Teaching and Learning Mathematical Modelling with Technology

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Abstract: *In the last few decades, there have been abundant discussions among mathematicians and mathematics educators on promoting mathematical modelling (a process of using mathematics to tackle real world problems) as a classroom practice. Mathematics educators and curriculum planners have been advocating the teaching of mathematical modelling in schools for some time now. Despite the consensus on its importance and relevance, mathematical modelling remains a difficult activity for both teachers and learners to fully engage in. In this paper, we examine some of these difficulties and discuss how technology can play a pivotal role in providing the essential support to make mathematical modelling a more accessible mathematical activity amongst students. Through a series of examples drawn from different fields and topics, we illustrate how a range of technological tools may be successfully and efficiently utilized in modelling tasks. In addition, we discuss the need for an optimal use of technology to balance between achieving the objectives of the tasks and attaining the goals of learning mathematics.*

1. Introduction

Mathematical modelling may be loosely defined as a process of representing real world problems in mathematical terms in an attempt to understand and find solutions to the problems. A mathematical model can be considered as a simplification or abstraction of a (complex) real world problem or situation into a mathematical form, thus converting the real world problem into a mathematical problem. The mathematical problem can then be solved using familiar mathematical techniques. The solution obtained is then interpreted and translated into real terms. Although there may be several interpretations of mathematical modelling, the process of mathematical modelling may be represented as a flow of events illustrated in Figure 1.1.

As depicted in the figure, we begin with a real world problem and we wish to find a real world solution to this problem. This may be difficult to achieve directly in the real world. We thus make an attempt to understand the problem, and then describe it in mathematical terms. At this stage, it is often necessary to identify the variables in the problem and construct relationships between or amongst these variables. Next, we develop a basic framework for the model. Here, assumptions about the model may need to be made to keep the problem tractable and simple so that we are able to solve the model using known methods.

Based on these assumptions, we construct a model, which could be a single equation, or a set of equations, or a set of rules or simply an algorithm governing how values of the variables may be found or assigned. This is the most crucial stage during which one would usually justify the formulation of the model based on the real physical meanings of the variables in the problem. Very often, model formulation is the most challenging stage for students (and teachers) as it requires fairly high order thinking, inter-disciplinary knowledge and modelling experience.

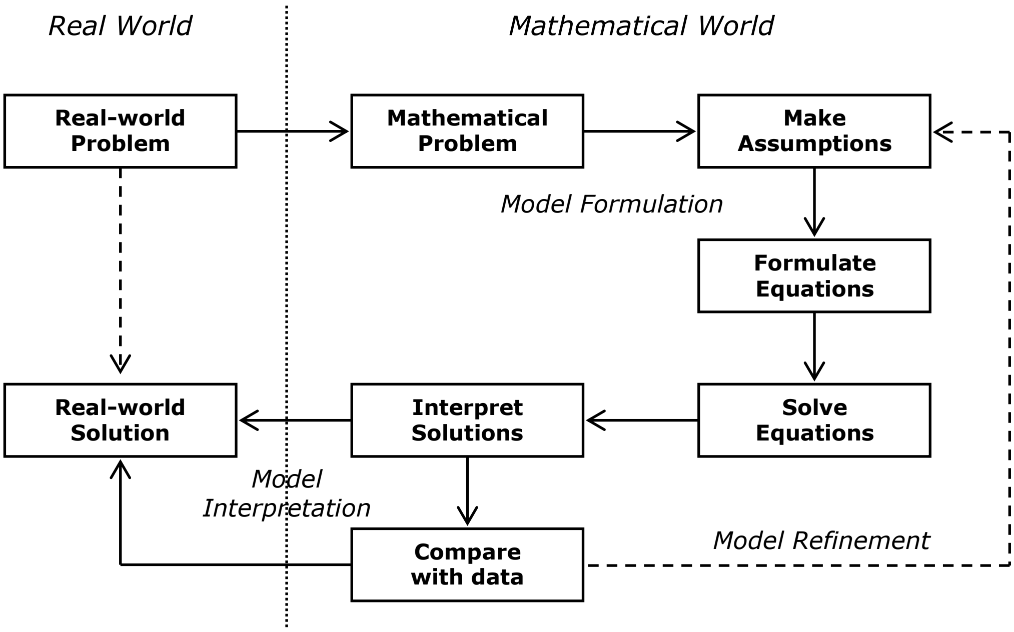


Figure 1.1: The modelling process (adapted from Ang, 2006b)

