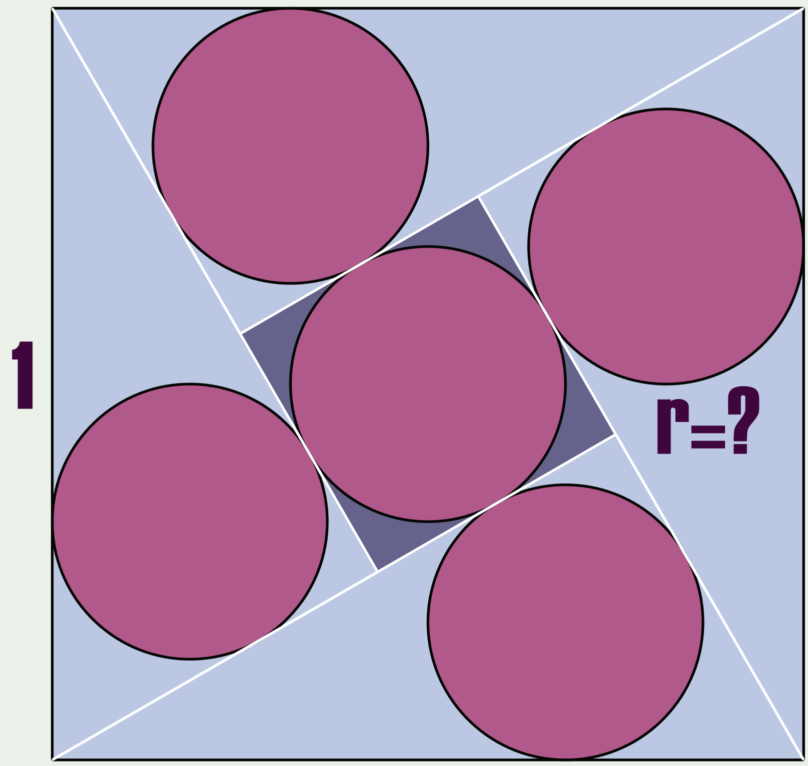
ATCM 2023, Pattaya, Thailand

# Creative, Innovative and Computational Thinking in STEM Through Technology

## Proceedings of the 28th Asian Technology Conference in Mathematics

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# Introduction

The ATCM 2023 (atcm.mathandtech.org) returned to Asia after being held in China in 2019. We are fortunate to hold the virtual format during the COVID-19 in 2020 and 2021. It was held for the first time in Europe in 2022. The ATCM 2023 is in a hybrid format. We proudly present speeches from 21 invited speakers, 23 contributed papers, 11 workshops, 15 presentations with abstracts, and 8 poster sessions. We are happy to see many presenters from Thailand this year, and this is no surprise since we planted the seeds that we held ATCM in Thailand in 2008, 2012, and 2016.

Given a Ph.D. or BS math degree program being eliminated, don't we have to consider why students must choose math as a major? We have seen many examples of how incorporating technological tools provides critical intuition and motivation to learners and makes challenging problems more accessible to more students. Integrating the computer algebra system with the dynamic geometry system will allow us to make conjectures, discover more mathematics, and provide us with an excellent methodology to deal with many real-life problems.

All authors and readers are encouraged to contribute their favorite ideas to the next ATCM or publish interesting articles in the Electronic Journal of Mathematics and Technology (eJMT: https://php.radford.edu/~ejmt/). Selected eJMT papers will be published in the Research Journal for Mathematics and Technology (RJMT: http://rjmt.mathandtech.org). It is always nice to renew old friendships and exciting to make more new friends at an ATCM. We hope you will invite a few of your colleagues to experience future ATCMs for themselves.
We look forward to seeing everyone in person at ATCM 2024 in Yogyakarta, Indonesia.

Wei-Chi Yang, on behalf of ATCM 2023
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PROCEEDINGS OF THE 28TH ASIAN TECHNOLOGY CONFERENCE IN MATHEMATICS

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# Graphs of Uniform Convergence on Iteration of Loci generated by Special Convex Combinations of Curves and Surfaces

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**Abstract**

*We extend the convergence of locus discussed in the paper [5], which originated from a practice problem for the Chinese college entrance exam. In this paper, we are interested in the limit of a recursive sequence of loci built on a special convex combination of vectors involving curves or surfaces. We shall see many interesting graphs of uniform convergence of sequences generated by parametric curves and surfaces, which will inspire many applications in computer graphics, and other related disciplines.*

## 1 Introduction and Motivation

In the paper [5], the problem is to find the locus that is determined by two fixed vectors using bisection theorem. In this paper, we discuss the proposed question of what will happen when we iterate the locus sequentially, and would like to find the limit of such locus. In short, we shall see a continuous deformation of an initial shape into a target shape, which is an interesting subject in computer graphics. We shall see the limit of a recursive sequence of convex combinations of vectors that involve curves or surfaces.

**Original College Entrance Practice Problem:** *Given a unit circle centered at* $(0,0)$ *and a fixed point at* $A=(2,0)$. *Let* $Q$ *be a moving point on the unit circle* $C$. *Find the locus* $M$ *which is the intersection between the angle bisector* $QOA$ *and line segment* $QA$.

It is an easy exercise to verify that the locus of point $M$ is a circle, which we leave as an exercise for the readers. Moreover, it is natural to imagine when DGS and CAS tools are available for students in a classroom as a project to explore, they may quickly pose 'what if' scenarios. We briefly state the following Exploratory Activity has been discussed in [4] and [5]. We then extend it to what we will focus on in this paper.

**Exploratory Activity (**[4] and [5]**):** Given an ellipse $C$: $[x(t),y(t)]=[a\cos(t),b\sin(t)], t\in[0,2\pi]$, and a fixed point $A=(p,q)\notin C$. Let $Q$ be a moving point on the ellipse (shown in green in Figure 1). Find the locus of the point $M$ which is the intersection between the bisector $QOA$ and line segment $QA$.

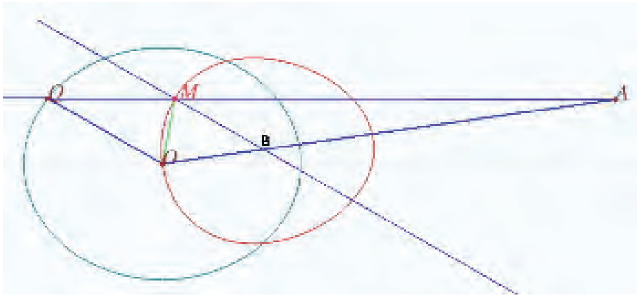


Figure 1. Locus, bisection and an ellipse

We derived that

$$\overrightarrow{OM} = \frac{OQ}{OA+OQ}\overrightarrow{OA} + \frac{OA}{OA+OQ}\overrightarrow{OQ}, \tag{1}$$

where $OQ = \left\|\overrightarrow{OQ}\right\| = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$ and $OA = \left\|\overrightarrow{OA}\right\| = \sqrt{p^2+q^2}$. We see that the parametric equation for the locus $M(t)$ can be plotted directly from Eq. (1) (see the red curve in Figure 1 above) with the help of a computational tool. It should cause no confusion throughout the paper that when $t \in [0, 2\pi]$, we often use $\overrightarrow{OM}$ to denote the vector $\overrightarrow{OM(t)}$, $OQ$ stands for the magnitude of $\left\|\overrightarrow{OQ(t)}\right\|$ when $Q(t)$ is a parametric curve, and $OA$ stands for the magnitude of $\left\|\overrightarrow{OA}\right\|$ if $A$ is simply a point.

It is natural to extend our exploration and ask what would happen to the plot of

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{OQ_n}{OA+OQ_n}\overrightarrow{OA} + \frac{OA}{OA+OQ_n}\overrightarrow{OQ_n}, \tag{2}$$

when $n \to \infty$, where $\overrightarrow{OQ_n} = \overrightarrow{OM_n}, OQ_n = OM_n = \sqrt{x_n(t)^2 + y_n(t)^2}, n \in \mathbb{Z}^+$, and $OA = \sqrt{p^2+q^2}$. Consequently, consider the following extension with extra weights of coefficients $r$ and $s$ as follows: We therefore, consider the following scenario with extra weights of coefficients $r$ and $s$ as follows:

**Theorem 1** *Given a non-zero closed curve $C$: $[x(t), y(t)]$, and a non-zero fixed point $A = (p, q) \notin C$. Let $Q$ be a moving point on $C$. For $r, s > 0$, and $\overrightarrow{OM_1} = \frac{s \cdot OQ}{r\cdot OA + s\cdot OQ}\overrightarrow{OA} + \frac{r\cdot OA}{r\cdot OA+s\cdot OQ}\overrightarrow{OQ}$, if we write the Eq. (2) as*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{s\cdot OQ_n}{r\cdot OA + s\cdot OQ_n}\overrightarrow{OA} + \frac{r\cdot OA}{r\cdot OA+s\cdot OQ_n}\overrightarrow{OQ_n}. \tag{3}$$

*Then $\overrightarrow{OM_{n+1}}$ converges for some $t \in [0, 2\pi]$ when $n \to \infty$ if and only if either $OQ_n = \sqrt{x_n^2(t) + y_n^2(t) + z_n^2(t)} \to 0$ or $\overrightarrow{OM_n(t)} \to \overrightarrow{OA}$ for some $t \in [0, 2\pi]$ when $n \to \infty$.*

**Proof:** First, if $\overrightarrow{OM_{n+1}}$ converges for some $t \in [0, 2\pi]$ when $n \to \infty$, then $\overrightarrow{M_nM_{n+1}} = \overrightarrow{OM_{n+1}} - \overrightarrow{OM_n} \to 0$ for some $t \in [0, 2\pi]$ when $n \to \infty$. Moreover, since

$$\begin{aligned} \overrightarrow{M_nM_{n+1}} &= \overrightarrow{OM_{n+1}} - \overrightarrow{OM_n} \\ &= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OM_n} - \overrightarrow{OM_n} \\ &= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \overrightarrow{OM_n}\left(\frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n} - 1\right) \\ &= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \overrightarrow{OM_n}\left(\frac{-s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\right). \\ &= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\left(\overrightarrow{OA} - \overrightarrow{OM_n}\right). \end{aligned} \tag{4}$$

Hence, $\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}$ converges for some $t \in [0, 2\pi]$ if either $OQ_n = \sqrt{x_n^2(t) + y_n^2(t) + z_n^2(t)} \to 0$ or $\overrightarrow{OM_n(t)} \to \overrightarrow{OA}$ for some $t \in [0, 2\pi]$ when $n \to \infty$. The other direction is clear. ■

We describe a special convex combination of vectors in the vector space $\mathbb{R}^n$ below.

**Definition 2** *Given a finite number of vectors $v_1, v_2, \ldots v_n$ in $\mathbb{R}^n$, a conical combination of these vectors is vector of the form*

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n,$$

*where $\alpha_i > 0, i = 1, 2, \ldots n$. A set of conical combination of vectors is called a **convex combination** [2] if in addition the coefficient satisfying the following condition*

$$\sum_{i=1}^{n} \alpha_i = 1.$$

In this paper, we shall discuss a special weighted convex combination of vectors that involve a recursive sequence. For example, if

$$\begin{aligned} \overrightarrow{OM_{n+1}(t)} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}v_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}v_2 \\ &\quad + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\overrightarrow{OM_n(t)}, \end{aligned} \tag{5}$$

then $\alpha_1, \alpha_2$ and $\alpha_3$ are positive real numbers. Using the scaling techniques, without loss of generality, we assume $\alpha_1, \alpha_2$ and $\alpha_3$ are real numbers in $(0, 1)$. We shall see in later proofs that the coefficient $\alpha_3$ is irrelevant to the convergence of $\lim_{n\to\infty} \overrightarrow{OM_{n+1}(t)}$.

## 2 2D iterations on one curve and one fixed vector

For the rest of the paper, we assume the fixed point $A$ is not on the original curve $C$. In view of the Theorem (1), we further extend the knowledge of uniform convergence of sequences of functions, which students learn in Advanced Calculus. We begin with the domain $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ being a sequence of functions, and note that since the metric space $\mathbb{R}^2$ is complete, which means that every uniformly Cauchy sequence $M_n$ is convergent. We consider the following:

**Definition 3** *Suppose $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ is a sequence of functions. If we write $M_n(t) = [x_n(t), y_n(t)]$, with $t \in [0, 2\pi]$, $\{M_n(t))$ is said to* ***converge uniformly*** *to $M^*(t) = [p(t), q(t)]$ if $\forall \epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ (i.e. $N$ depends only on $\epsilon$ in this case) such that the Euclidean distance between two points, $M_n(t)$ and $M^*(t)$, $\|M_n(t) - M^*(t)\|$ or $\|M_n(t)\, M^*(t)\|$, is arbitrarily small:*

$$\|M_n(t) - M^*(t)\| = \|M_n(t)\, M^*(t)\| = \sqrt{(x_n(t) - p(t))^2 + (y_n(t) - q(t))^2} < \epsilon.$$

*Similarly, the sequence $\{M_n(t))$ is said to* ***converge uniformly*** *to a point $A = (p, q)$ if $\forall \epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ such that $\|M_n(t)\, A\|$ is arbitrarily small. In other words,*

$$\|M_n(t)\, A\| = \sqrt{(x_n(t) - p)^2 + (y_n(t) - q)^2} < \epsilon$$

*for all $n \geq N$ and all $t \in [0, 2\pi]$. Intuitively, there exists a positive integer $N$, such that the parametric curves $M_n(t)$ will shrink to the point $A$ for all $n \geq N$ and all $t \in [0, 2\pi]$.*

**Definition 4** *Suppose $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ is a sequence of functions. If we write $M_n(t) = [x_n(t), y_n(t)]$, with $t \in [0, 2\pi]$, $\{M_n(t))$ is said to be uniformly Cauchy if for every $\varepsilon > 0$, there exists a positive integer $N$ such that the inequality*

$$\|M_n(t)\, M_m(t)\| < \varepsilon$$

*holds whenever $m \geq N$, $n \geq N$, and for all $t \in D$. We take it for granted in this paper that the sequence $\{M_n : D \to \mathbb{R}^2\}$ converges uniformly to another $M$ on $D$ if and only if, the sequence $\{M_n)$ is uniformly Cauchy.*

**Remarks:**

1. We remark that definitions in (3) and in (4) can be extended to $\mathbb{R}^n$.
2. We remind readers to distinguish the difference between uniform convergence versus point-wise convergence.
3. Recall our original bisection problem (1) is such that $\frac{M_1A}{M_1Q_0} = \frac{AB}{OB} = \frac{AB}{M_1B} = \frac{OA}{OQ_0} = k_1(t)$, where the convergence in the case of (2) is a homothety (see [3]). We may denote the following:
$$\frac{M_nA}{M_nM_{n-1}} = k_n(t)\left(= \frac{OA}{OM_{n-1}}\right), \tag{6}$$
where $n = 1, 2, ...$, and $M_0 = Q$, which is a point on the given curve $C$.

4. On one hand, we usually prove how a sequence of parametric curves $\{M_n(t)\}_{n=1}^{\infty}$ converge uniformly directly in this paper. On the other hand, we note that $\{M_n(t)\}$ is a sequence from $D = [0, 2\pi]$ to $\mathbb{R}^2$, and since $\mathbb{R}^2$ is a complete metric space, if one can show that $\{M_n(t)\}$ is a uniformly Cauchy sequence, then $\{M_n(t)\}$ is uniformly convergent. Instead of proving that $\{M_n(t)\}$ is a uniformly Cauchy sequence theoretically in this paper, with the help of a CAS, we often demonstrate that the graph of square distance function $f_n(t) = \sup(\|M_n(t) - M_{n-1}(t)\|)^2$ or $g_n(t) = \sup(\|M_n(t) - A\|)^2$, for all $t \in D = [0, 2\pi]$, is decreasing to 0 uniformly, and use such observation to conjecture that $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly.

The next observation is natural:

**Theorem 5** *Let $C$ be a given simple closed curve $[x_0(t), y_0(t)]$, $A = (p_1, q_1) \notin C$. For $r, s \in (0, 1)$ and $r \neq s$, we let*

$$\overrightarrow{OM_1} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{s \cdot OQ}{r \cdot OA + s \cdot OQ}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ}\overrightarrow{OQ},$$

*where $Q$ is a moving point on $C$. Now for $n \in \mathbb{Z}^+$, we consider*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OQ_n}, \tag{7}$$

*where $Q_n$ is a moving point on $(x_n(t), y_n(t))$, and $\overrightarrow{OQ_n(t)} = \overrightarrow{OM_n(t)}$. Then $\overrightarrow{OM_n(t)} \to \overrightarrow{OA}$ uniformly as $n \to \infty$ for all $t \in [0, 2\pi]$, $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t \in [0, 2\pi]$. Consequently, $\{M_n(t)\}_{n=1}^{\infty}$ converges to $A$ uniformly.*

**Proof:** First, if $r = s$ and $r, s \in (0, 1)$, we refer to Theorem (1) for discussion. Now, for $r, s \in (0, 1)$ and $r \neq s$,

$$\overrightarrow{OM_1} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{s \cdot OQ}{r \cdot OA + s \cdot OQ}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ}\overrightarrow{OQ},$$

we first observe that $M_n = Q_n = (x_n(t), y_n(t))$ for $n \geq 1$, and

$$\begin{aligned}
\overrightarrow{OM_2} &= \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \frac{s \cdot OQ_1}{r \cdot OA + s \cdot OQ_1}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_1}\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} \\
&= \frac{s \cdot OQ_1}{r \cdot OA + s \cdot OQ_1}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_1}\left(\frac{s \cdot OQ}{r \cdot OA + s \cdot OQ}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ}\overrightarrow{OQ}\right) \\
&= \overrightarrow{OA}\left(\frac{(rs)\left[(OA)(OQ) + (OA)OQ_1\right] + s^2(OQ)(OQ_1)}{(r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)}\right) \\
&\quad + \overrightarrow{OQ}\left(\frac{r^2 \cdot (OA)^2}{(r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)}\right).
\end{aligned}$$

By induction, we see

$$\begin{aligned}
\overrightarrow{OM_{n+1}} &= \overrightarrow{OA}\left(\frac{(r \cdot OA + s \cdot OQ_n)\cdots(r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ) - r^n \cdot (OA)^n}{(r \cdot OA + s \cdot OQ_n)\cdots(r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)}\right) \\
&\quad + \overrightarrow{OQ_n}\left(\frac{r^n \cdot (OA)^n}{(r \cdot OA + s \cdot OQ_n)\cdots(r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)}\right)
\end{aligned} \tag{8}$$

Since $0 < r < 1$,

$$\frac{r^n \cdot (OA)^n}{(r \cdot OA + s \cdot OQ_n) \cdots (r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)} \to 0.$$

Furthermore, since $\overrightarrow{OM_{n+1}} = a\overrightarrow{OA} + b\overrightarrow{OQ_n}$, where $a$ and $b$ are coefficients of $\overrightarrow{OA}$ and $\overrightarrow{OQ_n}$ respectively as seen in Eq. (8) with $a, b \in (0,1)$ and $a+b = 1$, this implies that $\overrightarrow{OM_{n+1}(t)} \to \overrightarrow{OA}$ as $n \to \infty$ for all $t \in [0, 2\pi]$ . Since three points, $M_{n-1}(t)$, $M_n(t)$ and $A$ are collinear, and $M_n(t)$ is in the interior of $M_{n-1}(t)$ and $A$, we see $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t \in [0, 2\pi]$, which can be shown that $\overrightarrow{M_n(t)}$ is uniformly Cauchy, and hence $\{M_n(t)\}_{n=1}^{\infty}$ converges to $A$ uniformly. ■

We remark that the uniform convergence of $\{M_n(t)\}_{n=1}^{\infty}$ to the point $A$ does not depend on the curve $C$.

**Example 6** *We consider the curve $C$ of $[a\cos(t), b\sin t]$, $A = (p_1, q_1) \notin C$, For the convex combination of $r$ and $s$, we let*

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{s \cdot OQ}{r \cdot OA + s \cdot OQ} \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ} \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix},$$

*and*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OQ_n}.$$

*If we choose $a = 5, b = 4$, and convex combination for $r = \frac{1}{3}, s = \frac{2}{3}$, $A = (3, 2)$ , then $\{M_n(t)\}_{n=1}^{\infty}$ converges to $A$ uniformly. (See Figure 2)*

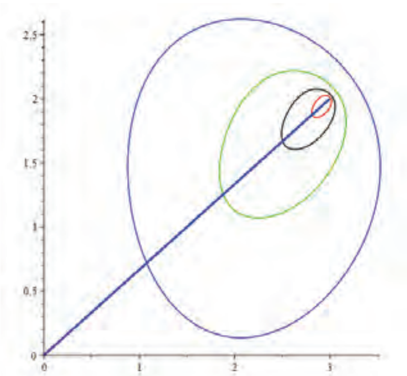


*Figure 2. Uniform converges to a point.*

**Exercises:** (1) If we use $r = s$ in Example (6), then we leave it to the readers to verify that $g_n(t) = (\|M_n(t) - A\|)^2$ does not converge uniformly to 0. In fact, the maximum value of $g_n(t)$ is the distance $(OA)^2$ at some $t \in (0, 2\pi)$. (2) If we replace $C$ by $[a\sin t, b\sin t\cos t]$, $a = 5, b = 4$, $r = \frac{1}{3}, s = \frac{2}{3}$, $A = (3, 2)$ in Example (6), then we may conjecture that $\{M_n(t)\}_{n=1}^{\infty}$ does not converge to $A$ uniformly by observing the graph of $f_n(t) = \|M_n(t) - M_{n-1}(t)\|$ does not converge uniformly to 0.

# 3 2D iterations on one curve and two fixed vectors

We consider convex combinations of three vectors below: Let $C$ be a given closed curve $[x_0(t), y_0(t)]$, $A = (p_1, q_1)$ and $B = (p_2, q_2)$ be two distinct points not lying on $C$. If $Q$ is a moving point on $C$, and $r_1, r_2$, and $r_3$ are real numbers in $(0, 1)$. For $n \in \mathbb{Z}^+ \cup \{0\}$, we consider

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OB} \\ &+ \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OM_n},\end{aligned}$$

where $M_0(t) = Q(t) \in C$, and $M_n(t) = Q_n(t)$ is a moving point on $(x_n(t), y_n(t))$. We are interested in $\lim_{n\to\infty} \overrightarrow{OM_{n+1}}$.

## 3.1 Generating sequence of shrinking curves due to convex combinations

Since the plot of the sequence $\overrightarrow{OM_{n+1}}$ in (9), where $r_1, r_2$, and $r_3$ are distinct real numbers in $(0, 1)$, is a convex combinations of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OM_n}$, the plot of $[x_{n+1}(t), y_{n+1}(t)]$ is generated by the following steps:

1. Connect three points of $M_n = (x_n(t), y_n(t))$, $A$ and $B$ to form the triangle $\triangle M_n AB$.
2. We view the point $M_n$ as the convex combination of three points $A, B$ and $M_{n-1}$, for $n \in \mathbb{Z}^+$, where $M_0 = Q$, which is a point on the curve $C$. Since $r_1, r_2$, and $r_3 \in (0, 1)$, the point $M_n(t)$ belongs to the interior of the triangle $\triangle M_{n-1}AB$ for each $t \in [0, 2\pi]$, and $n \in \mathbb{Z}^+$, see [2].
3. We shall see later in the proof of the Theorem (8) that the coefficient $r_3$ will not affect the final plot of $\overrightarrow{OM_n}$ when $n \to \infty$.
4. The convergence of $\overrightarrow{OM_n}$ will only depend on $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and will not depend on the curve $C$.

**Example 7** *We use closed curve $C$ to be $[a \sin u, b \sin u \cos u]$, $a = 5, b = 4$, $A = (3, 4)$, $B = (2, 5)$, $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$, and $r_3 = \frac{1}{6}$ for demonstrating how $[x_2(t), y_2(t)]$ is generated from $[x_1(t), y_1(t)]$. The graphs of $[x_1(t), y_1(t)]$ and $[x_2(t), y_2(t)]$ can be seen in black and purple respectively in Figure 4 (d) respectively.*

- Figure 4(a) shows when $t = 0$,the plot of $[x_2(t), y_2(t)]$ has not been generated yet.
- Figure 4(b) shows when $t \in [0, 0.9106]$, the plot of $[x_2(t), y_2(t)]$ is being generated in this interval and will be in the interior of $\triangle M_1 AB$ for each corresponding $t$.
- Figure 4(c) shows when $t \in [0, 3.1871]$, the plot of $[x_2(t), y_2(t)]$ is being generated in this interval and will be in the interior of $\triangle M_1 AB$ for each corresponding $t$, and finally, Figure

4(d) shows when $t \in [0, 2\pi]$, the plot of $[x_2(t), y_2(t)]$ is smaller than that of $[x_1(t), y_1(t)]$.

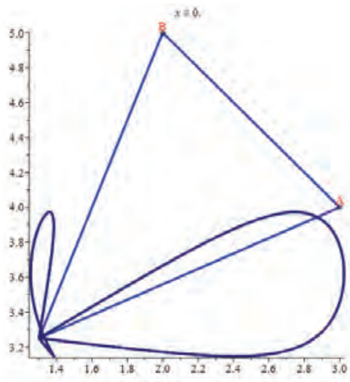


Figure 4(a), $t = 0$.

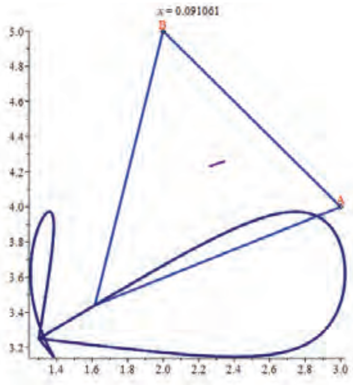


Figure 4(b), $t \in [0, 0.9106]$.

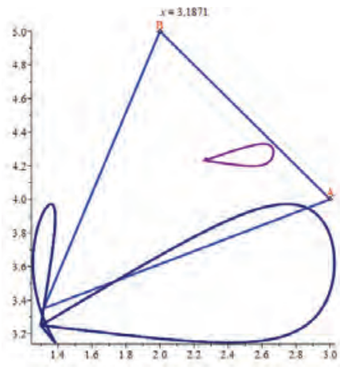


Figure 4(c), $t \in [0, 3.1871]$.

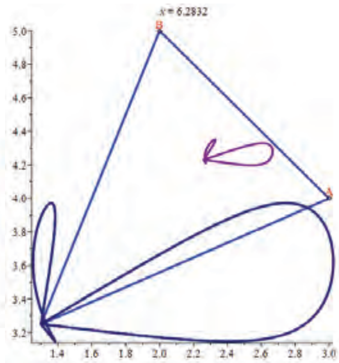


Figure 4(d), $t \in [0, 2\pi]$.

**Theorem 8** *Let $C$ be a given closed curve $[x_0(t), y_0(t)]$, $A = (p_1, q_1)$ and $B = (p_2, q_2)$ be two non-zero distinct points not lying on $C$. If $Q$ is a moving point on $C$, and $r_1, r_2,$ and $r_3$ are positive real numbers in $(0, 1)$, we let*

$$\overrightarrow{OM_1} = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OB} + \frac{r_3 \cdot OB}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OQ}.$$

*We further consider*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OB} + \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}, \tag{9}$$

*where $Q_n$ is a moving point on $(x_n(t), y_n(t))$ . Then $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to a point $D$, **which lies on the line segment** $\overline{AB}$. Consequently, $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t \in [0, 2\pi]$. We remark that the coefficient $r_3 \in (0,1)$ will not affect the location of the convergence $\{M_n(t)\}_{n=1}^{\infty}$.*

**Proof:** First, we observe

$$\begin{aligned}\overrightarrow{OM_2} &= \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \frac{r_1 \cdot OQ_1}{r_1OQ_1 + r_2OA + r_3OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1OQ_1 + r_2OA + r_3OB}\overrightarrow{OB} \\ &\quad + \frac{r_3 \cdot OB}{r_1OQ_1 + r_2OA + r_3OB}\begin{pmatrix} \frac{r_1 \cdot OQ}{r_1OQ+r_2OA+r_3OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1OQ+r_2OA+r_3OB}\overrightarrow{OB} \\ + \frac{r_3 \cdot OB}{r_1OQ+r_2OA+r_3OB}\overrightarrow{OQ} \end{pmatrix} \\ &= \overrightarrow{OA}\left(\left\|\overrightarrow{OA}\right\|\right) + \overrightarrow{OB}\left(\left\|\overrightarrow{OB}\right\|\right) + \overrightarrow{OQ}\left(\frac{r_3^2 (OB)^2}{(r_1OQ_1 + r_2OA + r_3OB)(r_1OQ + r_2OA + r_3OB)}\right),\end{aligned}$$

It follows from induction that

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \overrightarrow{OA}\left(\left\|\overrightarrow{OA}\right\|\right) + \overrightarrow{OB}\left(\left\|\overrightarrow{OB}\right\|\right) \\ &\quad + \overrightarrow{OQ}\left(\frac{r_3^n (OB)^n}{(r_1OQ_n + r_2OA + r_3OB)\cdots(r_1OQ_1 + r_2OA + r_3OB)(r_1OQ + r_2OA + r_3OB)}\right).\end{aligned}$$

Since $0 < r_3 < 1$, we see $r_3^n (OB)^n \to 0$, and

$$\overrightarrow{OM_{n+1}} \to m\overrightarrow{OA} + (1-m)\overrightarrow{OB},$$

when $n \to \infty$, where $m = \left\|\overrightarrow{OA}\right\|$, and $1 - m = \left\|\overrightarrow{OB}\right\|$. Let $D = m\overrightarrow{OA} + (1-m)\overrightarrow{OB}$, then $D \in \overline{AB}$, and $\overrightarrow{OM_{n+1}}$ converges uniformly to $\overrightarrow{OD}$. Hence $\overrightarrow{OM_{n+1}}$ converges uniformly to $\overrightarrow{OD}$, where $D \in \overline{AB}$. In view of the observations from section (3.1), we see $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to the point $D$, which lies on the line segment $\overline{AB}$. Moreover, it is clear that $\overrightarrow{M_{n-1}(t)M_n(t)} = \overrightarrow{OM_n} - \overrightarrow{OM_{n-1}}$ converges uniformly to 0 for all $t \in [0, 2\pi]$, ■

Computationally, we assume $\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} \to F = \begin{bmatrix} p \\ q \end{bmatrix}$, then the norm of the vector, $\left\|\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}\right\|$, converges to $\|F\| = \sqrt{p^2 + q^2}$, and we have

$$\left(1 - \frac{r_3OB}{r_1\|F\| + r_2OA + r_3OB}\right)\begin{bmatrix} p \\ q \end{bmatrix} = \frac{r_1\|F\|}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OA} + \frac{r_2OA}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OB}$$

$$\begin{aligned}\begin{bmatrix} p \\ q \end{bmatrix} &= \left(\frac{1}{\left(\frac{r_1\|F\| + r_2OA}{r_1\|F\| + r_2OA + r_3OB}\right)}\right)\left(\frac{r_1\|F\|}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OA} + \frac{r_2OA}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OB}\right) \\ &= \left(\frac{r_1\|F\|}{r_1\|F\| + r_2OA}\right)\overrightarrow{OA} + \left(\frac{r_2OA}{r_1\|F\| + r_2OA}\right)\overrightarrow{OB} \\ &= m\overrightarrow{OA} + (1-m)\overrightarrow{OB}, \qquad (10)\end{aligned}$$

where $m = \frac{r_1\|F\|}{r_1\|F\|+r_2 OA}$. To find the point $F$, it amounts to solve two equations in (10) for two variables $p$ and $q$ in terms of $t$; however, due to too many parameters that are involved, we are unable to express the solutions $p$ and $q$ in explicit form. Instead, we do the followings:

1. If $r_1, r_2$, and $r_3$ are real numbers in $(0, 1)$, we substitute the solutions $p$ and $q$ obtained (10) into the line equation $\overleftrightarrow{AB}$, we get the following equation from Maple after setting **the length of computations** to be 20,000 lines:

$$\frac{(q-q_2)\,p_1 + (q_1-q)\,p_2 - p\,(q_1-q_2)}{p_1-p_2} = 0$$

$$\begin{aligned} \Longrightarrow \quad & \frac{qp_1 - pq_1 + pq_2 - qp_2 - p_1q_2 + p_2q_1}{p_1-p_2} = 0, \\ \Rightarrow \quad & \frac{q(p_1-p_2) - p(q_1-q_2) - p_1q_2 + p_2q_1}{p_1-p_2} = 0. \end{aligned} \tag{11}$$

2. Assume $p_1 \neq p_2$ we deduce the numerator of (11) be to the following:

$$\begin{aligned} q(p_1-p_2) - p(q_1-q_2) - p_1q_2 + p_2q_1 &= 0, \\ \frac{q(p_1-p_2) - p(q_1-q_2) - p_1q_2 + p_2q_1}{p_1-p_2} &= 0, \\ q - p\left(\frac{q_1-q_2}{p_1-p_2}\right) - \frac{p_1q_2-p_2q_1}{p_1-p_2} &= 0. \end{aligned}$$

On the one hand, we see $F = (p, q)$ lies on the line of

$$y = \left(\frac{q_1-q_2}{p_1-p_2}\right)x + \frac{p_1q_2-p_2q_1}{p_1-p_2}. \tag{12}$$

On the other hand, we note that the line $\overleftrightarrow{AB}$ is with the slope $\frac{q_1-q_2}{p_1-p_2}$ and passes through the point $(p_1, q_1)$ :

$$\begin{aligned} y - q_1 &= \left(\frac{q_1-q_2}{p_1-p_2}\right)(x-p_1) \\ y &= q_1 + \left(\frac{q_1-q_2}{p_1-p_2}\right)(x-p_1) \\ &= \left(\frac{q_1-q_2}{p_1-p_2}\right)x + \frac{p_1q_2-q_1p_2}{p_1-p_2}. \end{aligned} \tag{13}$$

We see (12) coincides with (13) and hence $F$ lie on line segment $\overline{AB}$. We remark that when solving $p$ and $q$ symbolically if $r_1, r_2$ and $r_3$ are also considered to be variables, it is not possible to express using $p$ and $q$ due to too many unknowns when using [1], but numerical computations do show that the point $(p, q)$ lie on the line segment $AB$. We use the following Example for demonstration.

**Example 9** *We consider the closed curve* $C_1$ *with the parametric equation,* $[x_0(t), y_0(t)] = [\cos u(a - \cos(bu)) + 1, \sin u(a - \cos bu))]$, $A = (p_1, q_1)$, $B = (p_2, q_2)$, *and* $Q$ *is a moving point on* $C_1$. *We let* $r_1, r_2$, *and* $r_3$ *be three distinct real numbers in* $(0, 1)$ , *and*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OB} + \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}.$$

If we pick $a = 5, b = 4$, $p_1 = 3, q_1 = 4, p_2 = 2, q_2 = 5$, and $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$, and $r_3 = \frac{1}{6}$. Then we see

$$\lim_{n\to\infty} \{M_n (t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748) ,$$

see Figure 3(a) below for the convergence. In view of (10), we note that the convergence does not depend on the value of $r_3$.We also remark that convergence to the point $(2.60516252, 4.39483748)$ is irrespective to the curve $C_1$ we pick. For example, if we replace $C_2$ by $[a \sin u, b \sin u \cos u]$ , and use the same $a, b$, point $A$, and point $B$, we shall get the same convergence for $\lim_{n\to\infty} \{M_n (t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748)$ , (see Figure 3(b)). Similarly is true if we replace $C_3$ by $\left[4a \cos u (\sin u)^2 \cos u, 4a \text{ co}\right.$ see (Figure 3(c)).

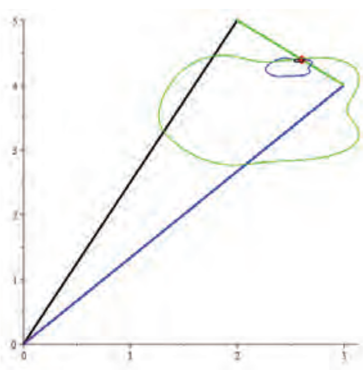


Figure 3(a). Convergence for $C_1$.

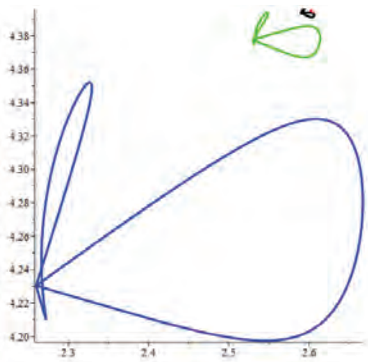


Figure 3(b). Convergence for $C_2$.

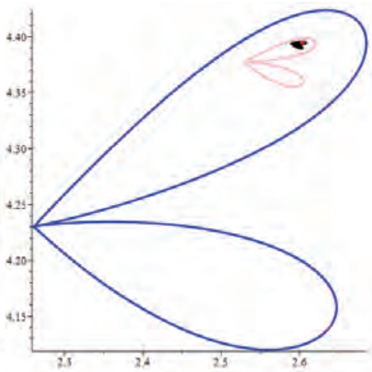


Figure 3(c). Convergence for $C_3$.

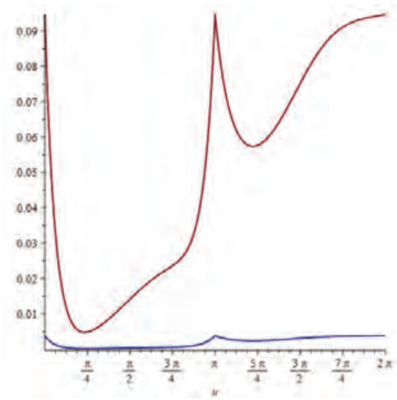
## 3.2 Uniform convergence using geometric constructions

In view of the Theorem (8) and observation from section (3.1), $C$ is a non-zero closed curve, $A$ and $B$ are two non-zero distinct fixed points, not lying on $C$, and $M_n(t)$ is in the interior of the triangle of $\triangle M_{n-1} (t) AB$ for each $t \in [0, 2\pi]$. We see the distance between $M_n(t) = [x_n(t), y_n(t)]$ and $M_{n-1}(t) = [x_{n-1}(t), y_{n-1}(t)]$ is decreasing and converges to 0 when $n \to \infty$, for all $t \in [0, 2\pi]$ . In other words, the square distance function

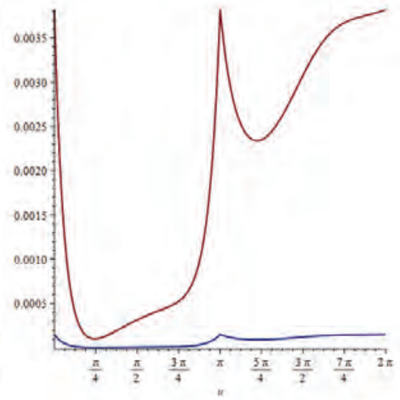
$$f_n(t) = (x_n(t) - x_{n-1}(t))^2 + (y_n(t) - y_{n-1}(t))^2$$

converges to 0 uniformly. Consequently, we see $\{M_n(t)\}_{n=1}^{\infty}$ converges to a point lying on the line segment $\overline{AB}$. In other words, if the graphs of $f_n(t)$ does not converges to 0 uniformly, then $\overrightarrow{OM_{n+1}}$ does not converge uniformly.

Suppose we adopt the Example in the section (3.1), we depict the pair functions $\{f_3(t), f_4(t)\}$ and $\{f_4(t), f_5(t)\}$ in the following Figures 5(a) and 5(b) with red and blue colors respectively:



Figures 5(a). Plots of $\{f_3(t), f_4(t)\}$ .



Figures 5(b). Plots of $\{f_4(t), f_5(t)\}$

In view of the plot of $f_5(t)$ (the blue in Figure 5(b)), we can see that if we pick $\epsilon = 0.0005$, for $n \geq 5$, $f_n(t) \to 0$ uniformly for all $t \in [0, 2\pi]$ . In view of the Example (9), the speed of the uniform convergence of $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748)$ is rather fast.

# 4 2D iterations on one curve, and two vectors on two respective curves

Now, we consider the plots of convex combinations of three vectors, one vector is iterated curve, and the two vectors are on two respective curves.

**Theorem 10** *Let $C$ be a given non-zero closed curve $[x_0(t), y_0(t)]$, $D$ and $E$ be two additional distinct closed curves of $[d_1(t), d_2(t)]$ and $[e_1(t), e_2(t)]$ respectively. Furthermore, we let $Q$ be a moving point on $C$. If $r_1, r_2,$ and $r_3$ are real numbers in $(0, 1)$, we let*

$$\begin{aligned} OQ &= \sqrt{x_0(t)^2 + y_0(t)^2}, \\ OE &= \sqrt{e_1(t)^2 + e_2(t)^2}, \\ OD &= \sqrt{d_1(t)^2 + d_2(t)^2}, \end{aligned}$$

*and*

$$\begin{aligned} \overrightarrow{OM_1} &= \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OD} \\ &\quad + \frac{r_3 \cdot OD}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OQ}. \end{aligned}$$

*In addition, for $n \in \mathbb{Z}^+$, we consider*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1 OQ_n + r_2 OE + r_3 OD}\overrightarrow{OD} + \frac{r_3 \cdot OD}{r_1 OQ_n + r_2 OE + r_3 OD}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}, \quad (14)$$

*where $Q_n$ is a moving point on $(x_n(t), y_n(t))$, for $n = 0, 1, ....$ Then $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to the curve $F(t^*)$ for $t \in [0, 2\pi]$, where the point $F(t^*)$ lies on the line segment $\overline{D(t^*)E(t^*)}$, for all $t \in [0, 2\pi]$. Furthermore, the real solutions of the following parametric curve is a subset of $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty}$. We remark that the coefficient $r_3 \in (0, 1)$ will not affect where $\{M_n(t)\}_{n=1}^{\infty}$ will converge to.*

**Proof:** In view of the Theorem (8) and (3.1), for each fixed $t^* \in [0, 2\pi]$, we consider two distinct fixed points $A_{E(t^*)}$ and $B_{D(t^*)}$, which lie on two distinct curves of $E = (e_1(t), e_2(t))$ and $D = (d_1(t), d_2(t))$, respectively. We see that $M_n(t)$ belongs to the interior of the triangle $\triangle M_{n-1}(t) A_{E(t^*)} B_{D(t^*)}$, for each $t \in [0, 2\pi]$, and $n \in \mathbb{Z}^+$. Since the the triangles $\triangle M_n(t) A_{E(t^*)} B_{D(t^*)}$ form a decreasing sequence, $\overrightarrow{OM_{n+1}(t)}$ converges uniformly to $\overrightarrow{OF(t^*)}$ for all $t \in [0, 2\pi]$, where $F(t^*)$ lies on $\overline{A_{E(t^*)} B_{D(t^*)}}$. Now we vary $t^* \in [0, 2\pi]$, since both $D(t^*)$ and $E(t^*)$ are closed curves, $M_{n+1}(t)M_n(t) \to 0$ for all $t \in [0, 2\pi]$, we see $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to the curve $F(t^*)$, where each of the point $F(t^*)$ lies on $\overline{D(t^*)E(t^*)}$.■

**Computationally**, we assume $\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}$ converges to a real solution of $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$, where $t \in [0, 2\pi]$. We see

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}\left(1 - \frac{r_3 \cdot OD}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}\right) = \frac{r_1\sqrt{p(t)^2 + q(t)^2}}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}\overrightarrow{OD}.$$

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}\frac{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD} = \left(\begin{array}{l} \frac{r_1\sqrt{p(t)^2+q(t)^2}}{r_1\sqrt{p(t)^2+q(t)^2}+r_2 OE+r_3 OD}\begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \\ +\frac{r_2 OE}{r_1\sqrt{p(t)^2+q(t)^2}+r_2 OE+r_3 OD}\begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} \end{array}\right)$$

It amounts to find the real solutions for $p\,(t)$ and $q\,(t)$ from the two equations (15) and (16) in terms of $t$, when $r_1, r_2, r_3$ are given.

$$\begin{bmatrix} p\,(t) \\ q\,(t) \end{bmatrix} = \frac{r_1\sqrt{p\,(t)^2 + q\,(t)^2}}{r_1\sqrt{p\,(t)^2+q\,(t)^2} + r_2\sqrt{e_1(t)^2+e_2(t)^2}} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \tag{15}$$

$$+\frac{r_2\sqrt{e_1(t)^2+e_2(t)^2}}{r_1\sqrt{p\,(t)^2+q\,(t)^2} + r_2\sqrt{e_1(t)^2+e_2(t)^2}} \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix}. \tag{16}$$

In the next Example, we shall see how the graphs of the square distance functions can be used as a conjecture if the convergence of $\{M_n\,(t)\}_{n=1}^{\infty}$ is uniform. Secondly, we will see how the real solutions from solving for $p\,(t)$ and $q\,(t)$ computationally from the two equations (15) and (16) can serve as partial solution for the parametric curve $F\,(t) = \begin{bmatrix} p\,(t) \\ q\,(t) \end{bmatrix}$ under the uniform convergence of $\lim_{n\to\infty}\{M_n\,(t)\}_{n=1}^{\infty}$.

**Example 11** *Let $C$ be the given ellipse curve $[x_0(t), y_0(t)] = [a\cos t, b\sin t]$, $D$ be the closed curve of $[d_1(t), d_2(t)] = [(\sin 2t + 2)\cos t, (\sin 2t+2)\sin t]$, and $E$ be the closed curve of $[(a - \cos(bt)\cos t +$ Let $Q$ be a moving point on $C$. We are interested in the plot of $\lim_{n\to\infty} M_n\,(t)$, see (14), when $n \to \infty$.*

1. We consider $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}, r_3 = \frac{1}{6}, a = 5, b = 3$. In addition, it is also worth noting that the square distance function

$$f_n(t) = (x_n(t) - x_{n-1}(t))^2 + (y_n(t) - y_{n-1}(t))^2$$

converges to 0 rather quickly in this case. We depict the pair functions $\{f_4(t), f_5(t)\}$ and $f_5(t)$ in the following Figures 6(a) and 6(b) respectively. Consequently, we may use these observations to conjecture that the convergence of $\{M_n\,(t)\}_{n=1}^{\infty}$ is uniform.

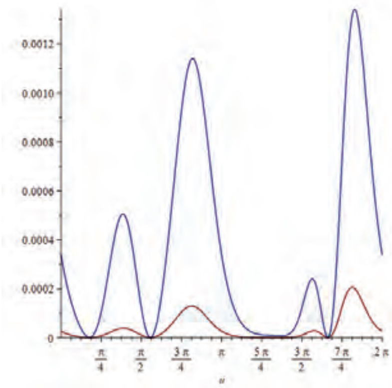



Figure 6(a). Plots of $\{f_4(t), f_5(t)\}$.

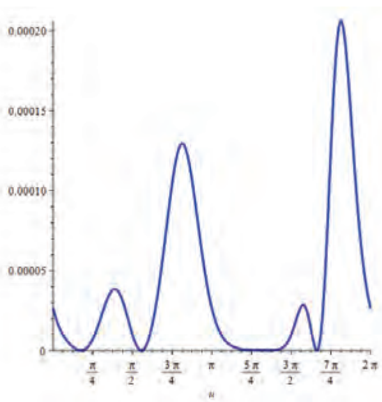



Figure 6(b). Plot of $f_5(t)$.

2. If we plot the real solutions of the branch 1, out of four branches when solving two equations (15) and (16), it coincides 'almost' exactly with that of $M_5(t) = \begin{bmatrix} x_5(t) \\ y_5(t) \end{bmatrix}$, see Figure 7 below, which we cannot tell them apart. See Supplementary Electronic Material [S1].

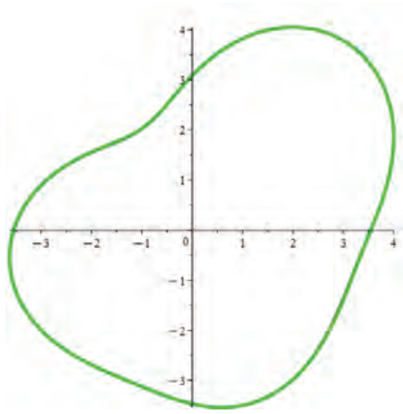



Figure 7. Graph of $M_5(t)$.

**Exercise:** We invite readers to explore that the plot of the curve $\lim_{n\to\infty} M_n(t)$, see (10), is invariant with the choice of curve $C = [x_0(t), y_0(t)]$.

## 5 3D Locus of one moving vector and one fixed vector

We invite readers to extend results in this paper from 2D to 3D accordingly. However, due to the limited length of the paper, we consider only the following 3D extension from our analogous scenario in 2D. First, we remind readers to interpret the uniform convergence in 2D (see (3) accordingly in 3D. The following observation is clear.

**Theorem 12** *Let $S$ be a given closed surface $[x_0(u_1,u_2), y_0(u_1,u_2), z_0(u_1,u_2)]$, and the point $A = (p_1, q_1, w_1)$ is fixed and is not on the surface $S$. For $r_1$ and $r_2$ being two distinct real numbers in $(0,1)$, we let*

$$\overrightarrow{OM_1} = \begin{bmatrix} x_1(u_1,u_2) \\ y_1(u_1,u_2) \\ z_1(u_1,u_2) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ + r_2 OA}\overrightarrow{OQ},$$

*where $Q$ is a moving point on $S$, and the locus $M_1$ is described in $(x_1(u_1,u_2), y_1(u_1,u_2), z_1(u_1,u_2))$. Now for $n \in \mathbb{Z}^+$, we consider*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(u_1,u_2) \\ y_{n+1}(u_1,u_2) \\ z_{n+1}(u_1,u_2) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA}\begin{bmatrix} x_n(u_1,u_2) \\ y_n(u_1,u_2) \\ z_n(u_1,u_2) \end{bmatrix},$$

*where $Q_n$ is a moving point on $[x_n(u_1,u_2), y_n(u_1,u_2), z_n(u_1,u_2)]$. Then $\overrightarrow{OM_n(u_1,u_2)} \to \overrightarrow{OA}$ as $n \to \infty$ uniformly, $\overrightarrow{M_n(u_1,u_2)\, M_{n-1}(u_1,u_2)}$ converges uniformly to 0, and $\{M_n(u_1,u_2)\}_{n=1}^{\infty}$ converges uniformly to the point $A$ for all for all $(u_1,u_2) \in [0,2\pi] \times [0,2\pi]$.*

**Proof:** The convergence of $\{M_n(u_1, u_2)\}_{n=1}^{\infty}$ follows directly from the corresponding 2D Theorem (5), which we omit here. ■

**Example 13** *Let $S$ be the given closed surface*

$$[x_0(u_1, u_2), y_0(u_1, u_2), z_0(u_1, u_2)] = [5\cos(u_1)\sin(u_2), 4\sin(u_1)\sin(u_2), 3\cos(u_2)],$$

*and the point $A = (1, 2, 3)$ be fixed. For $r_1$ and $r_2 \in (0, 1)$, and*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(u_1, u_2) \\ y_{n+1}(u_1, u_2) \\ z_{n+1}(u_1, u_2) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA}\begin{bmatrix} x_n(u_1, u_2) \\ y_n(u_1, u_2) \\ z_n(u_1, u_2) \end{bmatrix}.$$

*Then $\{M_n(u_1, u_2)\}_{n=1}^{\infty}$ converges uniformly to the point $A$.*

We depict the convergence for $r_1 = \frac{1}{3}$ and $r_2 = \frac{2}{3}$, and the plots of $\left\{\overrightarrow{OM_2}, \overrightarrow{OM_3}, \overrightarrow{OM_4}, \overrightarrow{OM_5}\right\}$ and the point $A = (1, 2, 3)$ in Figure 8:

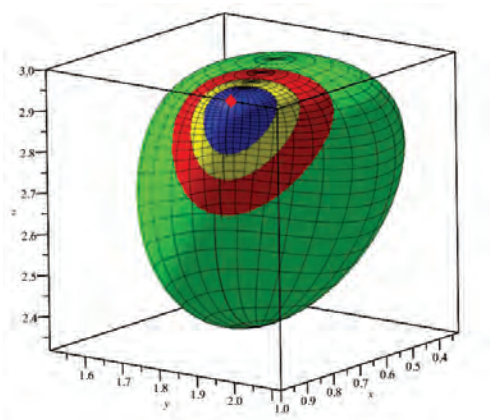


Figure 8. 3D convergence to a point.

