Geometric Ornament in Art and Architecture of Western Cultures

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Abstract

In this paper we will examine the role of geometry in selected examples of art and architecture from selected regions of the Western World. We will start from the Byzantine Empire. In the beginning we will briefly look at the role of geometry in Byzantine architecture and architectural decorations. Then we will explore other types of architectural decorations where geometry was used while creating these decorations. We will examine selected Cosmati designs in Roman architecture and then we will show the geometry behind the Gothic tracery (part 2 of this paper). Most of the designs discussed in this paper are based on circular and interlaced shapes. We will show how these designs were created. We will use Geometer's Sketchpad to recreate them.

Introduction

For most of us the word 'art' is a synonym of painting, sculpture and sometimes calligraphy. We consider also music as a form of art. For an average person art has nothing in common with mathematics or even geometry. However, if we look into the textbooks of history then we will find that ancient Greeks considered art and mathematics as tightly connected disciplines. There were many artists who have been inspired by mathematics and studied mathematics as a mean of complementing their works. The Greek sculptor Polykleitos recommended a series of mathematical proportions for carving the ideal male nude. Renaissance painters turned to mathematics and many of them became accomplished mathematicians themselves. We can find mathematics in creations of the middle century Islamic artists as well as in works of Gothic masons.

A glimpse into geometry in Byzantine architecture

Mathematics, in particular geometry, always played a major role in architecture. In early civilizations the tombs of leaders had shapes derived from a prism with a square base or half sphere. A real sophistication of geometric forms in architecture can be found in ancient Chinese, Indian or Greek architecture. Let us start our journey from Byzantine Constantinople and an example of mathematically sophisticated architecture. Byzantine Constantinople was for many centuries a capital of the Roman Empire[1], i.e. a significant place in Western civilization. Some temples build at this period of time in Constantinople achieved level of geometric complexity not known before. Let us examine one such example.

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[1] The Byzantine and Roman Empires have also quite an interlaced history. For centuries it was practically the same country. Depending on the location of the capital – Rome or Constantinople – we talk about the Roman Empire or the Byzantine Empire. Some historians prefer to use the names Western Roman Empire and Eastern Roman Empire. A separation of these two empires started around 330 (some historians point to the year 395) when Constantine I created the so called Second Rome on the site of Byzantium. Later the name changed to Constantinople and now it is Istanbul.

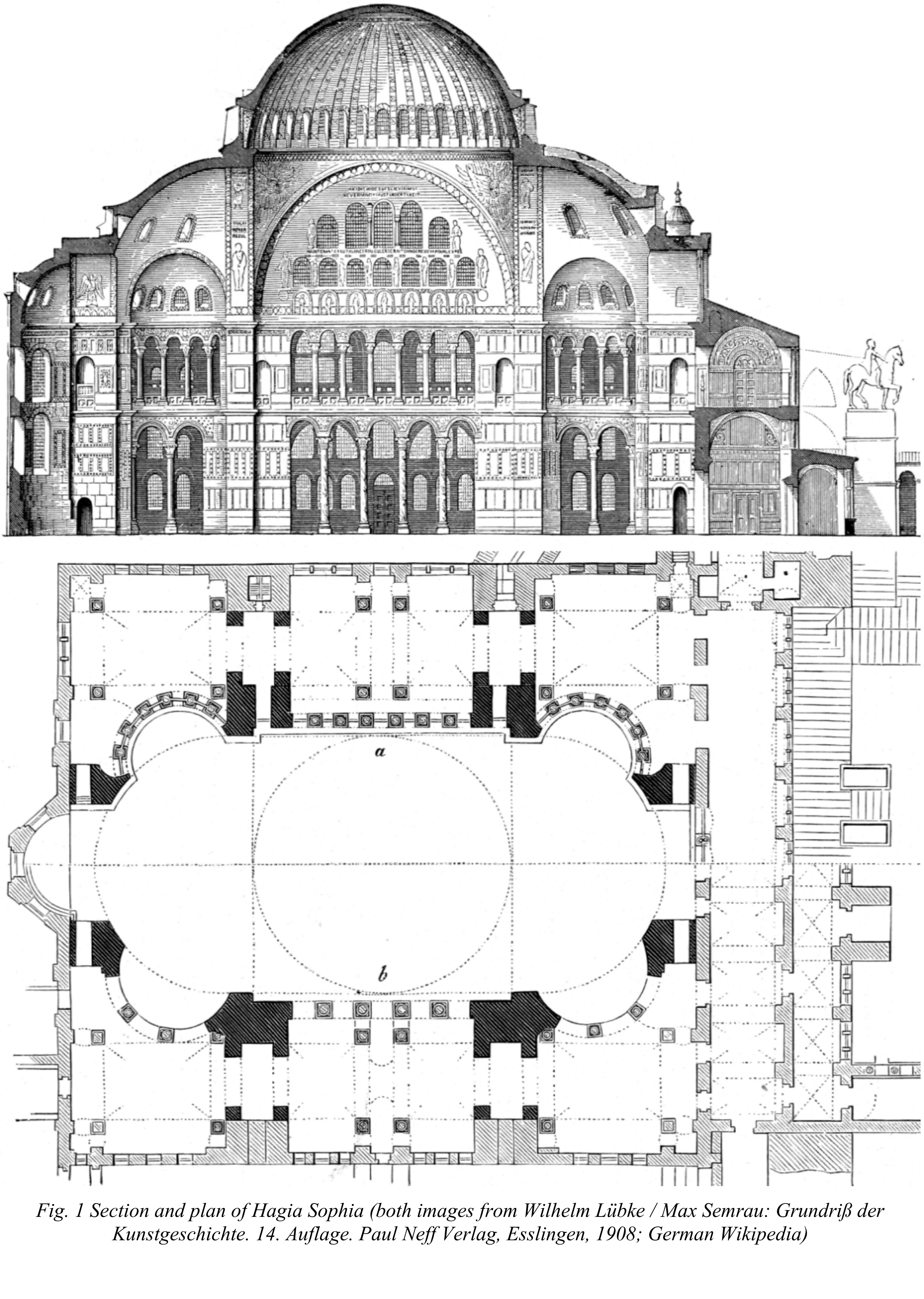


Fig. 1 Section and plan of Hagia Sophia (both images from Wilhelm Lübke / Max Semrau: Grundriß der Kunstgeschichte. 14. Auflage. Paul Neff Verlag, Esslingen, 1908; German Wikipedia)

Hagia Sophia as an example of mathematically complex architecture

The current building of Hagia Sophia was commissioned in February 532 by Emperor Justinian and constructed by two architects, the physicist *Isidore of Miletus* and the mathematician *Anthemius of Tralles*. Hagia Sophia is a construction where many geometric forms were used. We can look at it from a plane geometry point of view, examine the plan and section of the building and see what plane geometry figures are here and how they are combined. We can look at Hagia Sophia from the 3D geometry point of view and then we discover the 3D shapes represented in this building. There is no doubt that Hagia Sophia is one of the first buildings in the world with such a complex mathematical structure (see [4]). Later architects of many other churches and mosques would try to surpass this complexity – sometimes with good results, see for example the constructions of some Gothic cathedrals.