Once a model is constructed, the next stage requires the modeller to find ways to solve the model, using various mathematical techniques and tools. Very often, unless a model is particularly simple, some kind of technological or computing tool will be necessary. One also often finds that there can be a variety of ways of solving the same problem, making mathematical modelling a very enriching mathematical experience. We then interpret the results or solutions of the model in the context of the real world problem, and make attempts to compare the model solutions and any collected or known data. Sometimes, we wish to refine the model by revisiting and revising our assumptions.

2. Technology and Mathematical Modelling

Despite its importance and relevance to the real world, mathematical modelling is generally not the main approach to teaching and learning of mathematics in schools. For instance, although it has been proposed that mathematical modelling can be introduced to in Singapore schools (Ang, 2001), it was not until recently that such a suggestion was noticed by the local curriculum planners.

One reason could be the lack of readily available resources (lesson plans, modelling tasks, and so on) for the teacher, notwithstanding a recent attempt to develop resources for local teachers (Ang, 2009). Another is the teacher's lack of experience in mathematical modelling, leading to a lack of confidence and a general reluctance to embark on mathematical modelling in the classroom. At times, there is concern that students may not be mathematically ready for the tasks that teachers have painstakingly designed. As pointed out by Ang (2010), all these are stumbling blocks to an otherwise enriching and exciting approach to learning and teaching mathematics.

Technology may be the bridge for the cognitive gap that hinders a student from carrying out a modelling task. However, it should also be noted that technology should never replace the mathematics, much less the teacher; it should be viewed as a timely, and sometimes temporary, means of overcoming a difficulty.

The approaches to teaching mathematical modelling have been influenced by the development and introduction of technologies such as graphing calculators and computer software (Ferrucci and Carter, 2003). Many researchers and teachers have reported the successful use of technology in introducing mathematical ideas through exploration and investigation. For instance, the use of a spreadsheet to explore mathematical concepts has been discussed by Chua and Wu (2005) for a secondary classroom, and by Beare (1996) at college level. Ang and Awyong (1999) reported that the use of computer algebra systems such as Maple in some tertiary courses has been well received. Not surprisingly, the use of technology continues to prevail in the mathematics classroom at all levels.

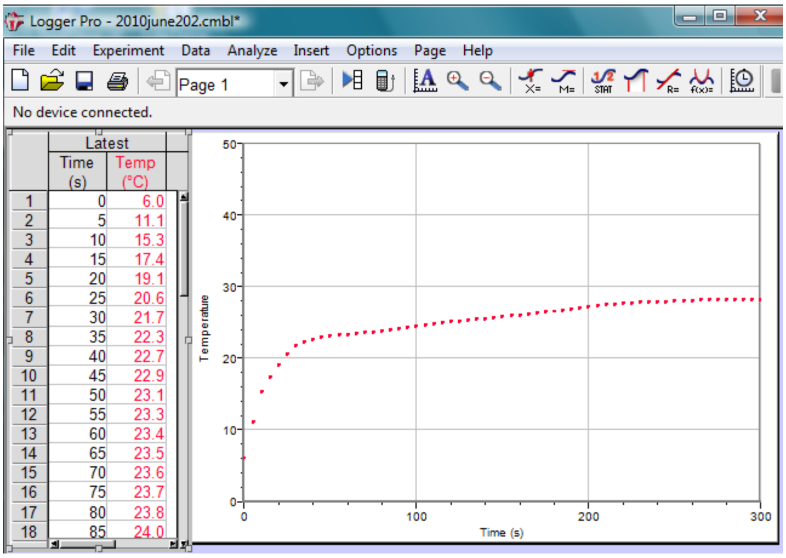
In the next section, we discuss four examples. In each example, we illustrate how technology can be successfully employed in the modelling task. In particular, the examples will show the role that technology can play in bridging gaps in mathematical knowledge or skills, without diminishing the student's mathematics learning experience by any large extent.

