium/75 PC for its weak center and local bifurcation of critical period. The obtained conclusions should be too complicated to derive without computer algebra systems. Section 2 is devoted to the theory of weak centers. In section 3 an algorithm for weak centers of reversible cubic systems is given. In section 4 a Maple computation program is listed and a sufficient condition for weak centers of order one is obtained. In section 5 we apply our method and program on two examples.

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2 Theory of Weak Centers

Let $V(x, y, \lambda)$ be a family of planar analytic vector fields parametrized by $\lambda \in R^n$ with a *nondegenerate center* at the origin, i.e., the vector field does not have a double eigenvalue zero at the origin. $P(r, \lambda)$ denotes the *minimum period* of the closed orbit passing $(r, 0)$, a point in a sufficiently small open interval $J = (-\alpha, \alpha)$ of x-axis.

Definition 2.1 . Let $F(r, \lambda_*) = P(r, \lambda_*) - P(0, \lambda_*)$. The origin is called a *weak center of finite order* k if

$$F(0, \lambda_*) = F'(0, \lambda_*) = ... = F^{(2k+1)}(0, \lambda_*) = 0 \quad \text{and} \quad F^{(2k+2)}(0, \lambda_*) \neq 0, \tag{2.3}$$

where the derivatives indicated are taken with respect to the first argument of the function F. The origin is called an *isochronous center*, i.e., all closed orbits surrounding the origin have the same period, if $F^{(k)}(0, \lambda_*) = 0, \forall k \geq 0$.

Definition 2.2 . *Local critical period* is a period corresponding to a critical point of the period function which bifurcates from a weak center.

Lemma 2.1 (Period Coefficient Lemma)[15]. *If $P(0, \lambda) = 2\pi, \forall \lambda \in R^n$, then for any given $\lambda_* \in R^n$,*

$$P(r, \lambda) = 2\pi + \sum_{k=2}^{\infty} p_k(\lambda) r^k, \tag{2.4}$$

which is analytic for $|r|$ and $|\lambda - \lambda_*|$ sufficiently small. Moreover, $p_k \in R[\lambda_1, ..., \lambda_n]$, the noetherian ring of polynomials; for $k \geq 1, p_{2k+1} \in (p_2, p_4, ..., p_{2k})$, the ideal generated by $p_{2i}, i = 1, ..., k$ over $R[\lambda_1, ..., \lambda_n]$; the first $k > 1$ such that $p_k(\lambda) \neq 0$ is even.

Definition 2.3 . For $\lambda_* \in V(p_2, p_4, ..., p_{2k}) := \{\lambda | p_{2i}(\lambda) = 0, \quad i = 1, ..., k\}$ and $p_{2k+2}(\lambda_*) \neq 0$, the period coefficients $p_2, p_4, ..., p_{2k}$ of F are said to be *independent with respect to p_{2k+2} at λ_** when the following conditions are satisfied: Every neighborhood of λ_* contains a point λ such that $p_{2k}(\lambda) \cdot p_{2k+2}(\lambda) < 0$; for any $j = 1, ..., (k-1)$, if $\lambda \in V(p_2, p_4, ..., p_{2j})$ and $p_{2j+2}(\lambda) \neq 0$ then every neighborhood of λ contains a point $\sigma \in V(p_2, p_4, ..., p_{2j-2})$ such that $p_{2j}(\sigma) \cdot p_{2j+2}(\lambda) < 0$.