It is natural to observe that the uniform convergence of $\{M_n(u_1, u_2)\}_{n=1}^{\infty}$ to the point $A$ will be invariant when starting with difference surfaces, which we demonstrate this using difference closed surfaces next.

**Example 14** *If we replace $S$ to be the closed surface of $S_2 = [\cos(u_1)\sin(u_2), \sin(u_1)\cos(u_2), \cos(u_2)+1]$, and the point $A = (1, 2, 3)$ be fixed. Furthermore, we pick $r_1 = \frac{1}{3}$, and $r_2 = \frac{2}{3}$, we depict the nested plots of $\{M_2(u_1, u_2), M_3(u_1, u_2), M_4(u_1, u_2), M_5(u_1, u_2)\}$ and the point $A = (1, 2, 3)$ below on Figure 9(a). The plot of $M_5(u_1, u_2)$ and the point $A$ (shown in red) is depicted in the Figure 9(b). We also plot the Figure 8 together with Figure 9(a) in Figure 9(c) below, which* *we can see both sequences of closed surfaces converge to the same point A.*

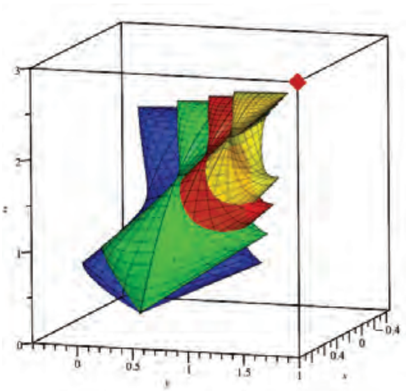


*Figure 9(a). Sequence of surfaces converge to the point A.*

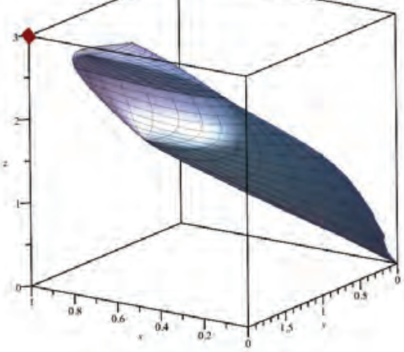


*Figure 9(b). The plots of $M_5(u_1, u_2)$ and A.*

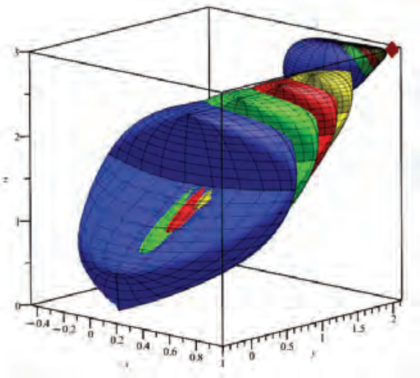


*Figure 9(c). Convergences do not depend on the original surface C.*

**Exercise**: If we use the same point $A$, and same coefficients $r_1 = \frac{1}{3}$ and $r_2 = \frac{2}{3}$, but use the surface $S_3$ of $\begin{bmatrix} 2\cos(u_1)\sin(u_1)\cos(u_1)\sin(u_2)+1 \\ 2\cos(u_1)\sin(u_1)\sin(u_1)\sin(u_2)+2 \\ 2\cos(u_1)\sin(u_1)\cos(u_2)-3 \end{bmatrix}$ as expected, we should see another sequence of surfaces converge uniformly to the same point $A$ (shown in red in Figure 10).

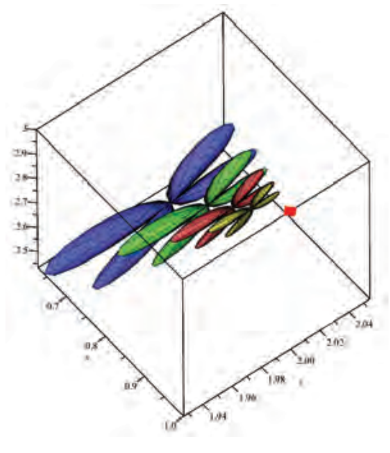


Figure 10. Convergence of $S_3$ and $A$.

# 6 Conclusions

We first remark that there are many other areas that readers can extend from this paper. For example, there are several other 3D scenarios that we have not explored from the corresponding 2D cases. In addition, we can also extend the plots of convex combinations to the plots of conical combinations both in 2D and 3D. Nevertheless, in this paper, we have seen some interesting graphics that resulted from sequence of convex combinations of vectors in 2D and 3D. Also, readers should have gained some insights how we can comprehend a complex concept of uniform convergence of sequences of parametric curves or surfaces. As a reminder, we indeed extended a simple college exam practice problem on locus into various interesting exploratory activities, both in 2D and 3D settings. Consequently, these exploratory activities have led to many interesting areas of computer graphics by integrating mathematical knowledge in Multivariable Calculus, Advanced Calculus, and Linear Algebra. We thus propose that a math curriculum should include proper components of exploration with the help of technological tools, especially where real life applications can be found.

It is common sense that teaching to a test can never promote creative thinking skills, it could even lose potential students who might pursue mathematics related fields in the future. We know that addressing the importance and timely adoption of technological tools in teaching, learning and research can never be wrong. Access to technological tools has motivated us to rethink how mathematics can and should be presented more interestingly and also how mathematics can be made a more cross disciplinary subject. There is no doubt that evolving technological tools have helped learners to discover mathematics and to become aware of its applications.

## 7 Acknowledgements

Author would like to express sincere thanks to Harald Pleym of Norway for writing sequence of plots using Maple [1].

## 8 Supplementary Electronic Materials

[S1] A Maple file for Example 10:

$$https://atcm.mathandtech.org/EP2023/invited/22003/ATCM2023.mw$$

## References

[1] Maple: A product of Maplesoft, see http://maplesoft.com/.

[2] Convex combination: https://en.wikipedia.org/wiki/Convex_combination.

[3] https://en.wikipedia.org/wiki/Homothety

[4] Yang, W.-C. Locus, Parametric Equations and Innovative Use of Technological Tools (pp. page 120-133). Proceedings of the 21st ATCM, the electronic copy can be found at this URL: https://atcm.mathandtech.org/EP2016/invited/4052016_21300.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.

[5] Yang, W.-C. From Static Locus Problems to Exploring Mathematics with Technological Tools (pp. page 67-88). The Electronic Journal of Mathematics and Technology, Volume 11, Number 2, ISSN 1933-2823, Mathematics and Technology LLC.

# Analysis of Progressive Casino Game Betting Systems

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***Abstract:*** *This work is primarily the product of the first author (who is also the presenting author), a student who completed the work under the direction of the secondary authors. We analyze three progressive betting strategies, each applied to three casino games, aimed at identifying optimal strategies after a given number of bets. The strategies analyzed are Martingale, Paroli, and Fibonacci, each of which are applied to the casino games blackjack, roulette, and craps, with bets placed that pay 1:1. The purpose of this work is not to try to discover methods for beating the house, which are known to not exist, but rather to search for methods for advancing gameplay through a maximum number of bets while retaining the possibility of earning a profit. Programming in the computer algebra system Maple will be used for the calculations.*

## 1 Introduction

Casino gamblers have been attempting to beat the house since organized gambling was introduced between the 7th and 10th centuries during the Chinese Tang dynasty [6]. Players have attempted to increase their odds of winning through both legal and illegal means, using playing strategies such as choosing when to make a particular decision or how much to wager on a specific outcome. The motivation for this paper came from a run of bad luck by one of the authors at a casino over several months. The hope was that by using some mathematical analysis, we would be able to see how to use betting strategies to skew outcomes in our favor over the short or long term.

Please note that we do not wish to give the impression that anything can be done to change the odds to players or the house. Rather, we only wish to try to identify a betting strategy that allows players to limit risk while retaining the potential for success. With a focus on games with around 50% probability of success, or, as it is known, "1:1 (one-to-one) odds," by applying the betting systems, we propose that a player could play a chosen number of bets with only a minimal loss or even a potential profit. The three games which will be played through simulation are blackjack, craps, and roulette, to which the three betting systems Martingale, Paroli, and Fibonacci will be applied.

# 2 The Games

## 2.1 Blackjack

Blackjack has a convoluted origin, with researchers unsure of its inception. The most common theory gives credit to early 18th century France with the invention of the game *Vingt-et-Un*, which translates to *Twenty-One*. Under the reign of King Louis XV, the game was played at French Royal Court, and by the early 19th century it had made its way into the streets of New Orleans, though played with a slight variation on the rules as compared to today. When gambling became legalized in Nevada during the 20th century, the game began being hosted by casinos and gambling halls. Its name was changed to blackjack with the hopes of growing its popularity, and additional payouts were given to players who won with a black Jack (the Jacks of clubs or spades) or the Ace of spades. These bonus payouts eventually became less common after the game did indeed grow in popularity [3].

A few variations exist for the rules of blackjack, although the odds of winning change only a very small amount. The standard rules for blackjack and the parameters for our analysis are:

1. All cards are assigned point values. Cards numbered 2–10 are assigned the number of points that match their face value, while Kings, Queens, and Jacks are all assigned 10 points each. Aces can be assigned either 1 or 11 points, as chosen on each deal by the player, who presumably chooses whichever option benefits them the most.

2. The dealer deals the player two cards, while the dealer receives two cards with one facing up. The player then decides whether to "hit," meaning take an additional card, or "stand," meaning do not take an additional card. If the player hits, then they decide again whether to hit or stand, and they continue in this manner until they either stand or "bust," meaning they have more than 21 total points. The player loses if they bust. If the player stands with fewer than 21 points, then the dealer repeatedly draws cards until they either beat the player or bust. In this scenario the house loses only if the dealer busts.

3. For our analysis, we will assume eight full decks of cards are used, and the dealer stands on a soft 17, meaning that if the dealer has any combination with an ace in their hand that could be counted as either 7 or 17, the dealer stands. Also, if the player's first two cards are a "blackjack," meaning any Ace and any other card worth 10 points for a total of 21 points on the deal, the payoff to the player is 3:2, meaning any bet earns a profit of 150%. Finally, the player will follow the basic strategy of the betting systems analyzed, which dictate gameplay based on scenarios. Although electronics cannot be used at blackjack tables, basic strategies can be printed and used at tables legally.

Following these rules, the odds are in the dealer's favor, but by a margin of only 0.43096% [5]. That is, the dealer wins on average 50.21548% of the time, while the player wins 49.78452% of the time.

## 2.2 Craps

Craps is a dice game also with a convoluted origin. The most common origin story for craps is that it was invented during the Crusades, when it was known as *Hazzard*, and grew in popularity during the gambling boom in 17th century France. As with blackjack, craps came to the United States via New Orleans as a street game called *street craps*. The term *craps* comes from the French word *crapaud*, meaning *toad*, which a person tended to resemble as they crouched over to play the game on a floor or sidewalk. The American version of craps saw a few minor rule changes over the years, but the game has basically been played in a very similar manner for many years.

Craps consists of 27 bets that can be played in prediction of the sum of two dice rolled by a *shooter* [4]. A few bets are inverses of each other, so playing both during the same roll would not be logical, but any other combination of bets is reasonable. The parameters for our analysis are:

1. The player will only play the *Field* bet, meaning the sums 3, 4, 9, 10, and 11 are 1:1 winners, and the sums 2 and 12 are 2:1 winners since they each result from the roll of two dice with only one possible combination.

The odds for the Field craps bet are in the house's favor by a margin of 5.56% [4].

### 2.3 Roulette

Roulette is a game played with a small white ball, called the *pellet*, spinning around a horizontal wheel, on which it eventually settles into one of the following 37 or 38 numbered positions: ones labeled 1–36 split between red and black with 18 positions for each, one labeled 0, and, sometimes, also one labeled 00. The roulette wheel was the accidental invention of the famed French mathematician Blaise Pascal when he was attempting to create a continuous motion mechanism requiring no outside force to stay in motion [8]. While Pascal's intended experiment failed, he nonetheless succeeded in creating the roulette wheel. As with craps, there are many different ways one can bet on roulette, including betting on individual numbers, splitting a bet between two or four numbers, or dividing the table into thirds by betting on 1–12, 13–24, or 25–36. The parameters for our analysis are:

1. The player will make the same even-money bet on each spin, which can be the even/odd where a player choosing even wins if a nonzero even number hits and a player choosing odd wins if an odd number hits. The player can also bet the high/low, which pays for either 1–18 or 19–36. The player can also bet on black or red, which pays if the correct color is chosen. All of these bets pay 1:1.

2. The board has 38 spaces, including both 0 and 00 positions, since this is the most common board used today. With this board, all even-money bets will have probability of success of $\frac{18}{38}$.

The odds for an even-money roulette bet are in the house's favor by a margin of 5.26% [5].

## 3 The Betting Systems

### 3.1 The Martingale System

The Martingale system, which was introduced by the French mathematician Paul Pierre Lévy in the 18th century, is likely the most common progressive betting system used today. Legend states that the system derives its name from an early 18th century London casino owner named John H. Martindale, who encouraged patrons to apply it in his own casino, which ironically led, as claimed by some, to its bankruptcy. The Martingale system has transcended standard casino gambling though, and is now used in many nontraditional forms of gambling, including investment banking, stock brokering, and sports betting. The rules of the Martingale system are straightforward; with each loss, one doubles their bet, until they win. Although starting bet sizes of 5% of one's bankroll are more common in practice, we will use 1% in our analysis in order to hopefully more clearly see long-term trends [2].

The following table shows an example of the Martingale system in practice, with a beginning bankroll of $100, and a starting bet size of $1.

| Round | Bet | Win or Loss | Bankroll |
|---|---|---|---|
| | | | $100 |
| 1 | $1 | Loss | $ 99 |
| 2 | $2 | Loss | $ 97 |
| 3 | $4 | Loss | $ 93 |
| 4 | $8 | Win | $101 |
| 5 | $1 | Win | $102 |

Under the Martingale system with a beginning bankroll of $100, after six consecutive losses it would be not possible to make a seventh bet. Nonetheless, even though Martingale is indeed a high risk system, it is also true that for one whose beginning bankroll is sufficient to fund the next bet after $n$ consecutive losses, a win on that next bet would return the beginning bankroll plus the amount of the first bet. Exhausting one's bankroll is not the only drawback to the system though. Casinos also often have betting limits, so one might reach a point where doubling after a loss would not be allowed.

The following code written for the computer algebra system Maple is for analysis of Martingale applied to blackjack, stopping if 25 rounds are reached. The code is unique to Maple and Martingale, but could easily be altered for any programming language and any game as long as the player knows the house edge, the probability of winning a round, and the maximum number of rounds to be played.

```
randomize():
with(Statistics):
Success := Vector[row](1 .. 25):
Running_Total := Vector[row](1 .. 25):
End_Value := Vector[row](1 .. 1000):
Num_Turns := Vector[row](1 .. 1000):
for j to 1000 do
    for k to 25 do
        r := rand(1.0 .. 2.0):
        if r() < evalf(1 + 0.4978452) then Success[k] := "W"
           else Success[k] := "L"
        fi
    od:
    initial_value := 500:
    bet := 5:
    for i to 25 do
        if Success[i] = "W" then initial_value := initial_value + bet:
                bet := 5: Running_Total[i] := initial_value:
           else initial_value := initial_value - bet: bet := 2*bet:
                Running_Total[i] := initial_value:
           if initial_value < bet then break
           fi:
        fi
    od:
```

```
        Num_Turns[j] := i - 1:
        End_Value[j] := Running_Total[i - 1]:
        if Num_Turns[j] < 25 then End_Value[j] := Running_Total[i]:
           else End_Value[j] := Running_Total[i - 1]
        fi:
    od:
    Num_Turns; End_Value; Mean(Num_Turns);
    Mean(End_Value); max(End_Value); min(End_Value);
```

The following table shows the result of applying 1000 trials of the Martingale betting system to each game, stopping if 25, 50, or 100 rounds are reached, with a beginning bankroll of $500 and initial bet size of $5 for each. Within each cell of the table, the first number is the average number of rounds played before either the indicated number of rounds was reached or the next bet could not be made, and the second number is the average final bankroll.

| **Rounds** | **Blackjack** | **Craps** | **Roulette** |
|---|---|---|---|
| 25 | 23.59 / $509.10 | 22.76 / $482.60 | 23.33 / $502.25 |
| 50 | 42.95 / $504.99 | 40.96 / $473.82 | 40.56 / $475.34 |
| 100 | 72.56 / $498.88 | 65.69 / $442.77 | 66.99 / $441.38 |

As we see, blackjack shows the highest likelihood of success, with almost 2% in expected profit with at most 25 rounds, and less than 0.2% in expected loss with at most 100 rounds. The following graphs show the distribution of the 1000 trials for all three games with at most 25 rounds, all of which show a bimodal distribution with left skewness and a majority of trials ending in a profit.

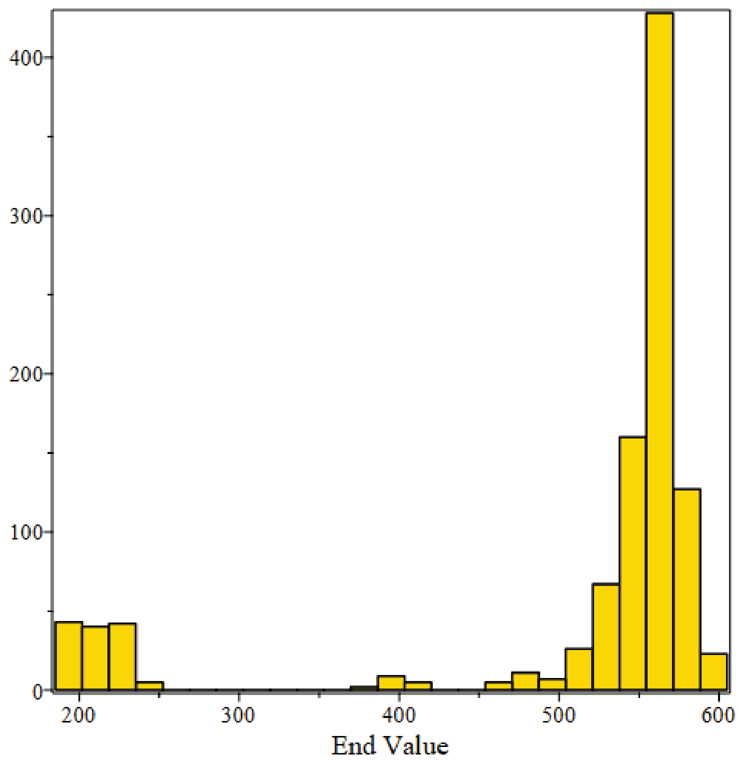


Figure 1: Martingale Frequency of Ending Values for Blackjack With at Most 25 Rounds

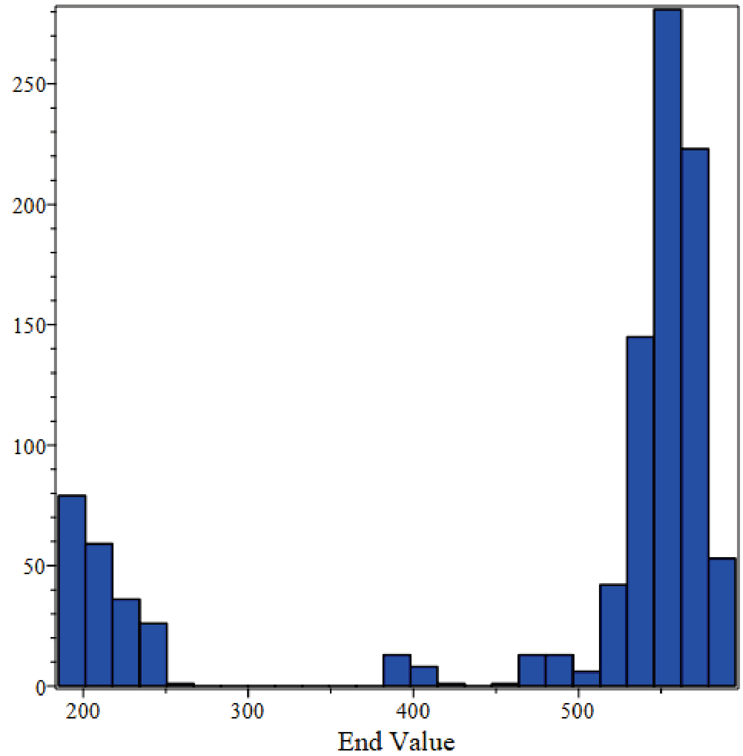


Figure 2: Martingale Frequency of Ending Values for Craps With at Most 25 Rounds

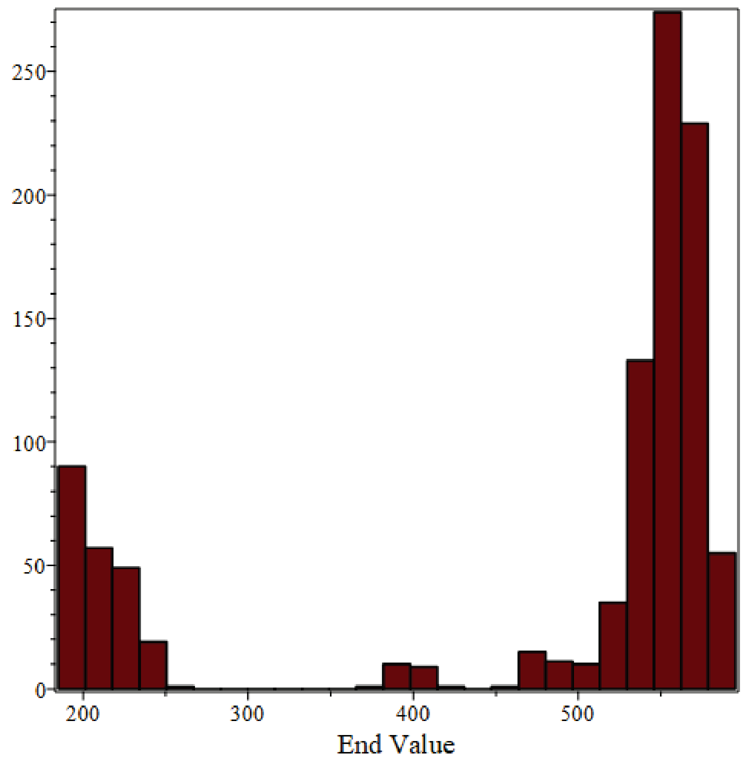


Figure 3: Martingale Frequency of Ending Values for Roulette With at Most 25 Rounds

### 3.2 The Paroli System

The Paroli betting system derives its name from the Latin word *par*, which carries the meaning *to break even*. This system has been credited as a 17th century invention of the French mathematician Blaise Pascal, who we have noted is also credited with inventing the roulette wheel. While Martingale is a high-risk betting system, the Paroli betting system, also known as "anti-Martingale," allows players to set a maximum amount of money they are willing to lose with each play [1]. Under the Paroli system, players double their bet after each round until they have achieved either three consecutive wins or a loss, after which they return to their initial bet amount [5]. For example, the following table shows an example of the Paroli system in practice, with a beginning bankroll of $100, and a starting bet size of $1.

| **Round** | **Bet** | **Win or Loss** | **Bankroll** |
|---|---|---|---|
| | | | $100 |
| 1 | $1 | Win | $101 |
| 2 | $2 | Win | $103 |
| 3 | $4 | Win | $107 |
| 4 | $1 | Loss | $106 |
| 5 | $1 | Win | $107 |

For a player with a starting bet size of $1, note that under the Paroli system, three consecutive wins results in a profit of $7. In gambling terminology, we would thus say that under the Paroli system, three consecutive wins pays "7 to 1." Since each round offers a chance of winning of approximately $\frac{1}{2}$, it follows that the probability of winning three consecutive rounds is approximately $\frac{1}{2^3} = \frac{1}{8}$. That is, under the Paroli system, players are taking a chance on earning a profit of 7 to 1 against odds of winning of 8 to 1.

The following code written for Maple is for analysis of Paroli applied to roulette with 50 rounds. The code is unique to Maple and Paroli, but could easily be altered for any programming language and any game as long as the player knows the house edge, the probability of winning a round, and the number of rounds to be played.

```
randomize():
with(Statistics):
Success := Vector[row](1 .. 50):
Running_Total := Vector[row](1 .. 50):
End_Value := Vector[row](1 .. 1000):
Num_Turns := Vector[row](1 .. 1000):
bet_placed := Vector[row](1 .. 50):
for j to 1000 do
    for k to 50 do
        r := rand(1.0 .. 2.0):
        if r() < evalf(1 + 18/38)
           then Success[k] := "W"
           else Success[k] := "L"
        fi
    od:
    initial_value := 500:
```

```
        bet := 5:
        for i to 50 do
            if bet > 20 then bet := 5
            fi:
            if Success[i] = "W"
               then bet_placed[i] := bet:
                    initial_value := initial_value + bet:
                    bet := 2*bet:
                    Running_Total[i] := initial_value:
               else bet_placed[i] := bet:
                    initial_value := initial_value - bet:
                    bet := 5:
                    Running_Total[i] := initial_value:
                    if initial_value < bet then break
                    fi:
            fi
        od:
        Num_Turns[j] := i - 1:
        End_Value[j] := Running_Total[i - 1]:
    od:
    Num_Turns;
    End_Value;
    Mean(Num_Turns);
    Mean(End_Value);
    max(End_Value);
    min(End_Value);
```

The following table shows the result of applying 1000 trials of the Paroli betting system to each game, stopping when 25, 50, or 100 rounds are reached, with a beginning bankroll of \$500 and initial bet size of \$5 for each. Within each cell of the table, the number is the average final bankroll after the indicated number of rounds was played.

| **Rounds** | **Blackjack** | **Craps** | **Roulette** |
|---|---|---|---|
| 25 | \$501.79 | \$488.11 | \$489.81 |
| 50 | \$500.77 | \$476.43 | \$478.01 |
| 100 | \$495.32 | \$455.11 | \$455.39 |

In comparison with the Martingale system, the Paroli system rarely returns a large profit. However, under Paroli players can guarantee that they would be able to play any specified number of rounds, since the most they could lose on any single round would be the value of their initial bet. For example, if a player wanted to guarantee that they would be able to play 25 rounds, then they could start by betting 4% of their bankroll, since this would require 25 consecutive losses for them to lose their entire bankroll.

The following graphs show the distribution of the 1000 trials of the Paroli system for all three games with 100 rounds. All of the graphs show a more normal distribution for the Paroli system than was seen with Martingale, with more predictability and less volatility. In particular, note that the ending values rarely dip below \$300.

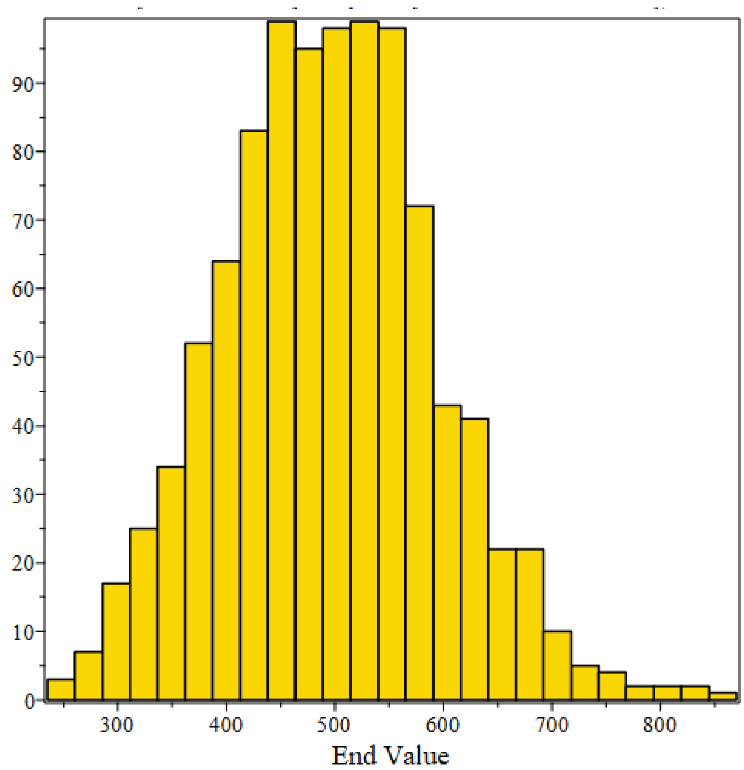


Figure 4: Paroli Frequency of Ending Values for Blackjack With 100 Rounds

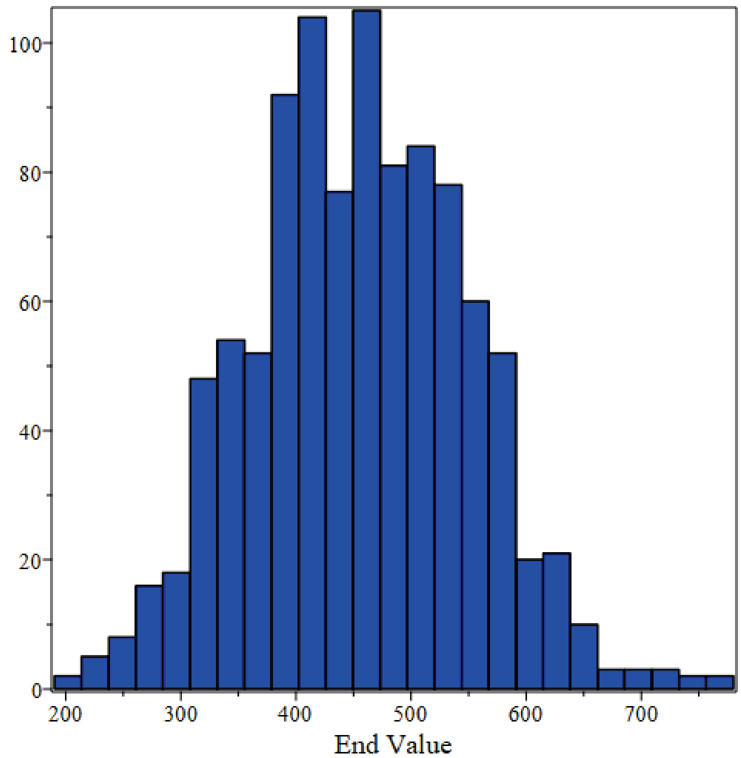


Figure 5: Paroli Frequency of Ending Values for Craps With 100 Rounds

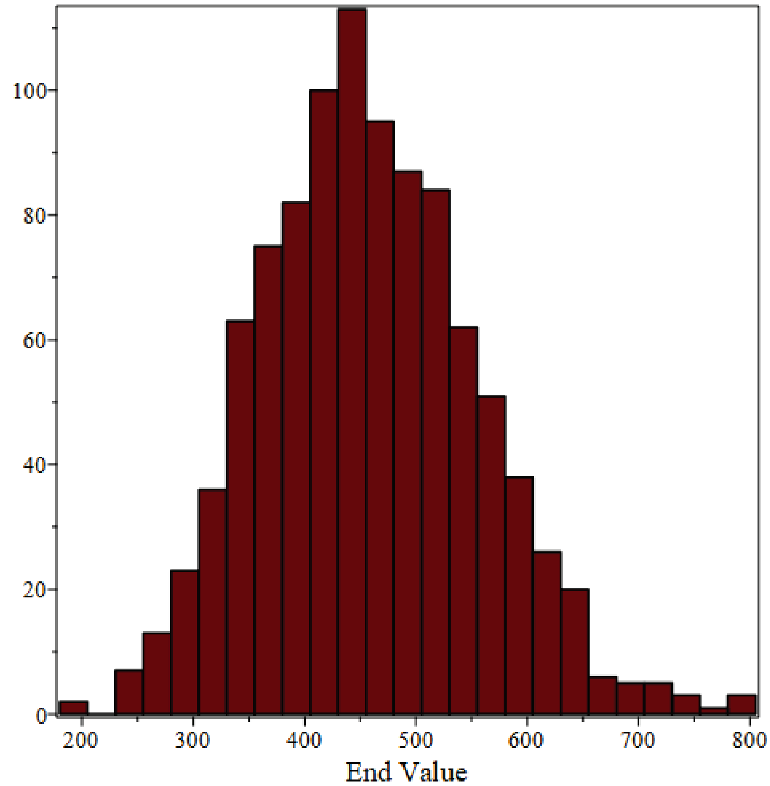


Figure 6: Paroli Frequency of Ending Values for Roulette With 100 Rounds

### 3.3 The Fibonacci System

The Fibonacci betting system connects naturally with a number sequence first studied by mathematicians in India going as far back as at least 200 BCE. This number sequence was brought to Western Europe by an Italian mathematician named Fibonacci, who included it in his 1202 publication *Liber Abaci* [7], after which it became one of history's most prolifically studied number sequences. In the 19th century the sequence began being referred to as the *Fibonacci sequence*. The Fibonacci sequence begins with the pair of terms $F_0 = 0$ and $F_1 = 1$. Subsequent terms $F_i$ after these first two are the sum of the previous two terms: $F_n = F_{n-1} + F_{n-2}$ (for $n > 1$). That is, the Fibonacci sequence is: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... .

Given the myriad ways in which the Fibonacci sequence appears in nature, it only makes sense that it should be applied to casino games and nontraditional forms of gambling. That being said, there is no certainty that the Fibonacci system has ever actually been used in practice. The Fibonacci sequence does lend itself perfectly to a progressive betting system though, since the goal of any such system should be to make up for as much loss as possible with a single win.

In the Fibonacci betting system, each bet a player makes is the starting bet multiplied by a multiplier. For the first bet, players multiplying their starting bet by the number in the third position of the sequence ($F_2 = 1$). After each loss, the bet multiplier moves forward one position in the sequence (from $F_i$ to $F_{i+1}$). After each win, the bet multiplier moves back two positions in the sequence (from $F_j$ to $F_{j-2}$), though never moving past the third number in the sequence [5]. For example, the following table shows an example of the Fibonacci system in practice, with a beginning bankroll of \$100, and a starting bet size of \$1.

| Round | Bet | Win or Loss | Bankroll |
|---|---|---|---|
| | | | $100 |
| 1 | $1 | Loss | $ 99 |
| 2 | $2 | Loss | $ 97 |
| 3 | $3 | Loss | $ 94 |
| 4 | $5 | Win | $ 99 |
| 5 | $2 | Win | $101 |

Note that in this example, after three consecutive losses, it took two consecutive wins in order for a profit to be earned. As with many other systems, the drawback to the Fibonacci system is that several consecutive losses can very quickly lead to very large bets. For this reason, literature on the system often recommends that users start by betting between only 1% and 5% of their beginning bankroll.

The following code written for Maple is for analysis of Fibonacci applied to craps, stopping if 100 rounds are reached. This code is unique to Maple and Fibonacci, but could easily be altered for any programming language and any game as long as the player knows the house edge, the probability of winning a round, and the maximum number of rounds to be played.

```
randomize():
with(Statistics): with(combinat):
Success := Vector[row](1 .. 100):
Running_Total := Vector[row](1 .. 100):
End_Value := Vector[row](1 .. 1000):
Num_Turns := Vector[row](1 .. 1000):
bet_placed := Vector[row][1 .. 100]:
for j from 1 to 1000 do
    for m  from 1 to 100 do
        r := rand(1.0 .. 2.0):
        if r() < evalf(1 + 0.4722) then Success[m] := "W"
           else Success[m] := "L"
        fi
    od:
    initial_value := 500: bet := 5: k := 1:
    for i from 1 to 100 do
        if k < 1 then k := 1:
        fi:
        if Success[i] = "W" then bet_placed[i] := bet:
                bet := 5*fibonacci(k + 1):
                initial_value := initial_value + bet: k := k - 2:
                Running_Total[i] := initial_value:
           else bet_placed[i] := bet:
                initial_value := initial_value - bet:
                bet := 5*fibonacci(k + 1): k := k + 1:
                Running_Total[i] := initial_value:
                if initial_value < bet then break
                fi:
        fi
```

```
        od:
        Num_Turns[j] := i - 1:
        End_Value[j] := Running_Total[i - 1]:
        if Num_Turns[j] < 100 then End_Value[j] := Running_Total[i]:
           else End_Value[j] := Running_Total[i - 1]
        fi:
    od:
    Num_Turns; End_Value; Mean(Num_Turns);
    Mean(End_Value); max(End_Value); min(End_Value);
```

The following table shows the result of applying 1000 trials of the system to each game, stopping if 25, 50, or 100 rounds are reached, with a beginning bankroll of $500 and initial bet size of $5. Within each cell, the first number is the average number of rounds played before the indicated number of rounds was reached or the next bet could not be made, and the second is the average final bankroll.

| Rounds | Blackjack | Craps | Roulette |
|---|---|---|---|
| 25 | 24.73 / $504.47 | 24.54 / $496.19 | 24.41 / $485.67 |
| 50 | 47.49 / $513.69 | 46.32 / $467.82 | 46.67 / $479.24 |
| 100 | 90.22 / $530.51 | 85.08 / $450.14 | 86.13 / $461.16 |

As we see, blackjack shows a profit for every number of rounds. The Fibonacci system also shows similar results to the Paroli system, with even with a better chance of higher profit. However, unlike Paroli, the Fibonacci system does not allow players to guarantee that they would be able to play any specified number of rounds, and could also allow bets to become very large. The following graphs show the distribution of the 1000 trials for all three games with at most 50 rounds.

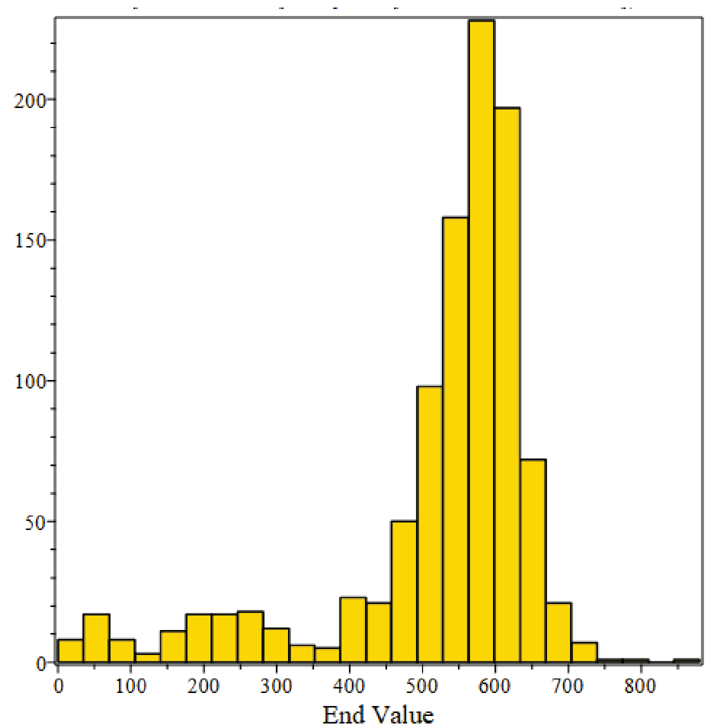


Figure 7: Fibonacci Frequency of Ending Values for Blackjack With at Most 50 Rounds

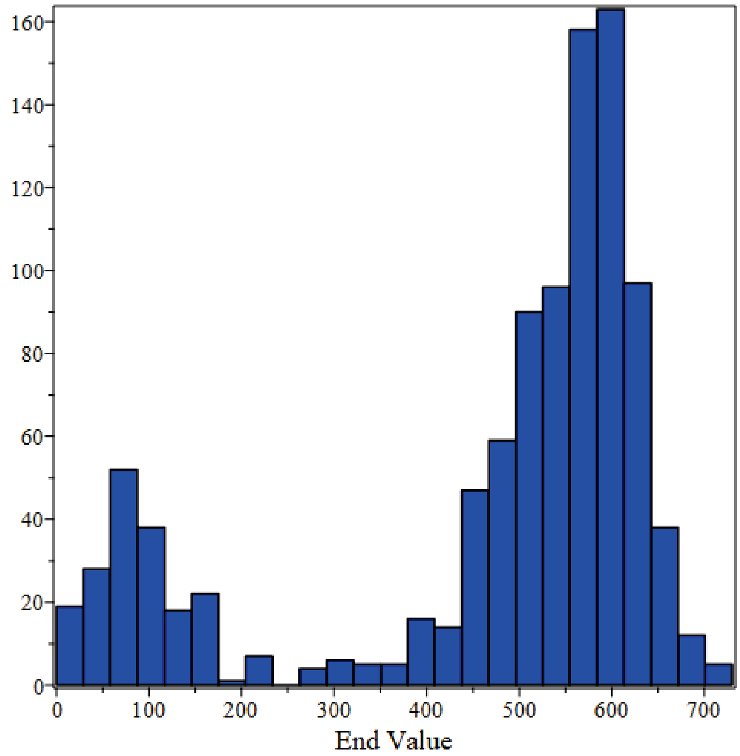


Figure 8: Fibonacci Frequency of Ending Values for Craps With at Most 50 Rounds

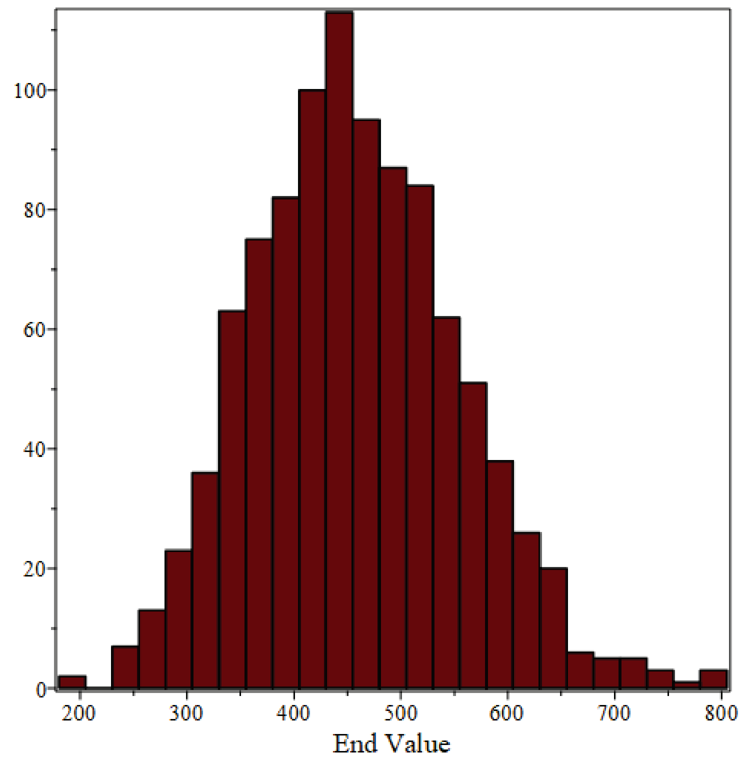


Figure 9: Fibonacci Frequency of Ending Values for Roulette With at Most 50 Rounds

## 4 Conclusions and Ideas for Future Work

As part of this project, simulations were also done using other number sequences such as Pi and Euler which show more randomness. These systems did not show promise from a gambling perspective though, as the key to a progressive betting system is the ability to regain losses as much as possible with a minimal number of wins.

For the analyses described in this paper, the Martingale betting system shows the most promise for potential profit in the short term, unless the probabilities of success on a given round are approximately 50%. However, under the Martingale system, if a player began by betting 1% of their beginning bankroll, then they would only be able to survive six consecutive losses before being unable to make the next bet. Even worse, if a player began by betting 5% of their initial bankroll, then they would only be able to survive four consecutive losses before being unable to make the next bet. As such, the player would need to decide what it is really worth to them to have a profit potential. An intelligent strategy for the Martingale system would be for a player to identify how much profit they would need before leaving the game. For example, if a player had the goal of earning $25 profit in at most 25 rounds, it would only take five wins when betting $5, and as soon as the player reached that goal, they could retire.

Contrary to the Martingale betting system, the Paroli betting system allows players to guarantee a specific number of rounds that they wish to ensure they will be able to play. More specifically, if the player chose to bet a certain percentage of their initial bankroll, then they would be able to predict a minimum number of rounds they would be guaranteed of being able to play. For example, starting by betting 1% of their beginning bankroll would guarantee that a player would be able to play a minimum of 100 rounds, and starting by betting 5% of their beginning bankroll would guarantee that a player would be able to play a minimum of 20 rounds (assuming the player completed 100 or 20 rounds, respectively, without three consecutive wins). Another favorable property of the Paroli betting system is that players can double their bets until they earn two wins in a row or four wins in a row or however many wins in a row they want to achieve. Recall though that regardless of however many consecutive wins a player wants to achieve in order to perceive the result as a success, the odds of reaching that number of wins would still need to be higher than the payoff. For example, two consecutive wins would result in a 3 to 1 payout, while the odds of winning two games in a row would be approximately 4 to 1.

Finally, recall that the Fibonacci betting system, like Martingale, allows bets to potentially become extremely high if players experience many consecutive losses. So, similar to Martingale, with the Fibonacci system it would be better for players to start by identifying a profit amount as the goal rather than a number of rounds to be played. The fact though is that all betting systems have benefits and drawbacks, and so it ultimately comes down to the goals of the player. This project only aims to investigate systems with the end goal of allowing players to play for as long as possible and to reach a reasonable number of rounds with the hope of spending a pleasant evening in a casino. The system that seems to best fit that criterion is Paroli, given that it is not for high-risk gamblers, and players can guarantee being able to play any specified number of rounds.

Further investigations could include developing an algorithm through which a player could choose a betting system from among the three presented in this paper (and possibly other systems as well), choose a game from among the three presented in this paper (and possibly other games as well), and then have the algorithm return to them the expected outcome of playing their chosen game under their chosen betting system. Understanding that no system can guarantee a profit, the player could at least make better choices that lead to a more pleasurable gaming experience.

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# Linear Algebra Computational Tool for LaTeX

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**Abstract**

*Linear algebra is used in different branches of science, engineering, and data science. There are many tools for doing computations on vectors and matrices. LaTeX is one of the most widely used typesetting systems for scientific publications, and there is often a need to type vectors and matrices inside LaTeX documents and perform different operations on them. We have developed a computational tool for linear algebra to deal with standard operations on vectors and matrices inside LaTeX. The standard practice of LaTeX users is to export computational results from other software and compile them inside LaTeX. This may be cumbersome when there are vast computations. The exported output from other software may need some editing before importing it into LaTeX as it may not be in LaTeX-compatible format or in the format that the user expects. The main aim of this paper is to give a brief introduction to the computational tool of linear algebra developed by us. This paper extends the series of basic computational tools that we developed [2, 3, 4, 5, 8, 10, 7]. It will reduce the dependence of LaTeX users on external software and can also be deployed for pedagogical uses.*

## 1 Background and Introduction

Lua programming language is a scripting language which can be embedded across platforms. With `LuaTeX` [9] and `luacode` [1] packages, it is possible to use Lua in LaTeX. TeX or LaTeX has scope for programming [12]. However, with the weird internals of TeX, there are several limitations, especially for performing calculations on numbers in LaTeX documents.

There is a good scope to perform basic mathematical computations in LaTeX using Lua. This approach is suitable for performing standard operations on real and complex numbers, sets, integers, vectors, and matrices. It involves some complex intermingling of Lua and TeX. We have developed the `luamaths` [4], `luacomplex` [2], `luaset` [8], `luagcd` [3], `luamodulartables` [5], `luanumint` [7] and `luatruthtable` [10] packages based on this approach. These packages are designed from a pedagogical perspective and enhance the computational aspect of LaTeX.

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We are extending the series by introducing a Lua-based linear algebra computational tool. It covers most of the numerical computations on vectors and matrices. No particular environment in LaTeX is required to use this tool. The time required for performing operations on vectors and matrices using the tool is not an issue due to dynamic memory management and garbage collection facility of Lua.

The article is organized in different sections. The methodology used for the development of the tool is discussed in the second section. The license and instructions for installation are given in the third section. The fourth section summarizes the key features of the tool. The pedagogical applications with a few illustrations are given in the fifth section. The resources used in the development of the tool are covered in the sixth session. The seventh section provides some limitations of the tool. The future plans and prospects of the research work are in the eighth section.

## 2 Development of the tool

The development of the linear algebra computational tool can be described in the following steps.

- The resources available in Lua are used for writing small blocks or chunks in Lua IDE. The abstraction of chunks is done by writing functions in Lua. These functions are put into a single Lua module.
- In the next step, the module is thoroughly tested in Lua independently of LaTeX. The necessary changes are made and restructuring of algorithms is done wherever required. The error handling mechanism is added to ensure the appropriate input from users.
- After testing stand alone Lua modules, the `luacode` package [1] is used to integrate the Lua module with LaTeX. This integration often gets complex as the intermingling of TeX and Lua is not straightforward. The customized environments and customized commands are written in the LaTeX package file to execute functions in the Lua module through LuaTeX. The `xkeyval` [14] package is also used in the development of the tool.
- Lua based LaTeX package is again thoroughly tested with its output in PDF file. TeX commands are modified as required.

## 3 Licensing and deployment of the tool

Lua is certified open-source software available. Its license is simple and liberal, which is compatible with GPL. This license allows users to freely copy, modify and distribute the file for any purpose and without restrictions. The tool is released under the LaTeX Project Public License v1.3c or later.

The installation of the linear algebra computational tool is similar to the plain latex package. It can be loaded with `\usepackage{lualinalg}` command in the preamble of the LaTeX document. The TeX file is to be compiled using the LuaLaTeX engine.

The `lualinalg` package is available on the CTAN repository and bundled with standard TeX distributions such as MikTeX and TeXLive. It can easily be used in the Overleaf online LaTeX editor. The source files of the tool need to be placed in the working directory of Overleaf.

# 4 Features of the tool

The tool mainly consists of two parts: operations on vectors and operations on matrices. This section summarizes key features of each of the part.

## 4.1 Operations on vectors

The vectors of reasonable size can be handled with ease. It supports vectors defined over the field of real and complex numbers. The meta-operations are used to evaluate complex expressions involving vectors. The following are some features in the vector part of the tool.

- Defining vectors : vectors are defined with the `\vectornew` command. The standard vector of dimension $n$ with $i$th coordinate 1 and other co-ordinates 0 can be produced with additional specifications in the same command.
- Operations on vectors: the following operations on vectors can be performed by using different commands available in the tool.
  - ☐ Define vectors with specified coordinates.
  - ☐ Create vectors with random coordinates.
  - ☐ Print vectors with optional arguments.
  - ☐ Parse and change coordinates of vectors.
  - ☐ Create multiple copies of vectors.
  - ☐ Perform addition, subtraction and scalar multiplication.
  - ☐ Find dot and cross product of vectors.
  - ☐ Calculate the Euclidean norm, p-norm, sum norm, and sup-norm of a vector.
  - ☐ Obtain angle between two vectors.
  - ☐ Evaluate complex expressions involving vectors using meta-operators.
  - ☐ Parse coordinates of vectors for plotting.
  - ☐ Perform Gram-Schmidt orthogonalization process with the option of producing computations step-by-step.

## 4.2 Operations on Matrices

The matrices of reasonable size can be handled with ease. It supports matrices with real and complex number entries. The meta-operations are used to evaluate complex expressions involving matrices. The following are some features in the matrix part of the tool.

- Defining Matrices: Matrices are defined with the `\matrixNew` command. The identity matrix can be defined by adding specific argument in the same command.

- Operations on matrices: The following operations on matrices can be performed by using different commands in the tool.