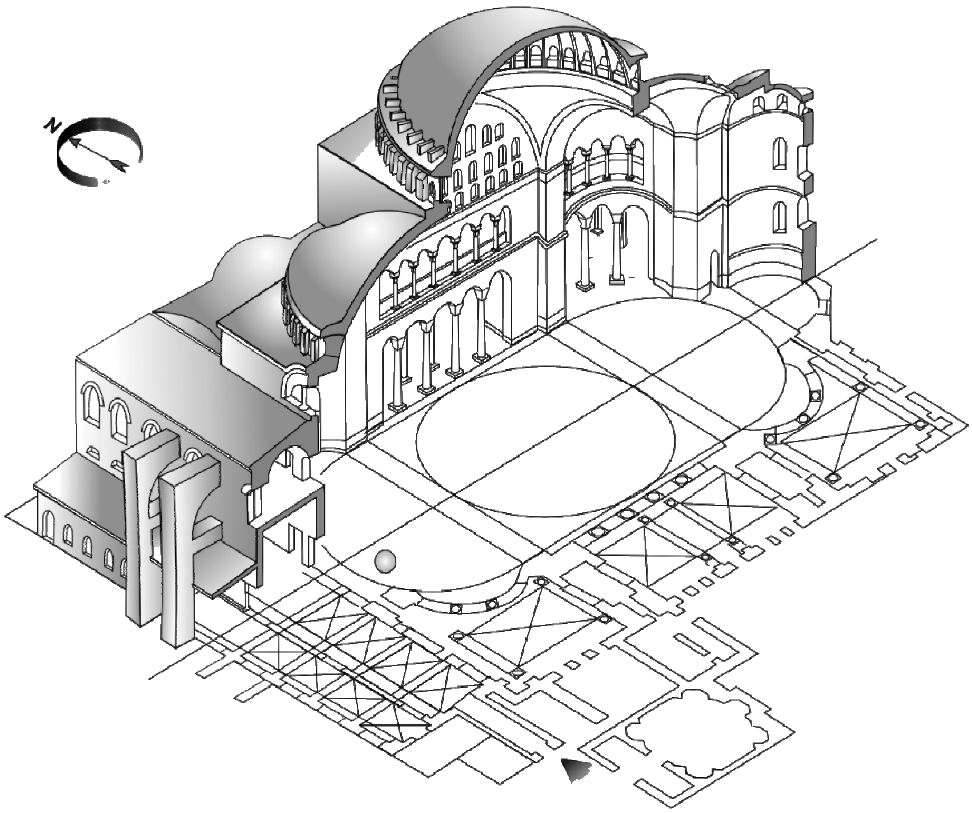


Fig. 2 Segment of Hagia Sophia shows all geometric solids used in this construction (image from Wikimedia Commons)

Let us see briefly what we can decipher from the plan, section and segment of Hagia Sophia shown in figures 1 and 2.

The central part of Hagia Sophia is a cubical shell obtained by removing the interior of the cube and parts of its walls. On top of it we have a flat cube in which two cylindrical shapes were removed. Finally on top of this element we have a half sphere from which the interior was removed. In a similar way the two parts adjacent to the central nave were constructed. Again we have here two halves of cubes and two quarters of a sphere. Overall, Hagia Sophia is a very complex and at the same time visually very clean assembly of geometric objects. There is also hidden geometry that a normal person may not see even after careful examination of the building. Locations and sizes of particular elements, e.g. entrances, columns, niches, arches, were established using Euclid’s mean and extreme ratio property (see [4]). Sizes of major part of the basilica were established using the side-and-diagonal numbers.

The construction of Hagia Sophia was copied many times and adapted both in the Islamic as well as in Western architecture including all geometric principles used in her construction.

The main objective of this paper is to investigate geometric patterns in the Western architecture. Therefore let us start from the patterns seen in Hagia Sophia and other Byzantine churches. First of all we have there a number of simple geometric mosaics that later were used as a source of inspiration for Islamic art. These are usually very regular patterns built out of small polygons painted on walls or assembled using small ceramic or stone tiles. The most frequently used shapes are squares, equilateral triangles and rhombi with angles 60° and 120°.

Images shown in the figure 3 are examples that we frequently see in Byzantine, Roman and Islamic decorations. Their specific feature is geometry of segments and simple convex polygons: equilateral triangles with one angle equal to 90°, squares and rhombi. Very frequent elements in these ornaments are fractal-like figures where one shape is built out of shapes similar to it, these shapes are again constructions of smaller and still similar shapes. The figure 3 shows a number of patterns that are a final approximation of the well-known in our times Sierpiński triangle.

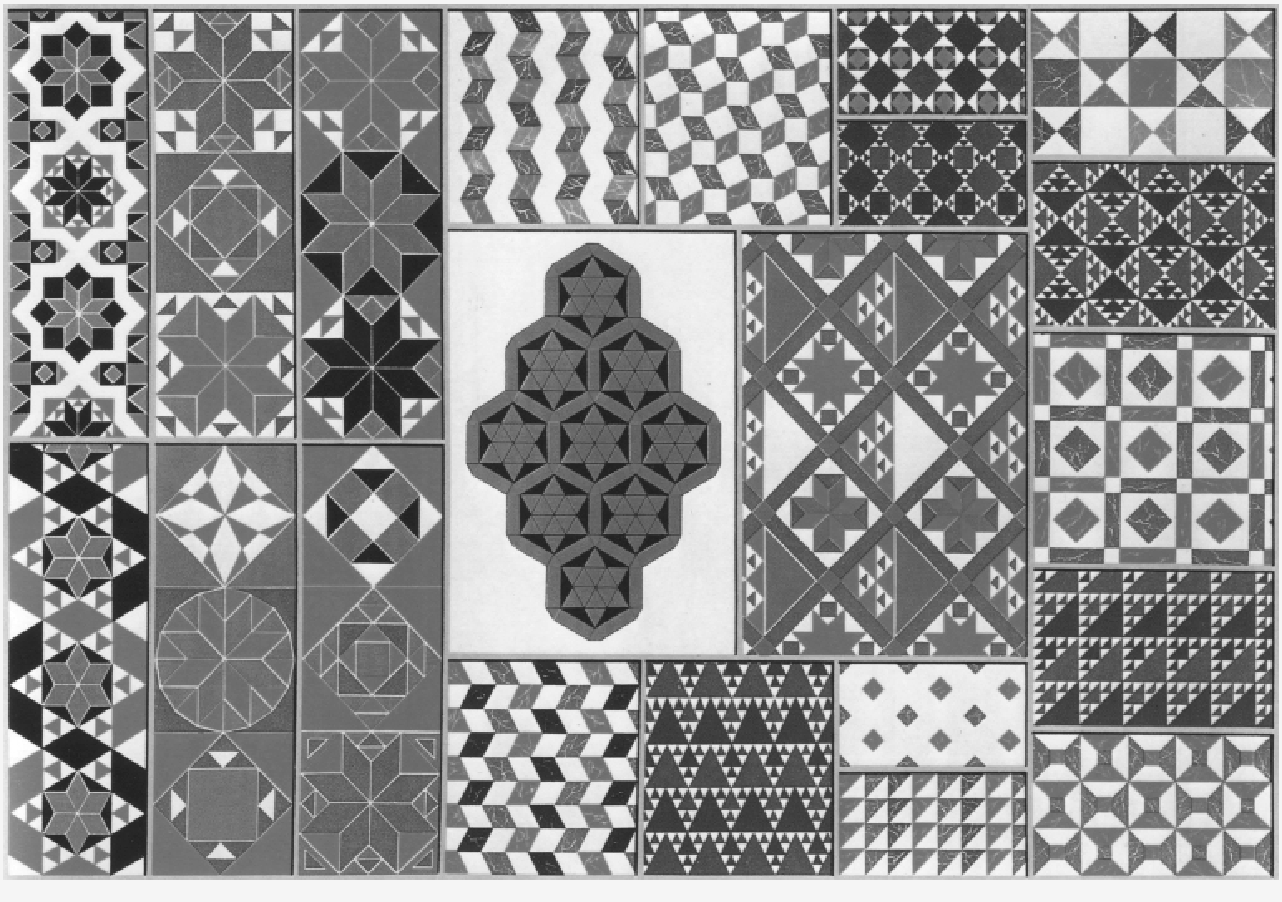


Fig. 3 Byzantine and Roman geometric ornaments (page from the book Le Ornement Polychrome, by A. Racinet, 1888)