Theorem 2.1 (Finite Order Bifurcation Theorem)[15]. *¿From weak centers of finite order k at parameter λ_* no more than k local critical periods bifurcate. Moreover, there are perturbations with exactly j critical periods for any $j \leq k$, if the coefficients $p_2, p_4, ..., p_{2k}$ of F are independent with respect to p_{2k+2} at λ_**

3 Algorithm for Reversible Cubic Systems

According to [17], a planar vector field is said to be *reversible* if it is symmetric with respect to a line. Consider the cubic system (1.2) which is symmetric with respect to y-axis. Clearly the right hands of (1.2), $X(x,y)$ and $Y(x,y)$, satisfy that $X(-x,y) = X(x,y), \quad Y(-x,y) = -Y(x,y)$, that is, only terms with x of even degree in $X(x,y)$ and that of odd degree in $Y(x,y)$ are retained. Such a reversible cubic system is therefore shown as

$$\begin{cases} \dot{x} = & -y + a_1x^2 + a_2y^2 + a_3x^2y + a_4y^3 \\ \dot{y} = & x + b_1xy + b_2x^3 + b_3xy^2 \end{cases} \tag{3.5}$$

with parameter $\lambda = (a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in R^7$. This symmetry ensures that Eq.(3.5) has a center at the origin, cf. [4].

Taking polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{3.6}$$

we have

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta = r^2G_2(\theta) + r^3G_3(\theta), \tag{3.7}$$

$$\dot{\theta} = (\dot{y}\cos\theta - \dot{x}\sin\theta)/r = 1 + rH_1(\theta) + r^2H_2(\theta), \tag{3.8}$$

where

$$\begin{aligned} G_2(\theta) &= a_1\cos^3\theta + (a_2+b_1)\sin^2\theta\cos\theta, \\ G_3(\theta) &= (a_3+b_2)\cos^3\theta\sin\theta + (a_4+b_3)\cos\theta\sin^3\theta, \\ H_1(\theta) &= (b_1-a_1)\cos^2\theta\sin\theta - a_2\sin^3\theta, \\ H_2(\theta) &= (b_3-a_3)\cos^2\theta\sin^2\theta + b_2\cos^4\theta - a_4\sin^4\theta. \end{aligned}$$

Thus

$$\frac{dr}{d\theta} = \frac{r^2G_2(\theta) + r^3G_3(\theta)}{1 + rH_1(\theta) + r^2H_2(\theta)}. \tag{3.9}$$

Lemma 3.1 . *The vector field defined by Eq.(3.9) is analytic and*

$$\frac{dr}{d\theta} = r^2G_2 + \sum_{k=3}^{\infty} r^k(G_2A_{k-2} + G_3A_{k-3}) \tag{3.10}$$

in a sufficiently small neighborhood of $r = 0$, where

$$A_0 = 1, \quad A_1 = -H_1, \quad A_k = -H_2A_{k-2} - H_1A_{k-1}, \quad \forall k \geq 3. \tag{3.11}$$

Proof. For all θ the functions $H_1(\theta)$ and $H_2(\theta)$ are uniformly bounded. Thus for sufficiently small r,

$$\frac{1}{1+rH_1+r^2H_2} = \sum_{k=0}^{\infty} r^k A_k \tag{3.12}$$

is analytic. By comparison of coefficients we obtain a recursive relation (3.11), which determines all A_k. The remainder of this proof is simple. $\square$

Consider the solution of (3.9) with $r(0,\lambda) = r_0 > 0$ in the form

$$r(\theta,\lambda) = \sum_{k=1}^{\infty} u_k(\theta,\lambda) r_0^k. \tag{3.13}$$

Then the initial condition implies

$$u_1(0,\lambda) = 1, \quad u_k(0,\lambda) = 0, \quad \forall k > 1, \lambda \in R^n. \tag{3.14}$$

Replacing r in (3.10) with the series (3.13) and comparing coefficients of $r_0^k, k = 1, 2, ...,$ we get the following differential equations

$$\begin{array}{ll} u_1' & = 0, \\ u_2' & = G_2 u_1^2, \\ u_3' & = (G_2 A_1 + G_3) u_1^3 + 2u_1 u_2 G_2, \\ \cdots\cdots & \cdots\cdots \end{array} \tag{3.15}$$

successively, where u_k' denotes $\frac{d}{d\theta} u_k(\theta,\lambda)$. Under the initial conditions (3.14) we can obtain their solutions

$$\begin{array}{ll} u_1(\theta) & = 1, \\ u_2(\theta) & = \int_0^\theta G_2(\xi)\, d\xi, \\ u_3(\theta) & = \int_0^\theta (G_3 + G_2(2u_2 - H_1))\, d\xi, \\ \cdots\cdots & \cdots\cdots \end{array} \tag{3.16}$$

one by one. That is what we often do for weak focuses.