  - ☐ Define matrices with specified entries.
  - ☐ Create matrices with random entries.
  - ☐ Obtain number of rows and columns of a matrix.
  - ☐ Fetch and change elements of matrices from specified rows and columns.
  - ☐ Obtain a submatrix of the matrix.
  - ☐ Augment matrix horizontally and vertically.
  - ☐ Perform Gauss-Jordan elimination with option of producing computations step-by-step.
  - ☐ Create copies of matrices.
  - ☐ Calculate the 1-norm, infinity-norm, max-norm and Frobenius-norm of a matrix.
  - ☐ Perform addition, subtraction, multiplication and scalar multiplication.
  - ☐ Determine inverse, transpose, conjugate, complex conjugate, rank and trace of a matrix.
  - ☐ Perform elementary row and column operations on matrices.
  - ☐ Obtain Reduced Row Echelon Form (RREF) of a matrix with the option of producing computations step-by-step.

# 5 Pedagogical applications

- The tool can assist in creation of interactive teaching modules in LaTeX when computations on vectors and matrices are involved. A variety of problems on vectors and matrices can be generated for assignments of students by using the tool. The final computational results of problems can also be provided by using available commands in the tool.
- The beamer is a document class in LaTeX to create presentations. The tool can be used in beamer to illustrate various concepts of linear algebra while teaching.
- The tool can also be used to illustrate step-by-step procedures to obtain the rref form of a matrix.

Lualinalg code 1: Step-by-step computations of rref

```
\def\z{{{lfrac(3,2),lcomplex(3,3),5},{6,7,8},{7,0,9}}}
\matrixNew{M}{\z}
\[M = \matrixPrint{M}\]
\matrixRREFSteps{M}
```

Lualinalg code 1 generates the following output.

$$M = \begin{bmatrix} \frac{3}{2} & 3+3i & 5 \\ 6 & 7 & 8 \\ 7 & 0 & 9 \end{bmatrix}$$

Step 1: Divide row 1 by $\frac{3}{2}$.

$$\begin{bmatrix} 1 & 2+2i & \frac{10}{3} \\ 6 & 7 & 8 \\ 7 & 0 & 9 \end{bmatrix}$$

Step 2: Multiply row 1 by 6 and subtract it from row 2.

$$\begin{bmatrix} 1 & 2+2i & \frac{10}{3} \\ 0 & -5-12i & -12 \\ 7 & 0 & 9 \end{bmatrix}$$

Step 3: Multiply row 1 by 7 and subtract it from row 3.

$$\begin{bmatrix} 1 & 2+2i & \frac{10}{3} \\ 0 & -5-12i & -12 \\ 0 & -14-14i & \frac{-43}{3} \end{bmatrix}$$

Step 4: Divide row 2 by $-5-12i$.

$$\begin{bmatrix} 1 & 2+2i & \frac{10}{3} \\ 0 & 1 & \frac{60}{169}+\frac{-144}{169}i \\ 0 & -14-14i & \frac{-43}{3} \end{bmatrix}$$

Step 5: Multiply row 2 by $2+2i$ and subtract it from row 1.

$$\begin{bmatrix} 1 & 0 & \frac{466}{507}+\frac{168}{169}i \\ 0 & 1 & \frac{60}{169}+\frac{-144}{169}i \\ 0 & -14-14i & \frac{-43}{3} \end{bmatrix}$$

Step 6: Multiply row 2 by $-14-14i$ and subtract it from row 3.

$$\begin{bmatrix} 1 & 0 & \frac{466}{507}+\frac{168}{169}i \\ 0 & 1 & \frac{60}{169}+\frac{-144}{169}i \\ 0 & 0 & \frac{1301}{507}+\frac{-1176}{169}i \end{bmatrix}$$

Step 7: Divide row 3 by $\frac{1301}{507}+\frac{-1176}{169}i$.

$$\begin{bmatrix} 1 & 0 & \frac{466}{507}+\frac{168}{169}i \\ 0 & 1 & \frac{60}{169}+\frac{-144}{169}i \\ 0 & 0 & 1 \end{bmatrix}$$

Step 8: Multiply row 3 by $\frac{466}{507} + \frac{168}{169}\mathrm{i}$ and subtract it from row 1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{60}{169} + \frac{-144}{169}\mathrm{i} \\ 0 & 0 & 1 \end{bmatrix}$$

Step 9: Multiply row 3 by $\frac{60}{169} + \frac{-144}{169}\mathrm{i}$ and subtract it from row 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Some illustrations

- The tool can be used to solve numerical problems on vectors and matrices, and to verify results. We mention a few of the computations for illustration.

Lualinalg code 2: Solving problems on vectors the linear algebra computational tool

```
\vectorNew{v1}{{3,8,5}} \vectorNew{v2}{{4,7,4}} \vectorNew{v3}{{5,4,3}}
\[v1=\left(\vectorPrint{v1}\right),\]
   \[v2=\left(\vectorPrint{v2}\right),\]
   \[v3=\left(\vectorPrint{v3}\right)\]
\vectorGramSchmidtSteps[brckt=round,truncate=3]{{'v1','v2','v3'}}
```

Lualinalg code 2 produces a step-by-step computation of Gram-Schmidt orthogonalization process on vectors $v1 = (3, 8, 5)$, $v2 = (4, 7, 4)$ and $v3 = (5, 4, 3)$. Following is the output of this procedure.

$$v1 = (3, 8, 5),$$

$$v2 = (4, 7, 4),$$

$$v3 = (5, 4, 3)$$

Take given vectors as $v_1, \ldots, v_3$ in order.
Step 1:

$$u_1 = v_1 = (3, 8, 5)$$

$$e_1 = \frac{u_1}{||u_1||} = (0.303, 0.808, 0.505)$$

Step 2:

$$u_2 = v_2 - \sum_{j=1}^{1} proj_{u_j}(v_2) = (1.306, -0.184, -0.49)$$

$$e_2 = \frac{u_2}{||u_2||} = (0.928, -0.131, -0.348)$$

Step 3:

$$u_3 = v_3 - \sum_{j=1}^{2} proj_{u_j}(v_3) = (0.247, -0.66, 0.907)$$

$$e_3 = \frac{u_3}{||u_3||} = (0.215, -0.574, 0.79)$$

By using `\vectorEuclidNorm` and `\vectorDot` commands in the tool, it can be verified that $e_1$, $e_2$ and $e_3$ are pair-wise orthogonal unit vectors.

- The tool can be used to solve examples on matrices, and to verify results numerically. As an example, consider the following system of linear equations.

$$\frac{1}{4}x + y + z = 6, 2x - 3y - z = 9, 4x - y + z = 10 + \mathrm{i}$$

With the command `\matrixGaussJordanSteps` in tool, it is possible to produce step-by-step computation of Gauss-Jordan elimination of an augmented matrix.

Lualinalg code 3: Solving linear equations using the linear algebra computational tool.

```
\def\a{{{lfrac(1,4),1,1},{2,-3,-1},{4,-1,1}}}
\def\b{{{6},{9},{lcomplex(10,1)}}}
\matrixNew{S}{\a}
\matrixNew{T}{\b}
\matrixConcatH{W}{S}{T}
\[W = \matrixPrint{W}\]
\matrixGaussJordanSteps{S}{T}
```

Lualinalg code 3 generates the following output.

$$W = \begin{bmatrix} \frac{1}{4} & 1 & 1 & 6 \\ 2 & -3 & -1 & 9 \\ 4 & -1 & 1 & 10+\mathrm{i} \end{bmatrix}$$

Step 1: Divide row 1 by $\frac{1}{4}$.

$$\begin{bmatrix} 1 & 4 & 4 & 24 \\ 2 & -3 & -1 & 9 \\ 4 & -1 & 1 & 10+\mathrm{i} \end{bmatrix}$$

Step 2: Multiply row 1 by 2 and subtract it from row 2.

$$\begin{bmatrix} 1 & 4 & 4 & 24 \\ 0 & -11 & -9 & -39 \\ 4 & -1 & 1 & 10+\mathrm{i} \end{bmatrix}$$

Step 3: Multiply row 1 by 4 and subtract it from row 3.

$$\begin{bmatrix} 1 & 4 & 4 & 24 \\ 0 & -11 & -9 & -39 \\ 0 & -17 & -15 & -86+\mathrm{i} \end{bmatrix}$$

Step 4: Divide row 2 by $-11$.

$$\begin{bmatrix} 1 & 4 & 4 & 24 \\ 0 & 1 & \frac{9}{11} & \frac{39}{11} \\ 0 & -17 & -15 & -86+\mathrm{i} \end{bmatrix}$$

Step 5: Multiply row 2 by 4 and subtract it from row 1.

$$\begin{bmatrix} 1 & 0 & \frac{8}{11} & \frac{108}{11} \\ 0 & 1 & \frac{9}{11} & \frac{39}{11} \\ 0 & -17 & -15 & -86+\mathrm{i} \end{bmatrix}$$

Step 6: Multiply row 2 by $-17$ and subtract it from row 3.

$$\begin{bmatrix} 1 & 0 & \frac{8}{11} & \frac{108}{11} \\ 0 & 1 & \frac{9}{11} & \frac{39}{11} \\ 0 & 0 & \frac{-12}{11} & \frac{-283}{11}+\mathrm{i} \end{bmatrix}$$

Step 7: Divide row 3 by $\frac{-12}{11}$.

$$\begin{bmatrix} 1 & 0 & \frac{8}{11} & \frac{108}{11} \\ 0 & 1 & \frac{9}{11} & \frac{39}{11} \\ 0 & 0 & 1 & \frac{283}{12}+\frac{-11}{12}\mathrm{i} \end{bmatrix}$$

Step 8: Multiply row 3 by $\frac{8}{11}$ and subtract it from row 1.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-22}{3}+\frac{2}{3}\mathrm{i} \\ 0 & 1 & \frac{9}{11} & \frac{39}{11} \\ 0 & 0 & 1 & \frac{283}{12}+\frac{-11}{12}\mathrm{i} \end{bmatrix}$$

Step 9: Multiply row 3 by $\frac{9}{11}$ and subtract it from row 2.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-22}{3} + \frac{2}{3}\mathrm{i} \\ 0 & 1 & 0 & \frac{-63}{4} + \frac{3}{4}\mathrm{i} \\ 0 & 0 & 1 & \frac{283}{12} + \frac{-11}{12}\mathrm{i} \end{bmatrix}$$

These computations show that

$$x = \frac{-22}{3} + \frac{2}{3}\mathrm{i}, y = \frac{-63}{4} + \frac{3}{4}\mathrm{i} \text{ and } z = \frac{283}{12} + \frac{-11}{12}\mathrm{i}$$

is the solution. It also illustrates the Gauss-Jordan elimination to obtain the solution of a system of linear equations.

- Apart from computations on vectors, the tool can be used with other packages that can plot vectors defined over the field of real numbers in 2 or 3 dimensions. The tool thus facilitates graphical illustration of various concepts on vectors. As an example, the package can be used with the `tikz` package. Lualinalg code 4 illustrates the plotting of vectors in a 3-D plane using the `luavector` and `tikz` package. It produces figure 1.

Lualinalg code 4: Plotting vectors in 3-dimensions with the linear algebra computational tool

```
\documentclass{article}
\usepackage{tikz,tikz-3dplot,lualinalg}
\begin{document}
\tdplotsetmaincoords{60}{100}
\begin{tikzpicture}[scale=1,
   tdplot_main_coords,
   axis/.style={->,thick},
   vector/.style={-stealth,very thick},
   vector guide/.style={dashed,red,thick}]
\vectorNew{o}{{0,0,0}}
\vectorNew{e1}{{5,0,0}}
\vectorNew{e2}{{0,3,0}}
\vectorNew{e2n}{{0,-3,0}}
\vectorNew{e3}{{0,0,6}}
\vectorNew{f}{{1,2,0}}
\vectorNew{g}{{-2,1,2}}
% Axes
\draw [axis] \vectorParse{o}-- \vectorParse{e1} node [below left] {$x$};
\draw [axis] \vectorParse{e2n}-- \vectorParse{e2} node [right] {$y$};
\draw [axis] \vectorParse{o}-- \vectorParse{e3} node [above] {$z$};
% Plotting Vectors
\draw [vector, red] \vectorParse{o} --\vectorParse{f};
```

```
\draw [vector, blue] \vectorParse{o} --\vectorParse{g};
\vectorCross{h}{f}{g}
\draw [vector, green] \vectorParse{o} --\vectorParse{h};
% Labels
\node [below right] at \vectorParse{f} {$f$};
\node [above left] at \vectorParse{g} {$g$};
\node [left] at \vectorParse{h} {$f \times g$};
\draw[vector guide, black] \vectorParse{h} --
   (\vectorGetCoordinate{h}{1},0,0) node [below right]
   {$x=\vectorGetCoordinate{h}{1}$};
\draw[vector guide, black] \vectorParse{h} --
   (0,\vectorGetCoordinate{h}{2},0) node [below left, xshift = 0.3cm]
   {$y=\vectorGetCoordinate{h}{2}$};
\draw[vector guide, black] \vectorParse{h} --
   (0,0,\vectorGetCoordinate{h}{3}) node [right]
   {$z=\vectorGetCoordinate{h}{3}$};
\end{tikzpicture}
\end{document}
```

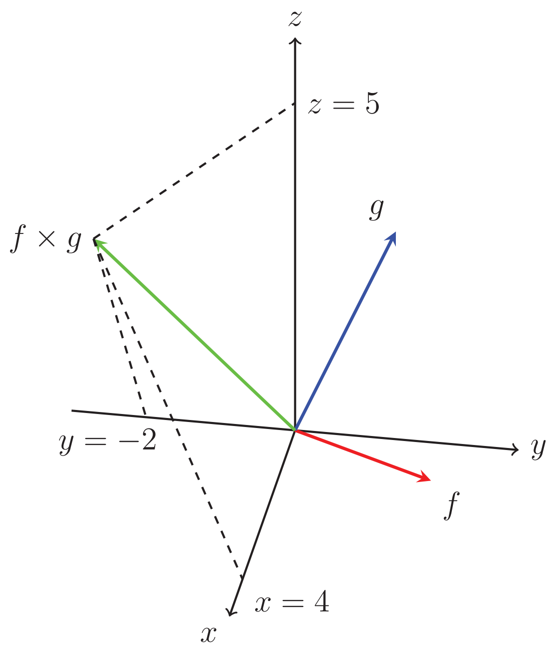


Figure 1: Plotting of 3-D Vectors with the luavector and tikz package

## Customized usage

The commands available in the package can be used for performing further operations on matrices and vectors. These commands can be modified or extended as well. The tool has no functions to check whether the matrix is involutory and to check the orthogonality of vectors. It is easily possible to construct these functions using the tool. Lualinalg code 5 illustrates this.

Lualinalg code 5: Customized usage of the tool

```
\documentclass{article}
\usepackage{lualinalg}
\begin{document}
\begin{luacode}
function matrix.checkInvolutory(m)
 if #m ~= #m[1] then error( "Matrix is not square.") end
 idn = matrix(#m, 'I')
return matrix.chqeql(m^2,idn)
end

function vector.checkOrthoVecs(v,w)
return lnumChqEql(vector.dot(v,w),0)
end
\end{luacode}

\newcommand\matrixChkInvolutory[1]{%
    \directlua{%
        tex.print(tostring(matrix.checkInvolutory(matrices['#1'])))
    }%
}%

\newcommand\vectorChkOrtho[2]{%
    \directlua{%
        tex.print(tostring(vector.checkOrthoVecs(vectors['#1'],vectors['#2']
            )))%
    }%
}%

\def\s{{{-5,-8,0},{3,5,0},{1,2,-1}}}
\matrixNew{m}{\s}
\(m = \matrixPrint{m}\)

It is \matrixChkInvolutory{m} that matrix \(m\) is Involutory.

\vectorNew{v1}{{lcomplex(0,1),lcomplex(-1,-1), 1}}
\vectorNew{v2}{{lcomplex(-1,1),lcomplex(0,2), lcomplex(1,1)}}
\(v1=(\vectorPrint{v1})\) and \(v2=(\vectorPrint{v2})\).

It is \vectorChkOrtho{v1}{v2} that vectors \(v1\) and \(v2\) are orthogonal.

\end{document}
```

Lualinalg code 5 generates the following output.

$$m = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$
It is true that matrix $m$ is involutory.
$v1 = (3, 8, 5)$ and $v2 = (-1 + \mathrm{i}, 2\mathrm{i}, 1 + \mathrm{i})$.
It is true that vectors vectors $v1$ and $v2$ are orthogonal.

# 6 Resources used

- Techniques and methods used in development of the tool do not need any special resources. All the techniques are platform-independent and work on the standard operating systems. The technique mainly uses Lua and LaTeX, which are platform-independent and do not need strong hardware resources.
- No proprietary software and tools are used in the development of the tool. The tool is made available as freeware and open source. This is important as the research work can further be extended without any restrictions. This is within the philosophy of open source and freeware tools that are available for the mathematics community.

# 7 Known issues and limitations

- The tool uses double precision for floating-point numbers. It represents each number with 64 bits, 11 of which are used for the exponent. The double-precision floating-point numbers can represent numbers with roughly 16 significant decimal digits. The tool supports small and big numbers. They can be input in the usual scientific notation. The math library in Lua defines constants with the maximum `math.maxinteger` and the minimum `math.mininteger` values for an integer. The result wraps around when there is a computational operation on integers that would result in a value smaller than the `mininteger` or larger than the `maxinteger`. It means that the computed result is the only number between the `miniinteger` and `maxinteger`. The bits after 64 bits are not taken into account.
- The symbolic computations on vectors and matrices are not supported at present.
- The error handling mechanism in the tool is not robust. There are some custom errors included in the package. However the package mostly depends on error handling mechanism of Lua. The error handling can be strengthened in future updates of the tool.

# 8 Future Plans and Prospects of the Research Work

- The tool currently supports only numerical computations on vectors and matrices. The table in a Lua is a data type that implements an associative array. This feature is used in packages to define and store vectors and matrices. This approach is close to object-oriented programming. It will allow easy conversion of algorithms in packages for symbolic computations. Future package updates will consider algorithm conversions to support symbolic calculations.

- The approach of embedding Lua in LaTeX can be extended in several other ways. With `luamplib` [6] and `metapost` [11] libraries, it is possible to plot graphs of functions in LaTeX in a native way. This sort of package can be developed in the future to enhance the graphical aspect of LaTeX using Lua.

- Lua supports procedural programming, object-oriented programming, functional programming, data-driven programming, and data description. It is thus possible to upgrade the developed tool with Graphical User Interface (GUI) to facilitate interactive computations on vectors and matrices. The tool can also have a facility to import and export computations in LaTeX-compatible format.

- Lua-based mini Computer Algebra System can be developed and integrated into LaTeX. It will consist of Lua packages designed for performing symbolic mathematical computations. There is good scope for developing such Computer Algebra System. There have already been efforts in this direction. There is a `symmath lua` project on GitHub. It is available on this link [13].

# References


[1] *luacode package page*. URL: https://ctan.org/pkg/luacode (visited on 03/10/2022).

[2] *luacomplex package page*. URL: https://ctan.org/pkg/luacomplex (visited on 12/29/2022).

[3] *luagcd package page*. URL: https://ctan.org/pkg/luatruthtable (visited on 12/30/2022).

[4] *luamaths package page*. URL: https://ctan.org/pkg/luamaths (visited on 12/27/2022).

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[6] *luamplib package*. URL: https://mirror.kku.ac.th/CTAN/macros/luatex/generic/luamplib/luamplib.pdf (visited on 02/22/2022).

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[9] *LuaTeX package page*. URL: https://ctan.org/pkg/luatex (visited on 01/22/2022).

[10] *luatruthtable package page*. URL: https://ctan.org/pkg/luatruthtable (visited on 09/18/2022).

[11] *Metapost language*. URL: https://www.tug.org/docs/metapost/mpman.pdf (visited on 02/22/2022).

[12] William M. Richter. "TEX and scripting languages". In: *TUGboat* 25.1 (80 2004), pp. 71–88. ISSN: 0896-3207. URL: https://tug.org/TUGboat/tb25-1/richter.pdf.

[13] *Symbolic Mathematics in Lua*. 2022. URL: `https://christopheremoore.net/symbolic-lua` (visited on 05/22/2022).

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# A Classification of Tritangent Conics: The Power of Geometric Macros in Dynamic Geometry

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**Abstract**: *Based on our knowledge of conics and my previous work, I will detail the algorithms for constructing conics tangent to the three sides of a triangle, internally and externally. These constructions developed in a dynamic geometry environment (here the new Cabri) largely using the "Macro Construction" tool (which is none other than a program of this environment) will make it possible to visualize all these conics in motion and to highlight evidence of some surprising properties of these families of conics: in particular, we will be led to conjecture a classification of conics tangent to the three sides of a triangle according to the position of one of their foci. This work requires for each type of conic an introduction concerning the construction algorithms of their characteristic elements as well as of their tangents lines within a dynamic geometry environment.*

## 1. First work around ellipses

### 1.1. Construction algorithms of the center, axes and foci of an ellipse

For an ellipse given in a dynamic geometry environment, here are the different constructions needed to obtain:

**1.1.1. The center of the ellipse** (Figure 1 left): ($E$) is a given ellipse and $A$, $B$ and $C$ are three points of this ellipse. An algorithm to construct the center $O$ of this ellipse is as follows:

1. Construct line ($AB$)
2. Construct line ($CD$) parallel to ($AB$) through $C$
3. Line ($IJ$) ($I$ and $J$ midpoints of [$AB$] and [$CD$]
4. Construct the points $L$ and $R$ intersection points between (E) and ($IJ$)
5. construct $O$ midpoint of [$LR$] which is the center of ellipse ($E$)

This construction is recorded as a macro construction called **center ellhyp** for which the initial object is an ellipse (or a hyperbola) and the final object is the center of the given conic.

**1.1.2**. **The axes of the ellipse** (Figure 1 center): ($E$) is a given ellipse and the previous macro **center ellhyp** is available. An algorithm to construct the axes of this ellipse is as follows:

1. Construct center $O$ of ellipse ($E$) in using the macro **center ellhyp**
2. Select any points $r1$ and $r2$ on $(E)$
3. Evaluate the distance between $r1$ and $r2$: $d$
4. Display number 90 as the result of $90 * \frac{d+1}{d+1}$
5. Ellipse $(E')$ image of ellipse $(E)$ with rotation centered at $O$ and which angle is the previous calculated number 90
6. Points $i1$, $i2$ and $i3$ *three first intersection points between* $(E)$ and $(E')$
7. $j1$ and $j2$, midpoints of [$i1$ $i2$] and [$i2$ $i3$]
8. Lines $(O\ j1)$ and $(O\ j2)$ are the axes of ellipse $(E)$

This construction is recorded as a macro construction called **axes ellipse** with the initial object an ellipse and the final objects the axes of that given ellipse. Note that this construction does not work to display the axes of a hyperbola.

**1.1.3**. **The two foci of the ellipse** (Figure 1 right): (*E*) is a given ellipse and the previous macros **center ellhyp** and **axes ellipse** are available. An algorithm to construct the axes of this ellipse is as follows:

| | |
|---|---|
| 1. Construct center *O* of ellipse (*E*) in using the macro **center ellhyp**<br>2. Construct the axes of ellipse (*E*) in using the macro **axes ellipse**<br>3. Create points *s2*, *r2*, *t1* and *t2*, intersection points between these axes and the ellipse<br>4. Display distances *Os2* (*a*) and *Ot2* (*b*) and evaluate $c = \sqrt{c^2}$ where $c^2 = \sqrt{(a^2 - b^2)^2}$ | 5. Create triangle *Os2t2* and *m* midpoint of [*s2t2*]<br>6. Create point *n* intersection between the triangle *Os2t2* and the perpendicular to *s2t2* at *m*<br>7. Create vector *On*<br>8. Measurement transfer of *c* (as calculated in the table below) to get point *f1* which is a focus of (*E*)<br>9. *f2*, symmetric point of *f1* with respect to *O* is the second focus |

This construction is recorded as a macro construction called **foci ellipse** with initial object an ellipse and final objects the foci of that given ellipse.

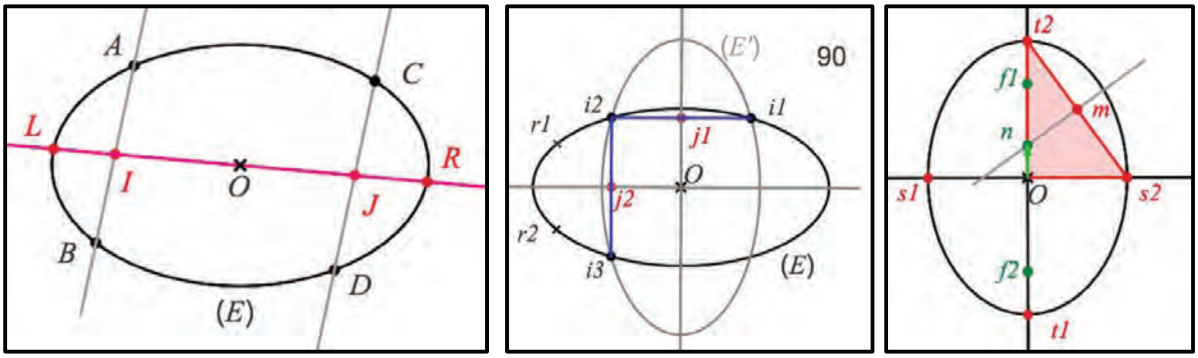


Figure 1: Center, axes and foci of an ellipse

**1.2. Construction algorithm of the tangent lines to an ellipse from a given point** (Figure 2 left)**:** (*E*) is a given ellipse and the previous macro **axes ellipse** is available. *M* is a given point outside the ellipse. An algorithm to construct the two tangent lines to this ellipse passing through *M* is as follows:

| | |
|---|---|
| 1. Construct the two foci *f1* and *f2* of ellipse (*E*) in using the macro **foci ellipse**<br>2. Construct line (*f1f2*) and points *e1* and *e2* its intersection points with (*E*)<br>3. Create segment [*e1 e2*] and cercle (*C1*) centered at *f2* of radius *e1e2*<br>4. Create circle (*C*) centered at *M* passing through *f1* | 5. Create points *h1* and *h2*, intersection points between (*C1*) and (*C*)<br>6. (*T1*) is the perpendicular bisector of [*f1h1*]<br>7. (*T2*) is the perpendicular bisector of [*f1h2*]<br>8. *t1* is the intersection point between (*T1*) and segment[*f2h1*]<br>9. *t2* is the intersection point between (*T2*) and segment[*f2k*] where *k* is the symmetric point of *f2* with respect to (*T2*) |

This construction is recorded as a macro construction called **tangent lines ellipse** with initial objects an ellipse and a point and final objects the two tangent lines to the ellipse passing through the given point.

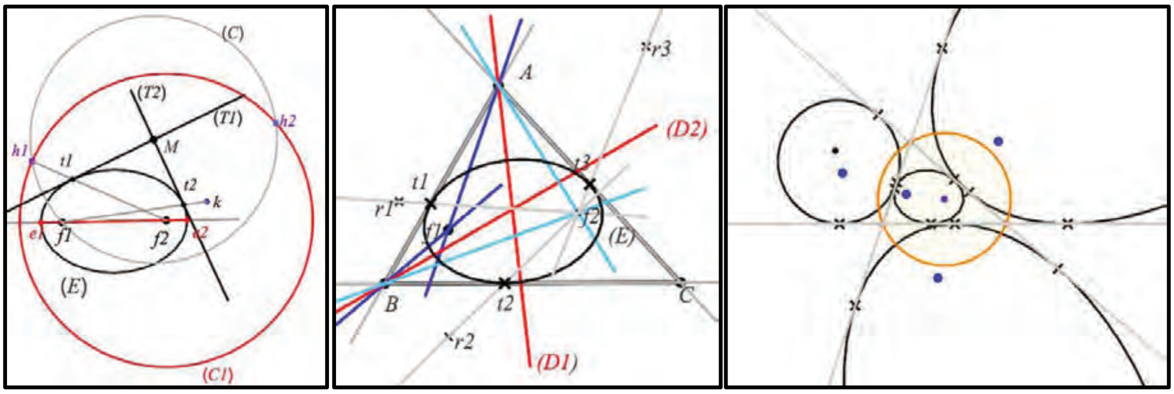


Figure 2: Tangent lines to an ellipse through a given point

### 1.3. Little Poncelet Theorem. Consequences

The Little Poncelet Theorem states that the angle bisector of *f1Mf2* (Figure 2 left) is also the angle bisector of the two tangent lines (*T1*) and (*T2*). Therefore, the angle bisector can be interpreted as an axis of symmetry. We can use this property for the following constructions.

**1.3.1. Ellipses tangent to the three sides of a given triangle** (Figure 2 center and right):
Here is an algorithm allowing such construction leading to a powerful macro construction

| | |
|---|---|
| 1. Construct the first focus *f1*, lines *(f1A)* and *(f1B)*, the angle bisectors of *CAB* and *CBA*, *(D1)* and *(D2)*<br>2. Construct line (*f1A*) and line (*f1B*)<br>3. Symmetric line of *(f1A)* with respect to *(D1)* and symmetric line of *(f1B)* with respect to *(D2)*<br>4. Create *f2* their intersection point | 5. Create points *r1*, *r2* and *r3* symmetric points of *f1* with respect to *(AB)*, *(BC)*) and *(CA)*<br>6. *t1* intersection point between (r*1f2*) and *(AB)*<br>7. *t2* intersection point between (r*2f2*) and *(BC)*<br>8. *t3* intersection point between (r*3f2*) and *(CA)*<br>9. *(E)* is the ellipse which foci are *f1* and *f2* passing through *t1* |

This construction is recorded as a macro construction called **ellipse tritangent** with initial objects a triangle and a point and final objects the ellipse tangent to the three sides of the given triangle, a second point which is the second focus of this ellipse (the first focus being the first given point) and the three contact points with the sides (in reality the lines supporting the sides).
*Important remark*: The proposed algorithm works when the first point lies **inside the triangle** and also **outside the triangle but only if outside the circumcircle** of the triangle. Figure 2 right displays the use of this macro for a point inside the triangle and three points outside the triangle but inside the circumcircle
*Another remark*: it seems that it is impossible to construct an ellipse tangent to the three sides of a triangle when the first given focus lies outside the triangle and inside the circumcircle.

**1.3.2. Ellipses tangent to two given rays**:
We show now how to construct the ellipse tangent to two given rays, one given focus and a given contact point on one of the two given rays. Here is an algorithm allowing this construction (Figure 3 left):

| | |
|---|---|
| 1. Construct two rays (*L1*) and of (*L2*), a point *t1* on (*L1*) and a point *f1*<br>2. Angle bisector *(B)* of the two rays (*L1*) and (*L2*)<br>3. Symmetric line of *(Of1)* with respect to *(B)*: *(S)*<br>4. Symmetric point of *f1* with respect to (*B*): *r1* | 5. Intersection point between (*r1t1)* and *(S)*: *f2*<br>5. Symmetric point of *f2* with respect to (*L2*): *r2*<br>7. Intersection point between *(f1r2)* and *(L2)*: *t2*<br>8. Ellipse *(E)* which foci are *f1* and *f2* passing through *t1* |

This construction is recorded as a macro construction called **ellipse tgt to 2 rays** with initial objects two rays, a contact point on one ray and the first focus of the expected ellipse, and final objects the ellipse tangent to the two given rays, the second contact point and the second focus of the ellipse.
Figure 3 right displays three red ellipses constructed with this macro: the given focus is a random point between the two rays and the chosen contact point is a contact point of one of the ellipses constructed with the macro **ellipse tritangent**.

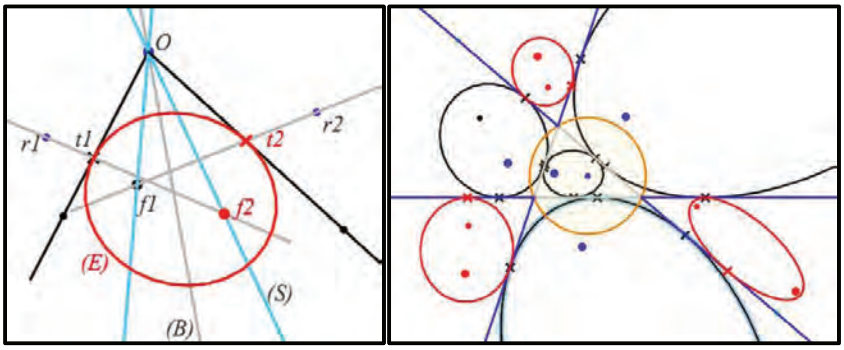


Figure 3: Ellipses tangent to two lines

**1.4. Tangent line to an ellipse. Conjugate directions**
In this paragraph, we show an algorithm to construct a tangent line to an ellipse at one of its points and then its conjugate directions (images by an affinity of two perpendicular diameters of a circle).
**The first case when the ellipse is defined by two foci and one of its point (Figure 4 left):** Here is the algorithm to construct a tangent line.

**1.4.1. Tangent line to an ellipse**

1. Construct line ((*f1f2)* and its intersection points with *(E)*: *e1* and *e2*
2. Segment [*e1e2*]
3. Point *m* on *(E)*
5. Circle *(C)* centered at *f1* and of radius *e1e2*
6. Ray *[f1m)* intersecting *(C)* at *n*
7. *Line Tm* perpendicular bisector of [*f2 n*]

This construction is recorded as a macro construction called **tangent ellipse 1** with initial objects the two foci and the point defining the ellipse and the point of the ellipse where we expect the tangent line and final object this tangent line.
**Second case when the ellipse is defined by five points:** the algorithm construction is exactly the same as the previous one on the condition of adding a preliminary stage of construction of the foci of the ellipse in using the macro **foci ellipse**.
Then, this construction is recorded as a macro construction called **tangent ellipse 2** with initial objects the five points defining the ellipse, and the point of the ellipse where we expect the tangent line and final object this tangent line.
**1.4.2. Conjugate directions (Figure 4 center):** Given an ellipse defined by five points and a point *m1* on this ellipse, here is below is an algorithm to construct ray [*Om1*) and its conjugate [*Om2*)
This construction is recorded as a macro construction called **conjugate directions** with initial objects the five points defining the ellipse and a point *m1* of this ellipse and the final object [*Om1*) and its conjugate [*Om2*) where *m2* lies on the ellipse.

| | |
|---|---|
| 1. We use the macro "foci ellipse" to construct the two foci of *(E)*: *f1* and *f2*<br>2. Line (*(f1f2)* and one of its intersection points with *(E)*: *e2*<br>3. Midpoint *O* of [*f1f2*] and circle *(C)* centered at *O* and passing through *e2*<br>4. Point *m1* on *(E)*. **Ray *[Om1)***<br>5. Line perpendicular to *(f1f2)* through *m1* intersecting *(f1f2)* at *h1* | 6. Ray *[h1m1)* intersecting *(C)* at *n1*<br>7. Evaluate distance *d* between *f1* and *f2*. Display number 90 as the evaluation of *d*-*d*+90<br>8. Rotation of *n1* around *O* (angle: the previous 90): *n2*. Ray [*On2*)<br>9. Line perpendicular to *(f1f2)* through *n2* intersecting *(f1f2)* at *h2*<br>10. Ray *[h2n2)* intersecting *(E)* at *m2*. **Ray [*Om2*)** |

1.4.3. Parallelogram circumscribed to an ellipse defined by five points and a point *m1* on this ellipse (Figure 4 right). Here is an algorithm to construct the parallelogram circumscribed to the ellipse containing a point m1 and with sides parallel to [*Om1*) and its conjugate direction.

| |
|---|
| 1. Create point *m1* on *(E)*.<br>2. Use the macro "**conjugate directions**" to construct the two conjugate directions *Om1* and *Om2*<br>3. Point *p1* symmetric of *m1* with respect to *O*<br>3. Point *p2* symmetric of *m2* with respect to *O*<br>4. Lines passing through *m1* and *p1* parallel to [*Om2)*<br>5. Lines passing through *m2* and *p2* parallel to [*Om1)*<br>6. Parallelogram defined by these four lines |

This construction is recorded as a macro construction called **circum parallelogram** with initial objects the five points defining the ellipse and a point *m1* of this ellipse and final object the parallelogram circumscribed to the given ellipse, tangent at *m1*.

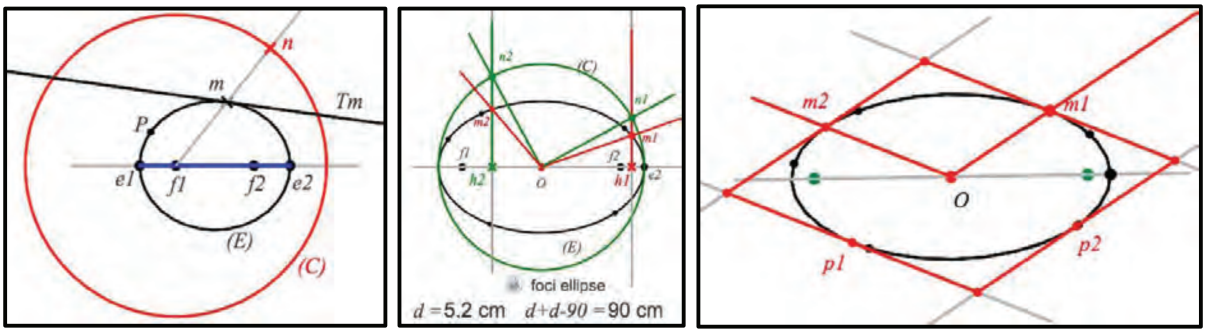


Figure 4: Tangent lines and conjugate directions of an ellipse

**1.5. Steiner ellipse**

**1.5.1. Reminder of the construction of the Steiner ellipse** (Figure 5 left)

The Steiner ellipse (*E*) of a triangle *ABC* is the image of the inscribed circle of an equilateral triangle, and justifies the algorithm of its construction detailed below:

| |
|---|
| 1. Create medians [*AA1*], [*BB1*] and [*CC1*]<br>2. Their intersection point *I*<br>3. *j* midpoint of [*IA*] and *k* midpoint of [*IB*] and *l* midpoint of [*IC*]<br>3. Conic (E) passing through *A1, B1, C1, j,* and *k*. *(E)* is the Steiner ellipse passing also through *l* centered at *I* |

This construction is recorded as a macro construction called **steiner ellipse** with initial object a triangle and the final objects the Steiner ellipse, its center (centroid of the given triangle) and the contact points (midpoints of the sides of the given triangle)

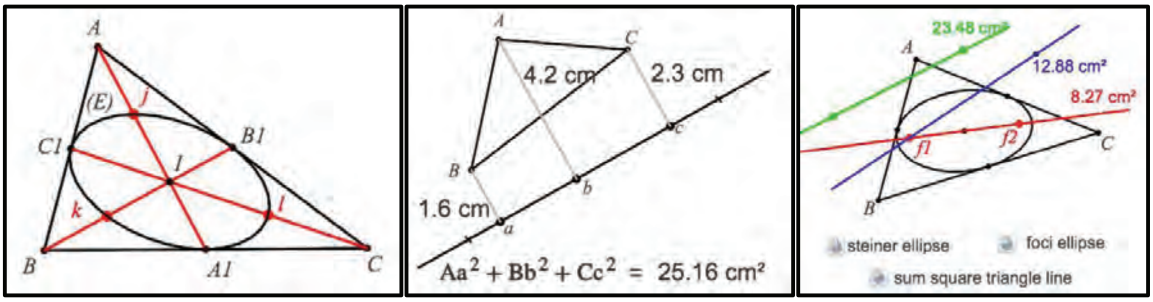


Figure 5: Steiner ellipse

### 1.5.2. A regression property of the Steiner ellipse

We create first a macro called **sum square triangle line** with initial objects a triangle and a line and the final object the number evaluating the sum of the squares of the distances between the vertices of the triangle and the given line (Figure 5 center).

Figure 5 (right) illustrates how it is possible to investigate in order to conjecture that the line joining the foci of the Steiner ellipse of the given triangle *ABC* is the one minimizing the sum of the squares of the distances between the vertices of *ABC* and a line. For the given triangle, use first the macro **steiner ellipse** to display the Steiner ellipse. Apply to this ellipse the macro **foci ellipse** to display its foci *f1* and *f2*. Create line (*f1f2*) to which we apply macro **sum square triangle line:** we obtain the number 8.27 $cm^2$. We create then a line with the number obtained with the same macro; Trying to decrease the value of this number in changing the position of the line leads to approach the position of (*f1f2*). This investigation can be conducted for any triangle and always leads to the same conjecture. In fact, the result stated by this conjecture is a known property of the Steiner ellipse.

## 1.6. Isoptic curves of an ellipse

### 1.6.1. Definition of an isoptic of an ellipse

Set of points from which an ellipse can be seen under a given angle.

**1.6.2.** Construction algorithm of the isoptics of an ellipse given by two foci, a point, and an angle beween 0° and 180°. Here is an algorithm for this construction:

1. Create a point *t1* on the given ellipse and the tangent line (*T1*) to the ellipse at *t1* in using the macro **tangent ellipse 1**
2. Measurement of *f1f2* and display number 90 as the result of *f1f2 - f1f2 + 90*
3. *p* image *of O* (center ellipse) by the rotation centered at *t1* and of angle the previous calculated number *90*
4. *p1,* orthogonal projection of *p* on *(T1)*
5. *Create vector t1p1*
6. Point *q*, measurement transfer of *f1f2* on the previous vector from *t1*
7. Create a slider whose boundaries are 0 and 180 commanding a number so so-called **angle**
8. *s1* image *of t1* by the rotation centered at *q* and of the angle the previous number **angle**
9. Create *Ray 1* = [*q,s1*)
10. *Ray 2* image of *Ray 1* by the translation mapping *q* onto *O*
11. *c1* intersection point between *Ray 2* and the ellipse
12. Hide all the previous constructions. Dont hide the ellipse, *t1*, (*T1*) and *c1*
13. Create 5 points on the ellipse in order to use macro **conjugate directions** and obtain rays [*O,c1*) and [*O,c2*)
14. (*T2*) is the parallel line to [*O,c1*) through *c2*
15. *m* intersection point between (*T1*) and *(T2*)
16. Measurement of angle *t1mc2* (equal to **angle**)
17. Locus of point *m* when *t1* moves along the ellipse provides the isoptic related to the **angle** commanded by the slider

Stages from 1 to 11 are illustrated by Figure 6 left. Stages from 12 to 16 are illustrated by Figure 6 center. Stage 17 is illustrated by Figure 6 right.

*Remark*: in Figure 6 right, changing the value commanded by the slider automaticaly changes the shape of the isoptic. Especially, when **angle** is equal to 90°, we obtain a circle. To check experimentally this result, construct a conic passing through five points of the displayed isoptic, then the software recognizes a circle.

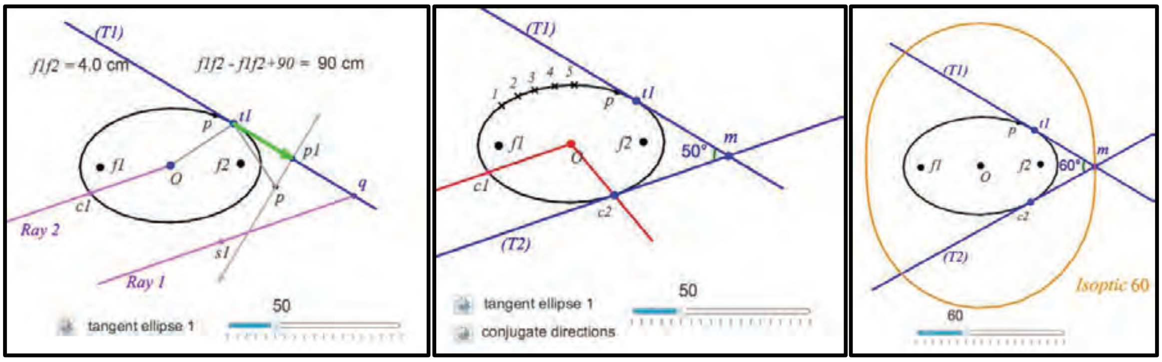


Figure 6: Isoptic curves of an ellipse

## 2. Second work around parabolas

### 2.1. Algorithm construction of the focus, the axis and the directrix of a parabola

Here is an algorithm for this construction (Figure 7 left):

| | |
|---|---|
| 1. Create three points *m1*, *m2* and *m3* on (*P*) and segment [ *m1*, *m2*]<br>2. Construct *m4* such as (*m1 m2*) // (*m3 m4*)<br>3. *i1* midpoint of [*m1, m2*] and *i2* midpoint of [*m3, m4*]<br>4. Line (*i1 i2*) intersects *(P)* at *t*<br>5. *(T)* parallel line to [*m1, m2*] through *t*<br>6. Line (*i1, i2*) and its perpendicular (*Pe*) through *m3* cutting *(P)* at *m5* | 7. Perpendicular bisector (*Pb*) of [*m3 m5*]<br>8. Line *(R)* symmetric line of (*i1 i2*) with respect to *(T)*<br>9. *f* (focus) of *(P)* intersection point of *(R)* and *(Pb)*<br>10. *s* intersection of *(P)* and *(Pb)*<br>11. *h* symmetric of *f* with respect to *s*<br>12. *(D)* (directrix of *(P)*) perpendicular to (*Pb*) at *h* |

This construction is recorded as a macro construction called **param parab** with initial object a parabola and final objects the focus, the directrix, the summit and the symmetric point of the focus with respect to the summit.

### 2.2. Algorithm construction of the two tangent lines to a parabola through a given point

In Figure 7 center, a parabola *(P)* is given by its focus *f* and its summit *s*. The following algorithm describes how to obtain the two tangent lines *(T1)* and *(T2)* to *(P)* through the given point *m*. We also obtain the contact points *t1* and *t2* of the tangent lines with *(P)*.
This construction is recorded as a macro construction called **2 lines tg parab** with initial objects the points *f*, *s* and *m* and final objects the two tangent lines through *m* to the parabola with focus *f* and summit *s* and the two contact points.

| | |
|---|---|
| 1. Create parabola (*P*) with its focus *f* and its summit *s*<br>2. Create *m* outside *(P)*<br>3. Create circle *(C)* centered at *m* passing through *f*<br>4. Perpendicular line (D*)* to *(fs)* through *h* symmetric point of *f* with respect to *s* | 5. *s1* and *s2* intersection points between *(C)* and *(D)*<br>6. *(P1)* perpendicular line to *(D)* through *s1*, cutting *(P)* at *t1*<br>7. *(P2)* perpendicular line to *(D)* through *s2*, cutting *(P)* at *t2*<br>8. *(T1)* perpendicular bisector of [*f s1]*<br>9. *(T2)* perpendicular bisector of [*f s2]* |

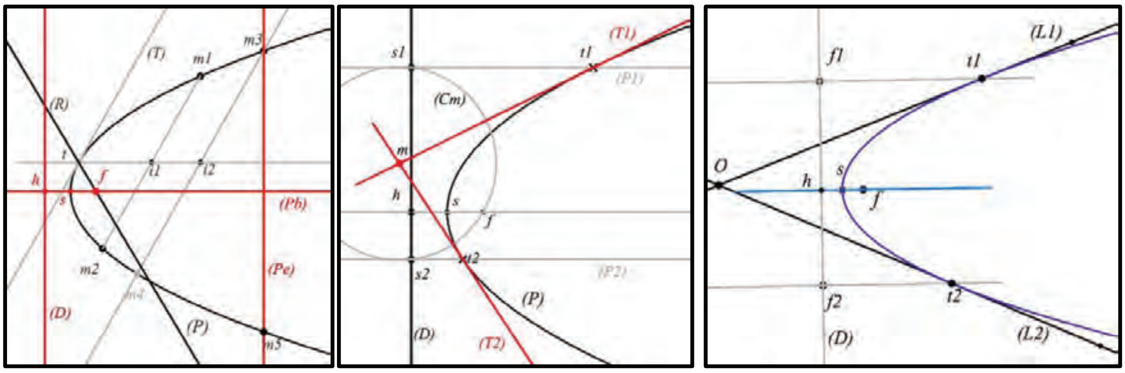


Figure 7: Parabolas

**2.3. Construction algorithms of the parabolas tangent to two given lines (Figure 7 right)**

Given two lines *(L1)* and *(L2)* crossing at *O* and a point *f* not lying on these lines, the following algorithm returns the parabola with focus *f* that is tangent to *(L1)* and *(L2)* at *t1* and *t2*.

| | |
|---|---|
| 1. Create lines *(L1)* and *(L2)* passing through *O* and a point *f* | 6. *(Axis)* cuts *(D)* at *h* |
| 2. *f1* symmetric of *f* with respect to *(L1)* | 7. Construct *s* midpoint of [*h f*] |
| 3. *f2* symmetric of *f* with respect to *(L2)* | 8. Construct parabola *(P)* with *f* as a focus and *s* as a summit |
| 4. Line (*D*) = *(f1f2)* | 9. Perpendicular line to *(D)* at *f1* intersects *(L1)* at *t1* |
| 5. Line *(Axis)* perpendicular line to *(D)* through *f* | 10. Perpendicular line to *(D)* at *f2* intersects *(L1)* at *t2* |

This construction is recorded as a macro construction called **parab bitg** with initial objects two lines *(L1)* and *(L2)* passing through point *O*, a point *f* and two points *t1* and *t2* on *(L1)* and *(L2),* and final objects the parabola with focus *f* tangent to these two lines at *t1* and *t2*.

**2.4. Construction algorithm of the parabolas tangent to the three lines supporting the three sides of a triangle (relation with the circumcircle and the Simson and Steiner lines)**

**2.4.1. *Reminder* (Figure 8 left):** Given a triangle *ABC*, its circumcircle *(C)* and a point *M*. Let us call *H1*, *H2,* and *H3* the orthogonal projections of *M* respectively on the three sides of the triangle and *M1*, *M2,* and *M3* the symmetric points of *M* with respect to the three sides of the triangle. We know this result ([5]):

Points *H1*, *H2,* and *H3* (respectively *M1*, *M2,* and *M3*) are colinear if and only if *M* belongs to *(C)*. In this case, the line joining *H1*, *H2,* and *H3* (respectively *M1*, *M2,* and *M3*) is called the **Simson line** of M (respectively the **Steiner line** of M) for the triangle *ABC*.

*Remark*: The Steiner line contains the orthocenter of the triangle

Creating Figure 8 left gives the opportunity to create two other macros:

Macro **Circumcircle** that returns the circumcircle of a triangle with its center.

Macro **Steiner line** that returns the Steiner line of a given triangle and a given point (chosen on the circumcircle).

**2.4.2.** The construction algorithm (Figure 8 center)

| | |
|---|---|
| 1. Create a point *f* on the circumcircle of the triangle *ABC* defined by the three given lines | 6. *s* midpoint of [*fr*] |
| 2. Create a triangle *ABC* | 7. *f1, f2,* and *f3* symmetric of *f* with respect to the three given lines (on the Steiner line) |
| 3. Use the macro **Steiner line** to obtain the Steiner line *(S)* of point *f* for *ABC* | 8. Perpendicular line to *(S)* at *f1* cuts *(L1)* at *t1* |
| 4. Perpendicular line *(Axis)* to *(S)* through *f* | 9. Perpendicular line to *(S)* at *f2* cuts *(L2)* at *t2* |
| 5. *r* intersection point between *(S)* and *(Axis)* | 10. Perpendicular line to *(S)* at *f3* cuts *(L3)* at *t3* |
| | 11. Parabola *(P)* which focus is *f* and summit *r* |

This construction is recorded as a macro construction called **parab tg 3 lines** with initial objects three lines (defining a triangle) and a point on the circumcircle of the triangle and final objects the parabola tangent to these three lines, its summit, and the three contact points.
Another macro can also be recorded in changing the initial objects of the three lines with a triangle. In this case, the macro is called **parab tg triangle**.
Figure 8 right displays three parabolas obtained thanks to macro **parab tg 3 lines** from the three points *focus 1*, *focus 2* and *focus 3* chosen on the circumcircle of the triangle defined by the three given lines.