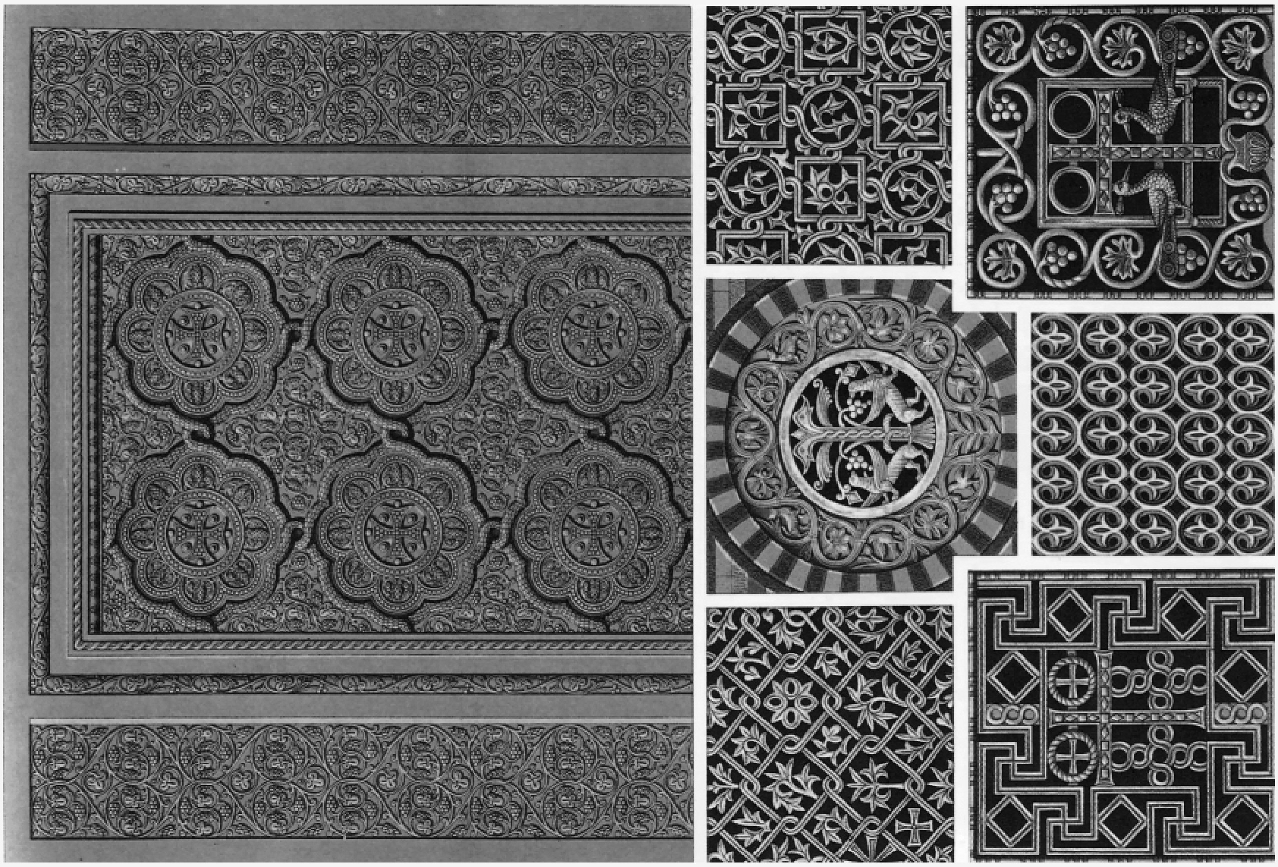


Fig. 4 Another set of ornaments specific for the Byzantine architectural decoration (page from the book Le Ornement Polychrome, by A. Racinet, 1888)

More complex and different in character ornaments are shown in the next figure (fig. 4). Ornaments shown here are not based on segment geometry and simple polygonal figures. All of them use circular shapes, often interlaced and decorated with floral forms. Their geometry can be very complex with multiple levels of overlapping curves. Here we frequently see various interconnected spiral shapes. This is interesting to notice that later these ornaments occur frequently in Islamic design in Central Asia – Samarkand (famous Afrasiab panels in the Samanids' palace in Samarkand, IX – X century), Bukhara, as well as in later Roman and Gothic architectural decorations.

Our first task in this paper will be to see how one can create some of the ornaments based on circular shapes. The floral decorations we will leave to the reader as a manual exercise.

Simple circular ornament from Hagia Sophia

Most of the ornaments in Hagia Sophia were destroyed by centuries of unpleasant events – fires, earthquakes, wars, etc. Some of them are not accessible to a visitor. In one of the very old books on history and architecture (see [3]) we can find the ornament shown in the next figure (fig. 5).

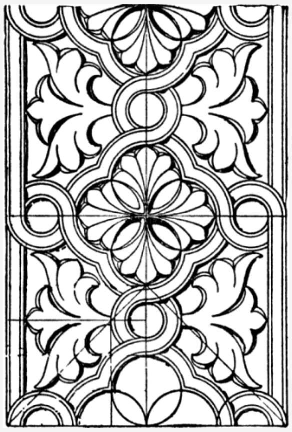


Fig. 5 Geometric ornament from the Hagia Sophia

The picture shows one tile of a larger ornament. We can easily see that the structure of this ornament is based on the rectangular grid of circles with the same radius, say R. In further calculations and constructions we will use R as a base unit for this ornament.

Four of these circles are located on the bottom edge of the pattern. In fact the width of the tile is equal to $4R$. Then we have a sequence of identical circles along the vertical symmetry line. We can easily prove that the height of the tile shown in the picture is equal to $6R$. Small circles shown here have radius close to $R/2$.

We will start our construction by creating a rectangular grid and then we will draw all necessary circles.

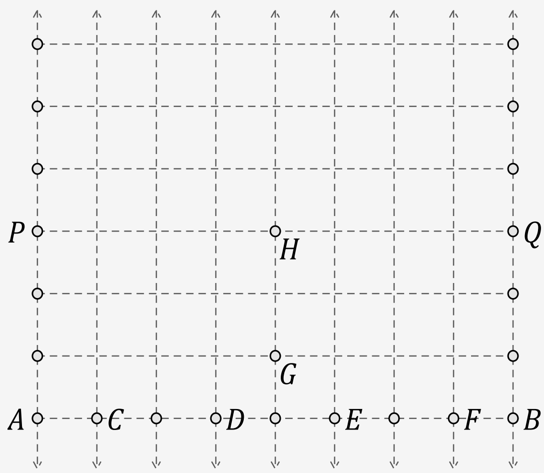
One could simplify this work by constructing only the left bottom part of the tile and then using reflections about its edge construct a larger part of the ornament.