3. Examples

3.1 Warming Up

One simple technological tool that can be used for capturing data for a modelling task is the data logger and the accompanying software, LoggerPro 3 (Figure 3.1). For instance, a modelling task could involve students using the temperature probe to capture the temperature of a cup of ice water as it warms to room temperature. As shown in Figure 3.1(b), a typical data set shows the temperature of water collected at 5-second intervals.

(a) Data logger



(b) A screenshot of LoggerPro 3

Figure 3.1: Modelling the warming of ice water using a data logger and LoggerPro

The task could be to "develop" or "discover" a mathematical model to describe the process of warming, in this case, of ice water. Of course, if we are aware of Newton's law of cooling/warming, we can immediately apply the law. However, the point of the task is to take the student through the process of modelling.

Questions that one may ask students performing this task include:
(1) What are the factors (variables) that can influence of affect the temperature of the water?
(2) What happens near the beginning and near the end of the experiment?
(3) What assumptions do we need to make about the warming process?
(4) How quickly or slowly does the temperature change at different times?
(5) What can you say about the rate of change of the temperature? Write down a word equation that describes the rate of change.
(6) Write down a differential equation that describes how the temperature changes with time.

Notice that these questions can be handled by students at different cognitive levels. For instance, a pupil in the primary (elementary) school may be able to handle the more basic questions like (1), (2) and (3), whereas a secondary (high school) or Junior College (pre-University) school student should be able to tackle all the questions, including higher order ones like (4), (5) and (6).

From these questions, hopefully, students can be led to "discover" that the rate of warming (or cooling) of an object is directly proportional to the difference between its temperature (θ) and that of the surrounding (s). Representing this as a differential equation, we obtain

$$\frac{d\theta}{dt} = k(\theta - S) \tag{3.1}$$

where k is some constant to be estimated from the data.

3.2 Queues

In our everyday life, we often experience queuing systems. Some examples include a bank-teller service, drive-through service at a fast-food restaurant, printing jobs for a computer network, and so on. How can we introduce a queuing system as a modelling activity in a mathematics classroom?

In a simple single-server queuing model, the three components involved are arrival process, service process, and queue structure. A good example for students to think about is a queue at an automatic teller machine or ATM. The ATM can be considered as a machine that serves one customer at a time, on a "first in first out" basis. Customers arrive at the ATM randomly over time, wait their turn in line, and spend a random amount of time at the machine before leaving.

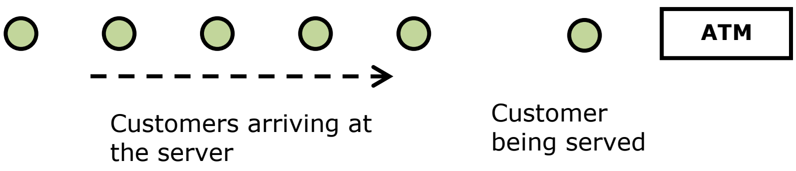


Figure 3.2: A single-server queuing system

The modelling task is to construct a model that can simulate such a queuing system. Like before, one can guide the student with questions like those in the previous example.

Here, two important variables are the arrival rate and service rate. Suppose we let a be the number of customers arriving per unit time, and b represent the number of customers served per unit time. Assuming that both customer arrivals and service are Poisson processes, then inter-arrival times and service times may be generated using an exponential distribution.

To generate random numbers from an exponential distribution with parameter a, we first generate a random number r from a uniform distribution in the interval $[0,1]$, (which can be easily achieved in a spreadsheet like Excel), and then compute

$$-\frac{1}{a}\log(r) \tag{3.2}$$

A simple simulation of this queuing model may be carried out on a spreadsheet such as MS Excel. Figure 3.3 shows a screenshot of a typical simulation run. Pressing the "Calculate Now" button, F9, generates another run.