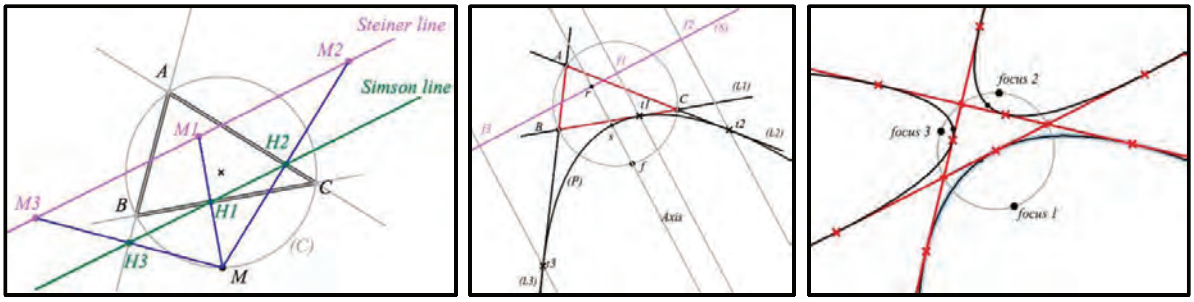


Figure 8: Parabolas tangent to three given lines

### 2.5. Isoptic curves of a parabola

The following algorithm will make it possible to construct the set of points from which a parabola can be seen under an angle given by a slider commanding a number that can vary from 0° to 180°. The parabola *(P)* is given with its focus *f* and its summit *s*.
This construction algorithm shows how to obtain two points *I1* and *I2* from where the parabola *(P)* is seen under a constant angle defined by the slider (Figure 9 left). Then we construct the loci of *I1* and *I2*, called *(H1)* and *(H2)*. By combining these two loci, we obtain the sought isoptic (Figure 9 center). A glimpse of the displayed result allows us to conjecture that this isoptic could be a branch of hyperbola. We will explain how to reach such conjecture and how to find the parameters of such a hyperbola (foci and directrix).

| | |
|---|---|
| 1. Ray *(R1)* = [*sf*) | 13. Create a slider (from 0 to 180) and call the number displayed ***slider*** |
| 2. Evaluate *sf* and then *90* as the result of *sf-sf+90* | 14. Evaluate *ang* = *sf - sf +slider* and then *-ang* |
| 3. *(R2)* image of *(R1)* by the rotation centered at *s* and of angle *sf-sf+90* | 15. *t2* image of *t1* by the rotation centered at *t* and of angle *-ang* |
| 4. *m* point on *(R1)* and *(R3)* image of *(R2)* by the translation mapping *s* onto *m* | 16. Ray *(S1)* = [*t t2*) intersecting *(P)* at *u* |
| 5. *t* intersection point between *(P)* and *(R3)* | 17. Ray *(S2)* image of *(S1)* by the translation mapping *t* onto *t1* |
| 6. *h* symmetric point of *f* with respect to *s* | 18. *v* and *w* intersection points between *(P)* and *(S2)* |
| 7. Perpendicular line (*D*) to *(R1) through h* | 19. *i* and *j* midpoints of [*t u*] and [*v w*] |
| 8. *r* orthogonal projection of *t* on *(D)* | 20. Line *(ij)* intersects *(P)* at *k* |
| 9. *(L1)* perpendicular bisector of [*rf*], (tangent line to *(P)* at *t* | 21. Line *(L2)* parallel to *(S1)* at *k* |
| 10. Evaluate 4.*sf.* Create a point *p* on *(L1)* | 22. *I1* intersection between *(L1)* and *(L2)* |
| 11. *q* obtained by measurement transfer of 4.*sf* on *(L1)* from *p* | 23. *(H1)* locus of *I1* (commanded by *m)* |
| 12. *t1* image of *t* by translation mapping *p* onto *q* | *24. (H2)* locus of *I2* (commanded by *m*) where *I2* is the symmetric *of I1 with respect to (R1)* |

In using the slider, we can check that for a value of 90° given by the slider, *I1* belongs to *(D)* which is a known result and the isoptic for 90° is not an hyperbola but a line, the directrix of the parabola.
In Figure 9 right, we have created point *e* intersection between *(H1)* and ray *(R1')* which is the symmetric ray of *(R1)* with respect to *s*. Then we create a point *f'* on the ray *(R1')*. At last we construct

the hyperbola *(H)* whose foci are *f* and *f'* and passing through *e*. We can state that, in changing the position of *f'*, there is a moment when *(H)* can be superimposed to *(H1)* and *(H2)* which means that the isoptic we have constructed is possibly a hyperbola. The idea we could have at this moment of our investigation is that this hyperbola has one of its foci which is the focus of the parabola and one of its directrix which is the directrix of the parabola. This conjecture will be corroborated below.

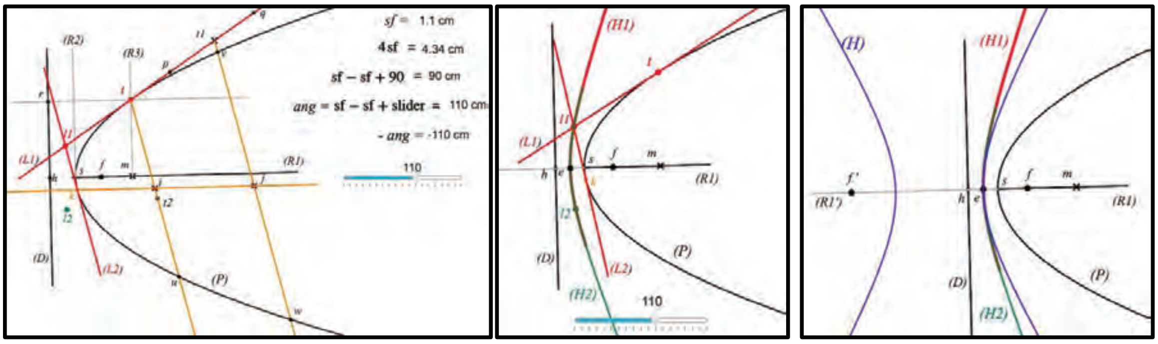


Figure 9: Isoptic curves of a parabola

Other investigations to grab the final property experimentally (Figure 10 left):

**A useful formula**: with the notations of Figure 10 left representing a hyperbola whose foci are *f* and *f'*, whose center is *O* and one summit *e*, we know that:

$oh = \frac{oe^2}{of}$ and as $oe = oh + he$ and $of = oh + hf$ we can deduce that $ho = \frac{he^2}{he-2h}$. This formula is available when *e* is on the right side of *h*. In the other case, the formula becomes $ho = \frac{he^2}{he+2hf}$. A formula encompassing the two previous cases could be $ho = \frac{he^2}{he+2.sl.hf}$ where $sl = sign(90 - \boldsymbol{slider})$. Note that **slider** is a number displayed by a slider of the software, more than 90 when *e* is on the right of *h* and less than 90 when *e* is on the left of *h*.

From that formula we can create a construction algorithm for the center *O* of an hyperbola knowing one focus *f*, one summit *e* and the foot of the directrix *h*.

**The construction algorithm**:

| | |
|---|---|
| 1. Create two points *h* and *f* and ray [*f h)*<br>2. Create *e* on this ray<br>3. Create expression *sign(90-x)*<br>4. Create a slider (bounds: 0 and 180). Number displayed: **slider**<br>5. Evaluate the previous expression for **slider** to get **sl**<br>6. Evaluate *he* and *hf* and then $ho = \frac{he^2}{he+2.\boldsymbol{sl}.hf}$ | 7. Construct ray *(R)* symmetric of ray [*hf)* with respect to *h*<br>8. Circle *(C)* centered at *h* and of radius *ho*<br>9. *O* intersection point between *(C)* and *(R)*<br>10. *f '* symmetric point of *f* with respect to *O*<br>11. *(H)* hyperbola defined by the two foci *f* and *f '* and passing through *e* |

This construction is recorded as a macro construction called **hyper from h f e** with initial objects three points *h*, *f*, *e*, a number between 0 and 180 and the expression *sign(90-x)* and final objects the hyperbola whose foci are *f* and *f'* and passing through *e* and *f'*.

**Final investigations**: in Figure 10 right, we start from a simplified version of Figure 9 center. We kept parabola *(P)*, its focus *f*, its summit *s*, its directrix *(D)* and the point *h* of *(D)* colinear with *f* and *s*. We kept also *(H1)* and *(H2)* which combination represents the isoptic of *(P)* corresponding to the number displayed by the slider (here 110). Eventually we also kept point *e* which is the intersection point between *(H1)* and line *(fs)*. Now we apply macro **hyper from h f e** to the points *h*, *e* and *f* to get the hyperbola *(H)* with foci *f* and *f'*, centered at *O* and passing through *e*. We can state immediately that *(H)* seems to be superimposed to *(H1)* and *(H2)*. This observation persists when we change the value of the slider. To increase the level of the corroboration we create a point $q$ on *(H)* to which we apply macro **2 lines tg parab** to get the two tangent lines to *(P)* passing through $q$. We measure and display the angle between these two lines and we obtain the same number as the one displayed by the slider; this observation persists when we move point $q$ along the right branch of hyperbola *(H)*. Same observations can be made for other values returned by the slider.

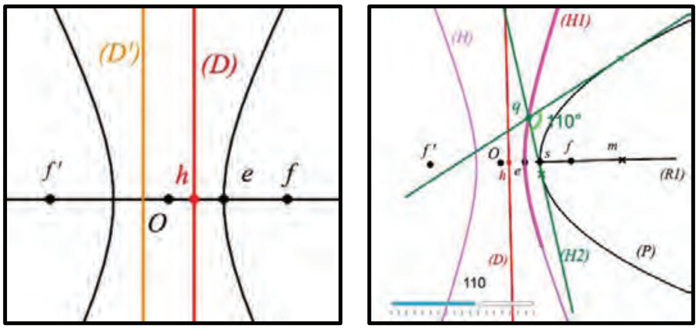


Figure 10: Corroboration of the conjecture about isoptic curves of a parabola

## 3. Third work around hyperbolas

### 3.1. Construction algorithm of the axes of a hyperbola

*(H)* is a given hyperbola, the following construction algorithm allows us to obtain its axes and its center (Figure 11 left).

1. Apply macro **center ellyp** to get the center $O$ of *(H)*
2. Circle *(C)* centered at $O$ passing through a point $l$ chosen on the left branch of the hyperbola
3. $n$ symmetric point of $l$ with respect to $O$
4. $m$ third point of intersection between *(C)* and *(H)*
5. $i$ midpoint of $[lm]$ and $j$ midpoint of $[mn]$
6. $Axis1$ is line *(Oi)* and $Axis2$ is line *(Oj)*

This construction is recorded as a macro construction called **axes hyper** with initial object a hyperbola and whose final objects are the two axes and the center of the given hyperbola.

### 3.2. Construction algorithm of the foci (and the center) of a hyperbola

For *(H)* is a given hyperbola, the following construction algorithm allows us to obtain its foci (Figure 11 center):

1. Apply macro **axes hyper** to get the two axes of *(H)* and its center $O$. Construct summits $A1$, $A2$
2. *(C)* centered at $O$ passing through a point $q$ chosen on the left branch of *(H)*
3. $Xq$ orthogonal projection of $q$ on $Axis1$
4. Measure $OXq = x$, $OA1 = a$ and $Oq = r$
5. Evaluate $a^2.\frac{r^2-a^2}{x^2-a^2}$ *which is* $\boldsymbol{b^2}$
6. Evaluate $\sqrt{b^2}$ which *is* $b$
7. $B$ intersection point between $Axis2$ and circle centered at $O$ and of radius $b$
8. $C$ image of $A2$ by translation mapping $O$ onto $B$
9. $f1$ and $f2$ intersection points between $Axis1$ and circle centered at $O$ *a*nd passing through $B$

This construction is recorded as a macro construction called **foci hyper** whose initial objects are a hyperbola and a point on its left branch and whose final objects are the two foci of the given hyperbola. We can check on Figure 11 right that line *(OC)* is an asymptotic line of the hyperbola. The second one is its symmetric with respect to $Axis1$.

So, we have recorded the macro construction called **asympt hyper** whose initial objects are a hyperbola and a point on its left branch and final objects the asymptotic lines of the hyperbola.

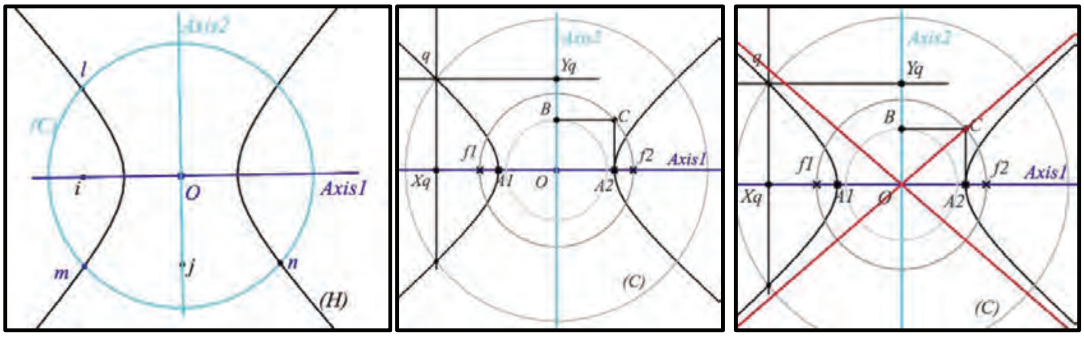


Figure 11: Construction of center, axes, foci and asymptotic lines of a hyperbola

### 3.3. Construction algorithms of the tangent line at a point of a hyperbola (two cases)

A hyperbola *(H)* is given by its two foci *f1* and *f2* and a point *q* defining the left branch of *(H)*.
Here are the algorithms for construction of the tangent line *(T2)* at a point *t2* of the right branch of the hyperbola (algorithm 1, Figure left) and the one of the tangent line *(T1)* at a point *t1* of its left branch (algorithm 2, Figure right).

| **Algorithm 1:** | **Algorithm 2** |
|---|---|
| 1. Line *(f1f2)* and its intersection points *a1* and *a2* with *(H)* | 1. Line *(f1f2)* and its intersection points *a1* and *a2* with *(H)* |
| 2. *g2* symmetric point of *f2* with respect to *a2* | 2. *g1* symmetric point of *f1* with respect to *a1* |
| 3. Circle *(C1)* centered at *f1* passing through *g2* | 3. Circle *(C2)* centered at *f2* passing through *g1* |
| 4. A point *t2* on the right branch of *(H)* | 4. A point *t1* on the left branch of *(H)* |
| 5. Ray *[f1 t2)* intersecting *(C1)* at *m1* | 5. Ray *[f2 t1)* intersecting *(C2)* at *m2* |
| 6. *(T2)* perpendicular bisector of [*f2 m1*] | 6. *(T1)* perpendicular bisector of [*f1 m2*] |

These constructions are recorded as two macro constructions:
The first one is called **tgt hyper right** with initial objects the two foci of a hyperbola, the point defining it (same side of the first foci: left side) and a contact point on the right branch of the hyperbola and final object the tangent line to the hyperbola at this last point.
The second one is called **tgt hyper left** with initial objects the two foci of a hyperbola, the point defining it (same side of the first foci: left side) and a contact point on the left branch of the hyperbola and final object the tangent line to the hyperbola at this last point.

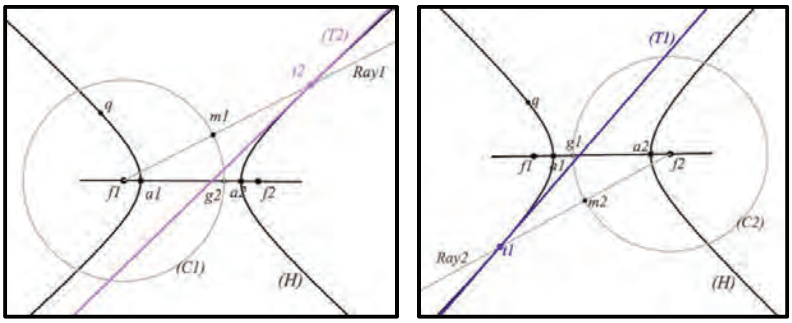


Figure 12: Tangent line at a point of a hyperbola

**3.4. Construction algorithms of the tangent lines to a hyperbola from a given point (two cases)**
Here are the two algorithms (Figure 13):

| Case 1 (from *U1*) | Case 2 (from *U2*) |
|---|---|
| 1. Hyperbola *(H)* from *f1, f2* and *q* | 1. Hyperbola *(H)* from *f1, f2* and *q* |
| 2. Apply macro **asympt hyper** to get the asymptotic lines *Asympt1* and *Asympt2* | 2. Apply macro **asympt hyper** to get the asymptotic lines *Asympt1* and *Asympt2* |
| 3. Circle *(C)* centered at *f1* passing through *g2,* symmetric point of *f2* with respect to *a2* | 3. Circle *(C)* centered at *f1* passing through *g2,* symmetric point of *f2* with respect to *a2* |
| 4. Circle *(C1)* centered at a point *U1 passing through f2* intersecting *(C)* at *r* and *s* | 4. Circle *(C2)* centered at a point *U2* passing through *f2* intersecting *(C)* at *r* and *s* |
| 6. *(T1)* perpendicular bisector of [*f2 r*] | 6. *(T2)* perpendicular bisector of [*f2 s*] |
| 7. *t1* intersection of *(T1)* and *(f1 r)* | 7. *t2* intersection of *(T2)* and *(f1 s)* |

These algorithms are recorded as two macro constructions:
Macro **tgt hyp 1 from pt** whose initial objects are the two foci of a hyperbola *(H)*, one of its point *q* on the left branch and a point *U1* and final objects a tangent line *(T1)* to *(H)* at *t1*, tangent to the right branch if *U1* is located under the right asymptotic line (*Asympt1*), to the left branch of *(H)* if *U1* is located above the right asymptotic line (*Asympt1*).
Macro **tgt hyp 2 from pt** whose initial objects are the two foci of a hyperbola *(H)*, one of its point *q* on the left branch and a point *U2* and final objects a tangent line *(T2)* to *(H)* at *t2*, tangent to the left branch if *U2* is located under the left asymptotic line (*Asympt2*), to the right branch of *(H)* if *U2* is located above the left asymptotic line (*Asympt2*).

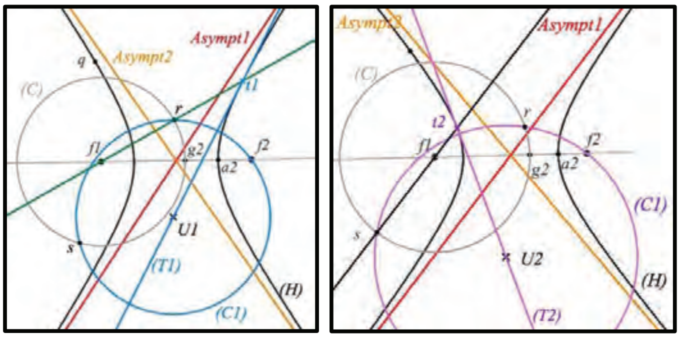


Figure 13: Tangent lines to a hyperbola from a given point

Below in figure 14, are represented the four different positions of the tangent lines of *(T1)* and *(T2)* regarding the position of point *U* relativelely to the asymptotic lines of the hyperbola

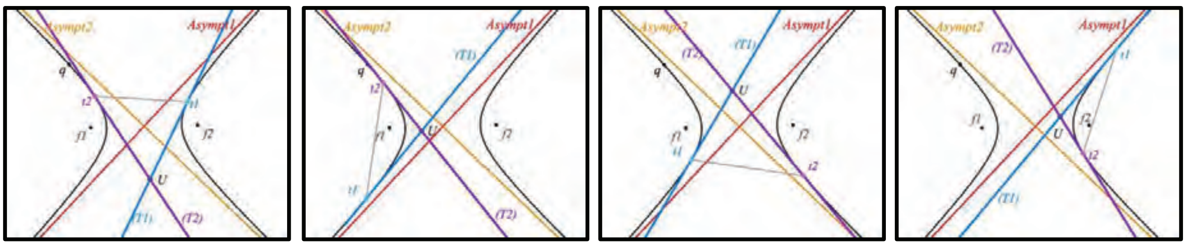


Figure 14: Tangent lines to a hyperbola according to the position of their origin

**3.5. Hyperbolas tangent to the three sides of a given triangle** (Figure 2 center and right):
3.5.1. Existence of such hyperbolas: using the previous construction algorithm (macro **tgt hyp right**), it is possible to construct three tangent lines to the same branch of a given hyperbola, these three lines defining a triangle *ABC*. It is easy to check that one of the foci of this hyperbola is always inside the circumcircle of triangle *ABC*. (See Figure 15 left).

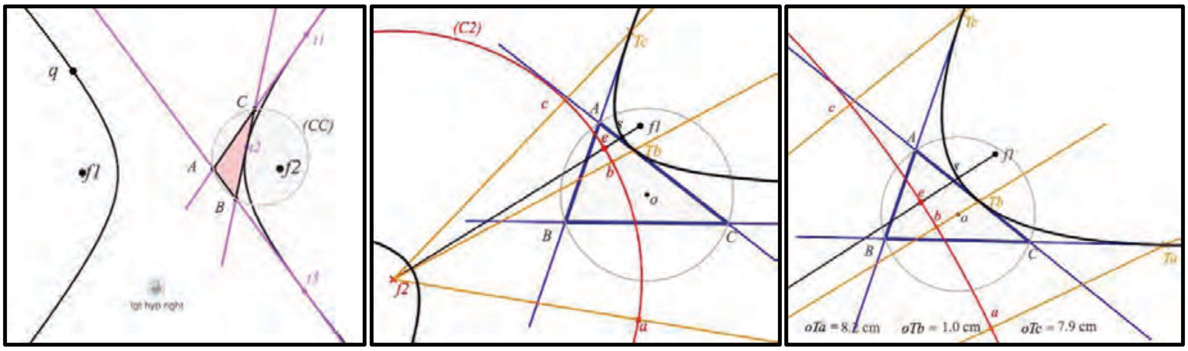


Figure 15: Hyperbolas tangent to the three sides of a triangle

**3.5.2.** Construction algorithm of a branch of hyperbola tangent to the three sides of a given triangle *ABC* is a given triangle and *f1* a given point outside the triangle (Figure 15 center). If a branch of hyperbola having *f1* as one of its foci is tangent to the three lines supporting the sides of ABC, necessarily the symmetric points of *f1* with respect to these lines, *a*, *b* and *c* belongs to the director circle *(C2)* ssociated with the second focus *f2* which is the center of this circle. Necessarily, *f1* must be inside the circumcircle of *ABC* because if not, *f*2 would be located inside (*C2*) which is impossible. If this branch of hyperbola exists the three contact points would be respectively the intersection points between ray [*f2 a*) and line (*BC*) for *Ta*, between ray [*f2 b*) and line (*AC*) for *Tb* and between ray [*f2 c*) and line (*AB*) for *Tc*.
Figure 15 right shows a case where the position of *f2* allows the construction of an expected hyperbola.
The hyperbola solution of our problem is defined by its two foci *f1* (given point) and *f2* (center of director circle) and one summit s (midpoint of [*e f2*] where *e* is the intersecion between [*f1 f2*] and *(C2)*.
To check positions of *f1* allowing the existence of the three points *Ta*, *Tb* and *Tc* and by the way the existence of a branch of hyperbola tangent to the thee lines *(AB)*, *(BC)* and *(CA)*, we measure and display the distances *oTa*, *oTb* nd *oTc* and move *f2* until a position where these three distances exist.

3.5.3. Construction algorithm (Figure 15 center)

| | |
|---|---|
| 1. Triangle *(ABC)* and point *f1*<br>2. Lines *(AB)*, *(BC)* and *(CA)*<br>*3. Points c,* a *and b* symmetric of *f1* with respect to these lines<br>*4.* Circle *(C2)* centered at a point *f2* passing through *a, b and c*<br>5. *e* intersection of *(C2)* and *[f1 f2]* | 6. *s* midpoint of *[e f1]*<br>7. Hyperbola with foci *f1* and *f2* passing through *s*<br>8. *Ta* intersection of *[f2 a)* and *(BC)*<br>9. *Tb* intersection of *[f2 b)* and *(AC)*<br>10. *Tc* intersection of *[f2 c)* and *(AB)* |

This construction is recorded as a macro construction called **tri tgt hyp** with initial objects a triangle and a point inside its circumcircle (but outside the triangle) and whose final objects are the hyperbola which first focus is the given point and the the triangle linking the contct points

**3.5.4. Possible locations of the first focus**

If we move point *f1* where it is allowed by the previous macro we can state quickly that the hyperbola does not always exist: in fact, this result can be reached by observing when the triangle linking the contact points appears or disappears. **I had the idea to move point *f1* along segments parallel to the sides of the given triangle.** That was an amazing idea because I could quickly conjecture that the hyperbola exists when *f1* is located inside the circumcircle but outside a special triangle which seemed to be similar to the given triangle. The measurements taken during my experiments led to obtain a ratio close to 1.60 (I suspected the golden ratio) and the center of the dilation transforming the given triangle onto this one being the orthocenter of the given triangle. To corroborate this conjecture, starting from a triangle *ABC*, I constructed its orthocenter *h* and transformed it by the dilation centered at *h* and with ratio equal to $\frac{1+\sqrt{5}}{2}$ (close to 1.618). Then, I applied the previous macro to ABC and a point *f1* in the previous suspected part of the plane where I expected the hyperbola to exist. And it works!

3.5.5. Final conjectures (Figure 16 right)

**About hyperbolas tangent to the three sides of a triangle**

*ABC* is a triangle, *(C)* its circumcircle, *h* its orthocenter, *A'B'C'* the image of triangle *ABC* by the dilation centered at *h* and of ratio, the golden ratio $\frac{1+\sqrt{5}}{2}$. Each point belonging to *(C)* but outside *A'B'C'* is one of the foci *f1* of a hyperbola tangent to the three lines supported by the sides of *ABC*.

**About conics tangent to the three sides of a triangle**

*ABC* is a triangle, *(C)* its circumcircle, each point *f1* of the plane except the points of the sides of the triangle and the points of the three portions of planes opposite to angles ∠*A*, ∠*B* and ∠*C* are the first focus of a conic tangent to the three lines supported by the sides of *ABC*. More precisely:

- →This conic is an ellipse when *f1* is inside *ABC* or outside its circumcircle
- →This conic is a parabola when *f1* is on its circumcircle
- →This conic is a hyperbola when *f1* is inside its circumcircle but outside the image of triangle ABC by the dilation centered at the orthocenter of *ABC* and of ratio, the golden ratio $\frac{1+\sqrt{5}}{2}$.

The plane portions corresponding to these three cases are visible in figure 16 right.

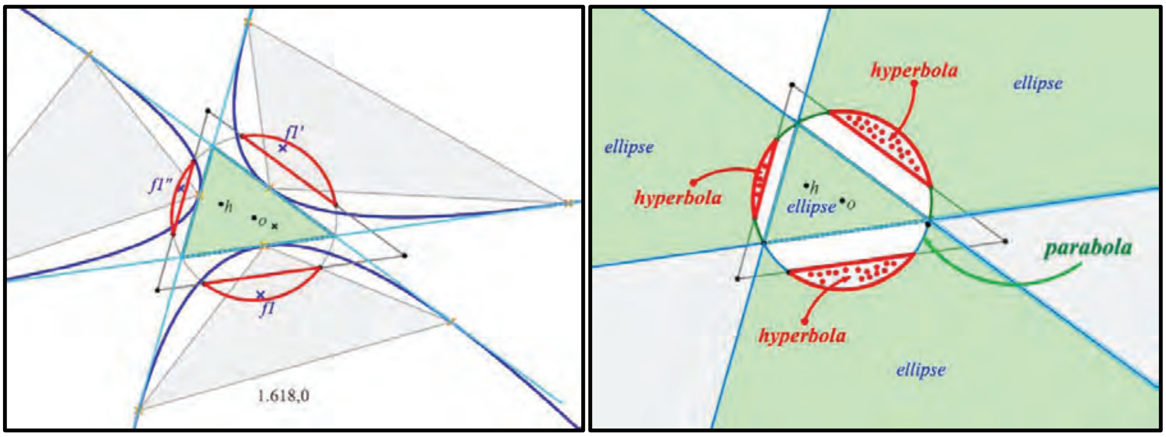


Figure 16: Locations of hyperbolas tangent to a "triangle" and final conjectures

**3.6. Isoptic curves of a branch of hyperbola**

The following algorithm will make it possible to construct the set of points from which a branch of hyperbola is seen under a given angle.

| | |
|---|---|
| 1. Points *A1* and *A2* intersection of *(H)* with *[f1 f2]*<br>2. Asymptotic lines in applying macro **asymp hyper**<br>3. Edit a number *d* and evaluate 2*d*<br>4. Create points *e* and *g* by measurement transfer of *d* and *2d* on vector *f1O*<br>5. Ray [*e g*) and a point *m* on it<br>6. *B2* intersection between *Asympt1* and the perpendicular to *(f1O)* at *A2*<br>7. Ray *[m b2*) where *b2* is the image of *B2* by translation mapping *A2* onto *m*<br>8. Apply macro **tgt hyp right** to get the tangent line (*T2*) to (*H*) at *t2* (on *(H)* and [*m b2*))<br>9. Evaluate and display distance between *f2* and (*T2*) called *h* | 10. Create a point *s2* on (*T2*) by Measurement transfer of *h* (on the right of *t2)*<br>11. Create a slider between 0 and 180<br>12. Evaluate the opposite of the number displayed by the slider<br>13. Use this last number to rotate *s2* around *t2* to get *w2*<br>14. Ray [*t2 w2*) and its intersection *t2'* with (*H*)<br>15. Midpoint *i* of [*t2 t2'*]<br>16. Intersection *j* between *(H)* and ray [*O i*)<br>17. *(T2)* parallel to (*t2 t2'*) through *j*<br>18. Intersection ***l*** between *(T2)* and *(T2')* |

Figure 17 left illustrates the stages from 1 to 11 and Figure 17 center the stages from 12 to 18.
The locus of point ***l*** is the part of the isoptic generated by the tangent lines to *(H)* associated to the points *m* of ray [*e g*). Points ***l*** exist until ray [*t2 w2*) becomes parallel to *Asympt 2*. The position of *e* (commanded by *d*) allows to avoid positions of *m* where (*T2'*) does not exist.
To be sure to obtain the complete isoptic related to the right branch of the hyperbola, we complete the locus of *l* by the locus of *l'* its symmetric point with respect to *(f1f2)*: see Figure 17 right where different isoptic curves are visible, obtained by changing the values of the slider without choosing a value superior to the angle between the two asyptotic lines and letting their trace be active.

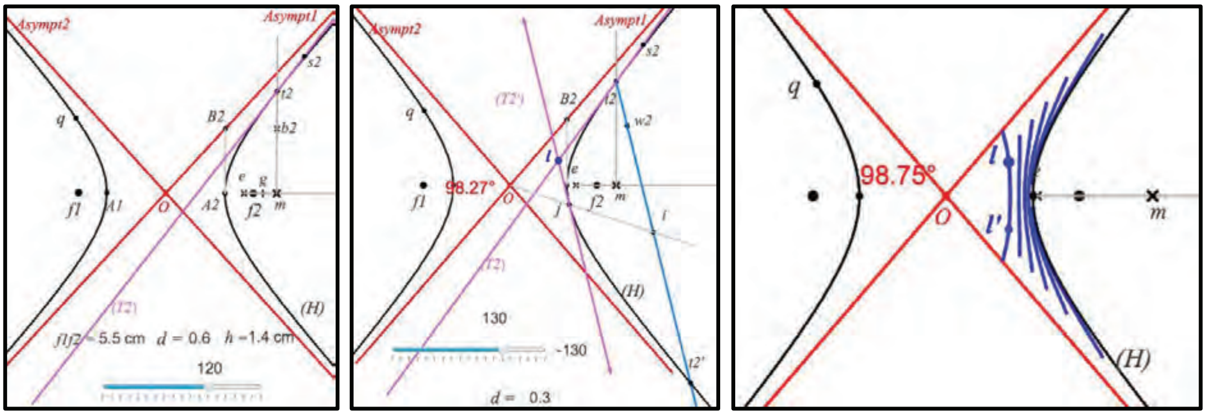


Figure 17: Isoptics of hyperbolas

**About the limit position of point *e*:**

The equation of the hyperbola *(H)* is $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ in a system of axes centered at O. This equation is equivalent to $y = \frac{b}{a}.\sqrt{x^2 - a^2}$ and we know that $\frac{dy}{dx} = \frac{b}{a}.\frac{x}{\sqrt{x^2-a^2}}$ . Therefore, the slope of the

tangent line at $M_0$ ($x_0$, $y_0$) is $\frac{b}{a}.\frac{x_0}{\sqrt{{x_0}^2-a^2}}$ = *tan(u)***.** Let us evaluate the slope of [*t2 w2)* (*v* is the angle displayed by the slider) which is:
*tan(u-v)*=$\frac{\tan(u)-\tan(v)}{1+\tan(u).\tan(v)}$. If we call $m$ the value - $\frac{b}{a}$ which is the slope of *Asympt2,* we want to find the position of $M_0$*,* when [*t2 w2)* is parallel *to Asympt2*, that is to say:
$\frac{\tan(u)-\tan(v)}{1+\tan(u).\tan(v)} = m$ or $tan(u) = \frac{m+\tan(v)}{1-m.\tan(v)}$ or $\frac{b}{a}.\frac{x_0}{\sqrt{{x_0}^2-a^2}} = \frac{-\frac{b}{a}+\tan(v)}{1+\frac{b}{a}.\tan(v)}$ , from which we obtain
$\frac{x_0}{\sqrt{{x_0}^2-a^2}} = \frac{-ab+a^2.\tan(v)}{ab+b^2.\tan(v)} = M$ equivalent to ${x_0}^2 = \frac{M^2a^2}{M^2-1}$ and $x_0 = \sqrt{\frac{M^2a^2}{M^2-1}}$ .

Eventually the limit value of $x_0$ to construct a point viewing the right branch under an angle of *v* is the previous value. $x_0$ is the distance *Oe.*

## 4. Conclusion

This article was the occasion of an original visit of the conics centered on the problem of their tritangency. Almost every construction gives the opportunity to create a detailed macro construction which will be used for the following investigations. The purely geometric construction of tritangent conics led to the discovery of two original conjectures on the classification of such conics, one including the golden ratio. Once again, the experimental approach mediated by dynamic geometry has shown its power for the illustration of known results with some of their lesser known consequences and the discovery of original and highly plausible conjectures. The reader will find there detailed all the algorithms of the purely geometric constructions used in order to realize that the initiation to programming can and must go through the stage of geometric macro constructions before approaching more complex formalizations.

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## Software

*Cabri Author (Version 4.10)* by Cabrilog at http://www.cabri.com

# Bridging the Mathematics Gap Through the Use of Mathematical Apps

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**Abstract**: *During the COVID-19 pandemic, school campuses worldwide were forced to close, and students had to learn primarily from home. This sudden disruption is estimated to have caused significant learning loss among learners. This paper reports the use of mathematical applications (apps) to bridge the mathematical learning gaps in Grades 1 to 11 in the Philippines after the pandemic, as part of a project funded by a national government agency. The apps include those that strengthen foundational concepts in number and fraction sense in grade school mathematics, develop proving skills in geometry, promote mastery in algebraic and trigonometry through drill and practice, and facilitate statistical understanding and reasoning. The description of the apps, their design, and pedagogical basis are discussed. Challenges encountered in the implementation of the project are also presented.*

## 1. Introduction

The COVID-19 pandemic disrupted education worldwide due to the closure of schools and the shift to online and blended learning modalities. As classes have returned onsite post-pandemic, educators, teachers, and school administrators are assessing the extent of the learning loss in students, and finding means to address this. In the school year 2022-2023, the Department of Education (DepEd) in the Philippines developed the *Basic Education Learning Recovery Plan* [1] to guide schools in addressing and identifying learning gaps. This was followed by the adoption of the *National Learning Recovery Program* [2] to address these gaps and learning setbacks. The program is currently being implemented across all grade levels by local DepEd school division offices, with their external stakeholders and partners.

Under a government-funded project [3] our team collaborated with some DepEd school divisions in providing mathematical resources to partner schools. These resources (also called *Mathplus* resources) consist of mathematical applications (apps), teaching guides/instructional videos, and performance tasks. The mathematical apps were designed to address the Most Essential Learning Competencies (MELCs) prescribed by DepEd [4] as well as to help narrow the existing learning gaps on competencies that were determined by the school administrators and teachers in our partner schools. The teaching guides and instructional videos were created as support to teachers towards the implementation of the apps in the curriculum. In partnership with the schools, capacity building seminars and webinars were given to teachers to guide them on the use of these resources.

The mathematical resources are made available through the *Mathplus* website https://mathplusresources.wordpress.com. Two new frameworks for sharing information are also used that are geared towards maximizing the reach of educational materials to locations with minimal internet connectivity. For selected schools, one way of distribution of the mathematical resources is through a community LTE network [5]. The community LTE (Long Term Evolution) network used is a mobile cloud network architecture called *Educloud* that facilitates a systematic and resilient way of distributing educational content that is ideal for distant and asynchronous learning. It is low cost, has a low technology requirement and is compatible with *Mathplus* resources for distribution [6]. A second way of distribution of instructional content is using a digital datacasting framework called *RuralCasting* [7]. Under this technology, a custom *set-top box* receives usual TV programming, along with the attached data content, providing multiple users access to the content thru Wi-Fi, and return information back using alternative transmission methodologies. The content can also be updated regularly using the datacasting transmission. The contents, namely, the mathematics applications, videos, text files, images are locally stored in the set-top box. A lightweight learning management system like Moodle and Canvas but specifically for the RuralCasting set-top box, named *Edukastv* was created. In *Edukastv*, the content creators or teachers can upload their subject outline, reading modules, quizzes, etc. for the students to access.

This paper discusses a selection of mathematical apps that were designed under [3] to bridge the mathematical learning gaps as communicated by partner schools to the *Mathplus* team. These apps are, for Grades 1-6: foundational concepts on place value, fraction and number sense; for Grades 7 to 10: algebra, geometry and trigonometry; and statistics for Grades 1 to 11. The design and pedagogical basis of the apps are discussed briefly. We also discuss the challenges in the implementation of the project.

## 2. General Design Considerations

The use of technology in mathematics education is at times criticized for perpetuating the "digital divide" or the gap between those with access to modern technology and those who do not [8]. However, the apps described in this paper were designed in consideration of under-resourced educational systems such as the Philippines where students and even teachers have limited access to the latest technology. First, the apps were meant to run on Android devices with basic specifications. Second, the apps were made freely available even for schools with little or no budget for purchasing software. Third, the portability and convenience of using mobile devices allow students to access educational materials more frequently [9], even when they are in their homes or when they are not under the guidance of their teachers.

The following key elements were incorporated in the design of the apps: the existing literature on mathematical learning, the benefits of technology in mathematics education and game-based learning principles.

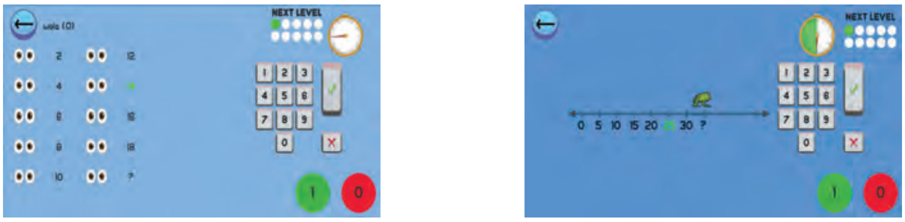
An important strategy that has been adapted in the aforementioned project is the use of game-based learning. Thus, several of the apps developed in the project are games that have clearly defined learning outcomes. The use of games for learning may lead to increasing students' motivation and engagement while allowing them opportunities for graceful failures and to customize their playing experiences according to their own situation [10]. Thus, such games may have a stronger potential to help with bridging learning gaps in mathematics, which many students find difficult and/or unenjoyable.

The design of our game-based apps is guided by Shi and Shih's Game-based Learning (GBL) Design Model, which includes macro-design concepts and 11 interrelated game-design factors: game goals, game mechanism, game fantasy, game value, interaction, freedom, narrative, sensation, challenges, sociality, and mystery [11]. The central game-design factor is game goals, which naturally affect the other factors of the game's design. In the case of our apps, the game goals always involve the learning or development of different mathematics learning competencies. The game mechanism is then usually guided by pedagogical research that promotes a more effective achievement of the learning competencies. To increase engagement, the games are designed to be interactive and responsive while allowing some freedom for players to customize their playing experience. The game fantasy, which includes its narrative and sensation, is mostly influenced by the game mechanics and the target grade levels for the game. In most cases, simple narratives are used and are complemented with colorful graphics and appropriate music and sound effects. To enhance the game value, most of the games have different modes and levels that contribute to the game's mystery while also setting up a progression according to the challenge or difficulty level. Usually, this progression is also aligned to the progression of the learning content. Since the apps are also intended for independent learning, they are mostly single-player games; however, opportunities for social interactions through the game can be possible in the classroom or virtual meetings.

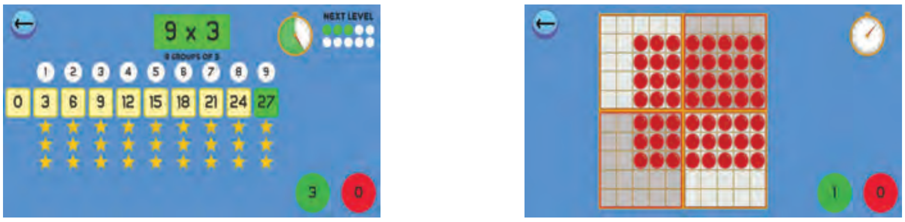
## 3. Apps for Grades 1 to 6

Remediation on the four fundamental operations has been continuously carried out in elementary schools in the Philippines even before the pandemic. Multiplication, for example, is one of the least learned competencies of students that has been emphasized by one of our partner school division offices. The *Multiplication Game* is one of the apps that we have designed to address this need.

The *Multiplication Game* facilitates mastery of the multiplication table through research-based representations such as repeated addition (modeled by equal groups), arrays and skip counting (modeled by a number). The range of representations is necessary because some representations are more accessible than others in promoting multiplicative thinking [12]. Further, illustrating multiplication using these representations is one of the MELCs prescribed by the DepEd [4] in Grades 1 and 2, The *Multiplication Game* addresses this, starting with the *Beginner* level where students carry out repeated addition by 2's, 5's, and 10's (Figure 3.1(a)) and in the *Advanced I* and *II* levels where students perform skip counting by 3's, 4's, 6's, 7's, 8's and 9's. The sequence of levels follows the learning trajectory and the official DepEd curriculum [13] which starts with multiples of 1, 2, 5, and 10, in Grades 1 and 2, followed by the other multiples, starting in Grade 3.




(a) (b)




(c) (d)

**Figure 3.1** Repeated addition, skip counting and use of arrays in the *Multiplication Game*

Skip counting is the recommended strategy by DepEd for Grade 4 learners [13] to identify the multiples of a given number up to 100. The game illustrates this using equal jumps in the number line. It is recommended that teachers first write multiplication equations on the board that match the repeated jumps on the number line. Using the *Multiplication Game*, students use skips of 5 (Figure 3.1(b)), for example, to help them multiply by 5. Students input the missing multiple in the number line. When students reach the last part of the *Beginner* and *Advanced* I and II levels, they write a multiplication equation using repeated addition and an array. As shown in Figure 3.1(c), students give the product of "9 x 3" and then the game will show that this is the same as "3 groups of 9". Consequently, the concept of commutativity is also reinforced by the *Multiplication* game. Lastly, in the *Expert I* and *II* levels (Figure 3.1(d)), students are shown red dots on an array and players must input the number of red dots. As the number of red dots becomes bigger, students will refrain from counting each dot one by one but will rather use a multiplication equation to get the number of dots.

After mastering the multiplication table by playing the *Multiplication Game*, students practice using divisibility and factoring rules by playing the *Divisibility Game* and *Factors Game*. Grade 5 learners are expected to use divisibility rules to find for 2, 5, and 10; 3, 6, and 9; 4,8,12, and 11 to find the common factors of numbers. In the *Divisibility Game* (Figure 3.2(a)), students can choose the dividends (3 or 4 digits), the divisors, for example 3, 6, 9, and the number of items (9, 12, or 16). Then the students click on the numbers that are divisible by the divisor. On the other hand, in the *Factors Game* (Figure 3.2(b)), the student first selects a number and then all its factors. The game helps students of varying grade levels in determining factors as well as multiples of a given number. This is helpful in providing them with a firm understanding of prime and composite numbers [14].

Some apps and games were also designed to study fractions, one of the least learned competencies for Grades 1 to 6. The learning of fractions may start with early experiences of 1/2 and 1/4, with the eventual aim of conceptualizing fractions as a number; that is, as a point on the number line [15].


(a) (b)

**Figure 3.2** The *Divisibility Game* and *Factors Game*

The apps designed for learning fractions provide various representations that are intended to help students connect fractions to the number line model. The first one we will mention here is the *Moving Fractions* game (Figure 3.3(a)). The following concepts can be visualized in a game-like setting: 1/2

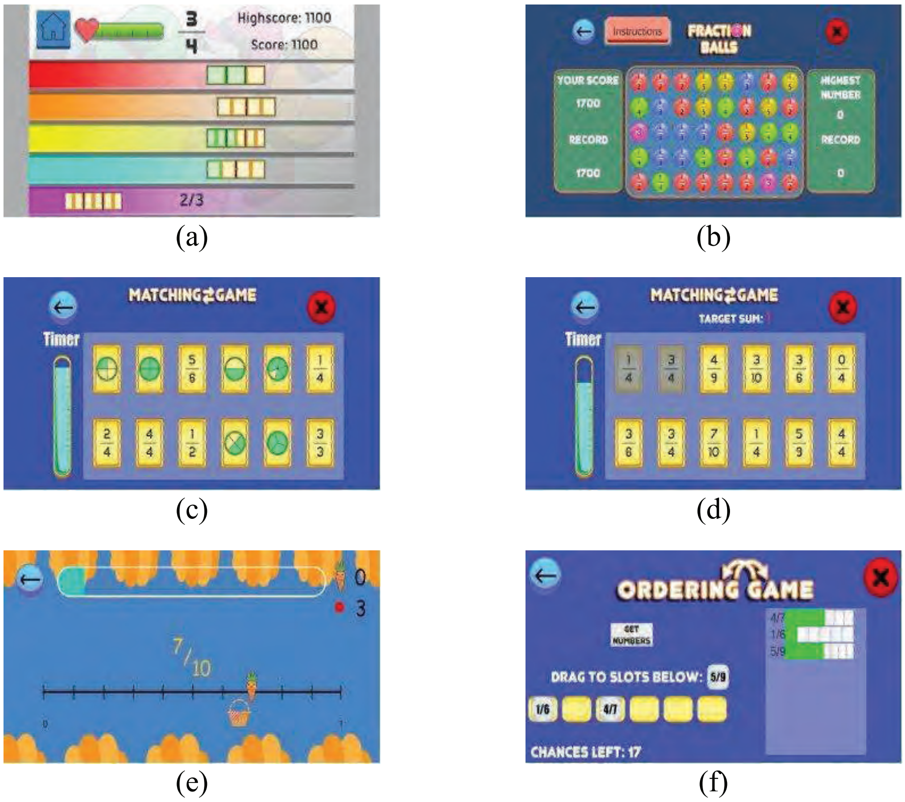



**Figure 3.3** Apps and games focusing on fractions

and 1/4, unit fractions, proper fractions, improper fractions, and mixed numbers. Students can envision how to add (join) or subtract fractions. This game is discussed in detail in [16]. Another game is the *Fraction Balls* game (Figure 3.3(b)) which shows addition (through joining) and subtraction (through missing addend) of similar fractions with colored visual supports. The user has the option to work with sums that convert to the lowest terms or not. In the *Matching Game*, a set of cards is shown (Figure 3.3(c)) and the student must match fractions and their corresponding pictures. Another option of this game asks students to match two fractions that add up to a certain target sum (Figure 3.3(d)). Students also have an option to play with whole numbers or decimals in the *Matching Game*. In the *Catch the Carrot* game, a random fraction and a number line are shown on screen (Figure 3.3(e)). The student must estimate where the fraction is located on the number line. The student's aim is to situate as many numbers as possible within the time given. Other options available in the *Catch the Carrot* game are whole numbers, decimals, integers or irrationals. The article [17] presents the design of C*atch the Carrot*. It also discusses the pedagogy of the use of number lines in developing the estimation and number sense skills of students. In the *Ordering Game*, a fraction is generated at random (Figure 3.3(f)), and the learner drags it to a row. The goal of the game is to fill up the entire row, so that the fractions appear in an ascending order, within the allowed number of chances, indicated at the bottom part of the screen. The game becomes more difficult as it progresses because it becomes more likely that a randomly selected number cannot be correctly placed in any of the remaining slots [18].

To summarize, a list of apps that are used by learners in different grades levels, based on their math curricula, are given in the table below.

**Table 1**. Applicable apps for different grade levels

| Level | Apps | Level | Apps |
|---|---|---|---|
| Grade 1 | Multiplication Game, Matching Game, Moving Fractions, Catch the Carrot | Grade 4 | Multiplication Game, Factors Game, Moving Fractions, Fraction Balls |
| Grade 2 | Multiplication Game, Matching Game, Moving Fractions. Ordering Game, Catch the Carrot | Grade 5 | Multiplication Game, Factors Game and Divisibility Game, Moving Fractions, Fraction Balls, Ordering Game, Catch the Carrot |
| Grade 3 | Multiplication Game, Matching Game, Moving Fractions. Ordering Game, Fraction Balls, Catch the Carrot | Grade 6 | Moving Fractions, Fraction Balls, Ordering Game, Catch the Carrot |

## 4. Apps for Grades 7 to 10

A selection of the mathematical apps developed for use in Grades 7 to 10 is meant to engage students in regular practice and drills on topics which the students have difficulty with. These apps, designed with game-like settings, are accessible, easy to play with and provide immediate feedback to students. This helps direct the students to either proceed to advanced levels in the app or simply give them opportunities to practice their math skills. Some mathematical skills can also be learned through deliberate practice [19]. Deliberate practice involves the learning of higher-order skills; drill and practice, meanwhile, is often associated with procedural skills [19]. Features of deliberate practice include well-defined goals to address weak points as determined for instance, by a teacher. Thus, the apps designed using the deliberate practice model can be used for remediation activities to provide students the opportunity to work on their weaknesses.

An important skill in mathematics is the ability to recognize number patterns. To this end, the authors designed an app called *Numberger* (Figure 4.1(a)). The app covers different topics: number identification (integers, rational numbers etc.), equivalent fractions, domain and range of functions and sequences. It aims to give opportunities for students to practice various math skills. The objective of *Numberger* is to let the user make burgers (hence the name “Numberger”) by being able to choose the correct answer to a given question. One of the topics included in this app is “Sequences”. There are two subtopics under “Sequences”: “Find the Next Term” and “Find the Term”. In the first subtopic, a student is shown some terms of a sequence and is asked what the next term is (Figure 4.1(b)). For this he needs to see the pattern or rule to be able to answer the question correctly. The questions are shown one at a time at the top part of the screen while an ingredient to make a burger is initially shown moving back and forth from left to right on the screen. Some options for the answer are shown at the bottom part of the screen and when a correct answer is chosen, the ingredient falls on the plate found at the bottom of the screen. In the second subtopic, an expression for the $n$th term, $a_n$, of a sequence is given and the student is asked to give the value of a particular term. Here the focus is on how to evaluate a term of a sequence, hence the sequence given may be an arithmetic sequence (Figure 4.1(c)) or a geometric sequence (Figure 4.1(d)).

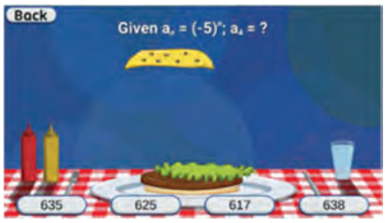
There are two modes for playing the app: Normal Mode-Complete 3 Burgers or Endless-Play without end (Figure 4.2(a)). A player who chooses the Normal Mode receives a “Good Job” message after completing three burgers (Figure 4.2(b)) and can choose to go play again or end the game.