Finally we should notice that the pattern presented here is a part of a border. It is linear pattern and can be expanded only in two directions (up and down). If we remove the vertical bars on its margin, we will be able to create an ornament covering a larger fragment of the plane.

EXAMPLE 1 CONSTRUCTION OF A BYZANTINE ORNAMENT FROM HAGIA SOPHIA

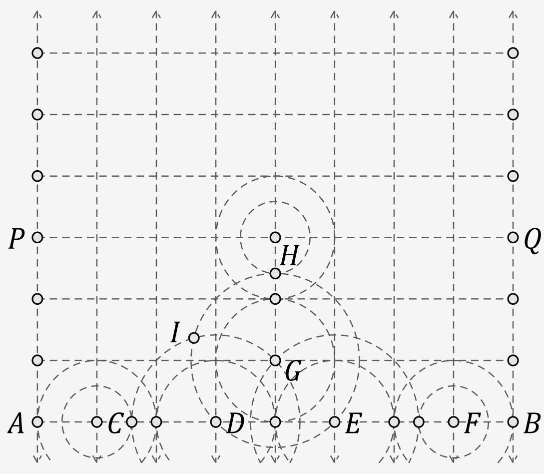
STEP 1. SQUARE GRID

Start by creating segment AB and dividing it into 8 equal parts. The length of each part will be considered as R. Now, construct a square grid with 6 rows of cells up from the segment AB. We will use vertices of these cells as centers for the circles in this construction. In the figure to the right some of these future centers of circles are already labeled as C, D, E, F, G, and H.



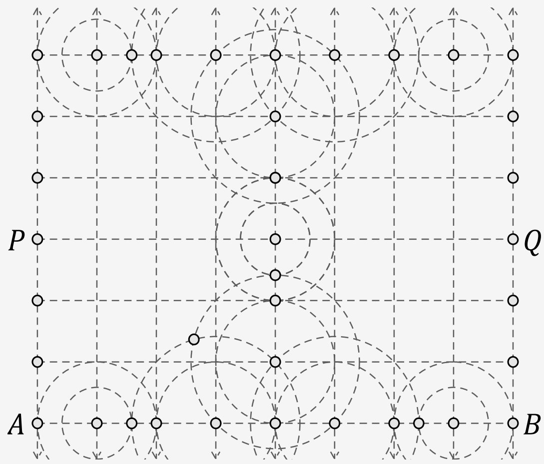
STEP 2. BOTTOM PART OF THE CIRCULAR GRID

Draw six circles with centers in C, D, E, F, G, and H. Each of them should have the same radius R equal to the length of segment AC=AB/8. Draw three large circles with radius DG and centers D, E and G respectively. Finally draw three small circles with centers in C, F and H. Note – the radius of large circles is equal to R√2, and the radius of a small circle is r = R(2-√2)= 0.5857R.



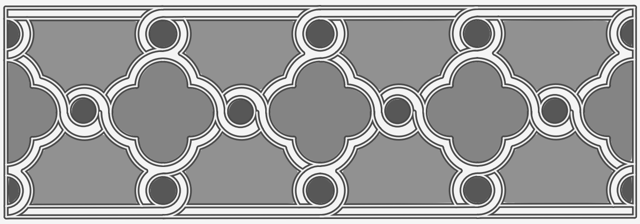
STEP 3. COMPLETE CIRCULAR GRID

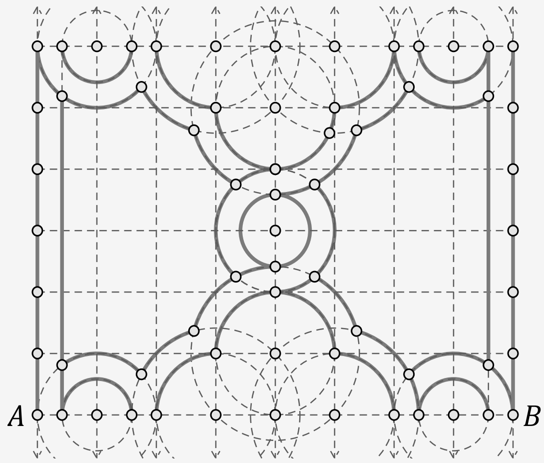
Construct identical pattern of circles in the top part of the grid. This pattern is a mirror reflection about the segment PQ of the pattern created in step 2.
The circular grid for the Hagia Sophia pattern is ready.
Now we can draw outlines of the ornament. In the enclosed picture (bottom-right) we used a thick red line to show them.



STEP 4. OUTLINES OF THE REPEAT UNIT FOR THE HAGIA SOPHIA ORNAMENT

The four vertical lines are boundaries of the two sides of the pattern. This way the final pattern will be linear. If we ignore them then we can create pattern covering large part of the plane.
Figure below shows one of the possible versions of this ornament.





Ornament from Hagia Sophia was a starting point for many ornaments both in the western as well as in Islamic counties. A very beautiful collection of such ornaments can be found on the mentioned earlier Afasiab panels in Samarkand. The next figure shows a more complex circular ornament from Orhan Cami museum. Formerly it was Hagia Sophia church (1065) in Byzantine Nicaea (currently Iznik) – one of many Byzantine churches in Turkey.

Fig. 6 Ornament from the floor in Hagia Sophia in Byzantine Nicaea (now Iznik in Turkey)

This image was created by removing perspective from a standard photograph of the floor. Therefore, the top part of the image is slightly out of focus and there are minor distortions in the ring of circles.

In order to recover the geometry of this pattern we may need precise measurements or make some rough suppositions. However, we can easily see that this pattern was obtained by wrapping Hagia Sophia ornament around a large disk.

Cosmatesque, quincunx and guilloche

The first time I saw a cosmatesque pattern I was sure that this is a type of art close to Islamic art. After short analysis a of a few cosmatesque patterns I realized that this is completely the other end of geometric constructions. Let us examine a few examples and see how we can reconstruct some of them.

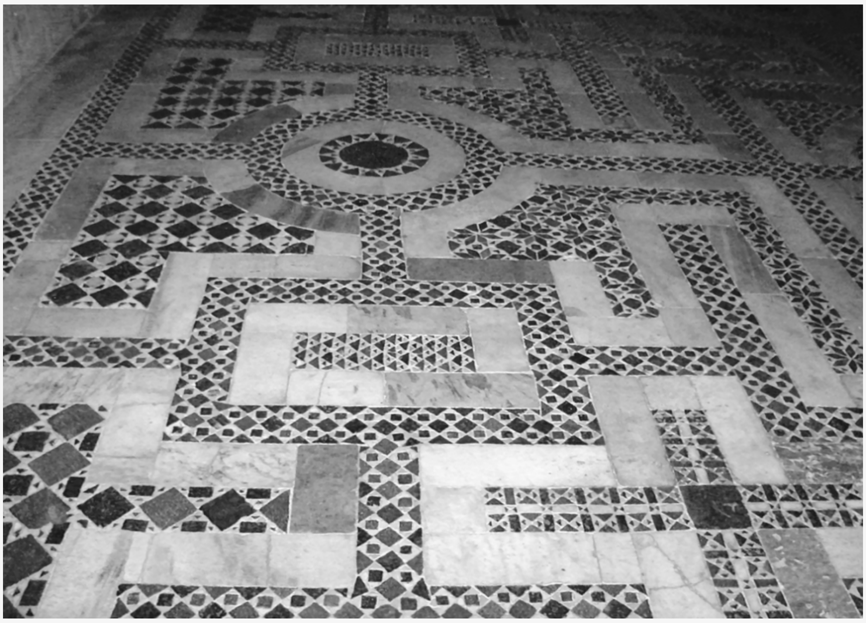


Fig. 7 Pavement in cosmatesque style usually contains two types of elements – the main pattern marked using a wide polygonal line formed out of segments, circles and arches; and a geometric mosaic filling the spaces between lines.