Finally, we compute the period $P(r_0,\lambda)$ of the closed orbit $C(r_0)$ through $(r_0, 0)$. From (3.8) and (3.12) we have

$$\begin{aligned} P(r_0,\lambda) &= \int_{C(r_0)} dt = \int_0^{2\pi} \frac{1}{1+rH_1+r^2H_2}\, d\theta \\ &= \int_0^{2\pi} (1 + \sum_{k=1}^{\infty} r^k A_k)\, d\theta \\ &= 2\pi + \int_0^{2\pi} \sum_{k=1}^{\infty} r^k A_k\, d\theta, \end{aligned} \tag{3.17}$$

meanwhile, from (3.13) we obtain the following power series expansion

$$\begin{aligned} \sum_{k=1}^{\infty} r^k A_k &= A_1 u_1 r_0 + (A_1 u_2 + A_2 u_1^2) r_0^2 + (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3) r_0^3 \\ &+ (A_1 u_4 + A_2(u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4) r_0^4 + ... \end{aligned} \tag{3.18}$$

Therefore

$$P(r_0, \lambda) = 2\pi + \sum_{k=1}^{\infty} p_k(\lambda) r_0^k, \tag{3.19}$$

where

$$\begin{array}{ll} p_1(\lambda) & = \int_0^{2\pi} A_1 u_1 \, d\theta = -\int_0^{2\pi} H_1(\theta)\, d\theta = 0, \\ p_2(\lambda) & = \int_0^{2\pi} (A_1 u_2 + A_2 u_1^2)\, d\theta, \\ p_3(\lambda) & = \int_0^{2\pi} (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3)\, d\theta, \\ p_4(\lambda) & = \int_0^{2\pi} (A_1 u_4 + A_2(u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4)\, d\theta, \\ & \ldots\ldots \quad \ldots\ldots \end{array} \tag{3.20}$$

and $A_k, u_k, k = 1, 2, ...$ are determined by (3.11) and (3.16).

By Lemma 2.1 and Definition 2.1 we have proved the following.

Theorem 3.1 . *If for a certain $\lambda_* \in R^7$ there exists an integer $k \geq 1$ such that*

$$p_2(\lambda_*) = p_3 = (\lambda_*) = ... = p_{2k+1}(\lambda_*) = 0 \quad \textit{and} \quad p_{2k+2}(\lambda_*) \neq 0, \tag{3.21}$$

then the origin is a weak center of order k. Otherwise, the origin is an isochronous center.

4 Computation of the Period Coefficients p_k

The following is a Maple program for computing the period coefficients p_n of the reversible cubic system (3.5).

```
#################### The main procedure
####################
pols:=proc(n)
    local A,u,H1,H2,i,j,p,e:
    H1:=(b1-a1)*cos(t)^2*sin(t)-a2*sin(t)^3:
    H2:=(b3-a3)*cos(t)^2*sin(t)^2+b2*cos(t)^4-a4*sin(t)^4:
    A.(-1):=0: A.0:=1:
    for i to n do
 A.i:=-H2*A.(i-2)-H1*A.(i-1)
    od:
    G2:=a1*cos(t)^3+(a2+b1)*sin(t)^2*cos(t):
    G3:=(a3+b2)*cos(t)^3*sin(t)+(a4+b3)*cos(t)*sin(t)^3:
    u1:=1:
    for i from 2 to n do
 u.i:=Int(du(i),t): ":=value("):
```

```
 e:=subs(t=0,"):
 u.i:=normal(""-e):
    od:
    Int(collect(coe(n),sin(t)),t=0..2*Pi):":=value("):
    p.m:=":
end:

###### Subprocedure for computing the coefficients of ######
###### the power series (3.18)
#########################
coe:=proc(n)
    local u,f,r,cd,ccd:
    f:=0:
    for i to n do
 f:=f+u.i*r^i:
    od:
    ccd:=0:
    for j to n do
 cd[j]:=f^j/(n!):
 for i to n do
     cd[j]:=diff(cd[j],r):
 od:
 f:=f-u.(n-j+1)*r^(n-j+1):
 ccd:=ccd+expand(subs(r=0,cd[j])*A.j):
    od:
end:

###### Subprocedure for differential equations (3.15) ######
du:=proc(n)
    local u,f,r,cd,ccd:
    f:=0:
    for k to n-1 do
 f:=f+u.k*r^k:
    od:
    ccd:=0:
    for j from 2 to n do
 cd[j]:=f^j/(n!):
 for k to n do
     cd[j]:=diff(cd[j],r):
 od:
 f:=f-u.(n-j+1)*r^(n-j+1):
 ccd:=ccd+expand(subs(r=0,cd[j])*(G2*A.(j-2)+G3*A.(j-3))):
    od:
end:
```