Another app that employs the deliberate practice model to hone algebraic skills is *Just Keep Solving.* The app focuses on helping students become more proficient in solving linear equations and inequalities. There are two topics: *Linear Equations* and *Linear Inequalities*. A sample opening screen of this app is shown in Figure 4.3(a) where the topic chosen is *Linear Inequalities*.


(a)


(b)


(c)




(d)

**Figure 4.1** (a) Numberger Topic: Sequences; (b) Finding the Next term; (c) Finding a term (arithmetic sequence); (d) Finding a term (geometric sequence)

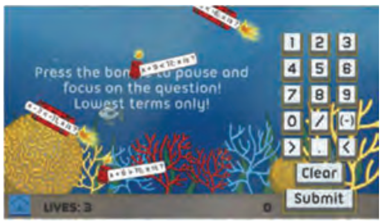

(a)


(b)

**Figure 4.2** (a) Modes: Normal or Endless; (b) Message after completing 3 burgers

When a student presses the play button, a screen such as the one shown in Figure 4.3(b) appears. The display is an underwater scene and bombs containing inequalities to be solved fall from the top of the screen. The goal is to be able to solve the inequalities before the bombs reach the ocean floor and cause destruction of the corrals. The mechanics for the topic of *Linear Equations* are like that of *Linear Inequalities*. More details about this app are given in the paper [20].


(a)

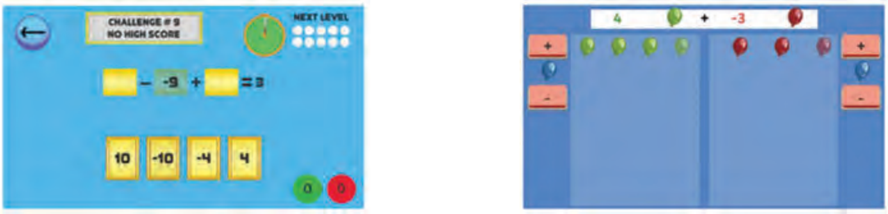



(b)

**Figure 4.3** The *Just Keep Solving* app

Disruption to education systems in the pandemic has led to many students experiencing learning loss and learning gaps as they enter high school. For instance, Grade 7 students have not mastered numeracy skills involving fractions, decimals, and integers. Some of the apps developed during the project were helpful in a high school remediation program to address this problem. The *Number puzzles* app, for instance, provides numerous opportunities for practice involving addition, subtraction, and multiplication. *Algeops* incorporates three models: objects, numerals, and number line in a visualization of integer operations. The app incorporates both the neutralization and number line models [21] which offer a more holistic representation of integers. *Moving Fractions and Fraction Balls* were also very helpful in addressing the difficulties of the students in fractions.

Students and teachers were given a survey through Google Form with the goal of evaluating the apps used in the remediation program in three (3) different categories: (a) usability; (b) game-like experience; and (c) alignment with learning objectives. Results indicate a positive response from students and teachers. Most of the students (73%) had no difficulty using the apps. The apps helped 74% of the students comprehend the lesson and its solutions. The apps also helped them find the answers easily and quickly. About 70% of the students noted that the apps provided a lot of opportunities for practice and expressed that with continued use, their math skills would improve. Moreover, the teachers found the apps useful in helping the students stay motivated and engaged throughout the lesson. They also commented that the numerous exercises provided by the apps helped the students master the foundational skills in mathematics.




(a) (b)

**Figure 4.4** The (a) *Number Puzzles* and (b) *AlgeOps* apps

One of the high school math subjects that requires the development of algebra and geometry abilities is trigonometry. These abilities are crucial for studying pre-calculus and calculus. In the course of our work in [3], teachers reiterated the need for technological tools involving trigonometry.

*Trigomatch* employs an interactive game with interactive feedback to help users master basic trigonometric concepts like reference angles, trigonometric ratios and trigonometric functions. It presents visual representations of these concepts to allow students to extend their understanding of trigonometry from the right triangle to the unit circle approach. When visual representations of trigonometry are presented using interactive technology, they are more accurate and allow the learners to proceed at their own pace [22]. Additionally, the accuracy of these presentations can reinforce images in a person's memory. This is not the case for hand-drawn images which tend to be distorted. Figure 4.3(a) and 4.3(b) display the *Trigonometric Ratio* and *Trigonometric Function* set-up, respectively. The objective is for the learners to match two equivalent representations of the trigonometric ratio and function within the allotted time.


(a) (b)

**Figure 4.5** (a) *Trigonometric Ratio* Level; (b) *Trigonometric Function* Level of *Trigomatch*

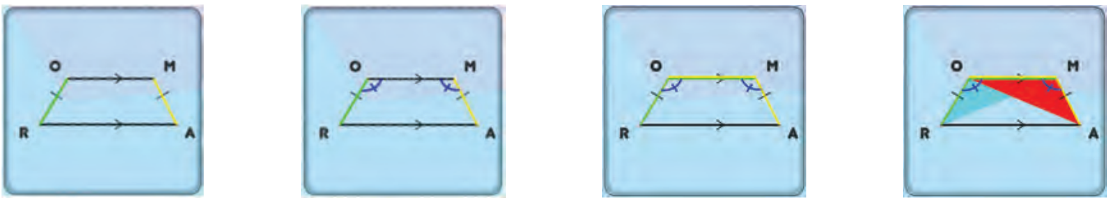
In high school geometry, one of the topics teachers find very challenging to teach is proving geometric statements. Students struggle with writing proofs and find the process abstract and uninteresting. This situation was exacerbated during the time of the pandemic when it was much more difficult to teach proving remotely. When teaching proofs, teachers rely on textbooks or modules, students' opportunities to learn proofs are limited to what is provided in these instructional materials. Two apps to meet the needs of teachers and students on proving are *Two Column Proof* and *ProveIt.* The design of both apps is grounded on the use of visualization and imagery in mathematics [23].

The *Two Column Proof* app is structured to support the students in writing a two-column proof, where a statement or reason in the proof is supported by a visualization element. The intent is to highlight properties and congruence relations of sides, angles and triangles through animation which will serve as visual clues to the students when completing the line of a proof. The visualization elements not only strengthen the student's understanding of the proof process but also stimulate their interest. In the Philippines, Grade 9 lessons on writing two column proof begin with properties of parallelograms. Much time is utilized discussing the first proving topic on parallelograms so oftentimes proving properties of other quadrilaterals such as kites and trapezoids are left for the students to work on individually. We designed the *Two Column Proof* app to include proving geometric concepts on parallelograms, kites, and trapezoids (Figure 4.6(a)).


(a) (b)

**Figure 4.6** (a) Main Menu of *Two Column Proof;* (b) Proving exercise on trapezoids

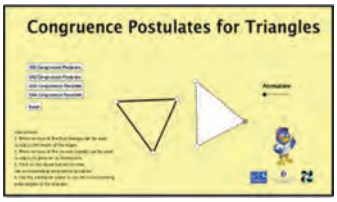
One proving exercise given in the *Two Column Proof* app is to prove the diagonal isosceles trapezoid congruency (Figure 4.6(b)). An example of a task is to write a mathematical statement in the "Statement" column where the justification in the "Reason" column is given. To guide the student, an animation is provided to visualize the property of a trapezoid or a congruence property in the "Reason" column. For instance, to visualize "Base angles of an isosceles trapezoid are congruent", the congruent base angles (highlighted in blue) of the isosceles trapezoid are flashed repeatedly (Figure 4.7(b)). Next, to visualize the Side-Angle-Side (SAS) congruence, an animation is provided in the following sequence: a pair of sides OR and MA from two different triangles (Figure 4.7(a)), a pair of congruent angles ROM and AMO (Figure 4.7(b)), the common side OM (Figure 4.7(c)). Then the two triangles are highlighted: triangle ROM (light green) followed by triangle AMO (red) (Figure 4.7(d)).



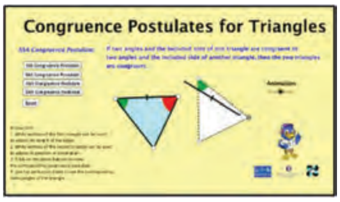

(a) (b) (c) (d)

**Figure 4.7** Animations in the *Two Column Proof* app

One of the essential tools when formulating proofs is knowledge of previously taught mathematical concepts that are needed in formulating the proof. For instance, in proving concepts on parallelograms, kites or trapezoids, one needs a firm understanding of the congruence postulates for triangles. Unfortunately, students either forget these concepts that have been taught earlier in high school, or these were not taught at all in the time of Covid. We also designed *GeoGebra* apps on triangle congruence for this purpose (Figures 4.8(a)-(b)). One advantage of *GeoGebra* is one can create applications in a much shorter time (than math apps that require some extensive programming) to fill in the learning gap, when necessary.



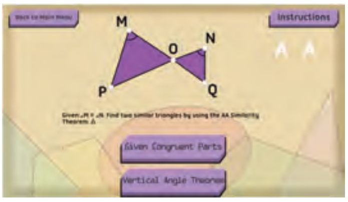

(a)



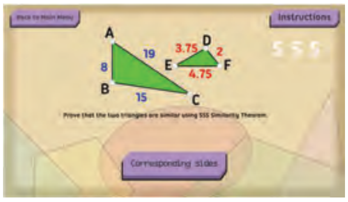

(b)

**Figure 4.8** GeoGebra applets on Triangle Congruence

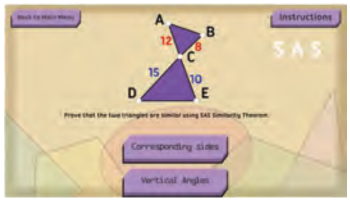
*Prove It* is another app that aims to facilitate students' proving competency. The first version of *Prove It* for the topic of triangle congruence is given in [24]. In [3], the app was extended to include the topic of triangle similarity, as a need to bridge the learning gap. A task requiring the Angle-Angle (AA) Similarity Theorem (Figure 4.9(a)), Side-Side-Side (SSS) Similarity Postulate (Figure 4.9(b)), or SAS Similarity Postulate (Figure 4.9(c)) is initially shown. The corresponding text AA, SSS, and SAS on the screen provides guidance to help students draw their attention to the relevant parts of the triangles. For AA, students need to select two pairs of congruent angles. For SSS, students need to select three pairs of proportional sides. For SAS, students need to select two pairs of proportional sides and one pair of congruent angles (which should be the included angle between the two proportional sides). Feedback is also provided to guide students towards selecting the correct pair. Once the student successfully identifies the correct pairs, an animation shows that the two triangles are similar (Figure 4.9(d)); for SSS and SAS, a corresponding ratio is shown (Figure 4.9(e)).




(a)

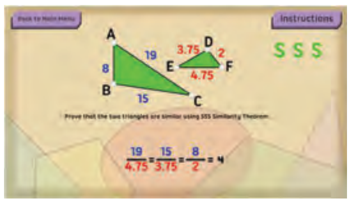



(b)




(c)


(d)




(e)

**Figure 4.9** The app *ProveIt* featuring tasks on Similarity

In the third quarter of the school year 2022-23, a short study was conducted to assess the effectiveness of the *Two Column Proof* app. Two pairs of control-experimental classes in a public high school in Quezon City, Philippines were given pretests and posttests covering the following competencies: (i) proves theorems on the different kinds of a parallelogram (rectangle, rhombus, square), (ii) proves theorems on trapezoids and kites. For the first pair of classes, the control class (38 students) recorded a mean gain score of 30.94% while the experimental class (40 students) recorded a mean gain score of 44.90%. The gap between mean gain scores was wider for the second pair of classes: 25.73% for the control (25 students) and 79.29% for the experimental (27 students). These results indicate the potential of the *Two Column Proof* app to help students learn geometry, particularly proving topics, better.

## 5. Probability and Statistics Apps

*Probability and Statistics* is identified as one of the five content strands in the Philippines' K to 12 Basic Education Curriculum [13]. The focus of the strand is on "developing skills in collecting and organizing data using charts, tables, and graphs; understanding, analyzing and interpreting data; dealing with uncertainty; and making predictions about outcomes" [13, p.5]. Competencies in probability and statistics primarily comprise the fourth quarter of the mathematics subject in Grades 1 to 10, except 9. Additionally, Grade 11 students take the core subject *Statistics and Probability* during the third and fourth quarters of the school year [25].

Even before the pandemic, there have already been some challenges in ensuring that students develop the learning competencies in probability and statistics. As the corresponding topics are scheduled to be covered during the last quarter of each school year, class delays or cancellations (e.g., due to typhoons, other disasters, or unexpected holidays) usually result in some of these topics being taught quickly or neglected altogether [26, pp. 300-301]. Another challenge in statistics education in the Philippines is the lack of resources [27]. Particularly, there is a need for accessible and available materials that can support or enhance the teaching and learning of probability and statistics that are also compatible with independent learning.

To bridge the learning gap in probability and statistics, we have developed different apps as part of [3]. These apps have been deliberately designed so that these can be used not only in actual classroom settings but also for remote or independent learning activities.

As emphasized in the Pre- K-12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II), "the focus on variability in data, the importance of context associated with the data, and the questioning of data, sets statistics apart from other mathematical sciences and makes it particularly relevant for all fields of study" [28, p.8]. Thus, it is important to have different data sets with clear contexts and with which learners can deal with variability. Further, a classroom study by Pfannkuch [29] has found that the contexts of data sets have the potential to assist students, up to a certain extent, in developing informal statistical inferential reasoning. Rossman and Chance [30], in their proposed introductory course for statistics, included activities where students are tasked to do probabilistic experiments such as flipping a coin. This is in alignment with Cobb [31], who has suggested using simulation- and randomization-based methods to develop statistical inference.

Given the above, we have developed the Android/PC app *Probability Simulator* and the web-based app *Senso Eskwela Pilipinas* (*SEP*). The *Probability Simulator* app (Figure 5.1) allows for the simulation of different probabilistic experiments (e.g., rolling a fair or unfair die, drawing a card from a deck). The number of trials of the experiments can be set in the app. The outcomes are displayed in the app or can be saved as a spreadsheet file for further processing or analysis.


(a) (b)

**Figure 5.1** Screenshots of *Probability Simulator*'s (a) home screen and (b) outcomes screen

*SEP* [32, 33] is an online platform that is based on the Census-at-schools program [34]. Students get access to *SEP* through their teachers and, once logged in, they can anonymously answer a 31-question survey that consists of varied questions that can appeal to young Filipinos' interests and experiences. The students' responses then become part of the *SEP* database that is accessible to *SEP* users. Teachers and students may access their own class data while any *SEP* user may access random samples from the database. Thus, *SEP* becomes a readily accessible source of authentic data that is relevant and relatable to the students. Initial data on the potential benefits of *SEP* with respect to students' learning of statistics are presented in [33].

With *Probability Simulator* and *SEP*, students have ready and easy access to data sets that are otherwise difficult, timely, or tedious to obtain. Moreover, these contextually clear data sets exhibit the variability that students need to handle when learning probability and statistics. Finally, both apps can be used with minimal supervision so these are ideal for independent learning and can be repeatedly used for learning activities across different competencies in probability and statistics.

Student worksheets have also been developed to complement the apps mentioned above. Each student worksheet corresponds to a learning activity; the materials needed, and step-by-step instructions are provided in the worksheet. The worksheets are also formatted in a way that students can readily write their answers on the worksheets, whether on a printout or electronically. For example, in the Grade 3 worksheet *Likely and Unlikely* (Figure 5.2), students are asked to use the

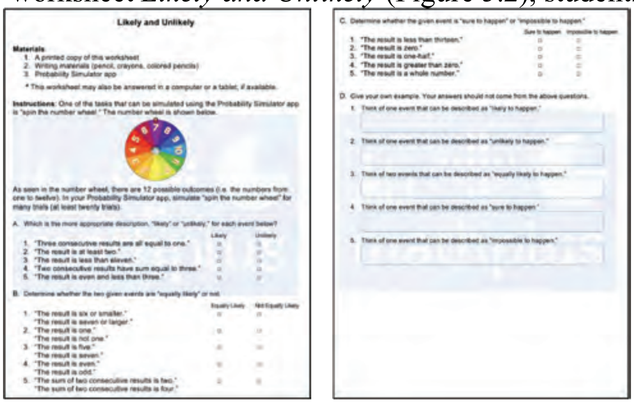


**Figure 5.2** Screenshots from the Grade 3 student worksheet *Likely and Unlikely*

*Probability Simulator* so that they can develop their sense of events that are likely, unlikely, sure to happen, or impossible to happen. Another example is a Grade 11 worksheet *Discrete Random Variables,* which outlines an activity in which students use an *SEP* question and data set to define a discrete random variable then construct its probability distribution and histogram. The student worksheets can be used for in-classroom activities and are also ideal for independent learning activities. Thus, together with the apps, these can be used to continue or to reinforce the learning of probability and statistics competencies that can no longer be fully accommodated within the last quarter of the school year.

## 6. Challenges in Implementation

Several challenges were encountered during the project [3] in the implementation of the use of technological tools in schools to bridge the learning gaps in mathematics.

Students' access to technology (smartphones, tablets, computers, etc.) remains one of the most challenging problems in the Philippines. Some schools under DepEd school divisions get support from the local government where each student is provided with a tablet, but this is not true for all schools. In this respect, it was difficult for students who did not have their own gadgets to have access to the apps and to use these to help in their math learning. Other students had to rely on the gadgets of their parents or older siblings but could only use these at certain times during the week.

Lack of teachers has also been a main concern. With the lack of teachers there are missed opportunities for a more focused and nurturing approach in addressing the least learned competencies of students. In a class for example, the learning gaps are different for each student, and the choice of mathematical app to remedy a particular learning competency may vary. Moreover, even if school administrators would want to conduct remediation programs outside of class hours, there are not enough teachers to carry out these programs.

Classroom shortage has also been a concern. There are insufficient classrooms that could be venues for remedial classes or learning spaces where students work on the mathematical apps during their free time in school.

Finally, families play a critical role in successfully integrating technology into learning. It has not been easy, for example, for teachers to collaborate with parents with regards to knowledge on the mathematical apps adopted in class so that their contribution at home could be more effective. For one, there is difficulty on the part of the parents understanding how the gadget and the mathematical apps work. Particularly for low-income families, there is really no time left for parents to learn the technology, as they must prioritize work to support their families. This can enhance the learning inequality further, particularly for young learners. For some students, there are no opportunities to carry out learning activities with the mathematical apps as parents must rely on them to carry out tasks at home.

## 7. Conclusion and Future Direction

The use of technological tools to help address the issue of major learning gaps in students after the pandemic is not unique to the Philippines. In India for example, the World Economic Forum launched the new 'Education 4.0 India Report' where digital tools will be utilized to enhance foundational literacy and numeracy skills [35]. The implementation of our project [3] to schools in collaboration with DepEd school division offices, is one step towards addressing this problem in the Philippines. In this paper, we showed the different mathematical apps that we have developed to address learning gaps in several content areas in the DepEd curriculum that have been identified by teachers and administrators in our partner schools. The design of the apps has been based on pedagogical principles and game design elements, as well as what is informed in the literature as benefits of technology in mathematical learning. The apps are free and run primarily on low-end mobile devices for accessibility. It is hoped that other educators can find these apps helpful in their work on the use of technological tools in bridging gaps in mathematics learning outcomes.

There is a lot of work to be done in terms of the mathematical apps-more updates can still be done in terms of adding questions and levels of difficulty. Collaboration is still ongoing with DepEd school division offices in terms of assessing the effectiveness of these apps in student learning. Furthermore, there are ongoing efforts soliciting support from NGOs and private sectors to bring the mathematical resources to more schools in the country.

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# A Particle Swarm Optimisation approach to the Generalised Fermat Point Problem: rethinking how a problem is solved

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**Abstract**

*This position paper claims that the way a mathematical problem is solved depends on the technology available to the problem-solver then. Drawing on the authors' mathematical experience of finding a new solution to an old problem – the Generalised Fermat Point Problem, salient observations are drawn to illustrate how a problem solver's experience can shaped by technological affordances.*

## 1 Introduction

A recent exploratory joint-work with the second author marks the genesis of this paper. The long and short of it is that we were re-visiting an old geometry problem called the Fermat Point Problem (original version involves 3 points in the 2D-plane) and could not satisfy ourselves with the well-known Torricelli's geometrical solution – it cannot be modified to solve the generalised Fermat Point Problem (which involves $n$ points, $n \geq 3$). Our probe for a solution to the generalised Fermat Point Problem led us to unearth various existing solutions – each emerged from a unique time period in history and supported by distinctive technology available at that time – and, finally, landed us on our own approach that makes use of Particle Swarm Optimisation, something of the present time.

The Fermat Point Problem and its generalisation are, after all, not new. Despite this lack of novelty, two things fuelled our exploration of this old problem. Firstly, we discovered that each existing solution in the literature fails to address some aspect of the problem. Secondly, we observed that what was available and accepted as a solution to the problem depended on the mathematics and technology available at that point of time in history. In this paper, we use the word 'technology' in a rather loose manner – that is, it is meant to encapsulate both the mathematical machinery and the physical technology associated to it. For example, geometry is the body of mathematical knowledge that deals with points, lines and curves in space, while the technology associated with geometry consists of straightedge, compass, and mechanical construction devices. [1] With these two points in mind, we set off to look for a new solution that would overcome the identified short falls. As we actively *create new solutions* to an old problem, our mathematical problem solving experience was continually enriched and shaped by the technological affordances available to us in this present computer age.

This paper (a) records our problem-solving experience as we look for a new solution to the Generalised Fermat Point Problem, and (2) draws out salient connections between *how a problem can be solved* and *the technology that avails to the problem solver(s) at that moment of problem-solving*. From these connections, we hope to gain some insights into how creative and innovative mathematical thinking take place through currently available technology.

We organize this paper as follows. Firstly in Section , we state clearly the Fermat Point Problem and recall Torricelli's geometrical characterisation of the Fermat point of a triangle. After a self-contained exposition of the existing solutions of the Fermat Point Problem (and its generalisation) in Section 3, we proceed to critique the pros and cons of these solutions. With these shortcomings in mind, we present in Section 4 our solution of the Generalised Fermat Point Problem using Particle Swarm Optimisation (as the present-day technology). In Section 5, we give a short exposition on the connection between the way a problem is solved and the available technology to the problem-solver then. Based on this connection, we discuss some implications on mathematics education, e.g. mathematics problem-solving and mathematics curriculum in schools.

# 2 The Fermat Point Problem (FPP)

In the early 17th century, Pierre de Fermat (1601–1665) posed a problem to Evangelista Torricelli (1608-1647), asking him for the location of the Fermat point $X$ of a given $\triangle A_1A_2A_3$, i.e., the point $X$ in the same plane as $\triangle A_1A_2A_3$ for which the sum of its distances from each of the vertices ($A_1$, $A_2$ and $A_3$) is the minimum (see Figure 1). We term this the *Fermat Point Problem*, or *FPP* for short.

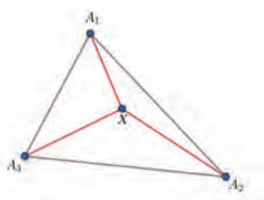


Figure 1: Determine $X$ for which $A_1X + A_2X + A_3X$ is minimum for $\triangle A_1A_2A_3$.

The existence of the Fermat Point of an arbitrarily given triangle was first established geometrically by Evangelica Torricelli. We state and prove this result.

**Theorem 1 (Fermat-Torricelli Point)** *Let* $\triangle A_1A_2A_3$ *be given.*

[1] We therefore caution all ATMC participants to be wary of this broader sense of the word 'technology' we are using here in this paper as opposed to the usual understanding that technology refers to computer technology.

1. *If* $\angle A_i \leq 120^\circ$ *for* $i = 1, 2, 3$*, then the Fermat Point* $X$ *satisfies the condition* $\angle A_1XA_2 = \angle A_2XA_3 = \angle A_3XA_1 = 120^\circ$.

2. *If* $\angle A_i > 120^\circ$ *for some* $i = 1, 2, 3$*, then* $X$ *is located at that obtuse angle.*

**Proof.** 1. First we establish that the Fermat point $X$, if exists, must lie inside $\triangle A_1A_2A_3$. Suppose on the contrary that $X$ lies outside $\triangle A_1A_2A_3$. Without loss of generality, assume that $X$ and $A_1$ lies on opposite sides of the line segment $A_2A_3$ (see Figure 2). Since $X'$ is the

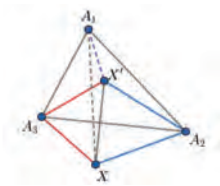


Figure 2: $X'$ is the reflection of $X$ in $A_2A_3$.

reflection of $X$ in $A_2A_3$, we have that $A_2X = A_2X'$ and $A_3X = A_3X'$. Because $A$ and $X'$ lies on the same side of $A_2A_3$ and $X$ is on the opposite side, $A_1X' < A_1X$. Consequently,

$$A_1X' + A_2X' + A_3X' < A_1X + A_2X + A_3X,$$

contradicting that $X$ is the Fermat point of $\triangle A_1A_2A_3$. Thus, the Fermat point $X$, if exists, must lie inside $\triangle A_1A_2A_3$.

Suppose that $\angle A_1$ is acute. Denote by $Y'$ the image of any given point $Y$ under clockwise rotation of $60^\circ$ about $A_1$. Then $A_1A_3 = A_1A_3'$ and $\angle A_3A_1A_3' = 60^\circ$. Thus, $\triangle A_1A_3A_3'$ is equilateral. Now, consider a variable point $X$ inside $\triangle A_1A_2A_3$. Then $A_1X = A_1X'$ and $\angle XAX' = 60^\circ$. Hence $A_1X = XX'$ (see Figure 3).

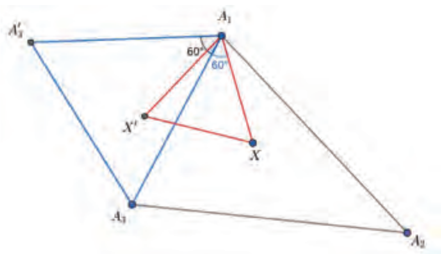


Figure 3: $\angle A_1 \leq 120^\circ$.

Thus, we have $A_1X + A_2X + A_3X = A_2X + XX' + A_3'X'$ because $A_3X = A_3'X'$. One may further assume that $\angle A_2 \leq \angle A_3$. Note that $\angle A_2A_3A_3' \leq 180^\circ$. Since $A_2$ and $A_3'$ are given points that are fixed, the straight line distance between $A_2$ and $A_3'$ is the shortest one and thus $A_1X + A_2X + A_3X = A_2X + XX' + A_3'X' \geq A_2A_3'$. In particular, equality holds if and only if $X$ and $X'$ lie on $A_2A_3'$. In other words, the minimum value of the sum $A_1X + A_2X + A_3X$ is achieved exactly when $\angle A_1XA_3' = 60^\circ$ since in that case $X'$ lies on $A_3'X$ (see Figure 4).

Since $\angle A_1XA_3' = \angle A_1A_3A_3' = 60^\circ$, by the converse of the inscribed angle theorem we have that $A_1XA_3A_3'$ is a cyclic quadrilateral. Thus, $X$ lies on $A_2A_3'$. In summary, $X$ is a Fermat point of $\triangle A_1A_2A_3$ if and only if $X$ is the intersection of the circumcircle $A_1A_3A_3'$ and

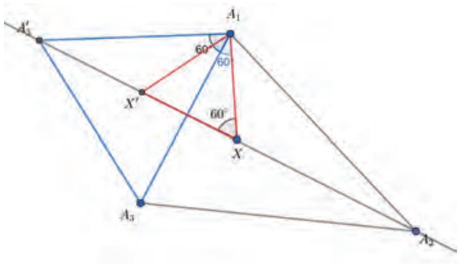

Figure 4: $A_1X + A_2X + A_3X$ is minimum exactly when $X'$ lies on $A_3'X$.

$A_2A_3'$. As $\angle A_1XA_3' = \angle A_3XA_3' = 60°$, it holds that $\angle A_1XA_3 = 60°$. Similarly, $\angle A_3XA_2 = 180° - \angle A_3XA_3' = 120°$. Thus $X$ is the required Fermat point exactly when $\angle A_1XA_2 = \angle A_2XA_3 = \angle A_3XA_1 = 120°$.

2. Without loss of generality, assume that $A_1$ exceeds $120°$. Again, denote by $Y'$ the image of any given point $Y$ under clockwise rotation of $60°$ about $A_1$. As in the preceding part, form $\triangle A_1A_3'A_3$ and for any point $X$ inside $\triangle A_1A_2A_3$ form $\triangle A_1X'X$ (see Figure 5).

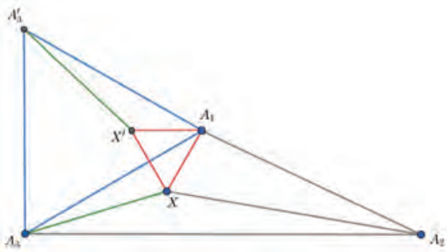

Figure 5: $\angle A_1 > 120°$.

Note that $\triangle A_1AA_3$ is congruent to $\triangle A_1X'A'3$. As before, we have that $A_1X + A_2X + A_3X = A_3'X' + X'X + XA_2$. Without loss of generality, assume that $\angle A_2 \leq \angle A_3$ so that $A_1A_3 \leq A_1A_2$. Since $X$ is inside $\triangle A_1A_2A_3$, it holds that $X'X + XA_2 = A_1X + XA_2 \geq A_1A_2$ and so $A_3'X' + X'X + XA_2 \geq A_3'X' + A_1A_2 = A_1A_3 + A_1A_2$. Now if $X$ lies outside $\triangle A_1A_2A_3$. Employ the reflection trick used earlier (see Figure 2), there exists a point $X'$ insides $\triangle A_1A_2A_3$ such that $A_1X + A_2X + A_3X > A_1X' + A_2X' + A_3X'$. In our preceding consideration, since $X'$ is inside $\triangle A_1A_2A_3$ it follows that

$$A_1X' + A_2X' + A_3X' \geq \underbrace{A_1A_1}_{=0} + A_1A_2 + A_1A_3.$$

Thus, we conclude that $A_1$ is the Fermat point of $\triangle A_1A_2A_3$. ■

# 3 How has the FPP been solved?

Having settled the existence of the Fermat Point, We now turn our attention to the various approaches of locating it.

## 3.1 Geometrical technology

What kind of solutions were availed to mathematicians of Fermat and Torricelli's times? *Geometry*! Geometry, along with arithmetic, stands as one of the oldest branch of mathematics. Until the 19th century, geometry is solely devoted to Euclidean geometry which dealt with notions of points, lines, planes, angles, surfaces, and curves. Euclidean geometry has a peculiar aspect in that it can be expressly implemented by classical geometry construction relying on just a compass and straightedge (i.e., an unmarked ruler). In a way, the mathematics of geometry is defined and shaped by both the then-accepted body of geometrical axioms and results together with the then-accepted technology of the geometrical tools of those times. Both pragmatic and aesthetic in nature, geometry is a powerful tool in solving practical problems in elegance.

To locate the position of the unique Fermat Point $X$ for the non-trivial case, that is, none of the angles $A_i$'s exceed 120°, Torricelli offered an elegant geometrical construction: (1) Construct an equilateral triangle on each of two arbitrarily chosen sides of the triangle. (2) Draw a line from each new vertex to the opposite vertex of the original triangle. (3) The two lines intersect at the Fermat Point $X$ (see Figure 6).

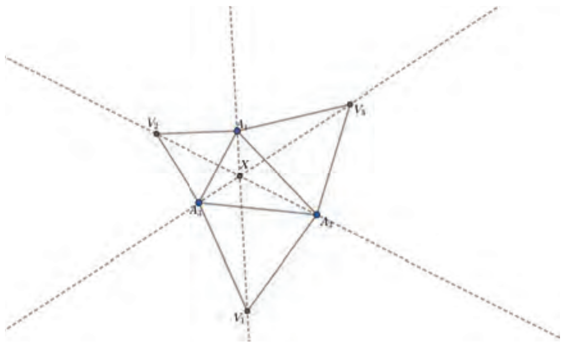

Figure 6: Constructing the Fermat Point $X$ of $\triangle A_1A_2A_3$.

The above construction makes sense only if it results in only one point $X$ *irrespective* of the choice of the two sides of the triangle, and this is justified below:

- $\angle V_2A_1A_2 = \angle A_3A_1V_3$; $V_2A_1 = A_3A_2$ and $A_1A_2 = A_1V_3$. $\therefore \triangle V_2A_1A_2 \equiv \triangle A_3A_1V_3$.
- $\therefore \angle A_1V_2X = \angle A_1A_3X$. By the converse of the inscribed angle theorem applied to the segment $A_1X$, $A_1V_2A_3X$ is a cyclic quadrilateral.
- $\angle A_1XA_3 = 180° - \angle A_1V_2A_3 = 180° - 60° = 120°$. By the inscribed angle theorem applied to the segment $A_2V_3$, we have $\angle A_1XV_3 = \angle A_1A_2V_3 = 60°$. So $\angle A_1XA_3 + \angle A_1XV_3 = 180°$. Thus, $X$ lies on the line segment $A_3V_3$.

Therefore, $X$ is the concurrent point of the lines $A_1V_1$, $A_2V_2$ and $A_3V_3$ (see Figure 7).

**Remark 2** *1. Lovers of Dynamic Geometry Softwares (DSG) will be delighted to know that Geogebra, together with its automated reasoning tools, can be used to establish mathematically that the three lines $A_1V_1$, $A_2V_2$ and $A_3V_3$ are concurrent.*

*2. There are many mathematical topics related to the Fermat Point Problem, for example, the Steiner Tree Problem. Readers who interested in the many geometrical musings related*

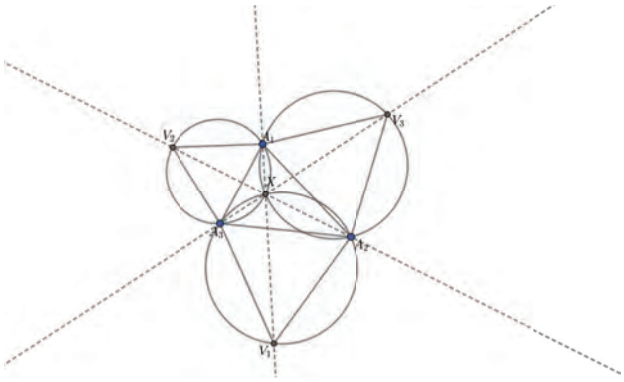

Figure 7: $X$ is the concurrent point of lines $A_1V_1$, $A_2V_2$ and $A_3V_3$.

*to the Fermat Point Problem and the Steiner Tree Problem may look up the recent 2010 HOGMAA Student Paper by S. Streck ([9]).*

We will run through a throughout geometry (Set 1) example now. Suppose we have 3 coordinates, $A_1$(1,2), $A_2$(3,0) and $A_3$(0,0).

To locate the coordinates for $V_2(x_2, y_2)$, we form the following two equations:

$$\sqrt{x_2^2 + y_2^2} = \sqrt{5} \quad (1)$$

$$\sqrt{(x_2 - 1)^2 + (y_2 - 2)^2} = \sqrt{5} \quad (2)$$

Solving them simultaneously, we have $x_2 = -\sqrt{\frac{13-4\sqrt{3}}{4}} = \frac{1-2\sqrt{3}}{2}$ and $y_2 = 1 + \frac{\sqrt{3}}{2}$. Now, we proceed to find $V_1$. As the $x-$coordinate of $V_1$ is $\frac{3}{2}$, we have the following equation:

$$(3 - \frac{3}{2})^2 + y^2 = 3^2 \quad (3)$$

giving us $y = -\frac{\sqrt{27}}{2}$. Hence, $V_1$ is $(\frac{3}{2}, -\frac{\sqrt{27}}{2})$.

Constructing a straight line $l_1$ passing through $V_1(1.5, -\frac{\sqrt{27}}{2})$ and $A_1(1, 2)$ and $l_2$ passing through $V_2(\frac{1-2\sqrt{3}}{2}, 1 + \frac{\sqrt{3}}{2})$ and $A_2(3, 0)$, we have the following

$$l_1 : y = (-4 - 3\sqrt{3})x + 6 + 3\sqrt{3} \quad (4)$$

$$l_2 : y = -\frac{4 + \sqrt{3}}{13}x + \frac{12 + 3\sqrt{3}}{13} \quad (5)$$

Solving equations 4 and 5 simultaneously, we obtained our Fermat point $X(\frac{234+195\sqrt{3}}{507}, \frac{351+39\sqrt{3}}{507})$, in exact form.

## 3.2 Mechanical technology

Moving ahead into the 18th century, mechanics has developed into a fairly mature field of mathematics. In particular, the Italian-French mathematician and astronomer Joseph-Louis Lagrange introduced *Lagrangian mechanics* in his 1788 work, Mécanique analytique, which founded classical mechanics on the *stationary-action principle* (also known as the *principle of least action*). Lagrangian mechanics characterises a mechanical system as a pair $(M, L)$ comprising a configuration space $M$ and a smooth function $L$ within that space called a *Lagrangian*. For most systems, $L := T - V$, where $T$ and $V$ the kinetic and potential energy of the system, respectively. The stationary action principle asserts that the action functional of the system derived from $L$ must remain at a stationary point throughout the time evolution of the system. This constraint then gives rise to the equations of motion of the system via the Lagrange's equations.

The FPP can be cast into a classical mechanics problem. Construct the $\triangle A_1A_2A_3$ on a horizontal frictionless plane and place holes at each vertex $A_1, A_2, A_3$. Consider a system of three separate equal masses, each connected by long massless strings to a pivot point in the triangle (see Figure 8). Pass one string through one of the holes that hangs a mass positioned under each vertex, and let the system come to equilibrium.

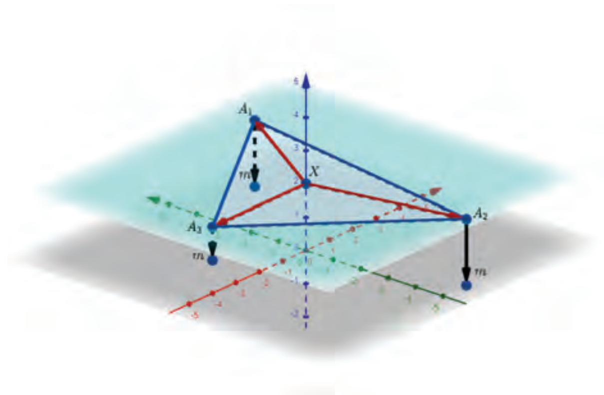


Figure 8: Mechanical set-up for FFP.

**Lemma 3** *At equilibrium, the pivot point is exactly at the Fermat point $X$ of $\triangle A_1A_2A_3$.*

**Proof.** By the stationary-action principle, the above mechanical system looks for the stationary state of $L$, where $L = T - V$, where $T$ and $V$ the kinetic and gravitational potential energy of the system, respectively. Since the system is at static equilibrium, $T = 0$. Hence the stationary value of $L$ is attained when the gravitational potential energy is minimized, which can be achieved by setting the weights closest to the ground. For each weight, the length of string below the vertex would be as long as possible, or equivalently, the length of string on the triangle is as short as possible. Hence the minimum $V$ occurs the total string lengths on the triangle is minimized – precisely when the pivoting point is at the Fermat point. ■

**Theorem 4** *The Fermat point $X$ of $\triangle A_1A_2A_3$ satisfies the condition*

$$\angle A_1XA_2 = \angle A_2XA_3 = \angle A_3XA_1 = 120°.$$

**Proof.** At equilibrium, the vector sum of the three forces acting on the pivot point is zero. Since each mass is equal, these forces must be exactly 120° away from each other. ■

### 3.3 Vectorial technology

A systematic study and use of vectors were a 19th and early 20th century phenomenon. With the vectorial machinery available to us, what type of solution for the FPP can emerge? A vectorial solution of the FPP was put forth by Titu Andreescu and Oleg Mushkarov ([1]). Herein, we first state a simple extension of the Fermat Point Problem to the so-called *Generalised Fermat Point Problem*, and then give a vectorial proof that extends Andreescu and Mushkarov's.

**Definition 5 (Generalised Fermat Point Problem)** *Let $n \geq 3$ be a positive integer. Given $n$ distinct co-planar points $A_1, \cdots, A_n$, locate a point $X$ (if it exists), co-planar with the points $A_i$'s, for which*

$$A_1X + \cdots + A_nX$$

*is minimum. We use the acronym* GFPP *for* Generalised Fermat Point Problem*.*

**Theorem 6** *The point $X$ co-planar to the given points $A_1, A_2, \ldots, A_n$, where $n \geq 3$, for which the unit vectors from $X$ to each of the $A_i$'s sum up to the zero vector is the Fermat point of $A_1, A_2, \ldots, A_n$.*

**Proof.** Let $X$ be such a point. Denote by $\mathbf{a}_i$ the position vector of $A_i$ with respect to $X$, and by $\mathbf{u}_i$ its corresponding unit vector. By the definition of dot product of vectors, it holds that $|\mathbf{a}_i| = |\mathbf{a}_i||\mathbf{u}_i| \cos 0 = \mathbf{a}_i \cdot \mathbf{u}_i$ for all $i = 1, 2, \cdots, n$.

Pick an arbitrary point $P$ coplanar with $A_1, A_2, \ldots, A_n$, and let its position vector with respect to the Fermat point $X$ be $\mathbf{p}$. Thus, for each $i = 1, 2, \ldots, n$, we have that

$$|\mathbf{a}_i| = \mathbf{a}_i \cdot \mathbf{u}_i = (\mathbf{a}_i - \mathbf{p}) \cdot \mathbf{u}_i + \mathbf{p} \cdot \mathbf{u}_i \leq |\mathbf{a}_i - \mathbf{p}||\mathbf{u}_i| + \mathbf{p} \cdot \mathbf{u}_i = |\mathbf{a}_i - \mathbf{p}| + \mathbf{p} \cdot \mathbf{u}_i.$$

Adding all these quantities together, we arrive at

$$|\mathbf{a}_1| + \cdots + |\mathbf{a}_n| \leq |\mathbf{a}_1 - \mathbf{p}| + \cdots + |\mathbf{a}_n - \mathbf{p}| + \mathbf{p} \cdot (\mathbf{u}_1 + \cdots + \mathbf{u}_n).$$

Since $X$ satisfies the condition $\mathbf{u}_1 + \cdots + \mathbf{u}_n = \mathbf{0}$, it follows that $|\mathbf{a}_1| + \cdots + |\mathbf{a}_n| \leq |\mathbf{a}_1 - \mathbf{p}| + \cdots + |\mathbf{a}_n - \mathbf{p}|$, which is equivalent to

$$A_1X + \cdots + A_nX \leq A_1P + \cdots + A_nP.$$

Since $P$ is an arbitrary point coplanar with $A_i$'s, $X$ is the Fermat point of $A_i$'s. ■

**Corollary 7 (GFPP for $n = 4$)** *Let $A_1, A_2, A_3$ and $A_4$ be four distinct coplanar points. Then the Fermat point $X$ of these four points is exactly the intersection of the diagonals of the quadrilateral $A_1A_2A_3A_4$.*

## 4 How can the FPP be solved now?

In this section, we briefly review each of above existing solutions of the FPP. Arguing that all these solutions were inspired and supported by technology available at specific points of time, we show how the GFPP can be solved in yet different ways by exploiting *given* technology of *present times*. The technology we use is that of Particle Swarm Optimisation (PSO).

## 4.1 Discussion about the existing solutions

Torricelli's solution relied on geometrical construction which is expectedly the approach used during his time. This is particularly so since a geometrical problem naturally asks for a geometrical solution. Indeed, Fermat anticipated this from Torricelli. Examining the details presented in Theorems 1 and 4, one must agree, in spite of the ingenuity of the proof and the aesthetics of the geometrical construction, that the manner of locate the Fermat point is way too complicated, and fairly tedious using compass and straightedge construction. More importantly the Torricelli's construction cannot be easily extended to solve the $n$-point GFPP ($n > 3$).

The proof inspired by Lagrangian mechanics is remarkably terse and leverages on the somewhat mystical prowess of theoretical mechanics. Furthermore, the technology is visible: you can set up a physical experiment using weights, strings and a triangle lamina to locate of the Fermat Point with a bit of trial-and-error. "Is this a mathematically legitimate solution?", you may ask. We do not have an answer here, but might just want to borrow a remark enunciated by Philip Davis in his paper "When Is A Problem Solved?":

> Is such and such really a solution? ... Apparently, the meaning of the word "solution" can be stretched quite a bit. The elastic quality of mathematical terms or definitions is remarkable, and is often achieved through context enlargement. ([3])

Unlike the two preceding solutions, the vectorial technology is pegged at a desirable level of abstraction that simultaneously veers us away from the geometrical intricacy and requires only a light mathematical overhead as compared to Lagrangian mechanics. Despite its advantages, the vectorial method is not constructive, i.e., it does not compute the position vector of the Fermat point $X$ in terms of the given points $A_i$'s – many would agree that this is a major defect of the vectorial method in contrast to Torricelli's geometrical method and the mechanical method. Nevertheless, the zero vector sum criterion provides us a computable test of accuracy for any candidate Fermat point.

Given the present-day computer technology, would we be able to create an alternative solution to the GFPP that can possibly help us overcome the shortcomings we have discussed so far. In the next subsections, we develop the theory and implementation of our proposed modern technology.

## 4.2 A short primer on PSO

*Particle Swarm Optimisation* (PSO, for short) refers to a characteristic class of mathematical algorithms – first created by James Kennedy and Russell Eberhart ([5]) – inspired by the movements of birds flying in a flock, where members of the flock benefit from the experience of other members. For instance, when searching for food a flock of birds can share their discoveries and the entire flock can use the information to locate the place where more food can be found. The individual bird will then fly in a vector which is based on information from the flock and information it has gathered individually. This vector changes every day, as the flock finds better hunts, and the individual bird also finds better hunts. Eventually, all the flock of birds will converge on a single point, which is the position of the best hunt obtainable. Our idea is to exploit the swarm movement to solve the GFPP, where particles move as a swarm and eventually converge to the Fermat point.

Certain mathematical terminologies used in PSO must first be established. Imagine a swarm of birds (swarm particles) flying in the same plane as the polygon $A_1 \dots A_n$, where $A_1, \dots,\ A_n$ are distinct points having position vectors $\mathbf{a}_k$ $(k = 1, \dots, n)$ in search of the Fermat point $X$ with position vector $\mathbf{x}$, i.e., where the *fitness function* or *objective function* $f : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f(\mathbf{x}) = \sum_{k=1}^{n} |\mathbf{x} - \mathbf{a}_k|$ is to be minimised. This function $f$ calculates the sum of the distances between $X$ and each of the vertices $A_1,\ A_2, \dots,\ A_n$.

A swarm consists of $N$ birds, and each $i$th bird archives an aerial log of its position vector $\mathbf{x}_i^d$ on the $d$th day. Given that the daily velocity $\mathbf{v}_i^d$ of the $i$th bird on the $(d+1)$th day, the daily position vector $\mathbf{x}_i^d$ of the $i$th bird is determined recursively by $\mathbf{x}_i^{d+1} = \mathbf{x}_i^d + \mathbf{v}_i^{d+1}$ for $i = 1, \dots, N$. The key business in PSO is to model the daily velocity of each bird, which is determined by three components: (i) *inertia* component, (ii) *cognitive* component, and (iii) *social* component. We explain this now.

**Inertia component.** The $i$th bird's velocity, denoted $\mathbf{v}_i^d$, gives rise to a certain scalar multiple $w\mathbf{v}_i^d$ of it which is then its *inertia component* of the velocity on the $(d+1)$th day. This inertia component represents the $i$th bird's degree of reluctance to change its previous day's velocity.

**Cognitive component.** Each $i$th bird keeps a record of its *personal best position*, i.e., the position vector $\mathbf{P}_i^d$ at which the smallest value of the objective function $f$ has been achieved up to and including the $d$th day. A randomized scaling of the directional vector $\mathbf{P}_i^d - \mathbf{x}_i^d$ models the *cognitive component* of the velocity of the $i$th bird on the $(d+1)$th day.

**Social component.** Each $i$th bird also keeps in its aerial log book the *global best position*, i.e., the position vector $\mathbf{G}^d$ of the location found by one of the $N$ birds in the course of the swarm's flight up to and including the $d$th day. A randomized scaling of the directional vector $\mathbf{G}^d - \mathbf{x}_i^d$ models the *social component* of the velocity of the $i$th bird on the $(d+1)$th day.

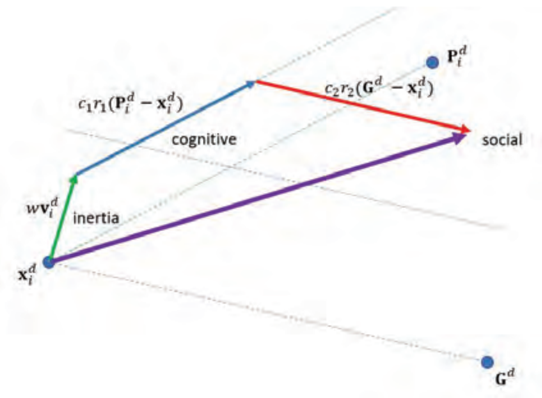


Figure 9: Daily velocity comprises three components.

Thus, the velocity $\mathbf{v}_i^{d+1}$ is the resultant of the aforementioned component vectors:

$$\mathbf{v}_i^{d+1} = \underbrace{w\mathbf{v}_i^d}_{\text{inertia}} + \underbrace{c_1 r_1(\mathbf{P}_i^d - \mathbf{x}_i^d)}_{\text{cognitive}} + \underbrace{c_2 r_2(\mathbf{G}^d - \mathbf{x}_i^d)}_{\text{social}}, \tag{6}$$

where $i = 1, \dots, N$ (see Figure 9). Certain constants, parameters, and randomized numbers appear in the above recursive equation, whose details are described below:

1. The bounded random constant $w$ in the term $w\mathbf{v}_i^d$ tunes the scope of exploration of the $i$th bird, modelling the distance it covers each day. A large value of $w$ signifies that the bird leans towards exploring further along yesterday's velocity.

2. The parameter $c_1$ in the term $c_1 r_1(\mathbf{P}_i^d - \mathbf{x}_i^d)$ is the impact factor of the cognitive component. An increase in the value of $c_1$ indicates a higher cognitive impact, i.e., the velocity of flight is influenced more by the bird's own belief system based on its own personal best records. The parameter $c_2$ in the term $c_2 r_2(\mathbf{G}^d - \mathbf{x}_i^d)$ is the impact factor of the social component. An increase in the value of $c_2$ indicates a higher social impact, i.e., the velocity of flight is influenced more by the swarm's global bests.

3. $r_1$ and $r_2$ are just random numbers uniformly distributed in the unit interval $[0, 1]$.

For more detailed account of the theory and applications of PSO, readers may refer to **??**.

### 4.3 Sample runs of PSO algorithm

Equation (6) is a recursive equation that can be easily implemented in PYTHON. The PYTHON codes[2] of the PSO algorithm for locating the Fermat point of the set of points $A_1,\ A_2, \ldots, A_n$ ($n \geq 3$) can be found at the Github repository (`https://github.com/howengkin/pso.io.git`).

We ran the PSO algorithm over many sets of data – different sets of points (with various sizes). Figures 10 and 11 shows animation clips of a swarm movement observed in a particular run of this PSO algorithm for two sample sets of data:

1. Set 1.

   - Number of points: 3
   - Data set: $A_1$(1,2), $A_2$(3,0) and $A_3$(0,0).
   - Number of swarm particles: 100
   - Number of iterations: 100
   - PSO Fermat point: $X(1.12771183, 0.8255423)$
   - Exact Fermat point: $\frac{234+195\sqrt{3}}{507}, \frac{351+39\sqrt{3}}{507}$
   - Percentage error by Euclidean metric:

$$x - \text{coordinate} : \left| \frac{1.12771183 - \frac{234+195\sqrt{3}}{507}}{\frac{234+195\sqrt{3}}{507}} \right| \times 100\% = 0.00000169\%$$

$$y - \text{coordinate} : \left| \frac{0.8255423 - \frac{351+39\sqrt{3}}{507}}{\frac{351+39\sqrt{3}}{507}} \right| \times 100\% = 0.00000846\%$$

   - Sum of unit vectors $\mathbf{u}_i$:

$$\mathbf{u}_{A_1X} + \mathbf{u}_{A_2X} + \mathbf{u}_{A_2X}$$

$$= \begin{pmatrix} 0.1081038411 \\ -0.9941396077 \end{pmatrix} + \begin{pmatrix} -0.9150020989 \\ 0.4034490787 \end{pmatrix} + \begin{pmatrix} 0.8068982404 \\ 0.5906904685 \end{pmatrix} = \begin{pmatrix} -0.0000000174 \\ -0.0000000605 \end{pmatrix}$$

[2]Readers may also make an email request for the PYTHON codes from the authors.