	A	B	C	D	E	F	G	H	I
1		Average arrival rate				Average service rate			
2		a =	1.00			b =	1.50		
3		inter-Arrival time	Arrival time	Service time	Finish time	Total time spent	Total Wait time	Average Total time Spent	Average Wait time
4	n	iAT	AT	ST	FT	Total	Wait	1.47	0.60
5	1	0.80	0.80	0.31	1.11	0.31	0.00		
6	2	0.04	0.84	1.27	2.38	1.54	0.27		
7	3	0.41	1.25	0.23	2.61	1.36	1.13		
8	4	1.33	2.58	0.05	2.66	0.08	0.03		
9	5	2.76	5.34	0.33	5.67	0.33	0.00		
10	6	0.69	6.02	1.22	7.24	1.22	0.00		
11	7	2.10	8.13	1.44	9.57	1.44	0.00		
12	8	0.32	8.45	1.22	10.79	2.34	1.12		
13	9	1.90	10.35	2.07	12.86	2.52	0.44		
14	10	0.39	10.74	1.42	14.28	3.54	2.13		

Figure 3.3: Screenshot of an Excel worksheet used to simulate a simple queue

The above simulation is carried out as follows.

1) Columns B and D are random numbers chosen from an exponential distribution with parameters a and b respectively using the method described earlier. These represent the inter-arrival times (IAT) and service times (ST) respectively for each customer.
2) Column C records the actual arrival time (AT) by adding the contents of the previous cell to the corresponding IAT (e.g. Cell C7 = C6 + B7).
3) The Finish Time (FT) in Column E is found by taking the larger of the sum of the previous FT and the corresponding ST and the sum of the corresponding AT and ST (e.g. Cell E7 = MAX(D7+C7, E6+D7)).
4) The total time spent by each customer (Column F) is the difference between the customer's FT and AT (e.g. Cell F7 = E7−C7).
5) The total wait time for each customer (Column G) is the difference between the customer's total time spent and ST (e.g. Cell G7 = F7 – D7).

Estimates for the parameters a and b may be obtained by experiment. One could video-record a simple queue at an ATM, and estimate customer arrivals and service durations for each customer. Average rates can then be calculated and fed into the simulation model. The simulation can also be automated and improved with the use of macros or VBA programs written in Excel.

In constructing a model for the simple queue, we may consider questions such as:
(1) How likely is it that a customer will need to wait to be served?
(2) What factors affect this likelihood?
(3) What is the average wait time?
(4) What options are there to reduce this wait time?

Queuing Theory is not an easy concept for primary or secondary high school students to grasp. However, queuing or waiting is a common experience in our everyday life. Using a simulation model, we can demonstrate how mathematics can be used to study such a complex process. Students will not need to understand the concept of Poisson process or exponential distribution until they are ready. In the mean time, the cognitive gap can be bridged by technology, giving them the opportunity to appreciate the power of mathematical modelling in an everyday life experience.

3.3 Falling rain

This example is inspired by a discussion on a modelling task, carried out by Oldknow (2003), who illustrated how Geometer's Sketchpad (GSP) may be used to study the motion of water spouting out from a Singapore landmark called the Merlion. A photograph of the Merlion showing the trajectory of water was used as the subject of study, and GSP, as well as Cabri Geometry (II Plus), was used successfully to examine the geometry of the motion.

It often pours in Singapore, and this year, we have seen floods at various times of the year at various places on the island. In fact, Orchard Road, a well known shopping district in the city state, turned into "Orchard River" on one occasion.

At the National Institute of Education, we have had our fair share of flooding experience as well. On one stormy day in June this year, rain water had gathered in the newly renovated terrace just outside our staff offices, and eventually overflowed into the corridors and flooded the offices. The estate and building management decided that the way to arrest this problem was to collect the water coming down from the gutter and draining it off in the manner shown in Figure 3.4. We can import the same image onto GSP and carry out a modelling task, as depicted in the screenshot in Figure 3.5.

Figure 3.4: Rain in the terrace

Based on the laws of particle motion, we can guess with reason that the trajectory of the water is a parabola. In other words, if we place the origin O at the point where the water just leaves the gutter, the function used to describe the trajectory is simply $f(x) = ax^2$, where a is a negative constant. Fixing $a = -1.5$, and using the slider technique as suggested by Oldknow, we can vary the unit distance of the coordinate system until the path of the water is as close to $f(x)$ as we can.