By running this program on a Pentium/75 PC we have obtained polynomials $p_2, p_4, \cdots, p_{10}$ defined by (3.20). The result is as follows.

$$p_2 = \frac{\pi}{12}(4\,a_1{}^2 + 3\,a_3 + b_1{}^2 - 5\,a_1\,b_1 - 3\,b_3 - a_2\,b_1 - 9\,b_2 + 10\,a_1\,a_2 + 9\,a_4 + 10\,a_2{}^2),$$

$$p_4 = \frac{\pi}{1152}(-468\,a_1\,a_2\,b_2 - 510\,a_1\,a_3\,b_1 - 444\,a_1\,a_2\,b_3 + 513\,b_2{}^2 + \cdots),$$

$$\cdots\cdots$$

Following table shows the total degree and number of terms of period coefficients $p_2, p_4, \cdots, p_{10}$.

	p_2	p_4	p_6	p_8	p_{10}
total degree	2	4	6	8	10
number of terms	10	49	168	462	1092

In particular, we obtain the following theorem.

Theorem 4.1 . *The origin is a weak centre of order one of the reversible cubic system (3.5) if and only if* $q_2 = 0$, $q_4 \neq 0$, *where*

$$\begin{aligned}
q_2 &= 4\,a_1{}^2 + 3\,a_3 + b_1{}^2 - 5\,a_1\,b_1 - 3\,b_3 - a_2\,b_1 - 9\,b_2 + 10\,a_1\,a_2 + 9\,a_4 + 10\,a_2{}^2,\\
q_4 &= 130\,a_2{}^3a_1 + 51\,a_1{}^2b_2 + 3\,a_2{}^2b_2 - 78\,a_1{}^3a_2 + 39\,a_1{}^3b_1 + 3\,a_1{}^2a_3 + 12\,a_1{}^2a_4\\
&\quad -36\,a_1{}^2a_2{}^2 + 33\,a_2{}^2a_3 + 282\,a_2{}^2a_4 + 42\,a_2{}^3b_1 - 12\,a_1{}^2b_1{}^2 + 3\,a_2{}^2b_1{}^2\\
&\quad +18\,a_3\,a_4 - 3\,b_1{}^2b_2 + a_1\,b_1{}^3 + 3\,a_4\,b_1{}^2 + 18\,a_3\,b_2 + 81\,a_4{}^2 - 28\,a_1{}^4\\
&\quad +120\,a_2{}^4 + 27\,b_2{}^2 + 246\,a_1\,a_4\,a_2 + 48\,a_1{}^2b_1\,a_2 + 36\,a_1\,a_3\,a_2\\
&\quad -6\,a_1\,b_1{}^2a_2 - 3\,a_3\,b_1\,a_2 + 69\,a_4\,b_1\,a_2 + 54\,a_2\,b_2\,a_1\\
&\quad +51\,a_2{}^2b_1\,a_1 - 3\,a_3\,b_1\,a_1 + 39\,a_4\,b_1\,a_1 + 6\,a_1\,b_2\,b_1
\end{aligned}$$