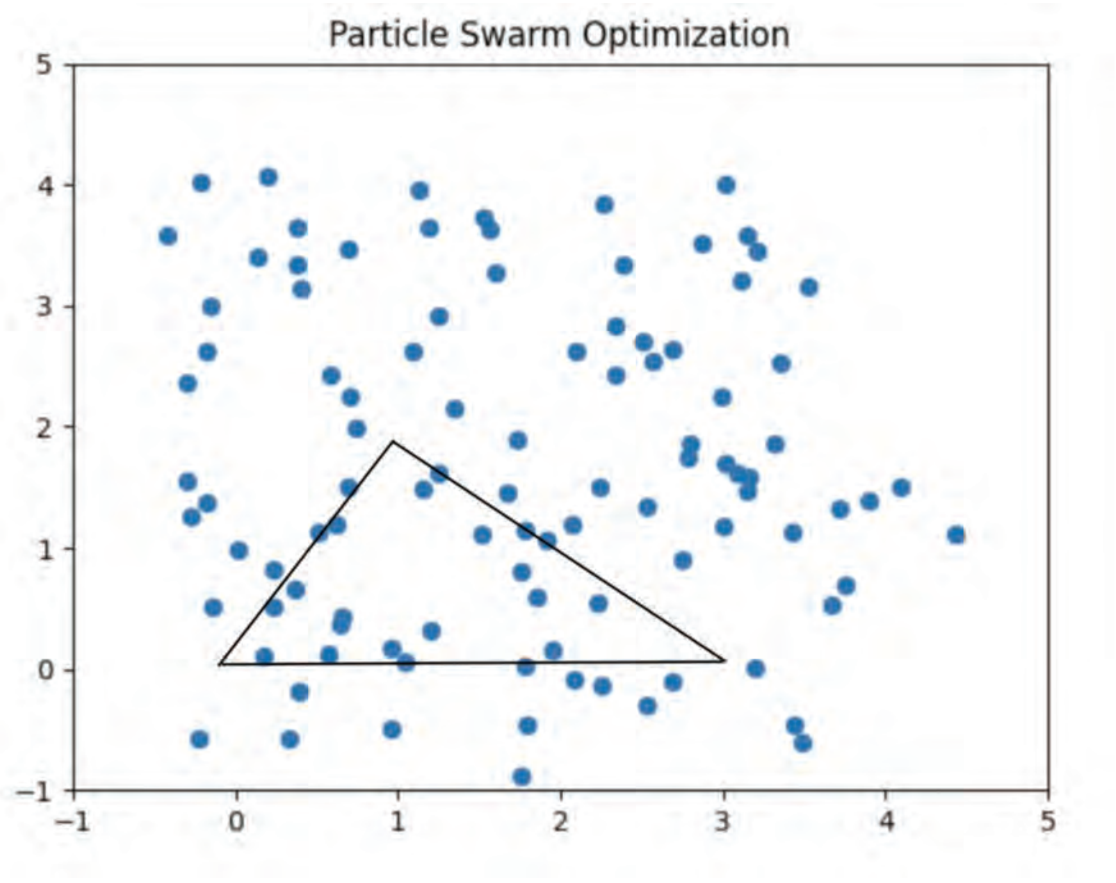


Figure 10: Swarm movement for Set 1.

2. Set 2.
   - Number of points: 5
   - Data set: $A(2,1), B(1,2), C(2,2.5), D(3,2), E(3,1)$.
   - Number of swarm particles: 100
   - Number of iterations:100
   - PSO Fermat point: $Y(2.26885941, 1.67413429)$
   - Exact Fermat point:
   - Percentage error by Euclidean metric:
   - Sum of unit vectors $\mathbf{u}_i$:

$$\mathbf{u}_{AY} + \mathbf{u}_{BY} + \mathbf{u}_{CY} + \mathbf{u}_{DY} + \mathbf{u}_{EY}$$

$$= \begin{pmatrix} 0.3704470089 \\ 0.9288536018 \end{pmatrix} + \begin{pmatrix} 0.9685688202 \\ -0.2487457348 \end{pmatrix} + \begin{pmatrix} 0.3095578354 \\ -0.9508806164 \end{pmatrix} + \begin{pmatrix} -0.9133871240 \\ -0.4070920802 \end{pmatrix} + \begin{pmatrix} -0.7351865730 \\ 0.6778648117 \end{pmatrix} = \begin{pmatrix} -0.0000000325 \\ -0.0000000178 \end{pmatrix}$$

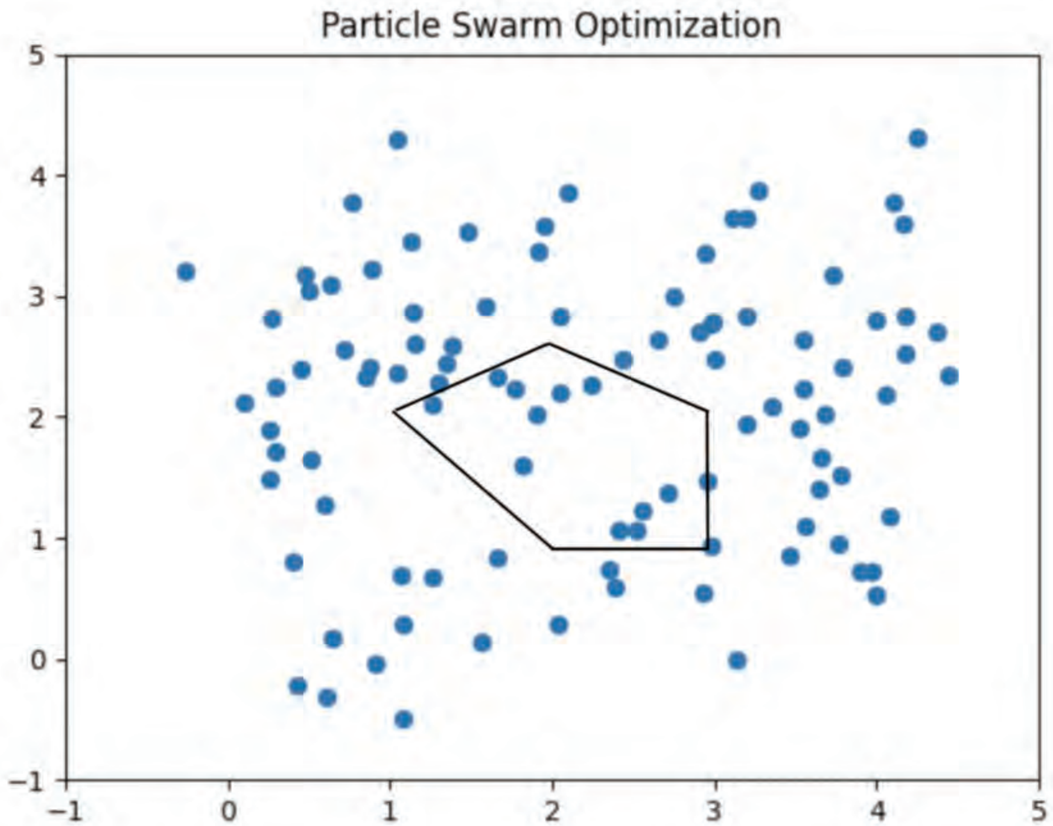


Figure 11: Swarm movement for Set 2.

## 4.4 Performance evaluation

We summarise the performance evaluation of the PSO algorithm for GFPP here.

**Few design parameters.** The PSO algorithm involves relatively few parameters, and this makes the programming task very simple. This has the advantage that anyone who wants to learn and apply the PSO method finds the codes easy to understand or write.

**Concurrent applications of efficient global search.** Running the PSO algorithm concurrently over many data sets of various sizes, we observed that the global search for the Fermat point in each of these runs is very efficient and accurate with convergence occurring under 100 iterations on the average. While the PSO method cannot be strictly classified as constructive because the point of convergence of the swarm is not the exact Fermat point in general, it nonetheless is very fast in locating a highly accurate approximation to the actual Fermat point, and would have sufficed in most practical situations.

| Problem Solving Component | Impact of (Computer) Technology |
|---|---|
| Resource | Dramatic increase in the accessible knowledge base |
| Heuristics | More opportunities for effective use of heuristic: making a table, drawing a graph |
| Control | Provides more effective management strategies |
| Beliefs | Develops beliefs that are specific to the context in which (computer) technology was used |

Table 1: Contribution of the (computer) technology to problem solving.

**Insensitive to scaling of design parameters.** In cases where the given set of points $A_1, A_2, \ldots, A_n$ are spaced far apart, appropriate scaling incorporated into the PYTHON codes did not dampen the speed of convergence to the Fermat point(s). This suggests that the PSO method is insensitive to scaling of the variables.

**Derivative-free and naturalistic.** The PSO solution is completely different in nature from the geometrical solution of Torricelli. Compared to the mechanical method, there is one commonality. That is, both rely on some physical phenomena to get as close to the exact Fermat point as possible: the PSO method exploits the collective wisdom of the swarm particles to search for the solution while the mechanical method relies on the forces in the system and some careful adjustments of the pivoting point. Another important characteristic of the PSO method is that it is derivative-free, i.e., it does not make use of any differentiation techniques in the whole process.

# 5 How can a problem be solved?

According to Alan Schoenfeld ([7]), four categories of knowledge and skills are required to be successful in mathematics problem solving: (1) resources – proposition and procedural knowledge of mathematics, (2) heuristics – strategies and techniques for problem solving such as working backwards, drawing figures, (3) control – decisions about when and what resources and strategies to use, and (4) beliefs – a mathematical “world view” that determines how the problem-solver approaches a problem. To these four categories, we propose to add on one more: *technology.*

Already, in [4], the impact of technology (in particular, computer technology) on each of these above categories has long been identified (see Table 5, reproduced/modified from [4, p.4]).

One important reality teased out from [4, p.5] is that “problem-solving strategies are dependent on the context of the problem, goal and motives of the problem-solver, and the *accessible tools*[3]” This reality is of particular relevance to us in this paper concerning our experience of solving the GFPP using PSO, and we develop it further.

[3] The emphasis has been added by the authors of this paper.

**Context of the problem.** The GFPP involves optimisation with a geometrical backdrop. The task of GFPP focuses on locating the Fermat point, but does not restrict the problem-solver to employing only calculus techniques and/or geometrical constructions. Additionally, optimisation problems occurring in practice demand efficiency in obtaining an approximate solution that satisfies a certain level of accuracy, not necessarily an exact one.

**Goals and motives of the problem-solver.** In the case of GFPP, we wanted to look for an alternative solution while having a relatively light theoretical overhead must be versatile and quick enough to locate the Fermat Point, given more than 3 points. It is the problem-solver's goal to overcome the shortcomings of the existing solutions using this new solution.

**Accessible tools.** To be able to meet the goals and motives of the problem-solver, it is of paramount importance that a suitable tool is in existence and made available to the problem-solver. Notice that the existing method of vectors already prepared the way for the PSO method since the search space can be taken as the 2D plane, and the swarm particles are nothing but probes moving in the search space and searching for the point of optimisation. The computer that processes the PSO algorithm gives the problem-solver the required speed and accuracy. The problem-solver did not develop the PSO method but knows of it – this knowledge is counted as problem-solving resource, and applies it as a powerful tool which is suitable to satisfy the requirements of the task set in the GFPP.

**Remark 8** *For more recent works of the role of technology on mathematics education, we point the reader to [2].*

# 6 Conclusion

Technology has improved the teaching and learning of mathematics over the past decades by actively transforming pedagogy, policy and practice. These transformations are informed by a growing body of meta-analytic research examining how the affordances of technology shape mathematical experiences ([11]). Our experience in applying the Particle Swarm Optimisation method in locating the Fermat point in the context of the Generalised Fermat Point Problem reinforces the claim that the way a problem is solved crucially depends on the accessible technology that came at the right time and the right place to meet certain goals and motives of the problem solver, given the context of the problem itself.

There remain skeptics who would question whether our way of solving the GFPP using PSO is even mathematically legitimate in the first place. After all, it looks like the PSO method is a guess-and-check method that has been performed quickly by a computer! Are students, teachers and mathematicians ready to enlarge the context of understanding what a solution to a mathematics problem so as to embrace the thesis that technology – when it becomes accessible to the problem-solver – will, in part, determine the way how a problem is solved? An answer to this question will fundamentally change the way we teach and learn mathematics at schools, allowing us to be more open-minded in applying available technology in creative and innovative ways!

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# Mathematics teacher training from the perspective of STEM – a particular case

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**Abstract**

*The ability to respond creatively and effectively to new challenges, whether it is acquiring new knowledge, solving problems, or the teaching process, is a key factor in determining the success of today's teachers. An important task of teacher training schools is therefore to create a suitable environment for the provision of education, as well as impulses for the development of the necessary knowledge and skills. This paper presents a specific project and corresponding activities for students implemented within mathematics teaching courses at the University of South Bohemia. Historical sources, both classical and local, from the area of present-day Czechia, are used. Emphasis is placed on the use of computers, especially dynamic geometry software, to model problems and their effective solution.*

## 1 Introduction

The content and methods of teacher training should reflect the complexity of knowledge and competences that current practice demands from teachers. The basic categories of teacher knowledge, subject matter content knowledge, pedagogical content knowledge and curricular knowledge, were identified by Shulman in his well-known study [18]. In addition, at the present time, with the increasing opportunities to learn about real-world phenomena in which the boundaries between educational disciplines overlap, the importance of skills that allow a teacher to interpret such phenomena to students through a STEM (Science, Technology, Engineering, and Mathematics) lens, of which creativity and flexibility are key, is becoming more and more important [15]. Recent research has also proved the importance of learning experience for future teachers [11]. In this paper, we present a specific case of a topic that provides opportunities to apply the above principles in mathematics teacher training.

The central topic of the article is the solution of the angle trisection problem. It is used as a supporting medium for illustrating the possibility of creative education of mathematics teachers. Historical materials devoted to this problem, both classical and local, from the area of present-day Czechia, are used as an impetus for the creative and research work of prospective mathematics teachers. Although the problem of trisection is currently perceived as a historical problem that is not the focus of attention in contemporary geometry, the paper brings a new discovery in this field, obtained and proved using tools offered by the GeoGebra software environment [5].

As is widely known, the trisection of an angle, i.e. division of an arbitrary angle into three equal parts, together with squaring a circle and doubling a cube, belongs to the classical problems of Greek mathematics, that were proved to be impossible constructions using just a straightedge and compass. The impossibility of this so-called Euclidean construction of a trisection of an angle was proved by French mathematician Pierre Laurent Wantzel in 1873. For more information see [10, 14].

# 2 The forgotten trisection method

In 1881, J. R. Vaňaus, a Czech grammar school teacher of mathematics and physics, presented in an article entitled *Trisektorie* (Czech term for trisectrix), which was published in the Czech *Journal for the Cultivation of Mathematics and Physics* [19], a previously unknown method of using an oblique strophoid [16] for the trisection of an angle. This article offers interesting possibilities to modern researchers.

## 2.1 Vaňaus' trisectrix

From the text of the article [19] it is apparent that Vaňaus probably did not know that the curve in question is a strophoid. He arrived at it when investigating the properties of algebraic curves given by the equation

$$Ax^3 + By^3 + Cxy^2 + Dx^2y + Ex^2 + Fy^2 + Gxy = 0, \tag{1}$$

which have a double point in the origin of the related Cartesian coordinate system. For a certain class of these curves he identified that they are sets of points equidistant from a circle and some of its secants. As we know today, this feature indicates a strophoid [12, 17]. Since he discovered that such curves were suitable for trisecting an angle due to this property, Vaňaus decided to call them *trisektorie*.

Besides determining the conditions for the values of the coefficients of equation (1), he defines it as a locus of points as follows (see Fig. 1): *Given a circle $c$ of arbitrary radius $r$ with diameter $OA$, where $O$ is the origin of the Cartesian coordinate system and $A$ lies on the $x-$axis, draw line $l$, the secant line of circle $c$, through $A$ at any direction angle $\alpha$ and place point $B$ on it. Then draw line $OB$ and mark its intersection with circle $c$ as point $D$. Finally, place $M$ on line $OB$ so that $|MD| = |DB|$. The locus of points $M$ for $B$ moving along line $l$ is the curve in question.* Based on this definition, Vaňaus derives the Cartesian equation of the resulting locus curve, which is

$$a(y^2(2r + x) - x^2(2r - x)) = y(y^2 + x^2 - 4rx), \tag{2}$$

where $r$ is the radius of the circle $c$ and $a$ is the slope of the line $l$, i.e. $a = \tan\alpha$. Because for him it was a hitherto unknown curve, which he discovered through his research, the author

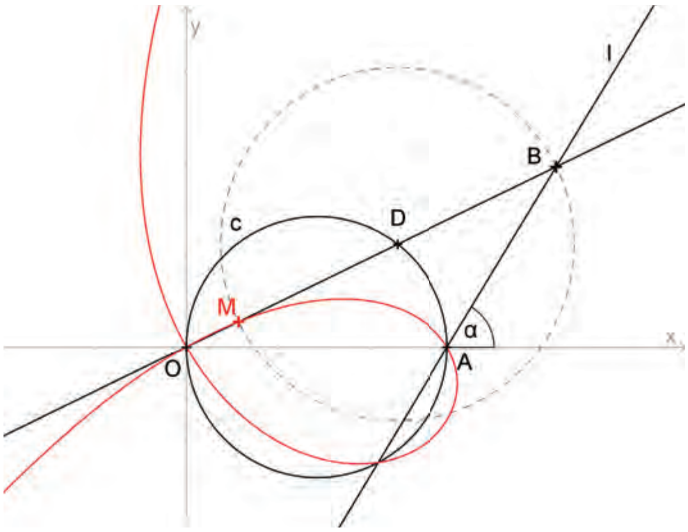


Figure 1: Vaňaus' trisectrix as the locus curve

was looking for analogies with already known curves. For the similarity of the definition as the locus of points he mentions the *cissoid of Diocles* [2], and for the similarity of the shapes, the *folium of Descartes* [4].

In the remainder of his article, Vaňaus presents the use of the introduced curve, which we will hereinafter call an *oblique strophoid*, for trisecting an angle. He proved both his knowledge of the theoretical impossibility of solving this problem with a straightedge and compass alone, as well as his wide overview of other non-Euclidean ways of doing so, adding the confident statement that the method based on the curve introduced by him is the least complex.

## 2.2 Vaňaus' trisection

Vaňaus' angle trisection method is described in detail in [6], as is proof of its correctness. In addition, the reader is confronted with an interesting problem that needs to be solved through the trisection of an angle. Vaňaus assigned this problem to the readers, grammar school students, of the *Journal for the Cultivation of Mathematics and Physics* in the problem section [20]. The methods of solving this problem are discussed both from the point of view of a student at that time and the present day in [6]. We will therefore present this method only briefly here for the purposes of the further direction of this paper.

In Fig. 2, the same objects are shown as in Fig. 1, but with point $B$ in a different position on line $l$, it lies on the chord $AF$, and with the ray $OA'$ and the circle $m$, with centre $O$ passing through $A$, added.

Vaňaus proved that for the position of point $B$ on line $l$, at which point $M$ coincides with the intersection of the strophoid and the circle $m$, the angle $A'OB$ is a third of the angle $A'OA$ [19, 6]. The proof is not complicated at all. The application of the basic properties of angles in a triangle and a circle is enough, see Fig. 3. A detailed description is given in [6].

The method used by Vaňaus for trisecting angle $u$ is clear, see Fig. 3. The rays $OA'$, $OA$ and $OM$ have a common point $O$, the vertex of the angle $u$, while points $A'$, $A$ and $M$ lie on the circle $m$. The angle $A'OM$ is a third of the angle $A'OA$ if, and only if, point $B$ lies on the perpendicular line from $A$ to $OA'$, $D$ being the midpoint of the segment $MB$.

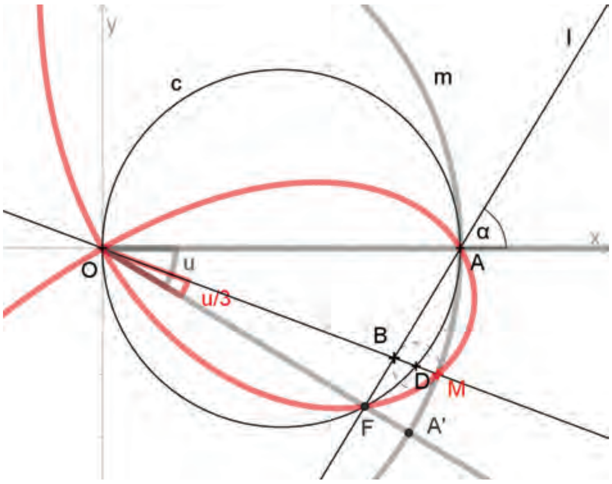


Figure 2: Configuration for trisecting an angle

Although the theoretical basis of the method is therefore well proven and quite understandable, how do we perform this trisection in practice? In this regard, Vaňaus left his successors with a mysterious task, the correct solution of which we will probably never know. At the end of his article, he states: *"In order to correctly draw the part of the trisectrix $AMF$ and, above all, to place the intersection point $M$ precisely, I assembled a very simple device, where the intersection of arc $AA'$ at point $M$ is created by means of a double forced movement."* [19].

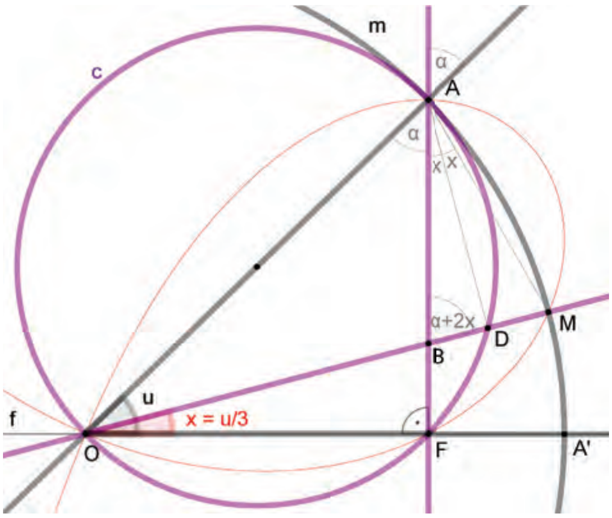


Figure 3: Vaňaus' method of trisection

Unfortunately, the device that Vaňaus mentioned has not survived, as have any written documents or sketches for its construction. One of Vaňaus' legacies, in addition to his written work, is therefore a challenge to us and our students to replace this lost artifact with our own solution. In the following passages of this paper, two intriguing and original results are presented. These arose from attempts to design and build a mechanism that could possibly correspond to the device that Vaňaus mentioned.

The first, is the design and production of a possible Vaňaus trisector by a student teacher of mathematics and technical education. The process ended with his successful participation in the national *Olympiad of Technology 2023* (`http://olympiadatechniky.cz`).

The second, is the discovery of an unsuspected connection between Vaňaus' method and *Ceva's trisectrix*, a curve described by Tomas Ceva in 1699 [1], which led to the discovery of a new and very simple application of this curve to angle trisection.

# 3 Possible Vaňaus trisector

The conditions defined by the essence of Vaňaus' method of trisection using an oblique strophoid, which we noted in the comment on Fig. 3 above, must be met by the device that we want to design for the purpose of performing the trisection. We can therefore try to construct a mechanism to comply with these conditions. With this task, student teachers of mathematics, prospective teachers at lower secondary school, were approached. One of them, Tomáš Randa, studying teaching of technical education as a second major, took this task to the stage we see in Fig. 4.

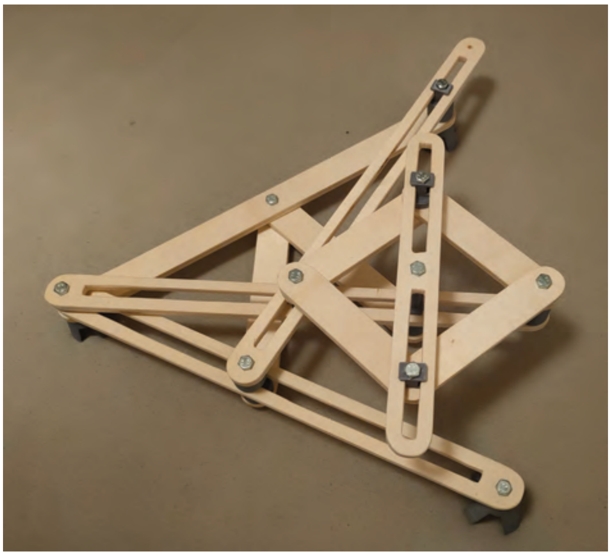

Figure 4: Vaňaus trisector – student work, T. Randa, 2023

It is a mechanism made by hand from poplar wood, with joints printed on a 3D printer. As stated by Vaňaus, the mechanism performs an angle trisection through a double forced movement. The original plan was to print the entire mechanism on a 3D printer, which would have been much easier. However, the student decided to enter the *Olympiad of Technology*, one of the conditions of which is that a significant part of the submitted object must be made by hand. His efforts paid off in the end, with him taking third place in the category "Didactic technical works" among students from all over the Czech Republic. The work was awarded not only for its technical implementation, but also for its geometric essence and the potential to recall, in a new way, one of the classic problems of geometry and bring the history of efforts to solve it closer to students.

# 4 New use of Ceva trisectrix

An unexpected consequence of both the analysis of the above mentioned Vaňaus' method of angle trisection and the effort to design the simplest possible device for its implementation, was the discovery of a new and simpler use of the Ceva trisectrix for the trisection of an angle.

## 4.1 Ceva trisectrix

In 1699, *Tommaso Ceva*, the younger brother of *Giovanni Ceva*, published the Latin book *Opuscula mathematica*, in which he presents, among other things, the curve he calls *cycloidum anomalarum* [1]. Among the properties of this curve, he mentions its suitability for angle trisection.

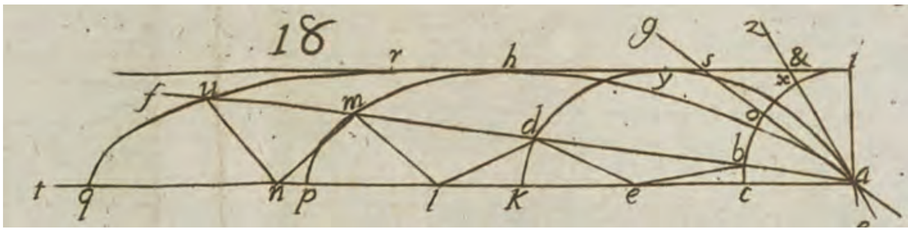

Figure 5: Cycloidas anomalas, T. Ceva, 1699 [1]

Ceva defines his cycloid as a curve that is described by the endpoint of a polyline with an odd number of segments of equal lengths, whose vertices lie alternately on two rays with a common point of origin, which is also the initial vertex of the polyline. In Fig. 5, which is Figure No. 18 from Ceva's book, we can see for example a curve drawn by point $d$, the endpoint of polyline $abed$ linking rays $af$ and $at$, when $af$ rotates around $a$. The other two curves in the figure are those drawn in the same way by points $m$ and $u$, endpoints of polylines $abedlm$ and $abedlmnu$, respectively.

An elementary knowledge of the relations between interior and exterior angles of a triangle is sufficient to prove that if $|\measuredangle cab| = \varphi$, then $|\measuredangle led| = 3\varphi$, $|\measuredangle nlm| = 5\varphi$ and $|\measuredangle qnu| = 7\varphi$, see also Fig. 6, where the relationships between angles essential to the proof are indicated. In what follows, we are only interested in the trisection of an angle, so we will focus mainly on the first of the curves, drawn by point $d$. This curve is referred to in available sources as the *Cycloid of Ceva* [21] or *Ceva trisectrix* [3]. It is a sextic, the algebraic curve of order 6, with the equation

$$(x^2 + y^2)^3 - r^2(3x^2 - y^2)^2 = 0, \qquad (3)$$

where $r$ is the length of a segment of the polyline forming it, i.e. the radius of the circle (hereinafter $k$) along which point $b$ moves in Fig. 5. Under the same definition of $r$, the polar equation of Ceva trisectrix is

$$\rho(\varphi) = r + 2r\cos 2\varphi. \qquad (4)$$

For the sake of completeness, the algebraic equations of the other two cycloids, the parts of which Ceva sketched in his picture are as follows

$$(x^2 + y^2)^5 - r^2(5(x^2 - y^2)^2 - 4y^4)^2 = 0,$$
$$(x^2 + y^2)^7 - r^2(7(x^2 - y^2)^3 - 2y^2(7x^4 - 3y^4))^2 = 0.$$

As is evident, only parts of the curves in question are presented in Fig. 5. The entire Ceva trisectrix is shown in Fig. 6. To verify that the curve is really drawn by a fixed point (point D) on a rolling circle (the green circle), the reader is invited to play the animation in the applet *Ceva trisectrix* [7].

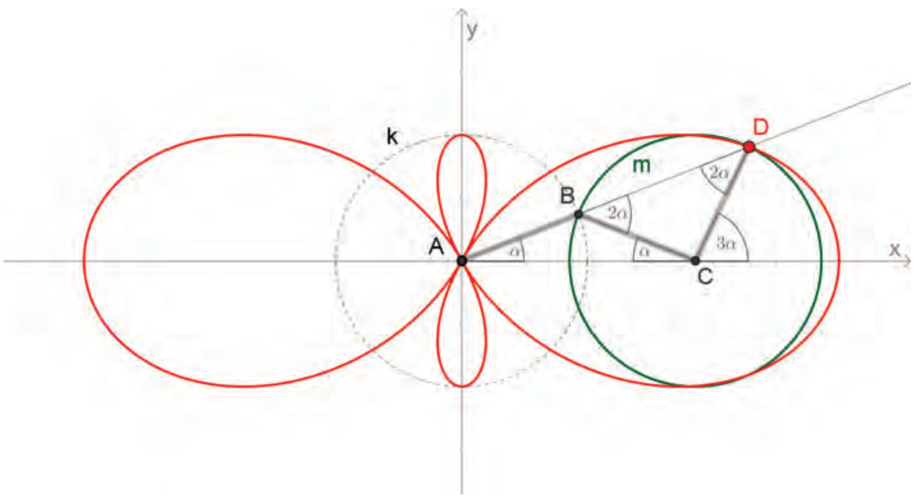


Figure 6: Ceva trisectrix

## 4.2 New way of use

As we already know, Ceva's curve is the closed curve of the sixth order, with two mutually perpendicular axes of symmetry, having four loops, of which the opposite are always congruent, one pair larger, one pair smaller, pairs not similar to each other, see Fig. 6.

The traditional trisection method based on this curve works with the larger of the loops, as shown in Fig. 6. In contrast, a new method, which we aim to introduce in the following text, works with the smaller of the loops, as shown in Fig. 7.

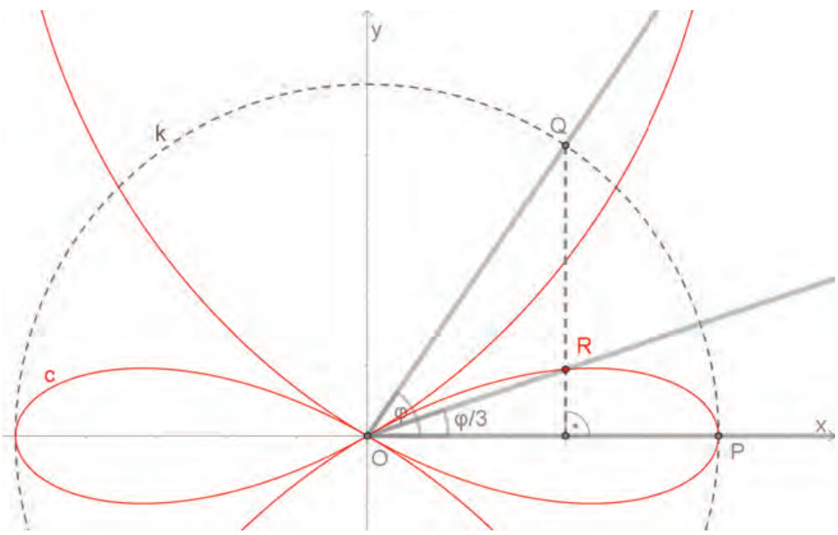


Figure 7: New use of Ceva's curve to trisect an angle

The step-by-step procedure for this new use of the Ceva curve for the trisection of angle $\varphi$ is as follows:

1. Place the curve in the Cartesian coordinate system, as shown in Fig. 7. Use the polar equation $\rho(\varphi) = r - 2r\cos 2\varphi$.
2. Draw a circle $k$, the defining circle of the Ceva curve, the radius of $r$ which is equal to the length of the shorter loop.
3. Construct the angle $\varphi$, whose third you are interested in, into the first quadrant, so that its first arm merges with the positive semi-axis $x$, and its second arm intersects $k$ at $Q$, within this quadrant. From $Q$, draw a perpendicular line to $x$, and mark its intersection with Ceva curve $c$ as $R$. Then the ray $OR$ is the arm of the angle $\measuredangle POR$, the third of $\measuredangle POQ$.

## 4.3 Genesis of the method

This new method of angle trisection appeared as an unexpected result of the analysis of the trisection method developed by Vaňaus. See section 2.2, or for a more detailed introduction [6]. The main impetus for this was the geometric simplification of the Vaňaus method, which would allow the design of a sufficiently simple device for its implementation.

The basic source of the findings for the analysis was a dynamic geometric model of the Vaňaus method created in GeoGebra [5]. This model is shown in Fig. 8. It is available in interactive form in the applet *Vaňaus' trisection – trace of B* [9]. If we change angle $\varphi$ by moving

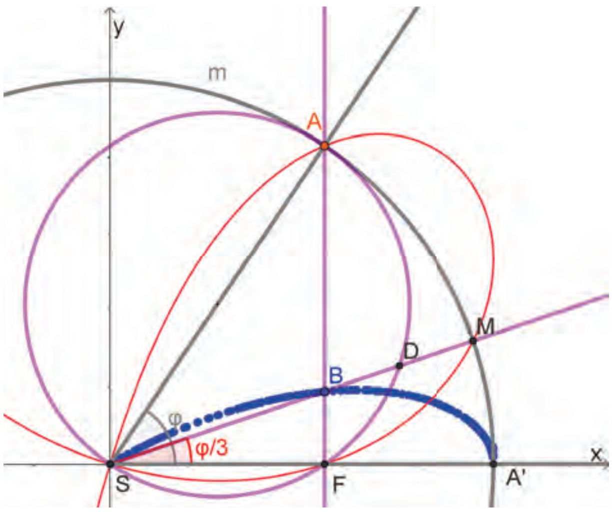


Figure 8: The locus of $B$ when $A$ moves along $m$

point $A$, the ray $SM$, where $M$ is the intersection of the corresponding oblique strophoid (it is different for each angle $\varphi$) with circle $m$, is the arm of the angle of size $\varphi/3$. The question arises, what is the trajectory of point $B$ when $A$ moves along $m$? The answer to this question could bring about the sought-after simplification. To initially examine the curve of this trajectory, we use the *Trace on* setting for $B$ in GeoGebra. It draws the blue trace, as can be seen in Fig. 8, or in the corresponding applet [9].

The structure shown in Fig. 8 defines the geometric relationship between $A$ and $B$ such that, for a particular position of $A$, the position of $B$ can be described by a system of algebraic equations. Using the *Eliminate* function in GeoGebra CAS, we can then derive the general equation of the locus curve of $B$. To derive the desired system of equations, we first assign coordinates to the decisive points in Fig. 8 as follows: $A[a_1, a_2]$, $B[x, y]$, $M[m_1, m_2]$. Then, as a preparatory step, we consider the family of Vaňaus' strophoids with parameters $a_1$, $a_2$, the coordinates of $A$, and derive the equations

$$s_1 : \; 2\,R\,r - a_1^2 - a_2^2 = 0, \tag{5}$$

$$\begin{aligned} s_2 : \; -2\,R\,a\,a_1^2\,m_1^2\,r + 2\,R\,a\,a_1^2\,r\,m_2^2 - 8\,R\,a\,a_1\,m_1\,r\,m_2\,a_2 + 2\,R\,a\,m_1^2\,r\,a_2^2 \\ - 2\,R\,a\,r\,m_2^2\,a_2^2 + 4\,R\,a_1^2\,m_1\,r\,m_2 - 4\,R\,a_1\,m_1^2\,r\,a_2 + 4\,R\,a_1\,r\,m_2^2\,a_2 \\ - 4\,R\,m_1\,r\,m_2\,a_2^2 + a\,a_1^3\,m_1^3 + a\,a_1^3\,m_1\,m_2^2 + a\,a_1^2\,m_1^2\,m_2\,a_2 + a\,a_1^2\,m_2^3\,a_2 \\ + a\,a_1\,m_1^3\,a_2^2 + a\,a_1\,m_1\,m_2^2\,a_2^2 + a\,m_1^2\,m_2\,a_2^3 + a\,m_2^3\,a_2^3 - a_1^3\,m_1^2\,m_2 - a_1^3\,m_2^3 \\ + a_1^2\,m_1^3\,a_2 + a_1^2\,m_1\,m_2^2\,a_2 - a_1\,m_1^2\,m_2\,a_2^2 - a_1\,m_2^3\,a_2^2 + m_1^3\,a_2^3 + m_1\,m_2^2\,a_2^3 = 0. \end{aligned} \tag{6}$$

defining $A$ and $M$ as the points of intersection of this family of curves with circle $m$, see lines 2 and 3, respectively, in the GeoGebra CAS code in Fig. 9. Subsequently, we apply the defining

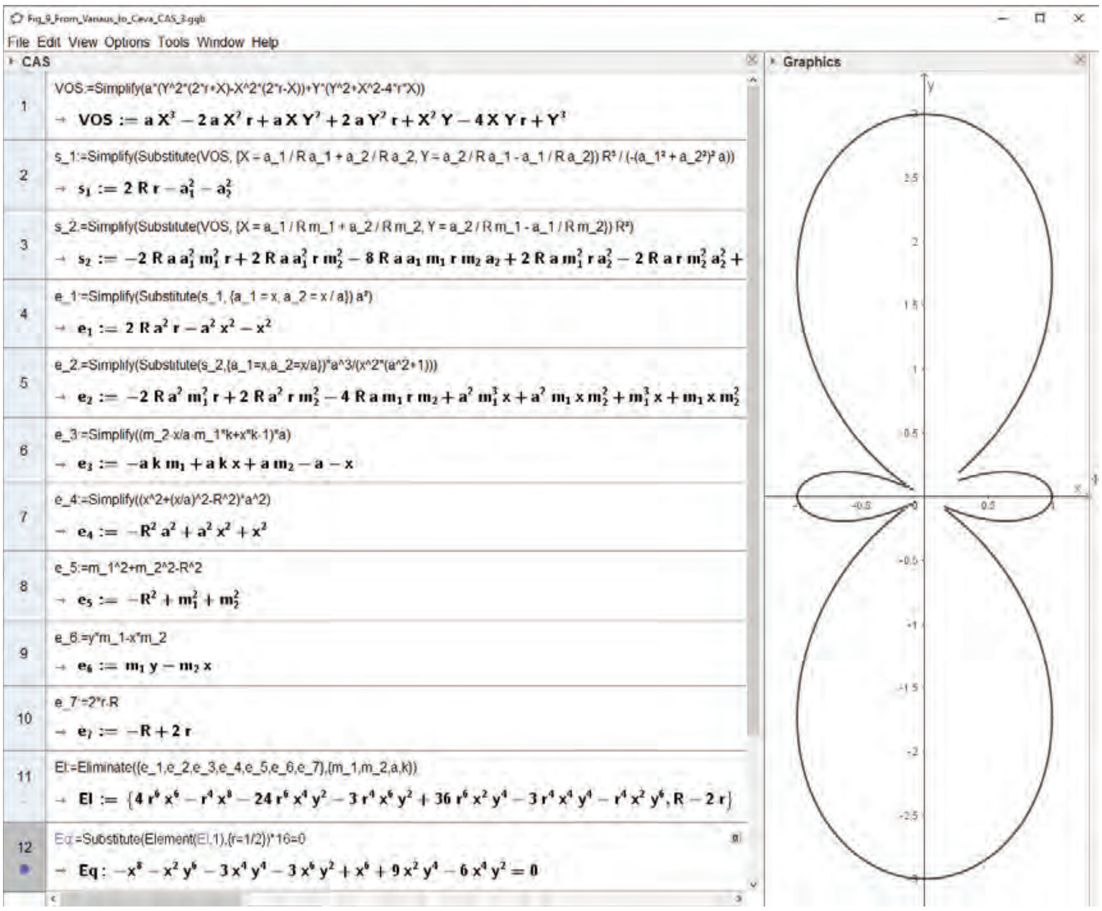


Figure 9: The locus curve of $B$ derived in GeoGebra CAS, plotted for $r = 1/2$

relation $a = \tan\alpha = \dfrac{a_1}{a_2}$ for parameter $a$, which was introduced in (2), together with the fact that $A$ and $B$ lie on the perpendicular to $x$, and write $a_1 = x$, $a_2 = x/a$, i.e. $A[x, x/a]$. The equations (5),(6) are thereby simplified to the first two equations of the required system

$$e_1 :\ 2\,R\,a^2\,r - a^2\,x^2 - x^2 = 0, \tag{7}$$

$$\begin{aligned} e_2 :\ -2\,R\,a^2\,m_1^2\,r + 2\,R\,a^2\,r\,m_2^2 - 4\,R\,a\,m_1\,r\,m_2 \\ + a^2\,m_1^3\,x + a^2\,m_1\,x\,m_2^2 + m_1^3\,x + m_1\,x\,m_2^2 = 0. \end{aligned} \tag{8}$$

To avoid the situation where $M[m_1, m_2] \equiv A[a_1, a_2]$, we set the non-degenerate condition $(m_2 - a_2) - k(m_1 - a_1) - 1 = 0$, where $k$ is an additional parameter, which leads to the third equation

$$e_3 :\ -a\,k\,m_1 + a\,k\,x + a\,m_2 - a - x = 0. \tag{9}$$

Then, we add two equations expressing that $A$ and $M$ lie on the same circle $m$ with radius $R$

$$e_4 :\ -R^2\,a^2 + a^2\,x^2 + x^2 = 0, \tag{10}$$

$$e_5 :\ -R^2 + m_1^2 + m_2^2 = 0. \tag{11}$$

The following equation determines that $B$ is the intersection of $SM$ with the line passing through $A$ perpendicular to the $x$-axis

$$e_6 :\ m_1\,y - m_2\,x = 0. \tag{12}$$

Finally, the last equation

$$e_7 :\ -R + 2\,r = 0 \tag{13}$$

expresses the relation between the radii $r$ and $R$.

Then, by eliminating parameters $m_1, m_2, a, k$ from the system of equations $e_1, \ldots, e_7$, utilising the *Eliminate* function, see line 11 in the GeoGebra CAS code in Fig. 9 (for the complete code from Fig. 9 visit [8]), we obtain

$$4\,r^6\,x^6 - r^4\,x^8 - 24\,r^6\,x^4\,y^2 - 3\,r^4\,x^6\,y^2 + 36\,r^6\,x^2\,y^4 - 3\,r^4\,x^4\,y^4 - r^4\,x^2\,y^6 = 0, \tag{14}$$

after factorisation

$$-\,r^4\,x^2\,\left(x^6 - 4\,x^4\,r^2 + 3\,x^4\,y^2 + 24\,x^2\,r^2\,y^2 + 3\,x^2\,y^4 - 36\,r^2\,y^4 + y^6\right) = 0, \tag{15}$$

where the third factor gives us the final equation of the locus curve of $B$

$$(x^2 + y^2)^3 - (2r)^2(x^2 - 3y^2) = 0, \tag{16}$$

which is the equation for Ceva's trisectrix, rotated by ninety degrees compared to (3), except that the value of radius $r$ is now half the value considered in (3).

We can thus take it as proven that the sought-after locus curve of B is a specific part of Ceva's trisectrix with equation (16). On this basis, it is possible to propose a new trisection method, as is clearly presented in Fig. 10. To use it, it is sufficient to make, for example on a 3D printer, the relevant part of the Ceva curve. This artifact will then, together with a ruler and compass, form a complete set of tools for trisection.

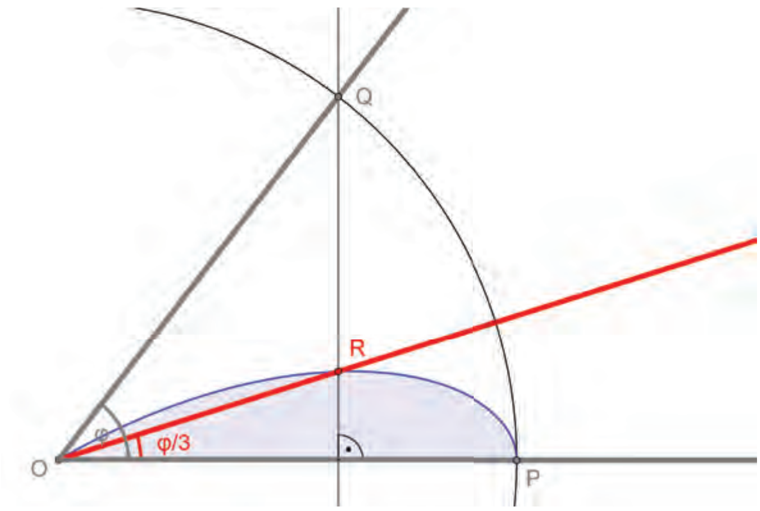


Figure 10: New method of trisection based on Ceva's trisectrix

# 5 Conclusions

As already stated, the skills and abilities that determine the success of a teacher include, among other things, creativity, problem-solving skills, and the ability to acquire new knowledge. We believe that with this paper we have succeeded in showing that this can be achieved through the effective use of suitable computer software, in this particular case, the dynamic mathematics program GeoGebra.

Current scientific studies into STEM education as a way of preparing students for an ever-changing world largely concur that concepts such as creativity, the ability to learn and apply new knowledge, problem solving skills, and the carrying out of authentic inquiry, play a significant role in its implementation [13, 15]. Among the ambitions of the paper was to show how these skills can be practiced and developed on the basis of a historical topic, namely solving a specific student project in which the basic mathematical curriculum is applied to solve problems based on appropriately collected and presented historical materials. It was shown that within the framework of such a project, students not only practice and develop the aforementioned skills, in conjunction with their digital competencies, but also carry out the technical design and production of the relevant artifact.

The ambition of the author was also to inspire colleagues to assign the presented problems to their students. Isn't it a captivating idea to collate as many ideas as possible to solve the Vaňaus trisector mystery?

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# Understanding Geometric Pattern and its Geometry Part 10 – Geometry lesson from Paigah Tombs

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**Abstract:** *Symmetry groups are a fairly recent development in modern geometry. Their origins can be traced from a paper by M. J. Buerger and J. S. Lukesh (see* [1] *). A very solid mathematical foundation of them can be found in Conway's "Symmetries of Things." We can also find there his Magic Theorem for plane symmetry groups, the so-called wallpaper groups, with a complete proof. Some mathematicians believe this theorem can be used to create any plane geometric pattern. Unfortunately, the Magic Theorem is not enough. It can help to determine the overall geometric structure of the pattern, but it does not handle what is happening inside the fundamental region of it. Thus we may have an infinity of geometric patterns within the same symmetry group. However, in many cases, symmetry groups can help us reconstruct an existing geometric design or create a new design.*

*This paper discusses a selection of patterns found in Paigah Tombs, or the Maqhbara Shams al-Umara, in Hyderabad, India. We will limit our discussion to a selection of hexagonal designs. Following this discussion, we will show how to analyze and reconstruct these patterns.*

## Introduction

Different classes of geometric patterns require different approaches and different methods. This paper discusses a specific selection of geometric patterns that we will refer to as hexagonal. Various authors provide different definitions of hexagonal patterns. Thus we need to specify what we mean by this term and how one can distinguish a hexagonal pattern from various other types.

For the purpose of this paper and any further investigations, we define hexagonal patterns by using plane symmetry groups and local symmetries of a pattern.

## Plane symmetry groups

Plane symmetry groups (in the following pages called wallpaper groups) allow us to classify all patterns by their transformation properties, i.e., reflections, rotations, translations, and glide reflections. Conway's Magic Theorem states that there are seventeen wallpaper groups. Thus, according to it, we may classify all patterns into 17 disjoint classes. From a geometric pattern design point of view, this information is not very useful. For example, all decagonal patterns can be split between two wallpaper groups only. But the same two groups also contain other types of geometric patterns, e.g., octagonal, dodecagonal, etc. However, wallpaper groups allow us to determine the primary cell[1] of any pattern. In some cases, this is enough to proceed with pattern design. Unfortunately, the most difficult task is still to be done in other cases.

***Definition[2]*** *– a geometric pattern is considered as hexagonal if it was created on a tessellation of regular hexagons or polygons (equilateral or equiangular), derived from a regular hexagon, and its highest local symmetry is less or equal D6 or C6.*

Hexagonal patterns can be located in a few wallpaper groups. The majority of them belong to groups *632 and 632. However, we can see them also in groups 333, *333, 3*3, and in other groups. All patterns discussed in this paper belong to wallpaper groups *632 and 632. Thus we will limit further discussion to these two groups only[3].

The wallpaper groups *632 and 632 are very large and contain many patterns that we do not want to consider as hexagonal, i.e., patterns with dodecagonal stars or rosettes, patterns with nonagonal stars or rosettes, patterns with mixed local symmetries larger than D6. Thus the D6 and C6 requirement is necessary to remove all unwanted geometric designs. Let us briefly discuss the two wallpaper groups *632 and 632.

## Wallpaper group *632

For a start, suppose that a pattern belongs to the wallpaper group *632. This means that we have in it points (kaleidoscopes) with 6 mirrors passing through them, kaleidoscopes with 3 mirrors passing through them, and kaleidoscopes with 2 mirrors passing through them. We will represent kaleidoscopes using the star symbol:  *6, *3, and *2. The only possible configuration of mirrors and kaleidoscopes in the wallpaper group *632 can be illustrated as follows.

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[1] Some authors call it a fundamental region of the pattern.