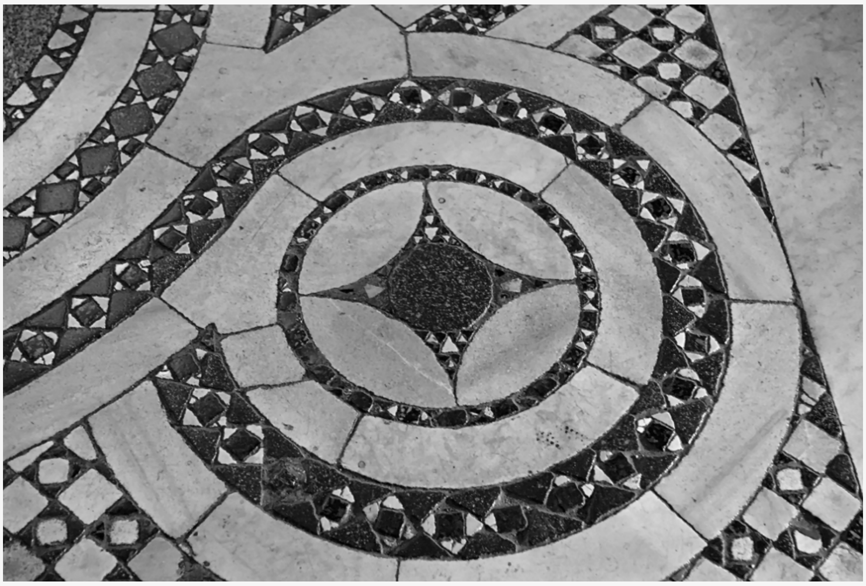


Fig. 8 Cosmatesque pavement from basilica Santa Maria Maggiore in Rome.

Here we see a small fragment of a pattern known as guilloche. This pattern was very popular in Medieval Italy in XII and XIII century.

Fot. Wikimedia Commons

First of all we will see some similarity to the Byzantine ornaments shown in figures 3 and 4. Exactly speaking we see there a combination of patterns shown in figure 3 with circular patterns from figure 4. The circular patterns are the focal points of a cosmatesque and the polygonal patterns are used to fill up empty spaces inside and between circular shapes. This combination makes very colorful mosaics with incredibly strong geometry.

Fig. 9 Another example of a pavement in cosmatesque style. The photograph was done in the cathedral church in Civita Castellana in Italy.

Photograph © Judith Moran and Kim Williams, used with permission of Kim Williams.

The name cosmatesque comes from the Cosmati family of stone masters that for a few generations (1190 to 1300) was involved in architecture, carving and decorative art in Rome. Their art is referred to as *opus sectile* where materials were cut in pieces and inlayed into pavements and walls. Works of the Cosmati family can be found mainly in Italy. There are very few examples outside of Italy with the most famous one in the Collegiate Church of St Peter at Westminster, known also as Westminster Abbey, which is a large Gothic church, in the City of Westminster (now part of London), located just to the west of the Palace of Westminster.

From a geometry point of view one can compare the cosmatesque with Islamic art. We can easily notice some important differences in geometry of these forms of art.

In both cases we deal with geometric constructions. In Islamic art we have precise geometric constructions using segments and sometimes circular shapes. The segment is a base element of the construction. In cosmatesque as well as in Byzantine art the base elements are triangles, rectangles, squares, discs, and rhombi. These elements are used to assemble the whole ornament. Although in Islamic art in the Maghreb we also have such elements, these are a side effect of the geometric constructions of complicated constellations of stars and rosettes. Simply, if we have a star or rosette pattern we have to fill the space between the lines. In a cosmatesque we create a skeleton of the ornament, then we inlay in the empty spaces some triangular or rectangular blocks filling spaces between them, then in such spaces we inlay smaller elements, filling again some empty space, and so on. This way we can create fractal-like patterns. We see there objects and relations that a few centuries later were discovered by Mandelbrot, Sierpiński and others – fractals, fractal symmetry and self-similarity.

In a cosmatesque we have also stars and simple rosettes. But these shapes are side effects only of filling in an organized way an empty space with simple blocks. These objects are not constructed at all; these objects were just assembled from simple building elements.

Geometry of cosmatesque ornaments

Although cosmatesque art had quite serious symbolic meaning in this paper we will concentrate only on its geometry.

In cosmatesque ornaments we have three major types of geometric ornaments – guilloche, a pattern that usually is a linear combination of interlaced circular shapes (fig. 9); quincunx – a pattern with five points organized in a form of a cross (fig. 12 and 13); and reasonably uniform mosaics (fig. 10).

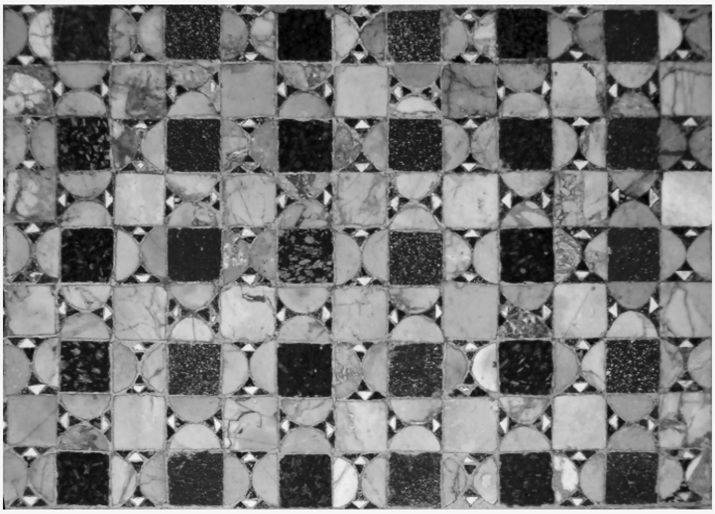


Fig. 10 Example of a typical mosaic in cosmatesque art

The chessboard of squares was created first, and then in the empty spaces between squares halves of circles were fitted, finally small triangles were fitted into the space between halves of circles.