One objective of this exercise could be to study the way rain shoots off the gutter, and evaluate the effectiveness of the current method of solving the flooding problem. We can also use the image and GSP to find the height of the roof where the gutter is, the angle at which the water hits the ground and so on.

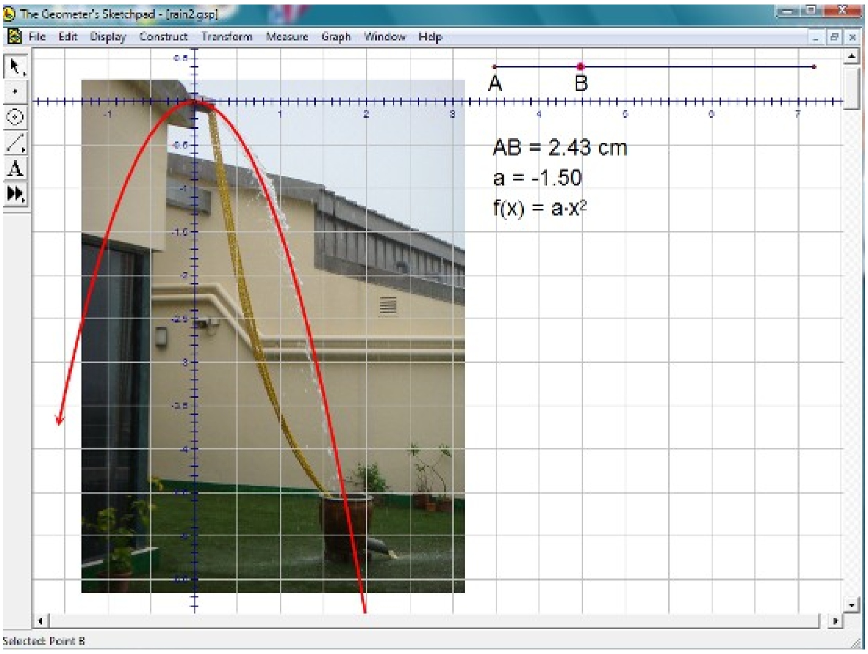


Figure 3.5: Modelling the path of rain water from the gutter using GSP

3.4 Disease Outbreak

In 2003, Singapore had to grapple with the outbreak of a deadly disease known as Severe Acute Respiratory Syndrome, or SARS in short. In a short span of 70 days, 206 cases were reported and of these, 31 lives were lost. Data for the SARS outbreak in Singapore are available in the public domain and reproduced below (Table 3.1). These can serve as useful material for a modelling task for students (Ang, 2003).

Given this set of data, students may be challenged to use a mathematical model for an epidemic to explain or describe the process of the outbreak. One simple epidemic model that may be suitable is the “S-I” model. In this model, the two compartments of populations are the susceptible individuals (S) and the infected (I) individuals of a community. For simplicity, we may need to assume that the community is closed.

Table 3.1: Number of individuals infected with SARS during the 2003 outbreak in Singapore (Heng and Lim, 2008)

Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)	Day (t)	Number (x)
0	1	15	25	29	101	43	163	57	202
1	2	16	26	30	103	44	168	58	203
2	2	17	26	31	105	45	170	59	204
3	2	18	32	32	105	46	175	60	204
4	3	19	44	33	110	47	179	61	204
5	3	20	59	34	111	48	184	62	205
6	3	21	69	35	116	49	187	63	205
7	3	22	74	36	118	50	188	64	205
8	5	23	82	37	124	51	193	65	205
9	6	24	84	38	130	52	193	66	205
10	7	25	89	39	138	53	193	67	205
11	10	26	90	40	150	54	195	68	205
12	13	27	92	41	153	55	197	69	205
13	19	28	97	42	157	56	199	70	206
14	23								

Suppose we let $x(t)$ be the number of infected (and hence infectious) individuals, and N be the total number of individuals in the community. Then, the S-I model reduces to the logistic equation