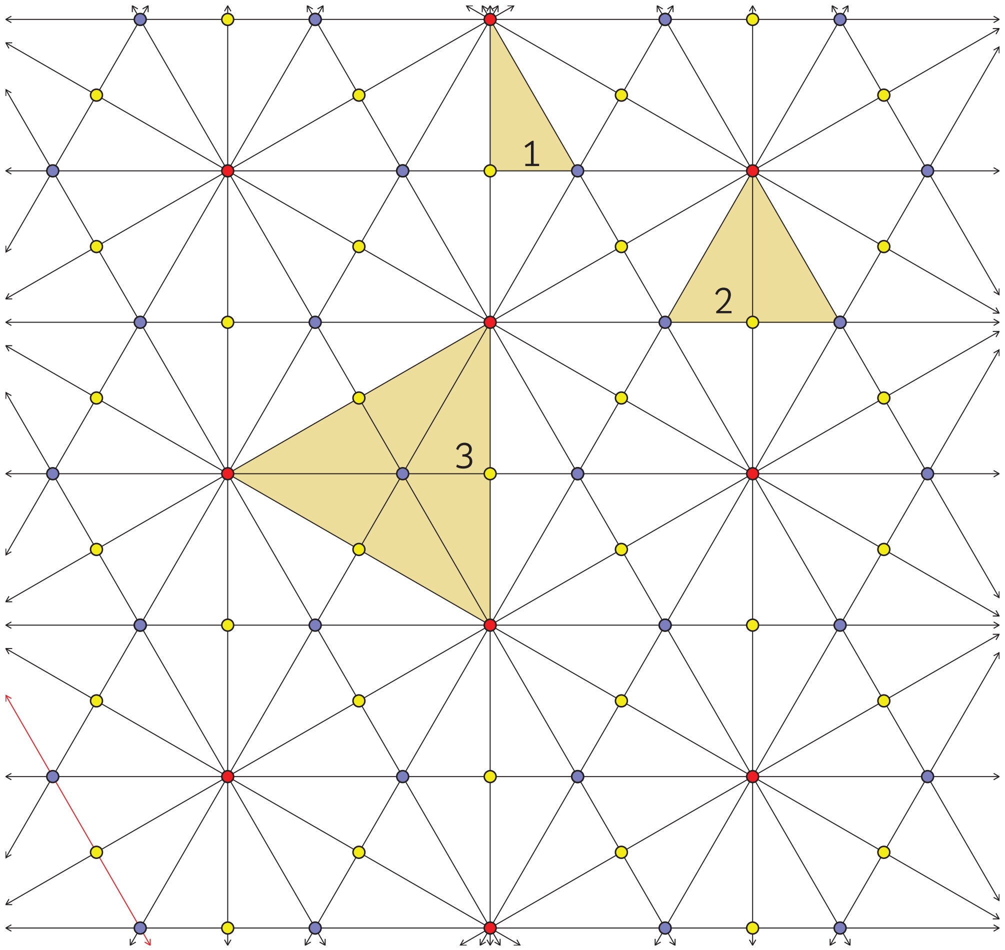
[2] Deciding whether a pattern is hexagonal, octagonal, or decagonal is more complex than many people think or teach. In fact, there are no precise mathematical definitions of these terms. Thus this definition should be treated as provisional, and further discussion is needed.

[3] According to our intuitions, we should expect a large number of hexagonal patterns in symmetry groups 333, *333, and 3*3. It is incredibly easy to develop geometric patterns with these three signatures. But some statistics show very few known hexagonal patterns with these signatures.

In the book „Islamic Design a Mathematical Approach" Brian Wichmann published the frequency of patterns in each symmetry group. According to his statistics in a collection of 1500 patterns (as for 2017), there are 198 patterns in group *632, 26 in group 632, 9 in group 3*3, and 1 in each group 333 and ∗333. We assume that since 2017 his collection grew up, and now these numbers are slightly different. But, still we may not expect many additions to groups 333, ∗333, and 3∗3.

In some other symmetry groups, there are also patterns that we would count as hexagonal. But there is no good way to extract them from these groups.

All known statistics do not reflect the coloring of patterns. They deal with segments only, ignoring colors of particular shapes. Thus a pattern that was considered as 632 with colors may change to 333.




**Network of mirrors and kaleidoscopes in any pattern with wallpaper group *632**

One can easily prove that this is the only possible configuration of kaleidoscopes where six mirrors intersect, three mirrors intersect, and two mirrors intersect.

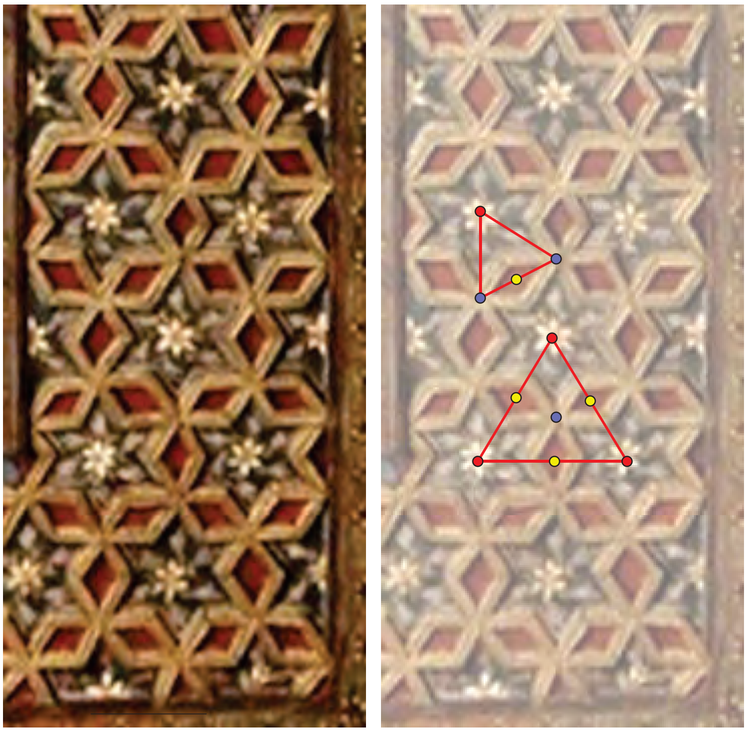
The **triangle 1** contains the smallest part of the pattern necessary to produce a larger design by mirroring it multiple times. We will call it the **primary cell of a pattern**.

**Triangle 2** was created using two copies of the fundamental region. We will call it a **small triangle** or **s-triangle**.

**Triangle 3** was created using six copies of the fundamental region. We will call it a **large triangle** or **L-triangle**.

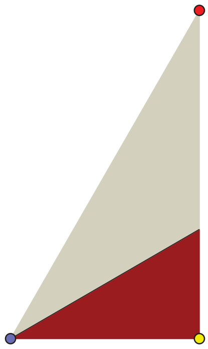
Although the primary cell contains the smallest part of the pattern, necessary to proceed with the pattern design; usually, it is more convenient to design patterns using larger structures. In practice depending on the pattern, one has to decide which triangle is more suitable to create his construction. In many cases, we can use hexagonal tiles built out of one of these triangles. In some cases, it is more convenient to construct a pattern using two or more different hexagonal tiles. Especially designing hexagonal patterns with two or more types of regular hexagonal tiles can be very helpful.

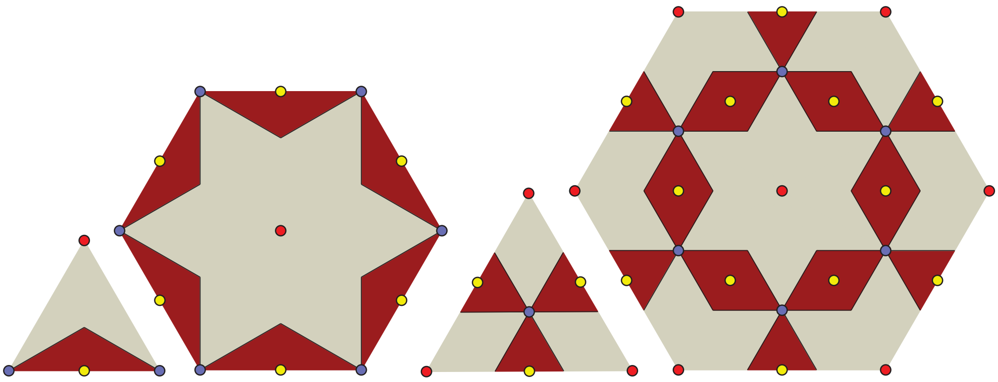
**Example 1. Pattern with wallpaper group *632**



The drawing shows a typical pattern with wallpaper group *632. Red points are points with six mirrors, i.e., kaleidoscopes *6, blue with three mirrors (*3), and yellow with two mirrors only (*2).

Thus the pattern can be created by using the following primary cell:





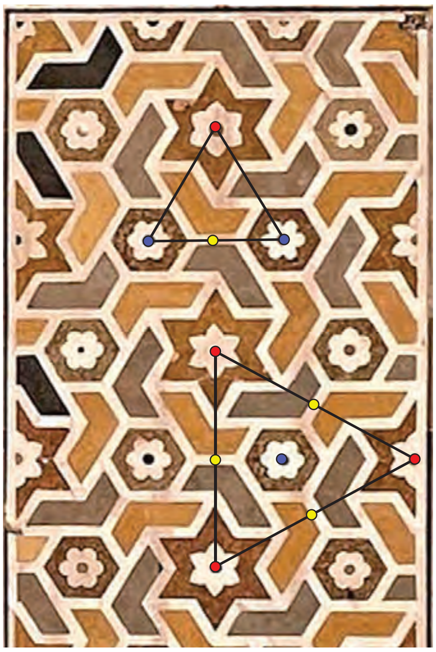
Small and large triangles and hexagonal tiles for the pattern

The pattern used in this example is often seen in many countries and places. The one presented here comes from Sabil, located at the intersection of al-Muizz and Tambakshiya streets in Cairo.

## Wallpaper Group 632

Patterns in wallpaper group 632 do not have mirrors but still have some characteristic points that we will call rotation points or gyrations (we mark them using the • symbol). These points form an identical network as the one for the *632 group. Let us examine one of such patterns. In this example, we will discuss a geometric design from Itimad-ud-Daula's tomb in Agra. The pattern treated as a construction of segments only has 632 signature. The situation changes when we start distinguishing tiles with the same shape but different colors. We no longer have gyrations •2, and the gyration •6 changes into a gyration •3.

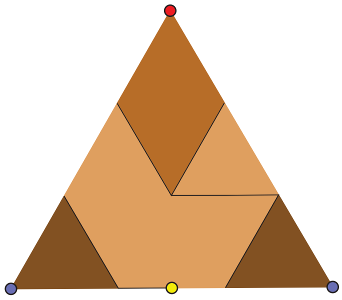
## Example 2. Pattern with wallpaper group 632



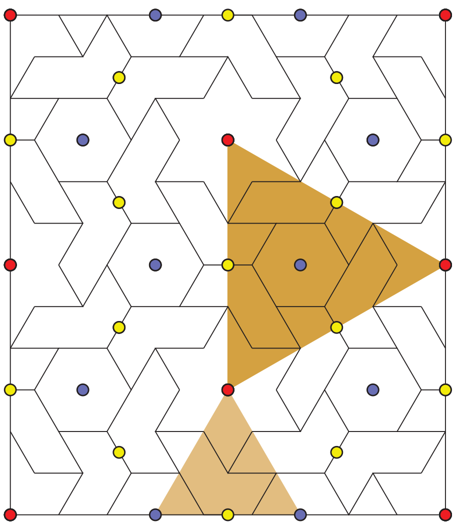
The pattern shown here can be seen in Agra at the Itimad-ud-Daula's Tomb. In this analysis, we ignore the colors of shapes.

The red points in this pattern are points where the pattern can be rotated 6 times 60 degrees (gyrations ●6), blue points where we can rotate it 3 times each time 120 degrees (gyrations ●3), and yellow points, mark rotations 180 degrees (gyrations ●2). Thus the signature of the pattern is 632.

Note – in this pattern, we do not have mirrors. We have only rotation points.



The primary cell for this pattern is an equilateral triangle, which is rotated 6 times around the red point. This way, we obtain a hexagonal tile filled with a pattern. A larger design can be obtained by translating the hexagonal tile.



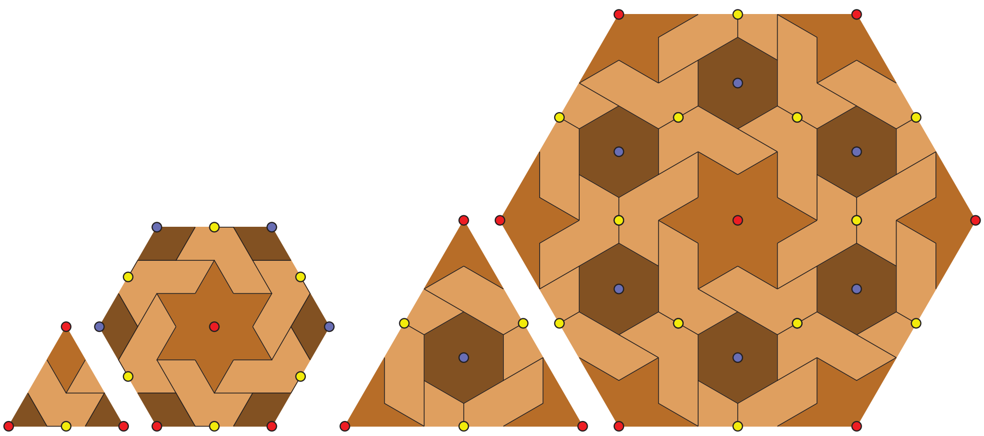
### Small and large triangles for the pattern

If we want to construct a large instance of the pattern, we need to create one of these two triangles and rotate it 6 times around the red point (rotation 60 degrees). This way, we will get a hexagonal tile, small or large. The whole pattern can be produced by translating this tile.

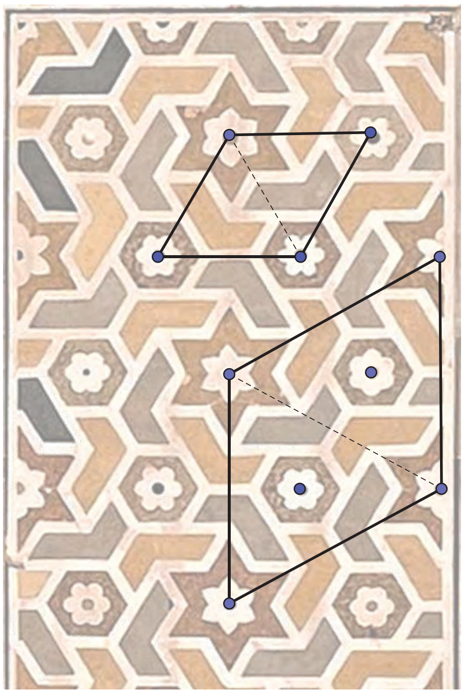
Depending on our needs, we can use small or large hexagonal tiles.

In the following construction, we ignore the colors of elements of the pattern. Therefore we treat the orange and gray V-shapes as identical.

**Below:** small and large triangles for the pattern and small and large hexagonal tiles



Assuming that shapes with different colors are different, we end up with a pattern with signature 333. This fact is presented in the following illustrations.

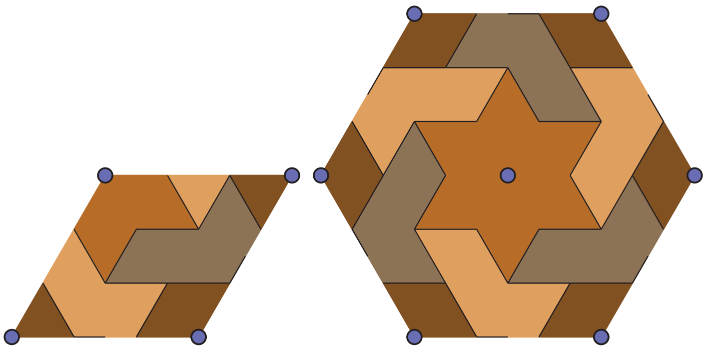


Example 3. Pattern with wallpaper group 333

The illustration shows the same pattern as in the previous example. But this time, we consider shapes with different colors as different. Thus we lost all yellow points. We do not have rotations about 180 degrees. Red points turned into blue. There are no longer rotations of about 60 degrees. All points marked on the photo are gyrations ●3.

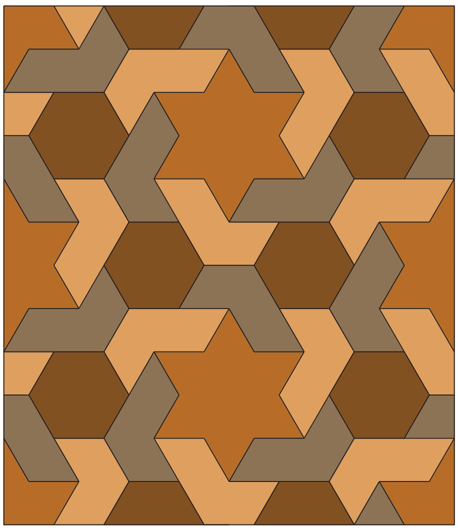
Important – the pattern can be reconstructed using two different triangles (here separated with dashed lines) or by constructing a pattern for one of the rhombi and rotating it 120 degrees around the gyration point near the wide angle of the rhombus. This way, we will produce two hexagonal tiles, small and large, and then a large pattern can be constructed by translating one of them.

The small rhombus is the primary cell of the pattern.



Up: Small rhombus tile and small hexagon tile

Left: Fragment of a large pattern created using the small hexagonal tile



The third example shows that the number of patterns with signature 333 can be significantly larger than provided by Brian Wichmann in his statistics. The same argument can also be used for patterns from the *632 wallpaper group.

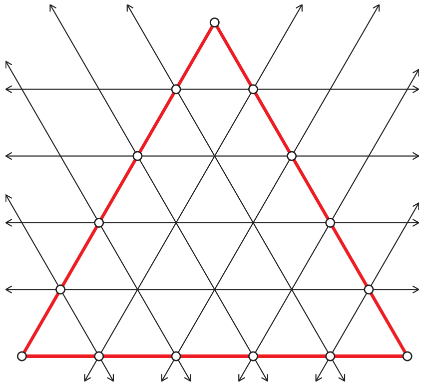
## On Paigah Tombs' patterns

*"Paigah Tombs, or Maqhbara Shams al-Umara, are the tombs belonging to the nobility of the Paigah family, who were fierce loyalists of the Nizams, served as states people, philanthropists, and generals under and alongside them. The Paigah tombs are among the major wonders of Hyderabad State, which are*

*known for their architectural excellence, as shown in their laid mosaic tiles and craftsmanship work. The Paigah's necropolis is located in a quiet neighborhood 4 km southeast of Charminar Hyderabad, in the Pisal Banda suburb, down a small lane across from Owasi Hospital near Santosh Nagar. These tombs are made out of lime and mortar with beautiful inlaid marble carvings. These tombs are 200 years old and represent the final resting places of several generations of the Paigah Nobles." (Copied from Wikipedia[4]).*

In Paigah tombs, we can find a large collection of geometric patterns carved in lime. The majority of them are hexagonal patterns. Almost all of them were created using a specific technique – triangular grids built on equilateral triangles. We will split them into three groups and discuss each group separately.

## Patterns created on equilateral triangles with parallel grids



**Parallel grid network**

The concept is simple and frequently used for patterns in Paigah tombs. It was also used in Iran and Anatolia.

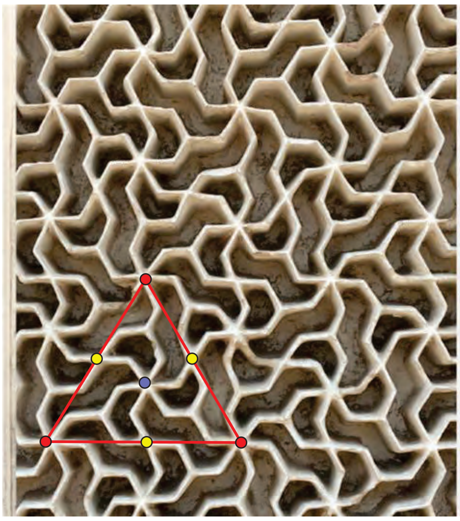
Take an equilateral triangle. Split each edge into $n$ equal parts (here, five parts). Draw lines passing through these points and parallel to one of the edges of the triangle.

Now we can use this network to draw a pattern with edges along grid lines.

This procedure can be used to create an s-triangle or L-triangle of the future pattern.

Below we discuss one of the most complex patterns from the Paigah Tombs using parallel grids.

### Example 4. One of many patterns from Paigah Tombs



The photo presents a pattern with a large equilateral triangle. All its vertices are points where the pattern can be rotated six times 60 degrees (gyrations ●6). The center of the triangle is ●3 gyration. Finally, the midpoints of the edges of the triangle are ●2 gyrations.

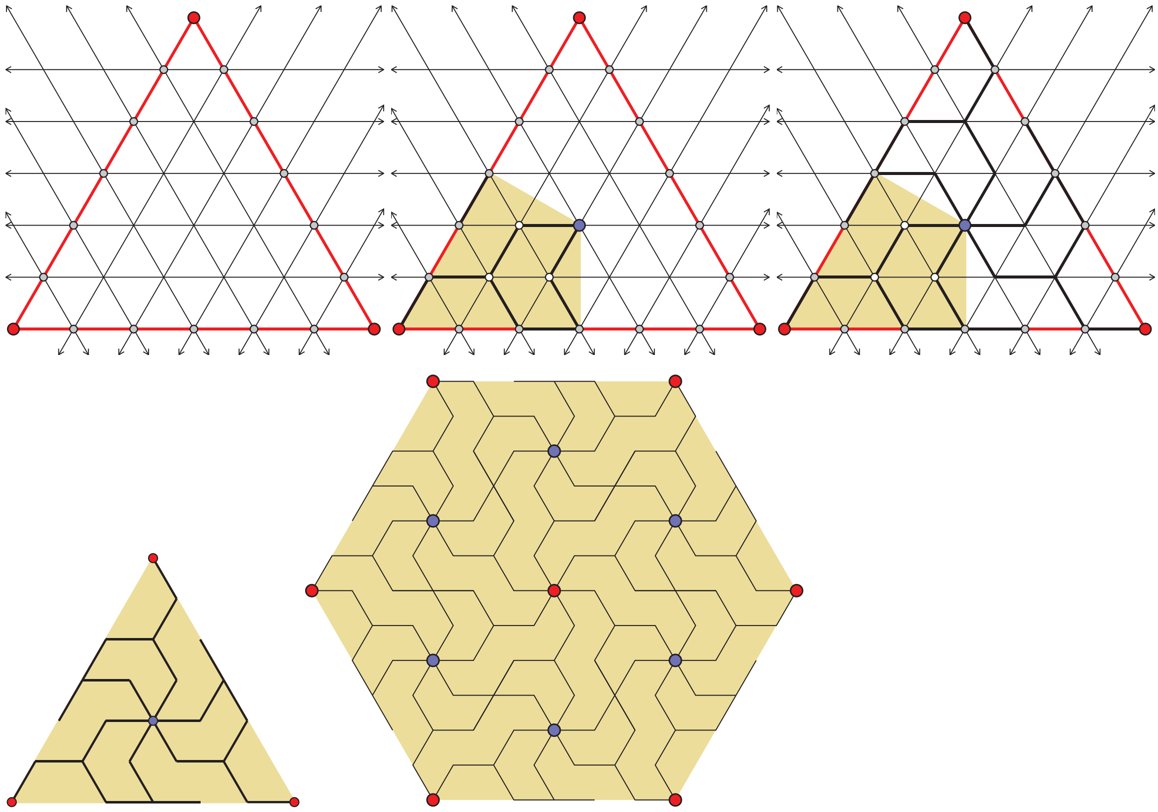
Consequently, we do not have mirrors. We have rotations only. Thus we have a pattern with 632 wallpaper signature.

One can easily notice that the pattern follows the parallel grid network with 6 lines parallel to each edge of the triangle.

This pattern has a few interesting properties that we will discuss later.

[4] https://en.wikipedia.org/wiki/Paigah_Tombs

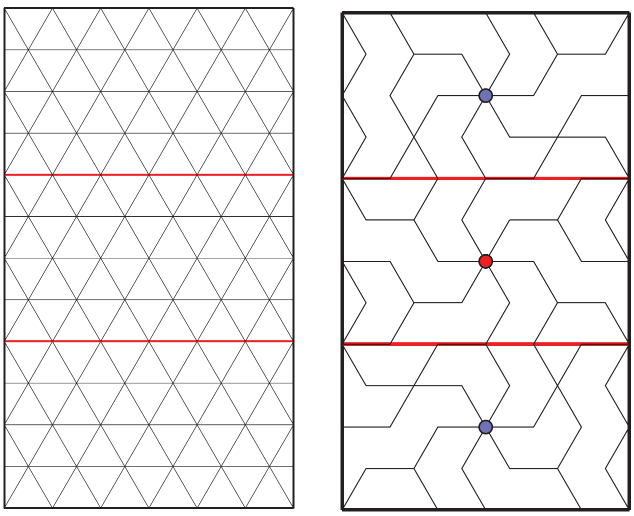
Construction of the pattern



By dividing each edge of the triangle into 6 equal parts, we produce a dense triangular grid. Then we design pattern in the shaded kite only. Finally, we rotated the pattern from the shaded area twice around the triangle center 120 degrees. This way, we get the large triangle of the pattern.

A large hexagonal tile is the last step in reconstructing the pattern.

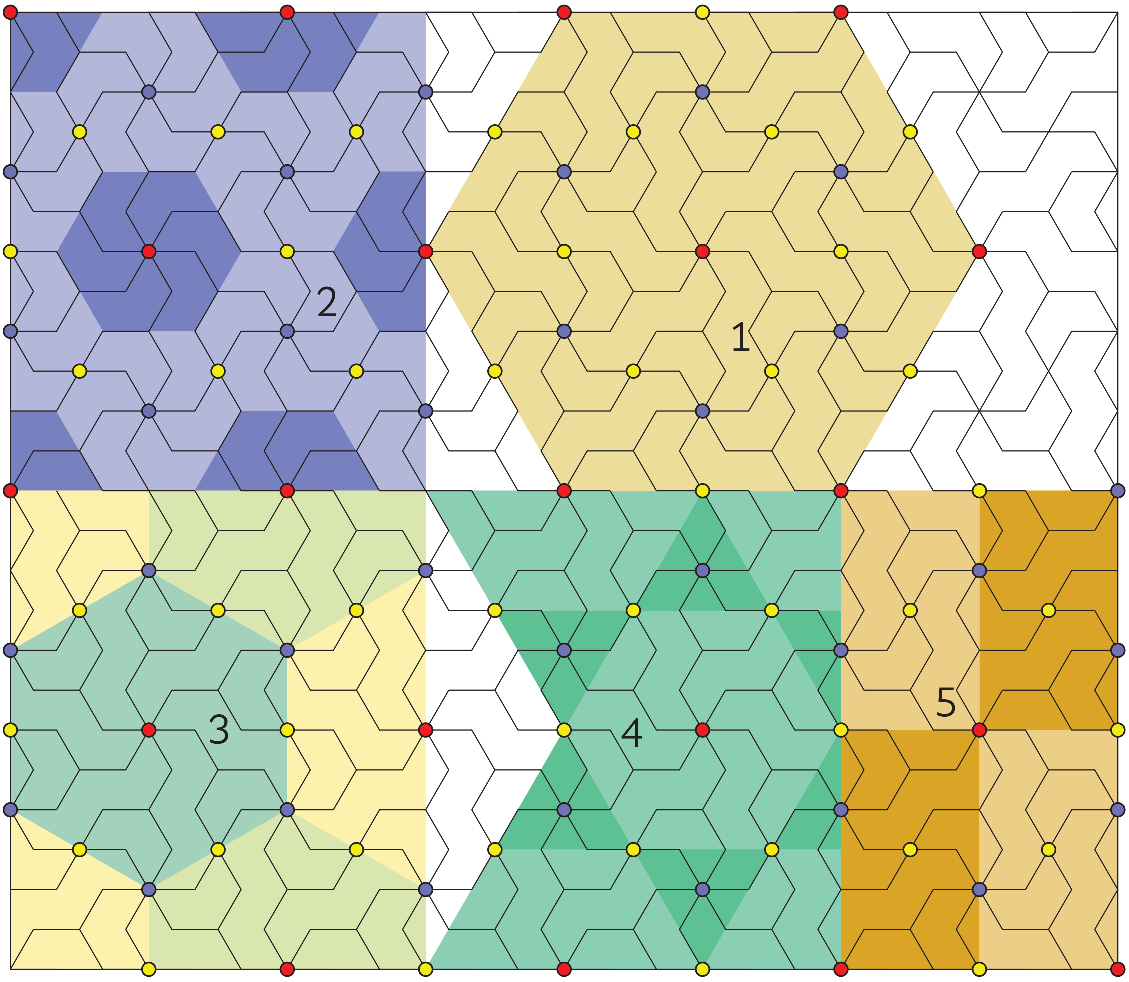
The pattern presented here is one of the most interesting patterns seen in Paigah tombs. It can be created in many other ways.



### Construction of the pattern from Example 4 using a rectangular template

This is one of the most popular methods for creating complex patterns. We create a rectangle with a grid shown in the left picture. Then we follow the grid lines to get the pattern (right drawing). This way, we produce a rectangular template.

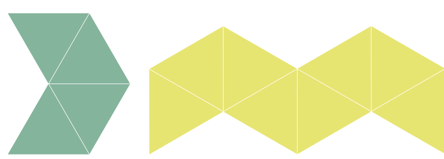
Using this template, we can create any larger design. Note – we should translate this template along its edges. No reflections should be used.




### Some other ways of creating pattern from example 4

The drawing illustrates a few other ways of creating this pattern.

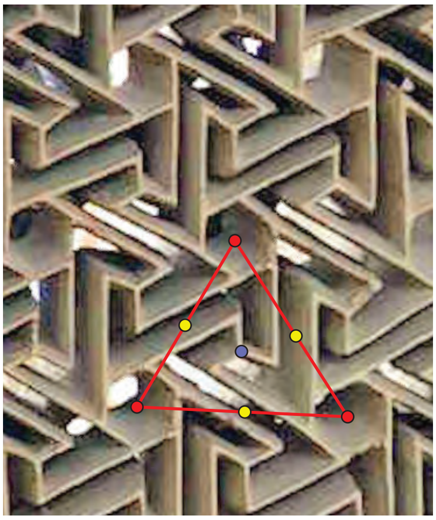
1. This is the method that we used in our solution. We created the L-triangle and then L-hexagon.
2. We use two hexagons with different rotations of the patterns around their centers. Note how different is the role of each type of hexagon. Hexagons with centers ●6 are surrounded by hexagons with centers ●3.
3. In this method, we create an s-triangle and then s-hexagon. Each of the hexagons shown here is the same. We use different colors to separate them.
4. In method 4, we use two different shapes a hexagon with center ●6 and vertices ●2 and an equilateral triangle with center ●3 and vertices ●2.
5. In this method, we use two rectangles of the same size but with different orientations of patterns inside them. Both rectangles should be placed on the plane, forming a chessboard-like structure.
6. The whole pattern is a mosaic built out of two shapes, and each is assembled from equilateral triangles. How many other patterns can be created using these two shapes?



## Patterns created on triangles and hexagons with perpendicular grids

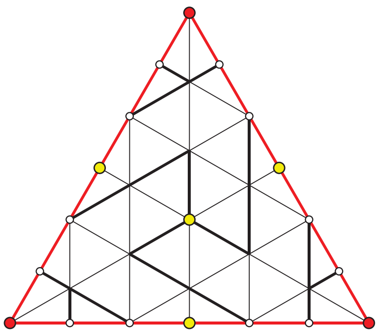
In the same way, we can use grids with lines perpendicular to the edges of the triangle. The following drawings show a pattern created using a perpendicular grid.

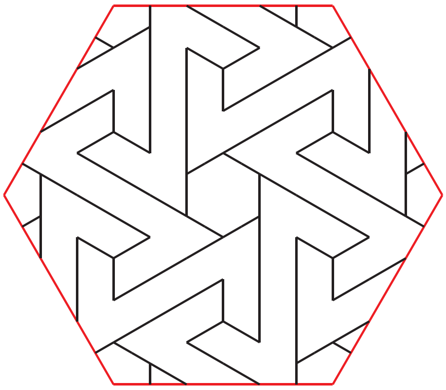
### Example 5. Another pattern from Paigah Tombs



In this photo, we have another hexagonal pattern from Paigah Tombs. Although the pattern looks very complicated, its construction is very simple.

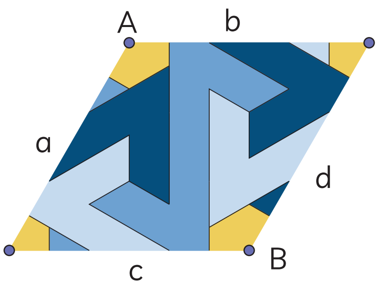
The L-triangle shown in the photo demonstrates how one can reconstruct the pattern. We use a triangular grid with lines perpendicular to the edges of the triangle.

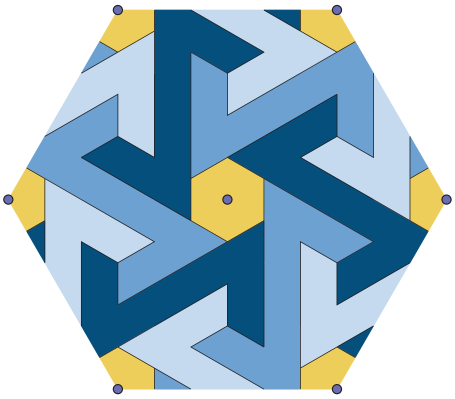




This pattern has signature 632, i.e., we have three gyrations: ●6, ●3, ●2.

We can easily turn the pattern into a pattern with signature 333 by using two different colors for Z-like shapes.



**Left:** the rhombus created from the original T-triangle and its rotated 180 degrees copy. The coloring was applied in such a way that colors along the edge *a* match colors along edge *b*.

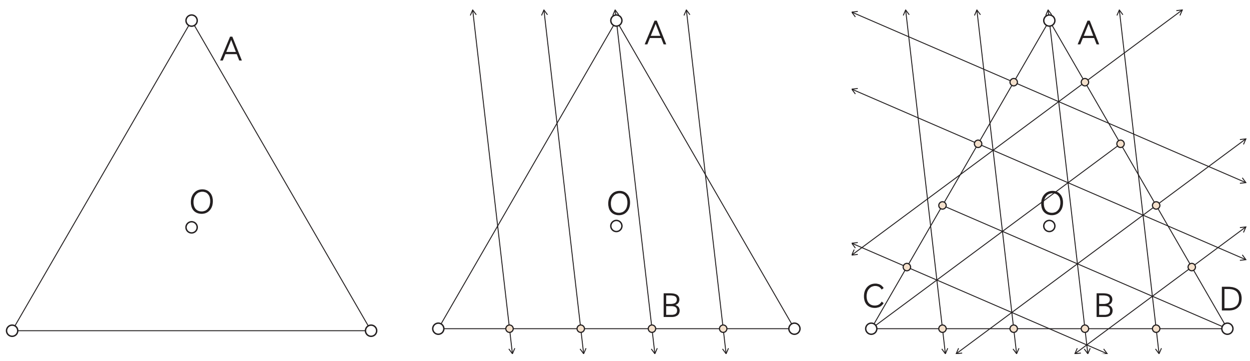
**Right:** The L-hexagon results from the rotation of the rhombus around point A three times 120 degrees.

One can easily notice that if an s-triangle of a pattern was created using a parallel grid, then its L-triangle can be created with a perpendicular grid. If an s-triangle of a pattern was created using a perpendicular grid, then its L-triangle can be created with a parallel grid. Thus these two methods are equivalent. One can choose an approach that is more convenient for him.

## Patterns created on triangles and hexagons with twisted grids

Persian and Anatolian Seljuk craftsmen used the concept of twisted grids for centuries. In Seljuk architecture in Anatolia, we can find very complex examples following this method. In Paigah tombs, there are also designs using twisted grids.

The twisted grids method



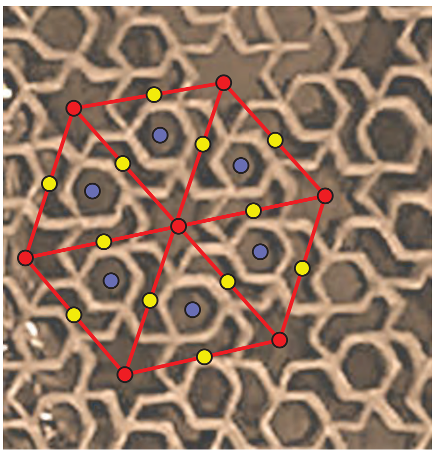

Description

1. Start with an equilateral triangle with its center shown.
2. Divide one of the triangle edges into any number of equal segments. Here we divided the bottom edge into five equal parts. Draw a line connecting the opposite vertex with one of the division points. Here it is, line AB. Draw lines parallel to it, passing through the remaining division points.
3. Rotate two times the set of parallel lines about the center of the triangle at 120 degrees. This way, we get the grid shown in the right drawing. We call it a twisted grid.
4. Use this grid to create any pattern. Note – the edge AC of the pattern should match the edge AD.

**COMMENTS:** To create a twisted grid, we must divide the edge of the triangle into at least three equal parts. Point B should not be a midpoint of the edge.

With twisted grids, we can construct patterns with signature 632. A colored version of such a design can often be turned into a pattern with signature 333. With a more complex grid, we can easily design patterns with signature 333 without using colored tiles.

### Example 6. Yet another pattern from Paigah Tombs

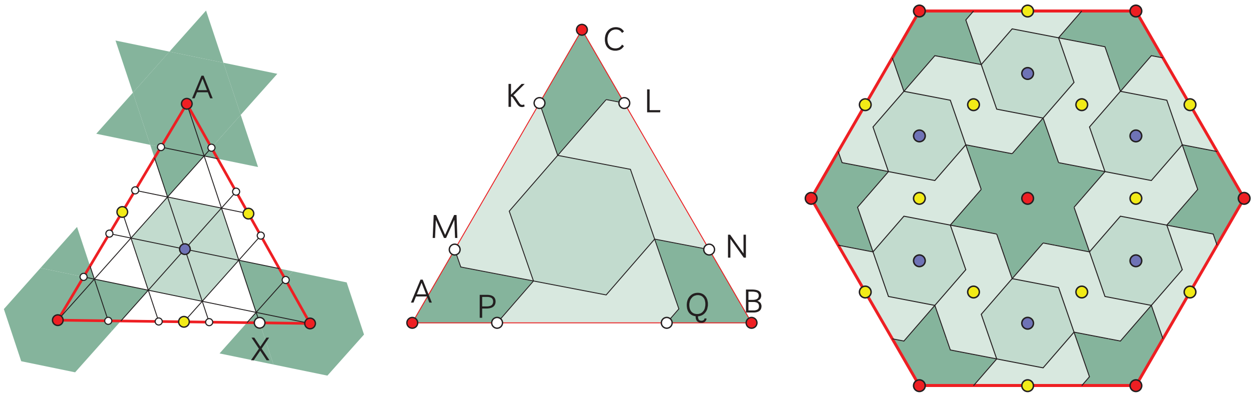


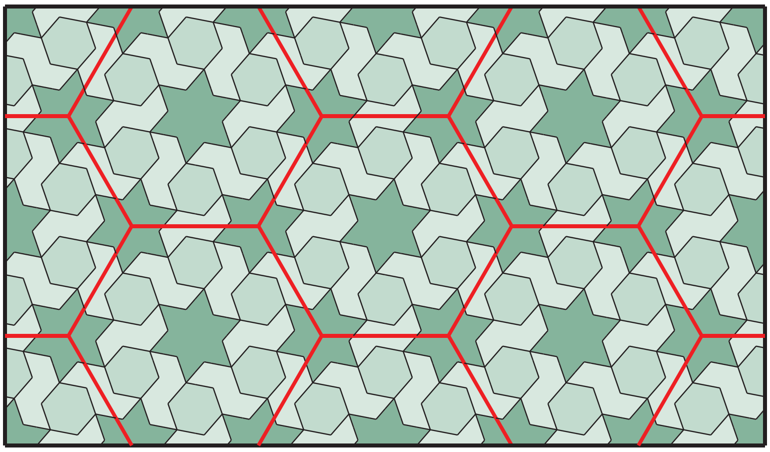
The pattern presented here uses a twisted grid. For creating an L-triangle, we will divide the edge of a triangle into five equal parts. We could also create the s-triangle, but such construction might be less intuitive.

In the next drawings, we show a brief construction of this pattern.

-

Construction of the pattern from Example 6



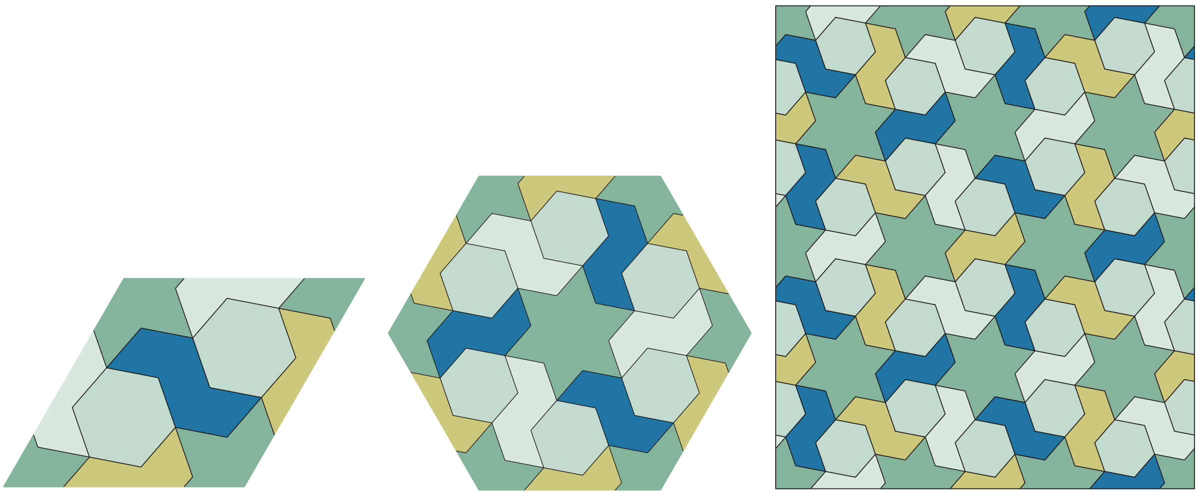



**Description:**

Divide the bottom edge of the triangle into 5 equal parts and draw a line through points A and X. Create a grid of lines parallel to AX, and rotate it two times 120 degrees about the center of the triangle.

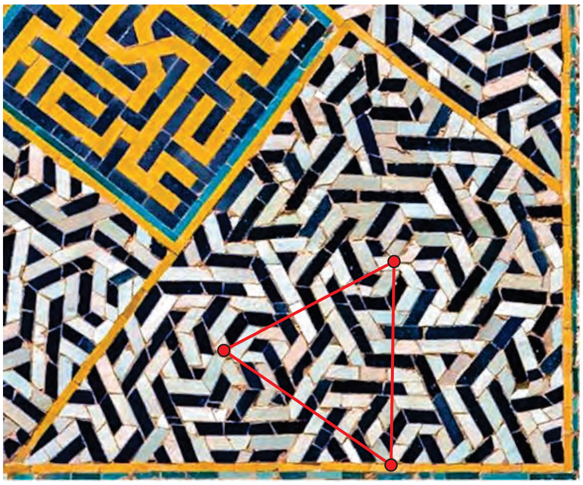
Note how the pattern was created. Points K and L must be at the same distance from the vertex C and points M and N should be at the same distance from vertices A and B, respectively.

Points P and Q should be at the same distance from A and B, respectively. This is the only way we can rotate the L-triangle around point A, and lines will flow without breaks from the L-triangle to its copy.



The drawings above show how we can modify the pattern from this example to obtain a pattern with signature 333.

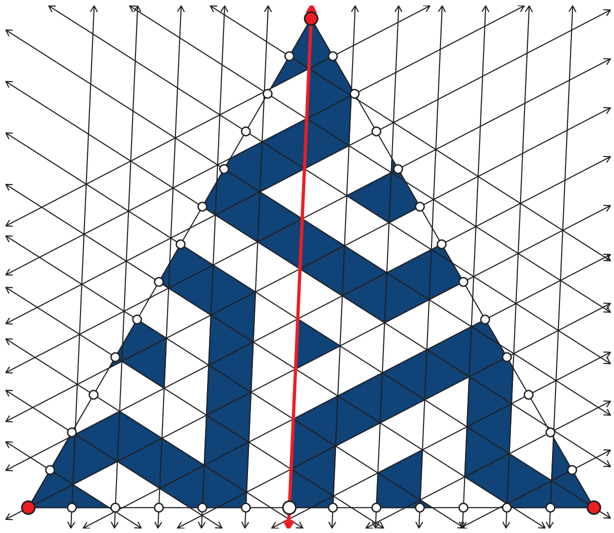
## Example 7. Pattern from Isfahan



With twisted grids, we can produce very complex geometric patterns. Here is one of them.

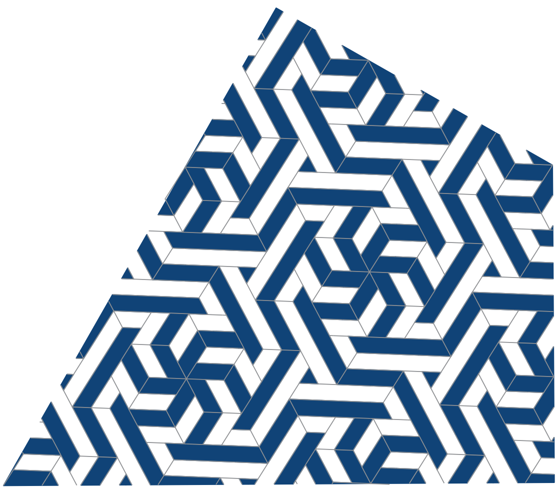
The photo shows a fragment of a pattern from Isfahan. The artwork is very rough, and it has many errors. Thus reconstructing this pattern is a challenging task.

In the photo, the L-triangle was displayed. Its vertices are gyrations ●6. The center of the triangle contains gyration ●3. Finally, gyrations ●2 are located in the middle of each edge of the triangle.



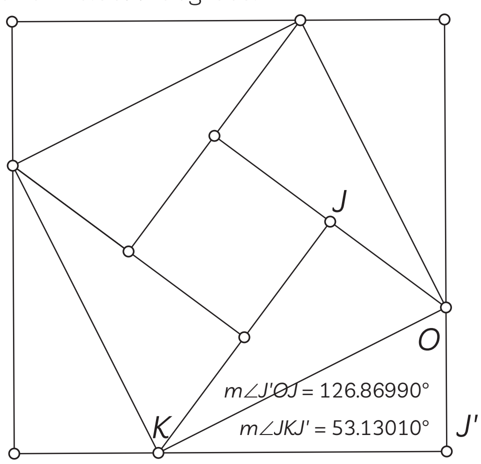
This drawing shows the twisted grid that was used to create the L-triangle. The bottom edge of the triangle was divided into 13 equal parts. The thick red line shows how the grid was started. The rest of this construction is easy to follow from the drawing.

**Right: final design of the L-triangle.**



A kite created using the L-triangle from our construction.

**IMPORTANT:** the kite presented here differs from the one in the photograph. Kite in our construction has angles of 60, 90, 90, and 120 degrees. Kite in the photograph has angles of 53.13010 deg, 90, 90, and 126.8699 degrees.



## Summary

Various types of geometric grids were used for centuries for creating geometric art. We see them in Japanese Kumiko artworks, in European art, in the Middle East, Iran, and Turkey. We can also find them in tribal arts of Africa and South Pacific islands. In this paper, we discussed only a few of them and methods for designing them – parallel and perpendicular triangular grids, and twisted triangular grids. We did not discuss multicentered grids since there are very few geometric patterns using them.

Each type of triangular grid is a powerful tool for reconstructing old patterns and designing many new ones.

## References

[1] M. J. Buerger and J. S. Lukesh (1937), *Wallpaper and atoms; how a study of nature's crystal patterns aids scientist and artist*[5].

[2] John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss, *The Symmetries of Things*, A K Peters/CRC Press; 1st edition (April 18, 2008),

[3] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020)

[4] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (part 1)*, eJMT, vol. 14, Nr 2, pages 87-106.



Last update July 31, 2023

[5] This paper is available only in a few scientific libraries. No electronic version exists.

# Education for the Future: Crafting 3D Geometric Models and Building Mathematics Knowledge with 3D Printing

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**Abstract**

*This paper addresses the creation of 3D geometric models using 3D printers and introduces a newly designed and 3D-printed construction set of polygons for educational purposes. The process of creating these geometric models encompasses steps such as design, 3D computer modeling with Constructive Solid Geometry (CSG), 3D computer modeling of parametric surfaces using principles of differential geometry, 3D scanning of real objects, and the process of manufacturing itself. Students can be involved in the entire process of crafting models for 3D printers, and these resulting printed models can be utilized in geometry education at all levels (university and secondary school in our scenario) as instructional aids. We explore the potential methods to design geometric objects using 3D computer modeling software; this covers both commercial options and open-source software like Tinkercad, a free web application for 3D design, electronics, and coding. We present the fabrication of a new construction set of polygons which consists of different shapes of regular and irregular polygons and it is intended for use in mathematics teaching to study polygon properties and create diverse types of tessellations at the secondary school level. The effects of using these instructional aids were tested with several groups of students. All steps of the model design for 3D printing, in combination with the physical 3D printed models, shed new light on mathematics education and more broadly, to education as a whole. This process engages students in solving real-world problems and enhances their understanding of geometry, while familiarizing them with 3D computer modeling and 3D printing technologies. Both 3D virtual models and 3D printed models can act as manipulative instructional aids.*

## 1 Introduction

The field of 3D printing technology continues to evolve, becoming increasingly user-friendly and reaching wider audiences. There is a significant growth in using 3D printers in various fields ranging from construction industries, medicine, cultural heritage, art designs to education [22].

Scholars and researchers in the field claim that 3D printing is a beneficial tool in facilitating learning, [6], enhancing skills, and increasing student engagement. This can be particularly applicable to areas such as STEM subjects, where 3D printing can provide tangible aids and together with the entire process of the model design including 3D computer modeling it can promote students' better and deeper understanding [3], [12], [19].

Learning geometry and mathematics is undoubtedly challenging at all stages of education. A variety of visual aids, such as virtual or concrete manipulatives, can significantly help in fostering an understanding of mathematical and geometrical concepts. Studies conducted worldwide have underscored the effectiveness of tangible instructional tools in mathematics education [17]. The use of virtual manipulatives also stimulates students' interest and enhances their performance [8].

Implementing 3D printing technology in education allows students to engage with both virtual and physical realities. Students have the opportunity to acquaint themselves with 3D computer modeling and design their own virtual models. Furthermore, this digital interaction is enriched by the ability to handle and explore 3D printed physical manipulatives. A significant portion of scientific literature indicates that 3D printing technology presents novel opportunities for innovative teaching practices by connecting engineering, technology, and the practical application of science subjects. International studies are focused on understanding the impact of integrating 3D printing technology within educational environments.

In their systematic review of literature, Novak and colleagues identified 78 publications focused on learning with 3D printing technologies [11]. Their conclusions indicated the positive influence of 3D printing on students' learning, its capacity to engage students in solving real-world problems, and the potential it provides for interdisciplinary research. Herrera et al. [7] found that 3D tools (and also augmented reality and virtual environments) have a positive effect on the development of spatial mathematical skills. 3D printing also supports students' understanding in concrete mathematics topics such as surface area [5]. Action-oriented training methods utilizing real physical models have shown good results in improving spatial abilities.

However, it is worth noting that there are studies indicating that manipulatives may not impact student achievement in mathematics [4]. It is important to remember that the use of manipulatives does not ensure success or meaningful comprehension.

The integration of digital fabrication, such as 3D printing, in education enriches the learning experience by mapping concrete items to abstract concepts and fostering problem-solving skills in geometry. This effective application of multiple representations is promising in enhancing the learning process. Therefore, it is beneficial to explore 3D printing further in educational contexts. Accordingly, this paper focuses on the processes of designing and creating 3D models on 3D printers and discusses learning experiences with them in a classroom context. This paper is a follow-up article dealing with 3D printing technology [20] and it furthermore explores other possibilities of digital fabrication and highlights its practical applications in mathematics education.

The paper is structured as follows. We begin with a brief overview of 3D printing technology. This is followed by a presentation of various methods for designing virtual geometric models emphasizing 3D computer modeling using constructive solid geometry. Subsequently, we introduce an educational initiative involving a newly created set of polygons produced through 3D printing. Finally, a gallery of selected models of tessellations is presented.

## 2 3D Printing Technology

Essentially, 3D printing transforms a digital model into a physical object, through an additive process where material is layered to form the final product. This technology, a multi-step process, combines designing, 3D computer modeling using CAD or other software platforms, and the actual fabrication.

There are numerous types of 3D printers that create objects through additive processes. In our research, we specifically utilize 3D printers that construct objects layer by layer, predominantly using a biodegradable plastic material known as Polylactic acid (PLA). We work with Original Prusa i3 MK3S+ 3D printer, which was awarded to us as a prize for the educational project that will be presented later in this article.