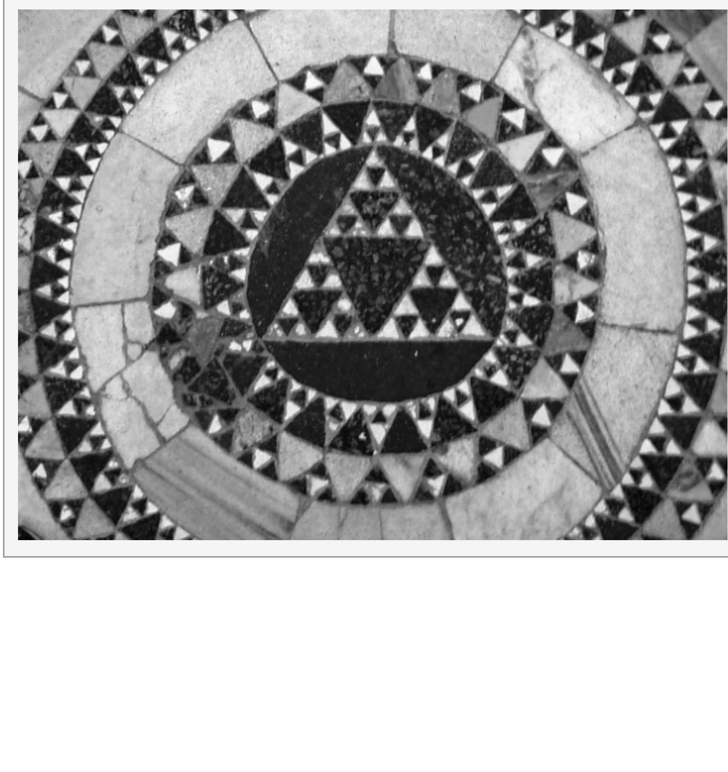


Fig. 11 Fragment of pavement from the cathedral Santa Maria Maggiore in Civita Castellana in Italy

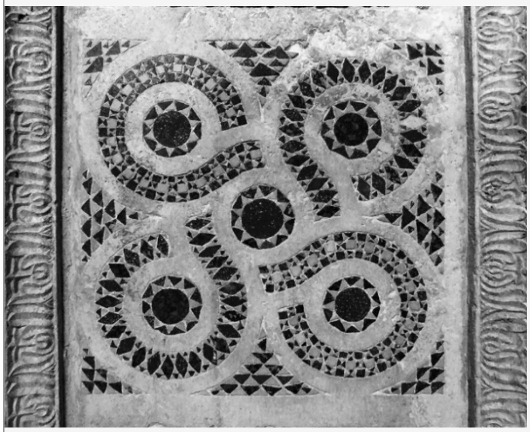
Here we see a shape that reminds us Sierpinski's triangle.

Foto © Judith Moran and Kim Williams, used here with permission of Kim Williams.

Geometry of quincunx

In churches and museums in Italy we can find a number of different versions of quincunx. We can easily notice that geometry of them is not as accurate as geometric constructions in Islamic art.

Usually we have there five interlaced circles inscribed in a square. However, the size of circles, the width of their "lines" can vary significantly. The two examples shown on fig. 12 and 13 show how it may look. The first pattern uses four circles inscribed into quarters of a square. The fifth circle fills the space between them. In the second example the size of the internal circle is much larger than size of other circles and it dominates the while design. Finding exact geometric rules to create these ornaments is a matter of examining measurements of thickness of lines and diameters of circles. Finally we have to consider the width of the bands of mosaics fitted into the whole construction.



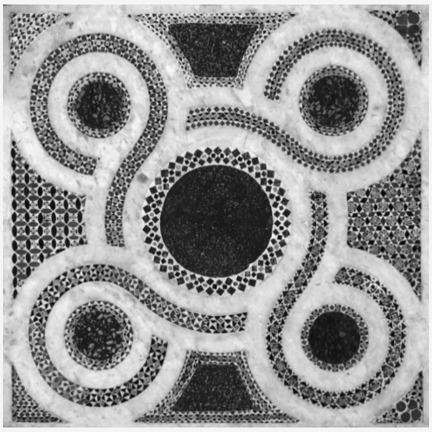


Fig. 12 and 13

Two examples of a quincunx – first (left) from Lugnano in Taverina, second (right) of unknown origin (fot. Wikimedia Commons)

Construction of quincunx from Lugnano in Teverina

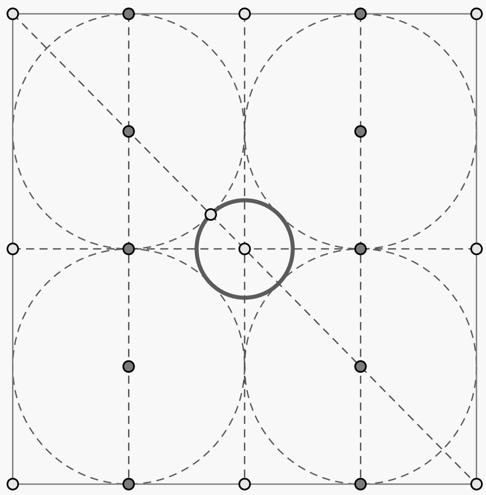
We will construct the quincunx shown in figure 12. We will split our construction into two parts: construction of the main pattern and constructions of mosaics filling up the space between lines of the main pattern.

STEP 1: Overall shape of the quincunx

Construct a square with a given side, and split it into four equal smaller squares.

*In each of these squares inscribe a circle. The radius of each circle will be **r=a/4**, where **a** is the length of the side of the large square.*

Finally draw a diagonal of the square and use it to construct the fifth circle inscribed in the space between the four circles.

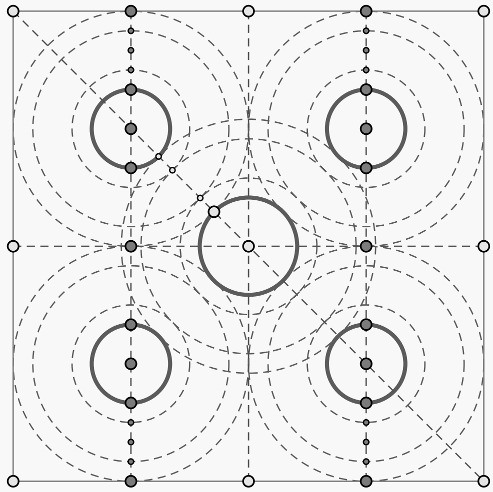


STEP 2: Shapes of main circles forming quincunx

Split radius of each large circle into six equal parts.

For each of the large circles draw three smaller circles concentric with it and with radii equal to ***2r/6*** *(thick line),* ***r/2*** *and* ***5r/6*** *(dashed lines).*

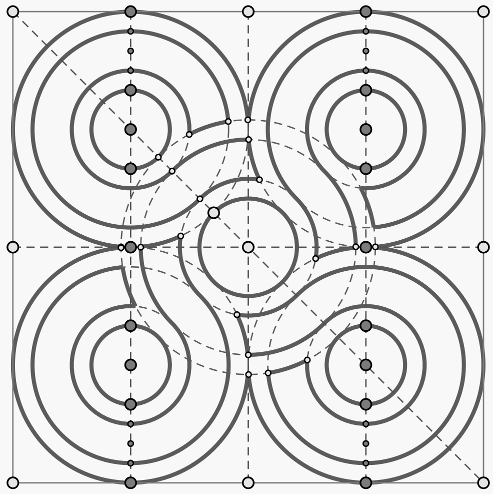
Use points of intersection of the diagonal line and the new circles to construct in the middle of the square three new circles as it was shown in the figure.



STEP 3: Construction of the main pattern

Use the points where the circles intersect in order to obtain the main pattern. Note, the existing grid of circles gives us a few possibilities. Therefore, in reality we can construct a few different versions of the quincunx. Pattern created in the picture to the right is identical with the one shown in figure 11.

Here we have four right-handed entanglements (starting from the center of the pattern). The pattern shown in figure 12 has four left-handed entanglements (again starting from the center of the quincunx).