Proof. $q_2 = \frac{12}{\pi}p_2$ and $q_4 = \frac{1152}{\pi}\mathrm{prem}(p_4, p_2, b_3)$, where *prem* is a *Maple* function[18] reducing p_4 modulo p_2 by substitution of the variable b_3. Then the result is given directly by Theorem 3.1. □

5 Analysis of Certain Reversible Cubic Systems

In this section we analyze the weak centres and local bifurcation of critical periods in two reversible cubic systems. First we consider a system in the following form

$$S_\lambda : \left\{ \begin{array}{rl} \dot{x} = & -y - ax^2 + ay^2 + a_3x^2y + a_4y^3 \\ \dot{y} = & x - 2axy + b_2x^3 + b_3xy^2, \end{array} \right. \tag{5.22}$$

where $\lambda = (a, a_3, a_4, b_2, b_3) \in R^5$ denotes the bifurcation parameters. Let

$$\begin{aligned} S_I &= \{\lambda \in R^5 | a_3 = b_3 = -3a_4 = -3b_2\}, \\ S_{II} &= \{\lambda \in R^5 | a_3 = b_3, a_4 = b_2 = 0\}, \\ S_{III} &= R^5 \setminus S_I \setminus S_{II}. \end{aligned}$$

Definition 5.1 *$(S_\lambda)_{\lambda \in R^5}$ has a center of type I (respctively II, III) if the system (5.22) is nonlinear and $\lambda \in S_I$ (respctively S_{II}, S_{III}).*

We prove the following theorem.

Theorem 5.1 *Consider the system (5.22).*

1. A centre of type I is an isochrone point.

2. A centre of type II is either a weak centre of order at least 13 or an isochrone point.

3. A weak centre of type III has order at most four. For any such centre of order $k \leq 4$ and each $j \leq k$, there exist perturbations with exactly j critical periods.

Proof . 1. For a centre of type I the corresponding system (5.22) has the following form

$$\left\{ \begin{aligned} \dot{x} = & \ -y - ax^2 + ay^2 - 3bx^2y + by^3 \\ \dot{y} = & \ x - 2axy + bx^3 - 3bxy^2. \end{aligned} \right. \tag{5.23}$$

It satisfies Cauchy-Riemann conditions, i.e., it can also be written into

$$\begin{aligned} dt &= \frac{dz}{iz - az^2 + ibz^3} \\ &= dz \left(\frac{-i}{z} + O(z) \right), \end{aligned}$$

where $z = x + i\,y$. By the residue theorem the period is constant.

2. For $\lambda_* \in S_{II}$, using the program in previous section we have obtained that

$$p_2(\lambda_*) = p_4(\lambda_*) = \cdots = p_{26}(\lambda_*) = 0,$$

thus the centre has order at least 13. This evidence suggests strongly that the corresponding system is an isochrone, although the further computation will be more complicated.

3. For $\lambda_* \in S_{III}$ we compute the period coefficients. Each coefficient, denoted by q_{2k}, is reduced modulo the ideal of the previous coefficients.

$$\begin{array}{rcl}
q_2 & = & -b_3 + a_3 - 3\, b_2 + 3\, a_4, \\
q_4 & = & 2\,(b_2 + a_4)a_3 + 9\, {a_4}^2 + 3\, {b_2}^2, \\
q_6 & = & (15\, a_4 + 8\, a^2)\, {b_2}^3 + (150\, {a_4}^2 - 28\, a_4\, a^2)\, {b_2}^2 \\
 & & +\,(15\, {a_4}^3 + 32\, {a_4}^2 a^2)\, b_2 + 12\, a^2 {a_4}^3, \\
q_8 & = & -132890625\, {a_4}^7 - 9071831250\, a^2 {a_4}^6 + 15239407500\, a^4 {a_4}^5 \\
 & & +2404621800\, a^6 {a_4}^4 + 724745440\, a^8 {a_4}^3 - 17954688\, a^{10} {a_4}^2 \\
 & & -1544704\, a^{12} a_4 + 184320\, a^{14},
\end{array}$$