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{N}\right) \tag{3.2}$$

where k is a positive constant representing the rate of transmission of the disease. Assuming that the initial condition is given as $x(0) = x_0$, and using a standard method of solution (such as separation of variables), the solution to Equation (3.2) may be written as

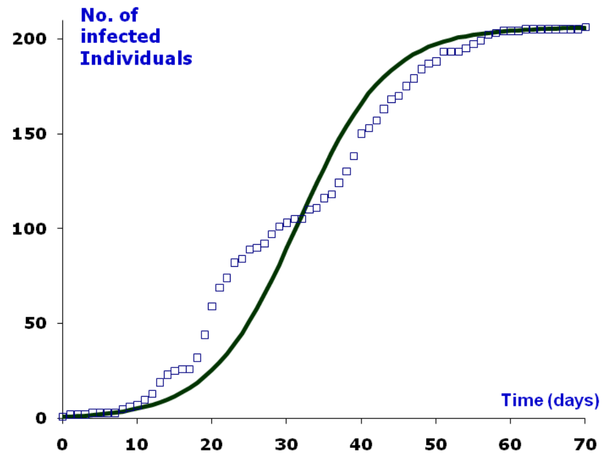
$$x(t) = \frac{Nx_0}{x_0 + (N - x_0)e^{-kt}} \tag{3.3}$$

From Table 3.1, it is clear that $x(0) = x_0 = 1$, meaning that the outbreak had started with one infected individual. In addition, it is assumed that the total number of individuals in this closed community is $N = 206$. Although this assumption is open to debate, it is a necessary assumption if we wish to use this model. With these values, what is needed to complete the model is a value for the transmission rate, k. Here is where the "Solver Tool" in MS Excel can be very useful. We first define an "average error",

$$E = \frac{\sqrt{\sum_{i=1}^{n}(\overline{x}_i - x_i)^2}}{n} \tag{3.4}$$

where $\overline{x}_i$ are the data values, x_i are the values generated from the logistic model. In this case, $n = 71$ (the total number of data points available). Using the "Solver Tool" in MS Excel, we can find a value for k that minimises the average error, E. In the present case, the minimum value of E is found to be 1.9145 when $k = 0.1686$. Plotting the graph of the solution given in (3.3) with this value of k, it is clear that the model follows a similar trend as the data (see Figure 3.6), but it can be improved. To improve the model, we observe that near the beginning and end of the outbreak, the model appears to be fairly reasonable. However, between $t = 15$ and $t = 50$, the model does not seem to fit very well with the actual SARS cases.

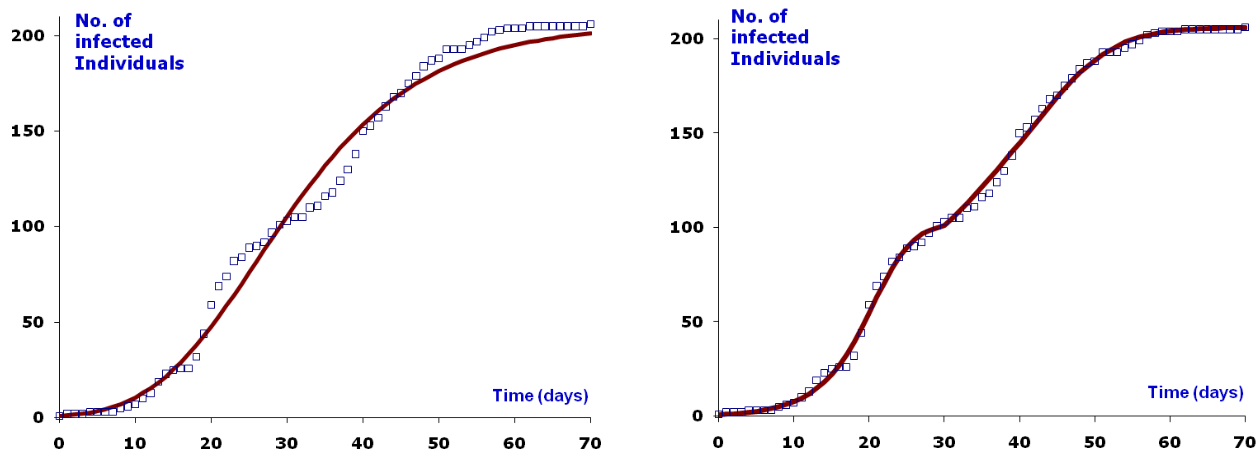
k =	0.1686	*E* =	1.9145
Day	SARS cases	Logistic Model	Squared Error
0	1	1.0000	0.0000
1	2	1.1826	0.6682
2	2	1.3982	0.3621
3	2	1.6529	0.1205
4	3	1.9536	1.0950
5	3	2.3083	0.4785
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
70	206	205.6838	0.1000