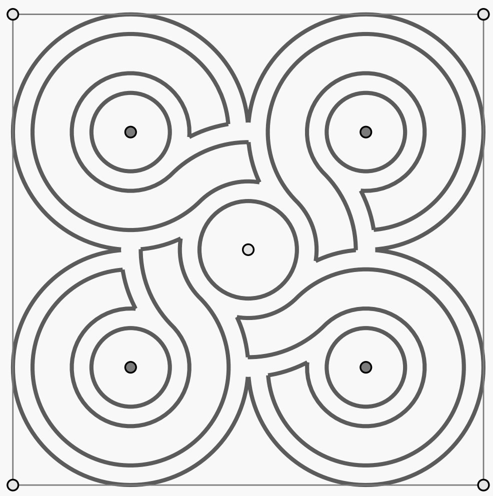


STEP 4: Final shape of the main pattern

Now we can remove all dashed lines leaving only thick lines and centers of circles. The main pattern is ready.

In next steps we will have to fit mosaic patterns into some of the empty spaces

This can be a very precise construction or just a nice drawing.

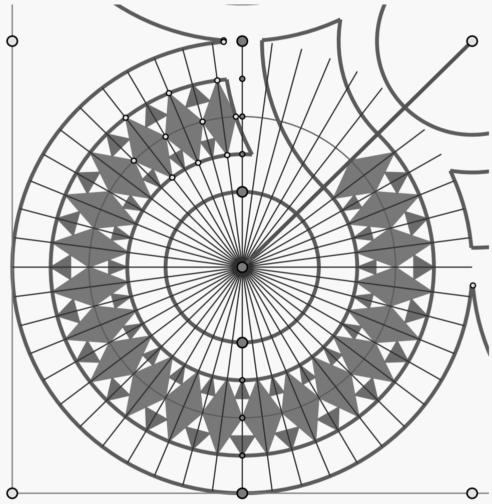


STEP 5: Mosaic filling large circles

Divide one of the four large circles into a large enough number of equal wedges. In my construction I use 48 equal wedges. A larger number of wedges will give us a more detailed pattern, smaller number will give us a less detailed and more clear pattern.

In order to obtain a very regular mosaic it is worth to draw first segment connecting the center of the circle with the center of quincunx (center of the square) and then draw wedges starting from this segment.

After constructing wedges we use their points of intersection with circles to construct a family of rhombi and triangles.

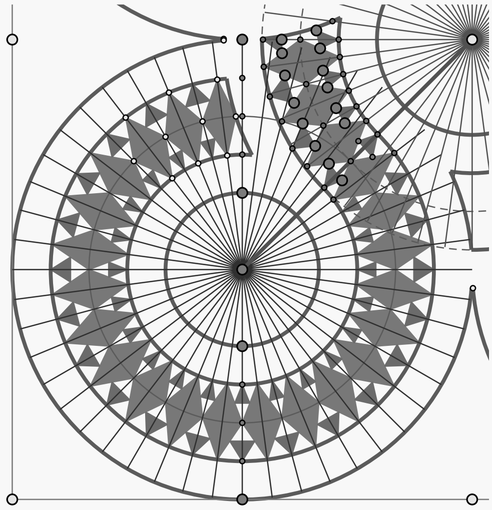


STEP 5a: Mosaic filling large circles (cont.)

Now do the same for the large circle in the center of quincunx – split it into wedges and fill them with rhombi and triangles as it was shown in the picture to the right. A difficult moment in this construction is the place where these two belts of rhombi meet.

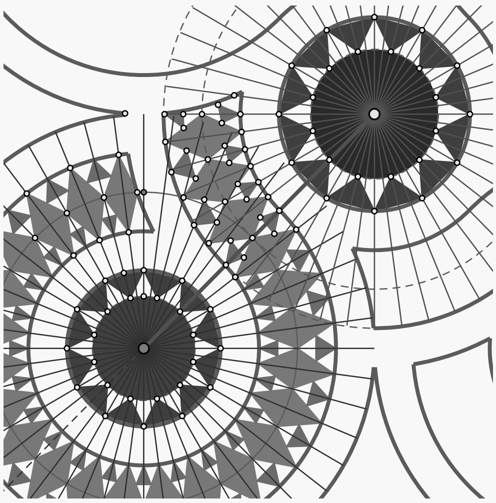
Repeat steps 5 and 5a for each of the remaining four large circles.

Construction shown here is one of the most popular in the Cosmati art.



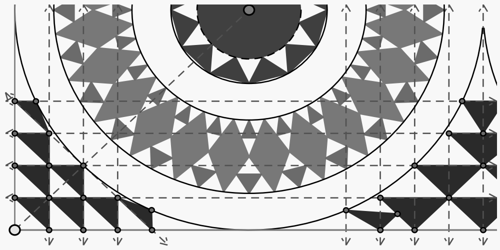
STEP 6: Stars filling centers of circles

For each of the five circles use the existing wedges to create two round belts of triangles. The internal one will form a star pattern and the external one a background for this star.



STEP 7: Mosaic filling the space between the main pattern and edges of the square.

The mosaic filling this part of the ornament can be constructed in many different ways. One of them is shown in the figure to the right.



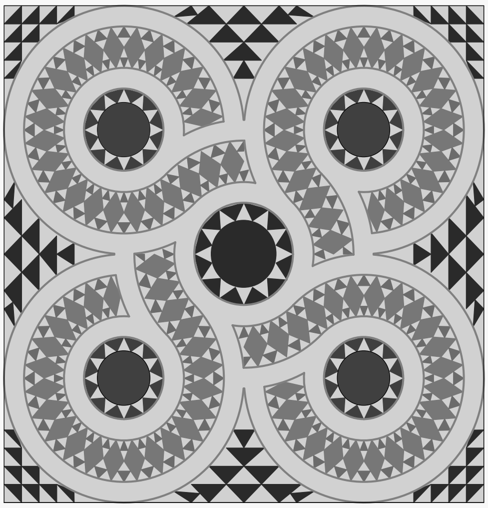


Fig. 14 Final outcome of the construction of quincunx from Lugnano in Taverina.

Geometry of a guilloche

Guilloche patterns can be reasonably simple or terribly complicated. The guilloche shown in figure 9 can be considered as a simple pattern. We have there a single path of interlaced rings forming a kind of marble carpet from the entrance of the church to the altar. The next figure (fig. 15) shows slightly more complex double guilloche with two rows of interlaced rings. Although this pattern is more complex than the previous one, we still can imagine a more complicated pattern with multiple rows of rings, e.g. 5x5 or even more. The way the rings will interlace can be also quite complex.