$$\begin{array}{rcl}
q_{10} & = & -19676131453125000\, {a_4}^{11} - 1653958596228750000\, a^2 {a_4}^{10} \\
 & & +2652999510044671875\, a^4 {a_4}^9 + 873772899636161250\, a^6 {a_4}^8 \\
 & & +48411776215735312 5\, a^8 {a_4}^7 + 49987526399075100\, a^{10} {a_4}^6 \\
 & & -3703103053643420\, a^{12} {a_4}^5 - 900204917868864\, a^{14} {a_4}^4 \\
 & & +159307105283584\, a^{16} {a_4}^3 + 1690694788608\, a^{18} {a_4}^2 \\
 & & -1155822981120\, a^{20} a_4 + 62108467200\, a^{22},
\end{array}$$

where a constant factor π in each formula is omitted for convenience. This implies immediately that

(1): $\Lambda_1 := \{q_2 = 0, q_4 \neq 0\} \neq \emptyset$ and if $\lambda \in \Lambda_1$ the origin is a weak centre of order one;

(2): $\Lambda_2 := \{q_2 = q_4 = 0, q_6 \neq 0\} \neq \emptyset$ and if $\lambda \in \Lambda_2$ the origin is a weak centre of order two;

(3): $\Lambda_3 := \{q_2 = q_4 = q_6 = 0, q_8 \neq 0\} \neq \emptyset$ and if $\lambda \in \Lambda_3$ the origin is a weak centre of order three;

(4): If $q_2 = q_4 = q_6 = q_8 = 0$ then $q_{10} \neq 0$ and the origin is a weak centre of order four.

Theorem 5.1 is proved. $\square$

The other system we discuss is the reversible cubic system with homogenuous nonlinearities of the third degree, i.e.,

$$\left\{\begin{array}{rl}
\dot{x} = & -y + a_3 x^2 y + a_4 y^3 \\
\dot{y} = & x + b_2 x^3 + b_3 x y^2,
\end{array}\right. \tag{5.24}$$

where $\lambda = (a_3, a_4, b_2, b_3) \in R^4$ denotes the bifurcation parameters. It is just a case considered by Rousseau and Toni[16] when

$$a_3 = -(e_6 - 3e_4), b_3 = (e_6 - 3e_4), a_4 = -(e_4 - e_5), b_2 = (e_4 + e_5), \tag{5.25}$$

for some real numbers e_4, e_5 and e_6. Those e_j's come from what Rousseau and Toni[16] used.

Theorem 5.2 *The system (5.24) has a weak center of order at most one. For any such center of order one there exist perturbations with exactly one critical period.*

Proof. Using the same program as before and taking the substitution (5.25) and $a_1 = a_2 = b_1 = 0$, we compute the period coefficients

$$p_2 = -\frac{\pi e_6}{2},$$

$$p_4 = \pi\{\frac{3\,e_4^2}{2} + \frac{3\,e_5^2}{2} - \frac{e_6\,e_5}{4} - e_6\,e_4 + \frac{9\,e_6^2}{32}\},$$

and

$$q_4 = \pi\{\frac{3\,e_4^2}{8} + \frac{3\,e_5^2}{8}\},$$

obtained from p_4 modulo the ideal of p_2. Obviously, if $p_2 = 0$, i.e., $e_6 = 0$, then $q_4 \neq 0$ and certainly $p_4 \neq 0$ except for that $e_4 = e_5 = 0$, which implies the system (5.24) is a trivial one without nonlinearities. Therefore, the origin is a weak center of order at most one. The proof of the theorem is completed. □

Clearly the equation (5.24) with restriction (5.25) is a reversible Sibirskii's form with a center of Rousseau and Toni's type I, defined in [16]. Our results is identical with Rousseau and Toni's in Theorem 3.3 of [16].

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