(a) Minimizing error using a spreadsheet (b) Graph of SARS cases and solution from model

Figure 3.6. Using Excel's Solver Tool in SARS outbreak model

Now, the logistic equation assumes a linear relationship between the fractional rate of change of $x(t)$ with $(1 - x/N)$. A more general model would be to assume that the fractional rate of change of $x(t)$ varies with $\left(1 - \left(\frac{x}{N}\right)^p\right)$ for some real constant p. We then apply the same procedure as before to find values of k and p that will minimize the average error. The result is a modified or generalized logistic model for the SARS outbreak and is shown in Figure 3.7.



(a) Modified logistic model with $p = 0.1988$ and $k = 0.4334$

(b) Double logistic model with four parameters (See Ang, 2004 for details)

Figure 3.7: Refined models for SARS outbreak

It is clear from Figure 3.7(a) that the modified logistic model is an improvement over the logistic model. Results can be further improved by using a "double logistic" model, as shown in Figure 3.7(b). Justifications for using a double logistic model and details on the approach can be found in Ang (2004).

This example illustrates the concept of model refinement in mathematical modelling. Moreover, it demonstrates the use of empirical data to estimate parameters, such as k and p. While k has a physical meaning, p can be considered as a parameter in a deterministic model.

4. Conclusion

It has been suggested that mathematical modelling can provide a "unifying framework" for teaching applied mathematics (Smith, 1996), without necessarily adding content to the curriculum. The way in which using and applying mathematics is presented may need some redesigning. Mathematical modelling could well be the approach to adopt as it provides opportunities to learn and apply mathematics at the same time (Warwick, 2007). Given proper guidance and scaffolding, students can indeed learn mathematical modelling and through the process, learn mathematics (Blum and Ferri, 2009).

It is clear from the examples discussed that mathematical modelling can provide very rich learning experiences in mathematics, and often contains inter-disciplinary elements. Problems can come from some other discipline and the mathematics teacher can capitalize on these opportunities to collaborate with other teachers in mathematical problem-solving. In addition, modelling exercises set in a local context adds authenticity to the task and arouse greater interest amongst students. It is for this very reason that the example on SARS model has received much attention and interest from teachers and students in Singapore when it was first discussed.

It is equally clear, as illustrated in the examples, that the use of technology plays an important role in mathematical modelling. Real life problems often involve real life data. These may need to be collected and manipulated at times, and technology can make it possible and convenient even for a young school pupil.

In some cases, technology can also help make the mathematics more accessible. For instance, in the SARS example, the Solver Tool in Excel helps us find parameters for the model. Although it is possible to work out the parameters by hand, the knowledge and skills required may be a little too advanced for the intended learners. Rather than having to struggle with complicated or tedious numerical computations, it may be better to use a tool so that one could focus on the model, the application and the mathematics. The use of this technological tool thus bridges the gap, which the student can fill in future.

As pointed out by Ang (2006a), when a mathematical modelling task is used as an approach to learning mathematics, technology may be employed in to help the student to "do more with less mathematics". This includes exploring possible graphical solutions of the problem, performing computational experiments or simulations in models, and manipulating or working with real data.

Using technology in the classroom for mathematical modelling activities can be a complex process. It is not a simple matter of pushing or offering computers, calculators and software to school children and teachers ad infinitum. One should be careful not to lose sight of the mathematics and the spirit of mathematical modelling even when technology provides all the conveniences, and sometimes, some of the answers.

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