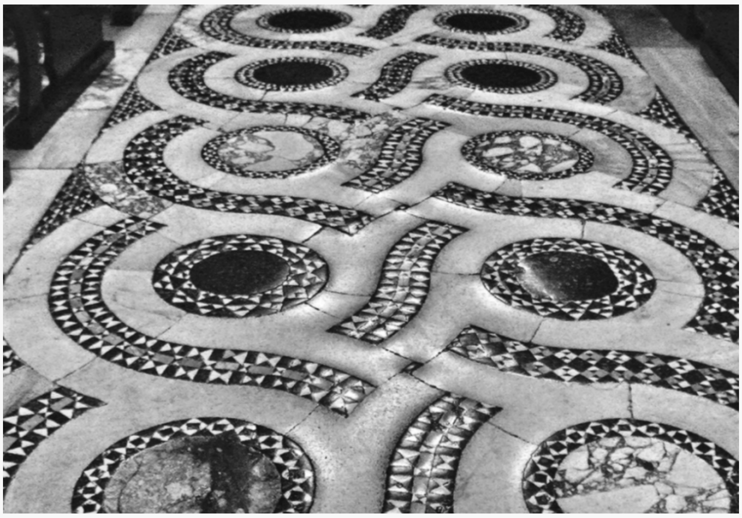


Fig. 15

Double guilloche from the cathedral of San Cesareo in Terracina in Italy

Construction of guilloche from the cathedral San Cesareo in Terracina

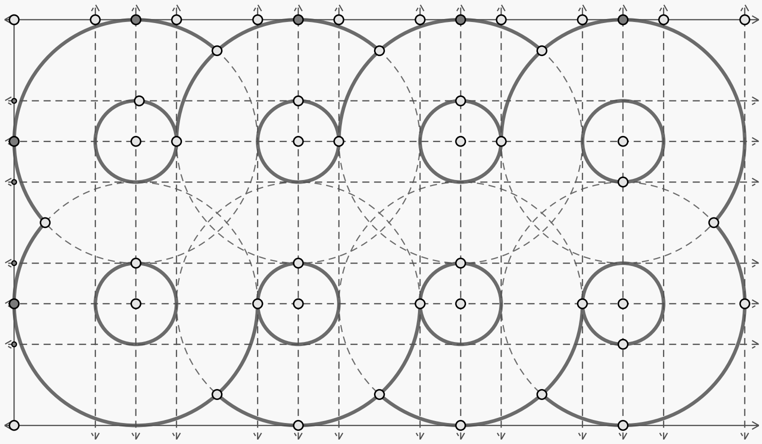
The construction of a double guilloche is similar to the construction of a quincunx. The most important task is to find out how the rings are interconnected. For the sake of simplicity of the construction we will construct a guilloche with four rings in a row and two rows only. Construction of a more complex guilloche will use the same techniques and a lot of work more.

STEP 1: Overall shape of guilloche

Draw a vertical segment, here the left edge, and split it into 5 equal parts. Parts 2 and 4 divide into two equal halves.

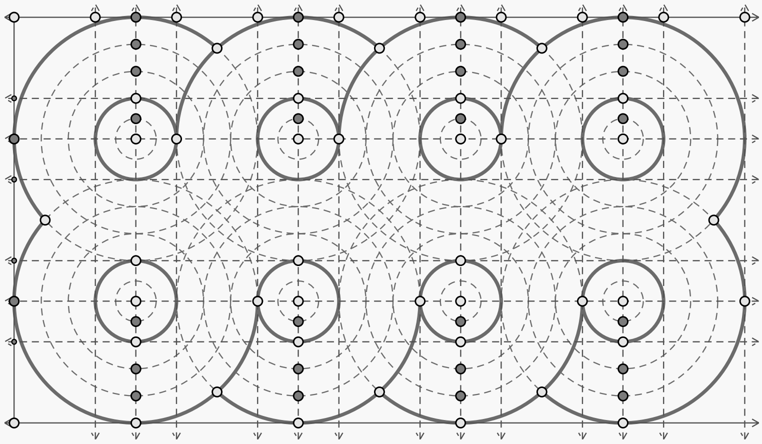
Now, from each point on the segment draw horizontal lines perpendicular to the segment.

In exactly the same way divide bottom and top horizontal lines. Draw vertical segments joining points of division. Use this grid to draw two rows of circles. We can have 4, 6 or more of them.



STEP 2: Grid of circles

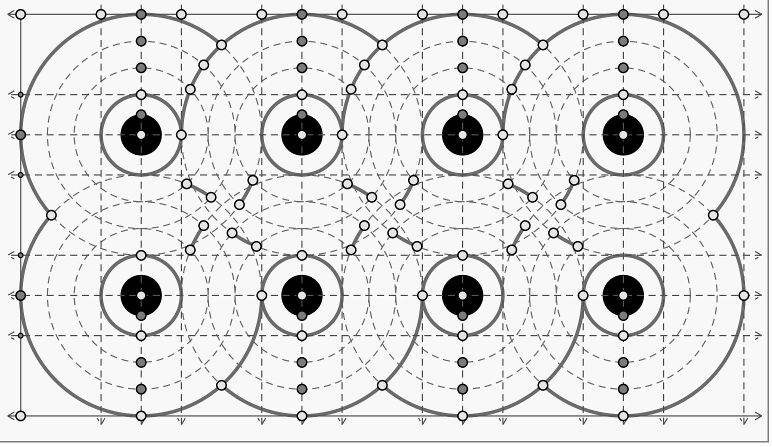
Divide radius of each circle into 5 parts as it was shown on here. Use these points of division to draw families of concentric circles. Each family should have 5 concentric circles.

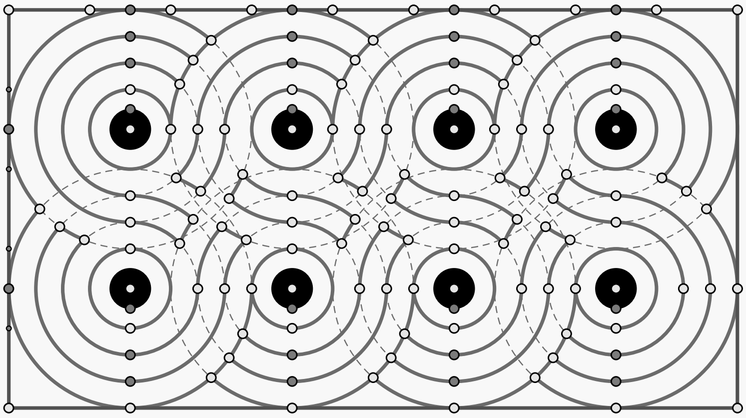
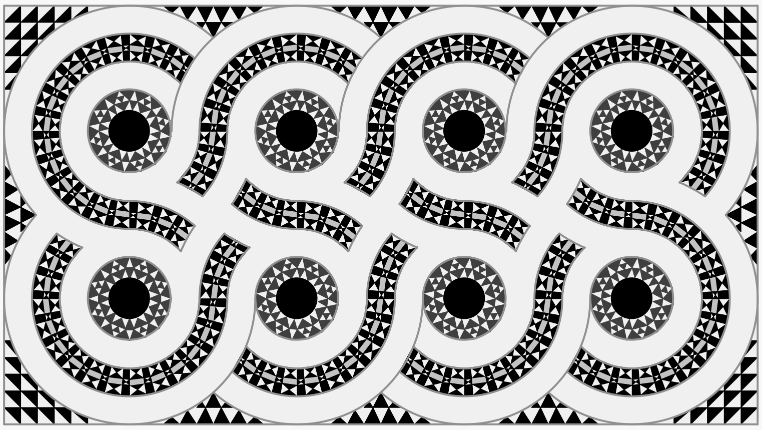


STEP 3: Marking starting points of internal belts

The figure next to this text shows points where internal belts of guilloche start and end.

Use thick lines to mark precisely short arches located between centers of circles. This is the most important step in this construction.



STEP 4: Construction of the main pattern *Starting from ends of short arches created in the previous step draw arches forming main pattern of guilloche.* *We should get pattern shown to the right.*	
STEP 4: Complete guilloche *The remaining part of the construction is similar to the construction of a mosaic filling empty spaces in quincunx.* *In my construction I created mosaic shown on the photograph (fig. 15).*	

Summary

Geometry and geometric constructions are the foundation not only of architecture but also architectural decorations. In most of them, including very old examples of buildings and architectural decorations, precise Euclidean geometry constructions were used. Foundations of these constructions are the same as those we learn from traditional textbooks of geometry, e.g. [2]. One can also apply to them some recently developed branches of geometry, e.g. tessellations and theory of symmetry. In fact architectural decorations gave a birth to some recent geometric theories – Penrose tiling theory or Ammann tiling theory.

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