The 3D printer transforms digital 3D models into physical objects using 3D printing files. There are several 3D printing file formats, for example, STL (a widely used 3D printing file type containing the surface geometry of a virtual 3D object), OBJ (which entails high-quality geometry, texture information, and full color), 3MF (which contains definitions for colors, materials, and precise shapes that are not present in STL files), G-Code (the format of output from slicer software used to describe the path of the tool in the 3D Cartesian coordinate system). We create digital models in STL format and convert them into G-Code format.

There are several 3D printer slicer software which can be used as control interfaces and most of them are free (for example *PrusaSlicer*, *Repetier-Host* [15], [16]). The software is used for *slicing*, i.e. converting the STL format, for instance, in which the virtual model is saved, into the G-Code format.

During the process of creating 3D models for printing, it is crucial to consider the concept of a "manufacturable object". In geometry, we deal with abstract concepts such as dimensionless points, infinitely thin lines, and surfaces without thickness. However, these abstractions present challenges from a practical standpoint. Consequently, in practical applications, we work with manifold objects, which are characterized by edges shared by exactly two faces. We craft digital models that can be fabricated from specific materials. In the case of a parametric surface, for example, we need to print both the surface and the offset surface, considering the volume in between these two.

3D printers construct three-dimensional objects by laying layer over layer of plastic. Every newly deposited layer relies on the support of the one underneath. If a portion of the model begins in mid-air without any underlying support, additional support structures must be implemented to guarantee a successful print. These parts are then removed after printing. The object can be also divided into parts and these parts are printed separately.

The geometric model placed in the interface of PrusaSlicer software is shown in Figure 1. It is the model of the helical surface, which already requires the printing of supports (shown in green color).

Each 3D printer is designed to print within a specified footprint size, and the precision of the particular printer must be considered. The size of the model determines the printing time and amount of material consumed. Unfortunately, even relatively small 3D models can take several hours per geometric model to complete.

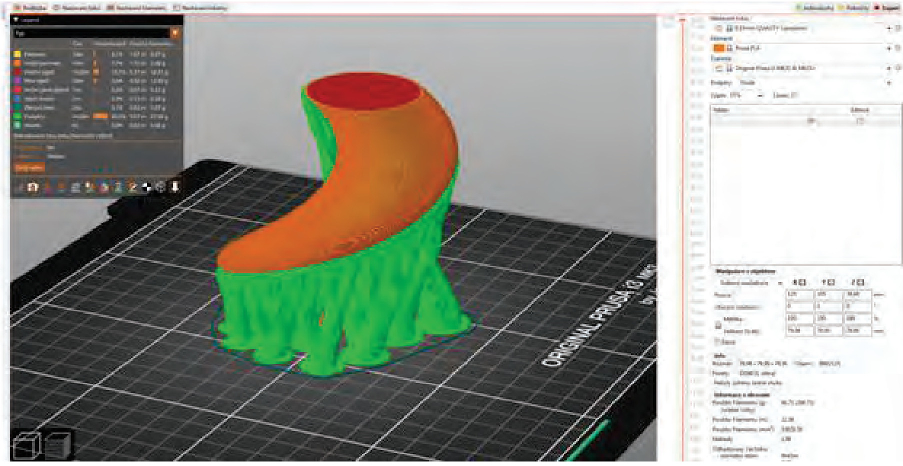

Figure 1: The sliced geometric model of a helical surface placed in the interface of PrusaSlicer and generated with organic supports (shown in green color).

# 3 The Journey from Abstract to Concrete with 3D Printing

We primarily use 3D printers to create geometric models that can be applied in mathematics education at all levels, at universities and secondary schools in our case. For example, 3D printed models of polygons and solid figures can be used in mathematics instruction at secondary schools. More complex geometric models, which are enclosed by an assembly of surfaces (not only parts of planes but also parts of other types of parametric surfaces), are intended for use in undergraduate geometry courses at universities.

In any case, the digital model for 3D printing has to be firstly created. There is always the possibility to download pre-made models on the Internet. There exist plenty of websites and the models are generally free to download [14]. For the certain educational purposes, it is always better to know how to create the geometric model. As previously mentioned, our plan is not only use the final products from 3D printers as instructional aids but we also want to encourage students (at any level of education) to participate in the process of digital fabrication.

The development of a geometric model for 3D printing requires an understanding of its definition and properties. There are various methods available to prepare a digital model for 3D printing. For example, 3D scanning can be an option, particularly when there is a need to digitize real objects. A mathematical description of a digital model is also acceptable. If we consider the digital model which is enclosed by an assembly of surfaces, these separate surfaces can be described in parametric forms.

A brief explanation of the 3D scanning process for real objects, as well as the use of parametric forms to define the closed volume of a solid, can be found in the previous paper [20].

## 3.1 3D Computer Modeling

It is not necessary to only use 3D scanning or describe a digital model through mathematical equations. One can take advantage of the modeling features of 3D computer modeling software.

There are numerous free software options available for creating 3D printable digital models. For example GeoGebra 6 is a free dynamic application including free tools for geometry, and it is user-friendly for both students and pupils. In GeoGebra 6, we can model solid figures or parametric surfaces (or parts thereof, to be precise), and this model can be directly exported to STL format. However, GeoGebra's built-in functions are limited. There are also professional CAD systems and 3D computer modeling software that include advanced modeling features. For example, Rhinoceros (a commercial NURBS-based 3D modeling tool commonly used in design processes), SketchUp, Blender (free), or FreeCAD (free). Arguably the easiest way to create a model is to use the online editor Tinkercad, a free web application for 3D design, electronics, and coding [2]. It is easy to use, and many video tutorials support users. Fusion 360, a more advanced tool, follows up Tinkercad and is free for educational use [1].

A simple geometric model of a mechanical component created in Tinkercad can be seen in Figure 2. It was modeled using the concepts of primitives which are basic geometric solid figures such as a cube (or cuboid), cone, sphere, cylinder and pyramid. These primitives can be combined and modified by using Boolean operations into more complex shapes. This technique for solid modeling is known as *Constructive Solid Geometry*. The process of creating a mechanical component using Constructive Solid Geometry is captured in Figure 3.

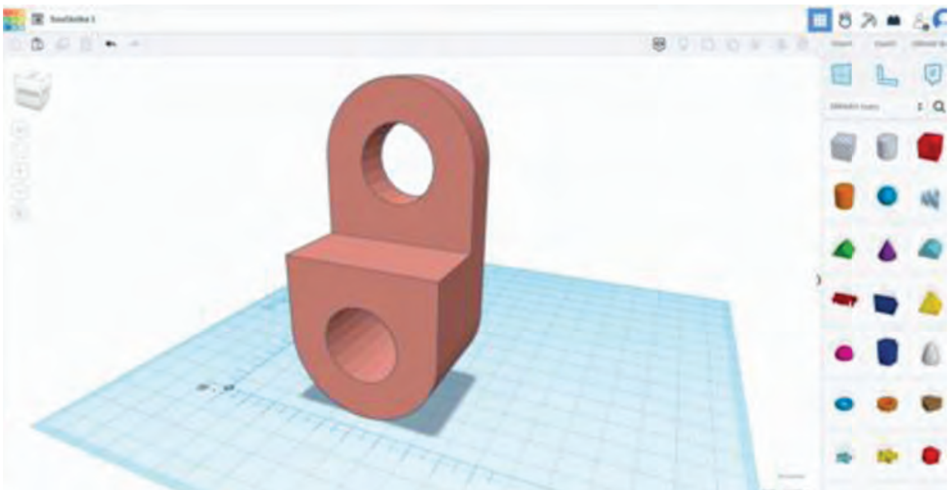

Figure 2: The geometric model of a mechanical component created in Tinkercad.

We have found that pupils at secondary schools and university students can easily use Tinkercad and create impressive models without any intervention from a teacher. Any digital model which is created in Tinkercad can be directly exported to STL format and is ready for slicing and 3D printing. Once again we would like to emphasize that working with any 3D computer modeling software requires familiarity with the specific tool, but also a thorough understanding of the geometric principles and properties that define the objects being modeled.

# 4 Educational Project with a 3D Printer

Programs supporting the use of 3D printers in schools exist in the Czech Republic. We also took advantage of this opportunity at the Grammar School where the author teaches. After

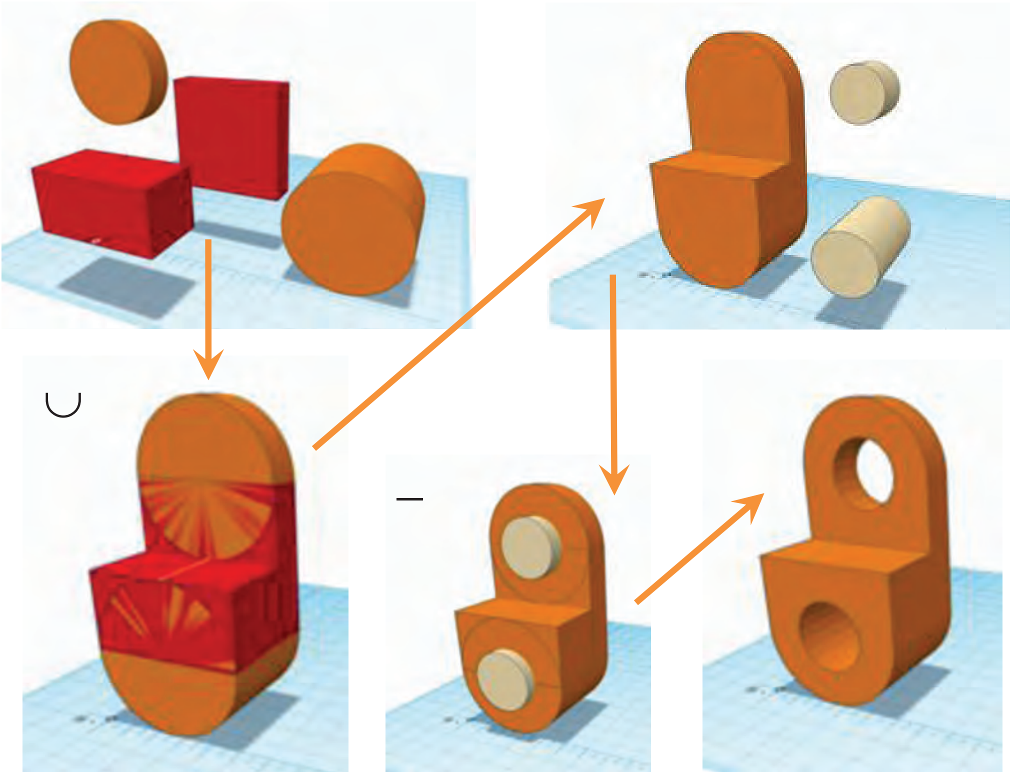


Figure 3: Generating a geometric model of a mechanical component with Constructive Solid Geometry.

proposing and defending the educational project aimed at enhancing the understanding of geometry, our school received a 3D printer at no cost. Together with the students, we designed and manufactured a construction set of polygons using the 3D printer as the educational project [18].

## 4.1 Polygons, Congruence Transformations, and Tessellations

A *polygon* is a plane figure defined by straight line segments acting as its boundaries. Typically, additional conditions are added to get only a simple polygon, which does not self-intersect. This geometric topic is standard in the curricula of elementary and secondary schools in the Czech Republic. The subject of polygons includes the exploration of their various types and properties. Other topics of planar geometry which are taught in the Czech Republic include geometric transformations such as translations, rotations, and axial or central symmetries, which are congruent, meaning they preserve shapes and distances.

A *tessellation* (or tiling), refers to the coverage of a plane using one or multiple types of tiles, which can be either regular or irregular polygons, leaving no overlaps or gaps. From a geometric perspective, tessellation is an interesting subject due to its reliance on the geometric properties of the polygons used in its construction, as well as the application of congruence geometric transformations. Tessellations can take various forms, for example, there are regular, semi-regular, or aperiodic tilings. There exist only three *regular tessellations*, consisting of equilateral triangles, squares, and regular hexagons. Moreover, an infinite number of tessellations can be generated from these three regular forms using congruence transformations. Figure 4 showcases several rules applied to a hexagonal tiling.

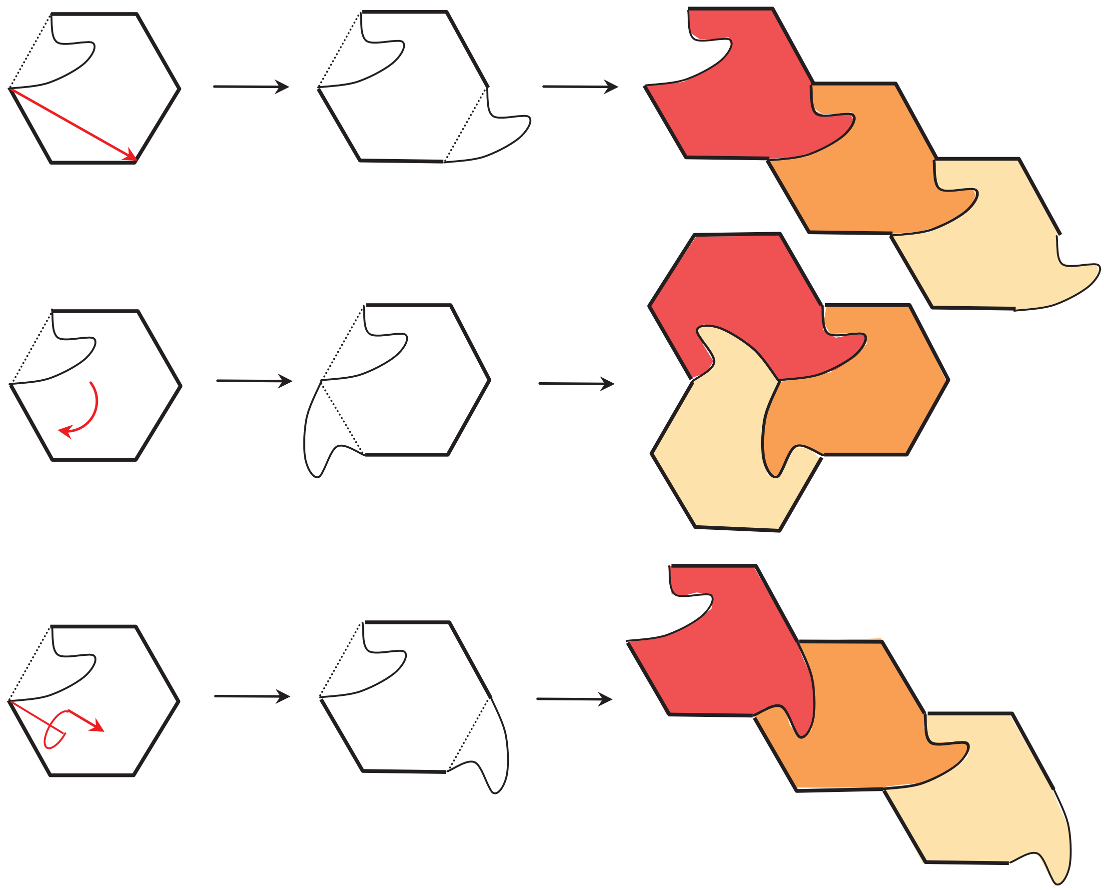

Figure 4: Three types of tessellations created from hexagonal tiling, based on geometric principles of translations, rotations, and glide reflections.

A tessellation created from hexagonal tiling and inspired by Escher's work [10] can be seen in Figure 5.

*Semiregular tessellations* (or Archimedean tessellations) refer to tessellations of the plane composed of two or more types of convex regular polygons. In these tessellations, each polygon vertex is surrounded by the same polygons in the same order. There are eight such tessellations in the plane.

Next to the various tessellations created by regular polygons, tilings by other polygons have also been studied. Any triangle or quadrilateral (even non-convex) can be used as a tile to form a tessellation using only one type of a tile. Such a tessellation is called *monohedral tessellation*. Figure 6 shows a monohedral tessellation made up of arbitrary triangles and a monohedral tessellation made up of arbitrary convex quadrilaterals.

Fifteen types of convex pentagons are known to form a monohedral tessellation of the plane.

Tessellations can also be formed by convex regular polygons that are not edge-to-edge. Such tilings can be considered edge-to-edge as nonregular polygons with adjacent collinear edges.

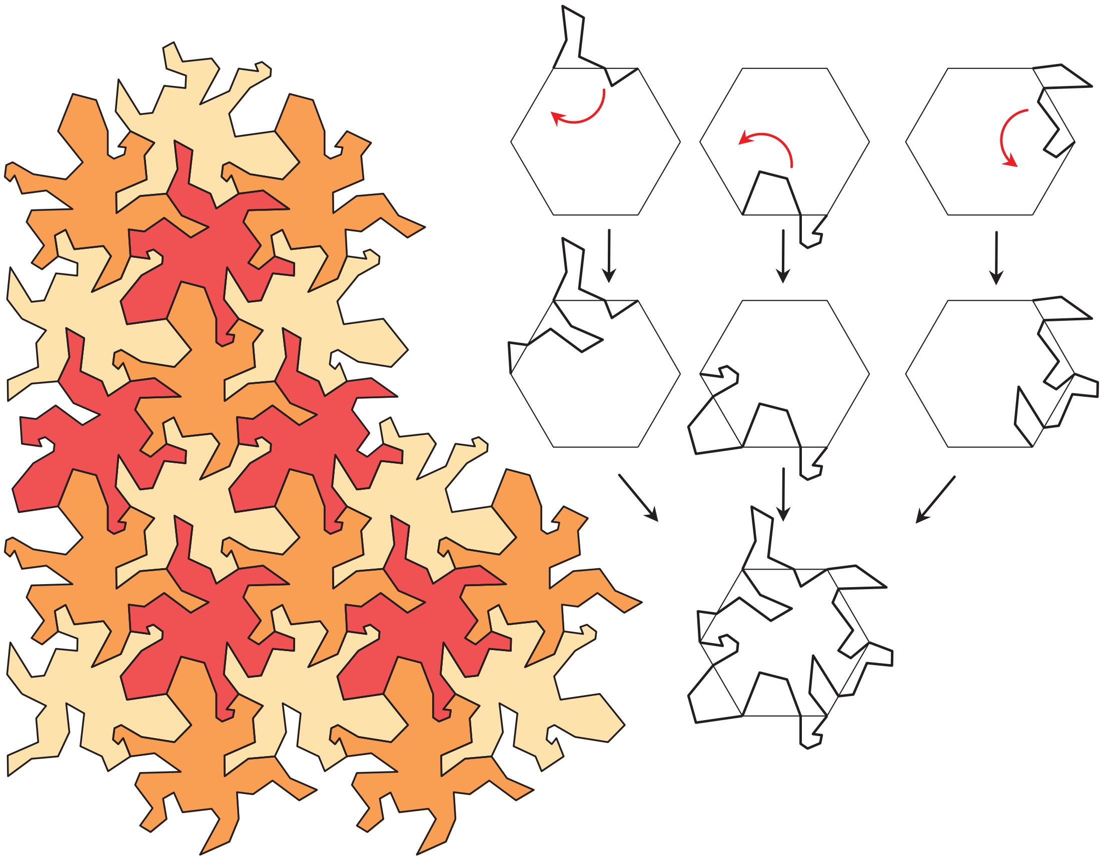

Figure 5: A tessellation inspired by M. C. Escher.

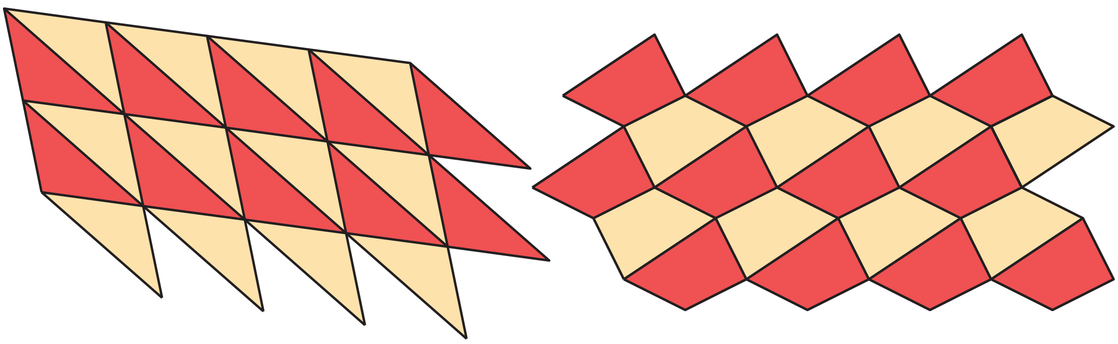

Figure 6: Monohedral tessellations: on the left, made up of arbitrary triangles, and on the right, made up of arbitrary convex quadrilaterals.

A tessellation is not typically included in the curriculum of elementary and secondary schools in the Czech Republic but it can serve as a subject for evaluating students' conceptual understanding of polygons, their properties, and congruence geometric transformations.

Further information on planar geometry, specifically regarding polygons, congruence transformations, and tilings, can be found in relevant literature [13].

## 4.2 A Construction Set of Polygons

The set printed on a 3D printer consists of different shapes of regular and irregular polygons and it is designed for use in mathematics teaching to study polygon properties and create diverse types of tessellations. As previously mentioned, for 3D printing we need to work with printable objects. Thus, we added some height to the polygons to create solid figures representing these polygons. We explained this modification to the students, who quickly understood and accepted it.

## 4.3 An Educational Experiment with 3D Printed Shapes

After creating a construction set of polygons, we also investigated the effects of using these instructional aids in students' understanding of geometry with several groups of students at secondary schools in the Czech Republic [21].

Our aim was to examine students' conceptual understanding of polygons through collaborative activities and how physical 3D printed manipulatives can help in solving geometry planar problems.

The experiment was repeated four times with four groups of students of different ages. In total 25 elementary school students and 34 secondary school students took part in the experiment. Elementary school students from one class (age 12-13 years) were one group (within the experiment they were divided into subgroups of 8, 8, and 9 students), three groups were secondary school students always approximately one half from one class – 14 students (age 15 years), 9 students (age 16 years), and 11 students (age 17 years). Since mathematics instructions are taught one a week divided into two. The experiments always took two mathematics instructions, i.e. 90 min, and were conducted in the school year 2021/2022 in the Czech Republic.

The experiment was conducted four times with students of varied ages: 25 from elementary school (ages 12-13) and 34 from secondary school (ages 15-17). Each session spanned 90-minute math lessons, held weekly, during the 2021/2022 school year in the Czech Republic.

Students were given tasks on polygon types, properties, and tiling applications. All students had prior knowledge on polygons from their previous studies. The study aimed to compare their success in solving geometric tasks with and without concrete manipulatives and to observe their collaborative activities. The experiment had four phases: (1) inquiring about polygon types (regular and irregular) and their properties, (2) identifying tessellations without any visual aids, (3) using drawing images, and finally (4) using concrete manipulatives from 3D printer. First and third tasks were solved individually of each student on a sheet of paper. Second and fourth tasks were discussions among students with the guidance by the author.

In the first phase, results indicated students had a strong factual knowledge of polygons and their characteristics. Almost all of them could identify and describe basic polygonal properties (number of sides, sizes of interior angles, regular and irregular, convex and concave, area of polygons ...), with older students outperforming the rest.

In the second phase, students immediately knew that tessellations can be made from equilateral triangles, squares, and rectangles. But they had uncertainty regarding other regular polygons, with some incorrectly suggesting tessellations from regular pentagons. A few students recalled seeing natural tessellations from regular hexagons. Irregular polygon tessellation concepts, however, remained elusive.

In the third phase, most students sketched basic tessellations using equilateral triangles, squares, and regular hexagons. Few identified the rule concerning the sum of interior angles of polygons in vertices where polygons meet.

The fourth phase was particularly illuminating: despite initial reluctance from secondary school students toward using concrete manipulatives —deeming them 'childish'—they quickly engaged and found value in them. Students discovered even semi-regular tessellations previously unidentified. Significantly, without teacher guidance, they began formulating general rules and mathematical proofs during their tessellation explorations. Notably, the older students formulated a statement about the sum of interior angles in irregular quadrilaterals (based on the division of quadrilaterals into triangles) and drew a geometric proof for tessellations with any quadrilateral.

The experiment showed that students retain facts but lack a deeper understanding of polygons. Using 3D printed manipulatives effectively concretizes abstract mathematical concepts and helps in comprehension. This is in compliance with literature, for example [9].

Students' activities and their collaboration when they were trying to find various types of tessellations and were using the construction sets of polygons can be seen in Figure 7.

Figure 7: Students' activities with the construction sets of polygons at Grammar school.

## 5 Gallery of 3D Printed Polygons for Tessellations

Several types of tessellations made up of various shapes of regular and irregular polygons are presented in the following pictures. See Figures 8, 9, 10, 11, 12.

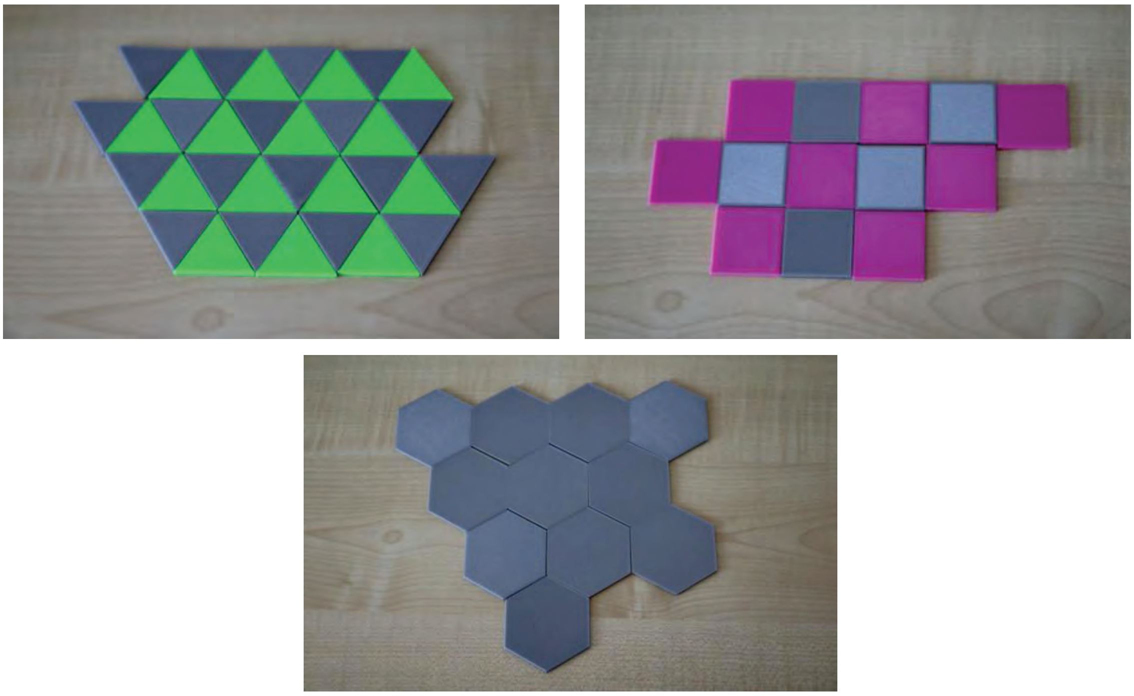

Figure 8: Three regular tessellations, consisting of equilateral triangles, squares, and regular hexagons (colors are used solely for clarity).

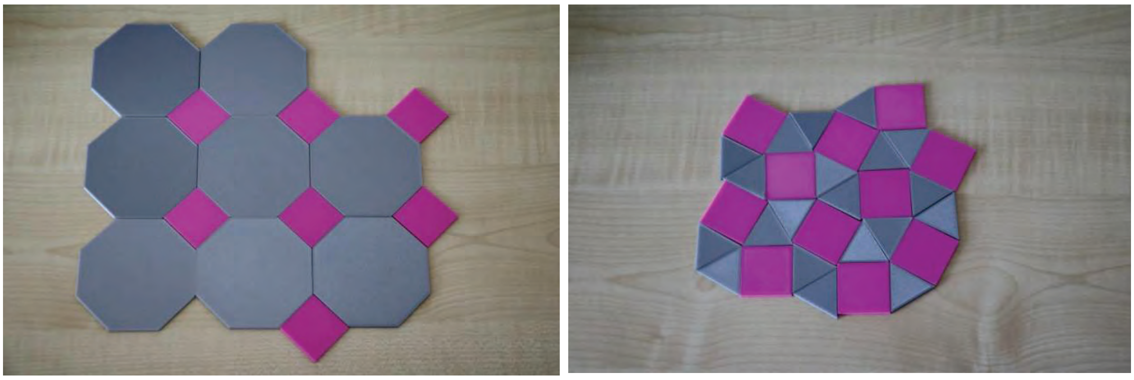

Figure 9: Two examples of semiregular tessellations: on the left, made up of regular hexagons and squares, on the right, made up of equilateral triangles and squares (colors are used to distinguish between different shapes).

# 6 Conclusion and Future Work

In this article, we have demonstrated various methods for creating a virtual geometric model suitable for 3D printing; we aimed especially at the simplest Tinkercad software. We have presented a new construction set of polygons produced through 3D printing. Furthermore, we have showcased examples of how these geometric models can be applied in educational contexts and have provided a collection of printed models for tessellations.

Regarding the future work, our aim is to expand our database of printed geometric models.

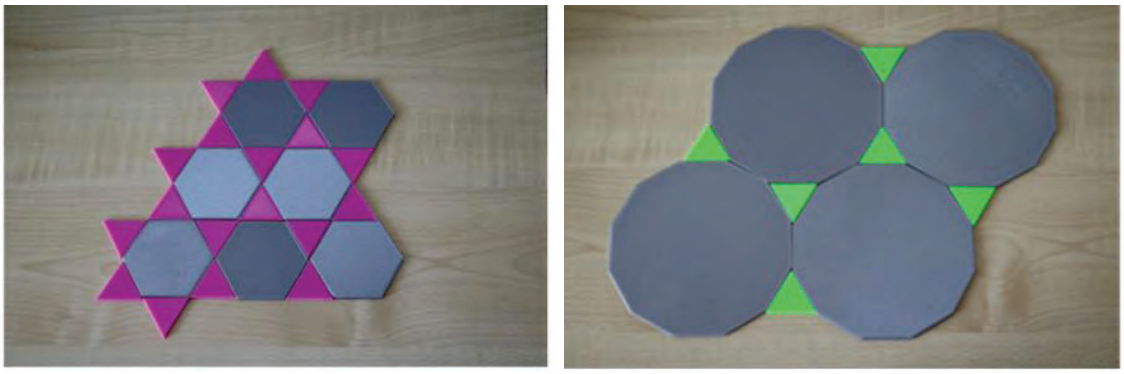

Figure 10: Two examples of semiregular tessellations: on the left, made up of regular hexagons and equilateral triangles, and on the right, made up of equilateral triangles and regular dodecagons (colors are used to distinguish between different shapes).

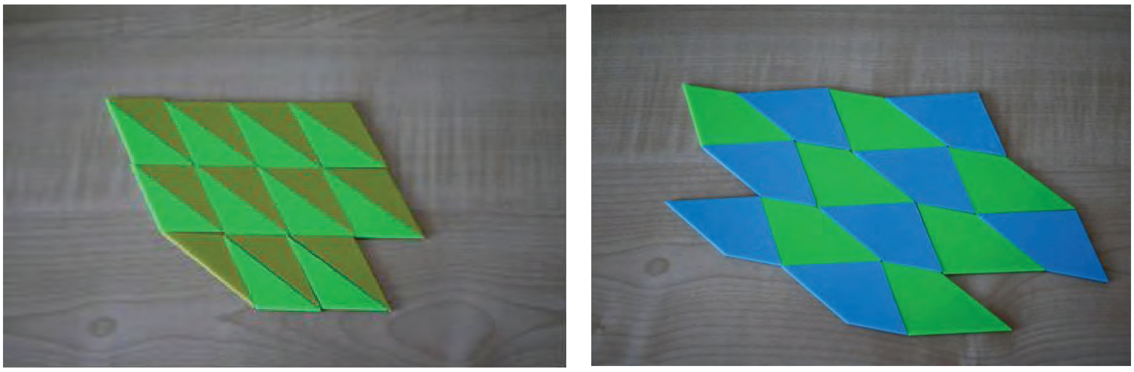

Figure 11: Monohedral tessellations: on the left, made up of arbitrary triangles, and on the right, made up of arbitrary convex quadrilaterals (colors are used solely for clarity).

Figure 12: A tessellation inspired by M. C. Escher.

We also intend to carry out research studies to investigate the impact of 3D printing technology and the utilization of printed models on students' geometric comprehension.

We plan to expand the use of 3D printing in education by creating models for other mathematical topics, such as solid volumes and deriving formulas for their calculations. We aim to explore the effectiveness of utilizing these 3D printed models in enhancing comprehension and engagement in these areas.

# 7 Acknowledgments

The paper was supported by the 'Cooperatio' Charles University Research program.

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# Exploring Creativity and Innovation in STEM Education: With and Without Technology

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**Abstract**

*In this work, we present a collection of geometric problems described in wooden tablets known as Sangaku. These original problems developed in Japan during the Edo period. Quadrilaterals, a subject of immense geometric richness, have been widely researched in the literature. They offer opportunities to explore constructions with ruler and compass, various representation techniques, proofs and theorems, with and without technology.*

*The main objective of this study is to demonstrate how Sangaku geometric problems can serve as valuable pedagogical resources, integrating different disciplines and enriching students' learning experiences. In addition to deepening the mathematical aspects of these problems, special attention was also given to the artistic elements present in the wooden tablets, stimulating students to the creative expression inherent to Geometry.*

*Adopting an interdisciplinary approach, the activity seeks to stimulate students' creativity, encouraging them to create visual representations of problems and explore various artistic materials, while developing their visual communication and mathematical skills.*

*This approach placed a strong emphasis on promoting transversal skills and attitudes, promoting the development of interdisciplinary skills. Consequently, it contributed to a broader and more integrated education in several fundamental disciplines.*

## 1 Introduction

Geometry serves as a means to describe, understand, and interact with the space in which we live, making it the most intuitive, concrete, and reality-related branch of mathematics. However, teaching and learning geometry can be challenging due to the cognitive complexity involved in geometric activities.

Geometry provides an easy connection between the abstract and the concrete, bridging the formal description of a given geometric construction with its tangible representation [9]. Euclid's Elements of Geometry written around 300 BC, stands as a prime example of mathematical deduction. Over the course of more than two millennia, mathematical proofs have become increasingly rigorous, and the language of mathematics has evolved to be more precise and symbolic. In the 17th century, Gottfried Wilhelm Leibniz (1646–1716) proposed the development of a universal symbolic language, known as *characteristica universalis* capable of expressing all statements, not limited to mathematics alone. He also suggested a universal logical calculus, the (*calculus ratiocinator*) to decide the truth of statements expressed in *characteristica universalis*. Leibniz's work on differential calculus, as presented in *Nova methodus pro maximis et minimis, itemque tangentibus, qua nec irrationales quantitates moratur* ("A new method for maxima and minima, as well as tangents, which is not hindered by irrational quantitie") in 1684, further contributed to the advancement of mathematical concepts [3].

Meanwhile, in Japan, the Edo period (1603-1867) was marked by national isolation, leading to the interruption of negotiations with the West, including Western mathematics. Despite this isolation, Japanese mathematics, known as Wasan, experienced significant development and a unique approach to presenting mathematical problems emerged, leading to a flourishing of Euclidean geometry in the country. Wooden painted tablets, called "Sangaku" were displayed in Shinto shrines and Buddhist temples for recreational fun and religious offerings. Although many of these tablets were lost during the modernization period that followed the Edo era, around 900 have been preserved.

The Sangaku tablets frequently feature simple figures, where the creativity of the forms plays a decisive role in selecting the problems. The majority of these problems were solved using analytic geometry and algebraic methods, collectively forming what is known as the "Geometry of Japanese Temples". This collection includes simple or regular polygons and polyhedra, circles, ellipses, spheres, and ellipsoids. Various conic sections, including the paraboloid, appear as well. The cylinder primarily intervenes in creating the ellipse through intersection with a plane, and affine transformations are used to transition from the circle to the ellipse. The collection contains several optimization problems with provided answers. However, the methods used to obtain these answers remain absent. Since the works of Newton and Leibniz were unknown to Japanese mathematicians at the time, and there is no evidence of contemporary Japanese mathematicians having a formal definition of the derivative, their resolution techniques for these problems remain unsolved [4].

## 1.1 The Interdisciplinary

The aesthetics of the forms on the Sangaku tablets play a crucial role in selecting the problems to be solved, demonstrating a direct connection between geometry and the visual arts. Sangaku tablets frequently present optimization problems, challenging students to find efficient solutions for practical real-world situations. This can open doors to interdisciplinarity with physics, engineering or other sciences, where optimization is a valuable tool for solving complex problems. In addition, some tablets may explore properties of the materials used in temple construction, relating to materials engineering and architecture. The use of affine transformations to create geometric figures is also related to mathematics and physics.

The historical aspect of Sangaku tablets can also promote a connection with Japanese history and culture, enabling interdisciplinary approaches in the areas of history, cultural studies and anthropology. In this way, the Sangaku tablets illustrate how geometry can act as a solid foundation to embrace interdisciplinarity, connecting to other areas of knowledge and enriching learning by providing a holistic and broad view of the relationships between different disciplines. By exploring and solving problems on the Sangaku tablets, students are encouraged to integrate knowledge from different areas, seeing mathematics as a powerful tool for understanding and solving problems in the world around them. This interdisciplinary approach can arouse students' curiosity and enrich their understanding of knowledge in diverse areas, preparing them to be more informed citizens capable of facing the complex challenges of contemporary society.

STEM (Science, Technology, Engineering, and Mathematics) education, it is an interdisciplinary approach to learning that integrates these four fields to promote critical thinking, problem-solving, and creativity among students. STEM education aims to provide students with real-world applications and hands-on experiences in these subjects, preparing them for future careers in various STEM-related fields. Integrating various areas of knowledge, interdisciplinary approaches inspire students to delve into novel concepts, seek inventive resolutions, and foster creative thinking. The rich significance of Sangaku, embracing historical, religious, mathematical, and artistic dimensions, renders it a compelling model for integrating STEM education among students, bridging diverse domains effectively. In this study aims to show how Sangaku geometric problems enrich students' learning through interdisciplinary integration and artistic exploration. It emphasizes both mathematical depth and artistic elements, fostering creative expression. Through an interdisciplinary approach, the activity boosts students' creativity, fostering visual problem-solving and artistic skill development.

## 2 Geometric Problems

Several geometric problems involving quadrilaterals and their respective solutions will be presented. In these problems, some are easy to solve, others are more difficult to solve, for basic/secondary education. They involve concepts of both Euclidean geometry, differential calculus or even complex numbers.

**Problem 1** Problem taken from a Sangaku displayed at Tagami Kannondo Shrine, Shin–Nagano, 1809 (Figure 1). In this problem, a rhombus $[ABCD]$ is given on side $a$ (Figure 2). It is known that $\overline{BD} = 2t$, depending on $t$. Let $S(t)$ be the area of the rhombus minus the area of the square of the diagonal $\overline{BD}$ and side $x$. Determine the value of $x$ as a function of $a$, when $S(t)$ is maximum [4].

**Figure 1** Tagami Kannondo Shrine, n.º 2 (Nakano, 1809)[1]

**Resolution:**

The problem to be solved is shown below:

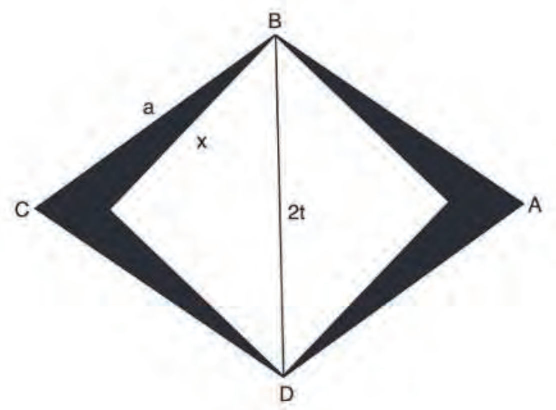


**Figura 2** A Square inscribed in a Rhombus

By the Pythagorean theorem we have

$$x^2 = t^2 + t^2 \Leftrightarrow x = \sqrt{2}t, x > 0 \tag{1}$$

The area of the rhombus $[ABCD]$, at Figure 2, is given by $A_{[ABCD]} = \dfrac{2t\sqrt{a^2 - t^2}}{2}$ and the area of the square is given by $A = x^2$.
It is $S(t)$ the difference between the area of the rhombus and the square, that is, $S(t) = \dfrac{2t\sqrt{a^2 - t^2}}{2} - x^2$, with $0 \leq t \leq a$, by (1) it comes,

$$S(t) = t\sqrt{a^2 - t^2} - 2t^2 \tag{2}$$

To determine the maximum value, derive $S(t)$ in order to $t$, i.e.,

$$S'(t) = \sqrt{a^2 - t^2} - \frac{t^2}{\sqrt{a^2 - t^2}} - 4t,$$

equaling zero, $S'(t) = 0$, results,
$\sqrt{a^2 - t^2} - \dfrac{t^2}{\sqrt{a^2 - t^2}} - 4t = 0 \Leftrightarrow \dfrac{a^4}{16t^2} - \dfrac{a^2}{4} + \dfrac{t^2}{4} = a^2 - t^2$, with $t \neq 0$, we have

$$20t^4 - 20a^2t^2 + a^4 = 0,$$

doing

$$t^2 = x \tag{3}$$

we have, $x = \dfrac{20a^2 \pm 8a^2\sqrt{5}}{40} \Leftrightarrow x = \dfrac{a^2}{2} \pm \dfrac{a^2\sqrt{5}}{5}$, replacing $x$ por (3), it comes,

[1]http://www.wasan.jp/nagano/tanoue2.html

$$t^2 = \frac{a^2}{2} \pm \frac{a^2\sqrt{5}}{5},$$

for $S'(t)$ be maximum, $t > 0$, $t = \sqrt{\frac{a^2}{2} - \frac{a^2\sqrt{5}}{5}}$. i.e., $t = a\sqrt{\frac{1}{2} - \frac{\sqrt{5}}{5}}$

By (1), it follows that, $\frac{x}{\sqrt{2}} = a\sqrt{\frac{1}{2} - \frac{\sqrt{5}}{5}}$. Therefore,

$$x = a\sqrt{1 - \frac{2\sqrt{5}}{5}}.$$

**Problem 2** A version of problem 1 is given as follows:

Let $[BCAD]$, be a square as shown in the Figure 3.1. Consider the diagonals $[AB]$, $[CD]$ and $M$ your point of intersection. On the line segment $[CD]$, let's take the points $P$, $R$ symmetrical with respect to the point $M$. We obtain a rhombus $[BPAR]$. Consider also the square $[PQRS]$. For what value or values of lengths $\overline{PQ}$ (sides of squares $[PQRS]$ ) do the values of the areas marked in red reach their maximum?

From the figure on the left, already described in the statement, the diagonal $[AB]$, $[CD]$ are symmetry axes and, therefore, the proposed problem is solved by determining what is the length value of $\overline{PQ}$ for which triangle $[PAQ]$ has maximum area.

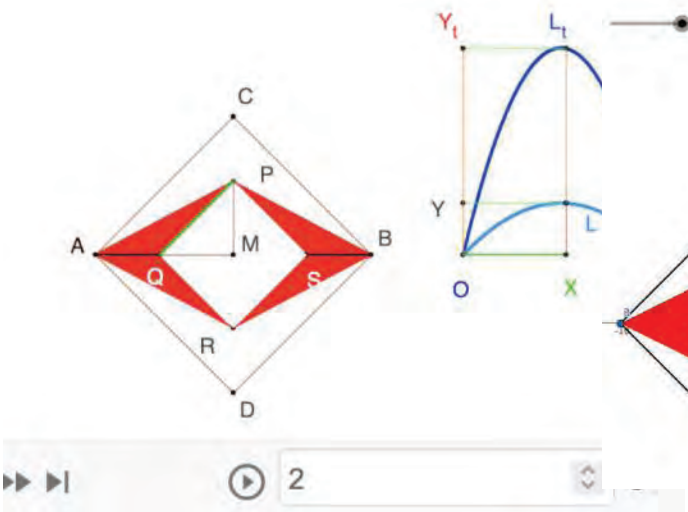


**Figure 3.1** Rhombus[2]

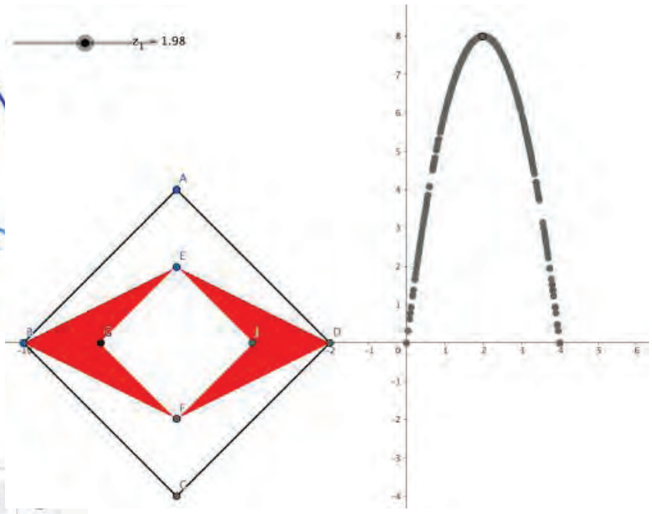

**Figure 3.2** Construction in GeoGebra of the Rhombus (author)

**Resolution**[3]**:**

Using Figure 3.1, what we are going to do is study the dependence of the values $y = \overline{OY}$ of the areas of triangle $[APQ]$ as a function of the values of the lengths of the sides $x = \overline{OX} = \overline{PQ}$ of square $[PQRS]$. The diagonals of the squares are equal $[AB] = [CD]$, $[PR] = [QS]$, bisect each other $[MQ] = [PM]$ perpendicularly $C\hat{M}A = P\hat{M}Q = 90^o$ , which is why

[2] *https://www.geogebra.org/material/iframe/id/nyfTHgGr/

[3] https://tinyurl.com/areaproblem

$\overline{PQ}^2 = \overline{PM}^2 + \overline{MQ}^2 \Leftrightarrow \overline{PQ}^2 = 2\overline{PM}^2 \underset{x=\overline{PQ}}{\Leftrightarrow} x = \sqrt{2}\ \overline{PM} \Leftrightarrow \overline{PM}^2 = \dfrac{x^2}{2}$ and, designating by $2a$ the fixed length of $[AB]$, and by $2d$ the value of the variable lengths of the diagonals of $[PQRS]$, over the area $y$ of the triangle $[PAQ]$ which is equal to the triangle $[PAM]$ subtracted from the triangle $[MPQ]$, we can write

$$y = \frac{a \times d}{2} - \frac{d^2}{2} = \frac{\sqrt{2}ax}{2} - \frac{\frac{x^2}{2}}{2} = \frac{2\sqrt{2}ax - x^2}{4}$$

When $P$ takes the position of $M$, $P \equiv M \equiv Q\ldots$ then $x = 0$. The largest value that $x = \overline{PQ}$ can reach is when $P = C$ and $Q = A$ then $\overline{PQ} = \overline{AC} = \sqrt{2}a$. For our problem, $x$ can take all values between $0$ and $\sqrt{2}a$ :

$$0 \leq x = \overline{OX} \leq \overline{AC} = \sqrt{2}a$$

and therefore as,

$$y = \frac{2\sqrt{2}ax - x^2}{4} = \frac{-(x^2 - 2\sqrt{2}ax + 2a^2) + 2a^2}{4} = \frac{1}{4}(2a^2 - (x - \sqrt{2}a)^2)$$

polynomial function of the second degree where $x^2$ has a negative coefficient $-\dfrac{1}{4}$

$$y = \frac{1}{4}(2a^2 - (x - \sqrt{2}a)^2) = 0 \Leftrightarrow\ x = 0 \vee x = \sqrt{2}a$$

$y$ reaches its maximum value for the average $x$ value of $[0, \sqrt{2}a]$ which is $\dfrac{\sqrt{2}a}{2}$ .

In GeoGebra, we can click on the animation button, in Figure 3.1, and visualize the traces of the abscissa points $x$ between $0$ and $\sqrt{2}a$ .

$L$ which has as ordinate $y = \overline{OY}$ the value associated with the area of the triangle $[PAQ]$ corresponding to each value of $x\ldots$ $L_t$ which has as ordinate $y_t = \overline{OY}_t$ the value associated with the area of the entire red surface $y_t = 4y = 4[PAQ]$ corresponding to each $x$ length of the side of the square $[PQRS]$.

**Problem 3** Given two rhombuses $[ABCD]$ and $[AECF]$ (Figure 4). Let $P$ be the area $A_{[ABCD]} - A_{[AECF]}$, $\overline{AD} = q$ and $\overline{AF} = r$. Determine the diagonal $\overline{AC}$ in terms of $P$, $q$, and $r$ [6].

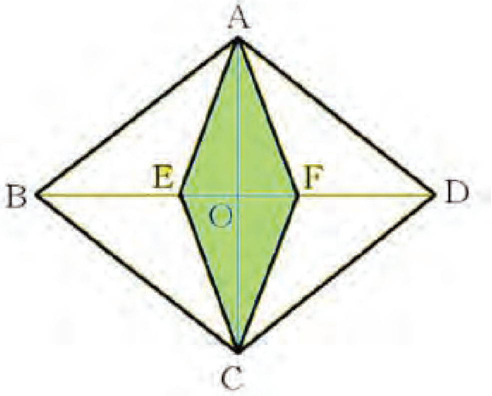


**Figure 4.** Rhombus

**Resolution:**
Let $\overline{OD} = a$, $\overline{OF} = b$ and $\overline{AC} = x$.
Knowing that $A_{[AFD]} = \dfrac{(a-b)\frac{x}{2}}{2}$ and that $P = A_{[ABCD]} - A_{[AECF]}$ , it follows that, $\dfrac{P}{4} = \dfrac{(a-b)\frac{x}{2}}{2}$, then

$$b = a - \frac{P}{x} \tag{4}$$

By the Pythagorean Theorem we know that,

$$q^2 = \frac{x^2}{4} + a^2 \tag{5}$$

and

$$r^2 = \frac{x^2}{4} + b^2 \tag{6}$$

we have,

$$a = \sqrt{q^2 - \frac{x^2}{4}}, \text{with } a > 0 \tag{7}$$

$$b = \sqrt{r^2 - \frac{x^2}{4}}, \text{with } b > 0 \tag{8}$$

By (1) and (4),

$$b = \sqrt{q^2 - \frac{x^2}{4}} - \frac{P}{x} \tag{9}$$

By (3) and (6), it follows that, $r^2 = \dfrac{x^2}{4} + \left(\sqrt{q^2 - \dfrac{x^2}{4}} - \dfrac{P}{x}\right)^2$

Simplifying a few steps, we see that:
$x^4[P^2 + (q^2 - r^2)] - 2P^2x^2(q^2 + r^2) + P^4 = 0$, solving for $x$, with $x > 0$, then,

$$x = P\sqrt{\frac{q^2 + r^2 \pm \sqrt{4r^2q^2 - P^4 - q^4}}{P^2 + (q^2 - r^2)^2}}.$$

Therefore,

$$\overline{AC} = P\sqrt{\frac{q^2 + r^2 \pm \sqrt{4r^2q^2 - P^4 - q^4}}{P^2 + (q^2 - r^2)^2}}.$$

**Problem 4** It was written on a tablet hanging in a shrine which reads as follows:

Let four points be $A$, $B$, $C$ and $D$ (Figure 5) on the same circle (the points are said to be concyclic). If $H$, $I$, $J$ and $K$ are respectively the centers of the circles inscribed in the triangles $[ABC]$, $[BCD]$, $[CDA]$ and $[DAB]$, show that the quadrilateral $[HIJK]$ is a rectangle.

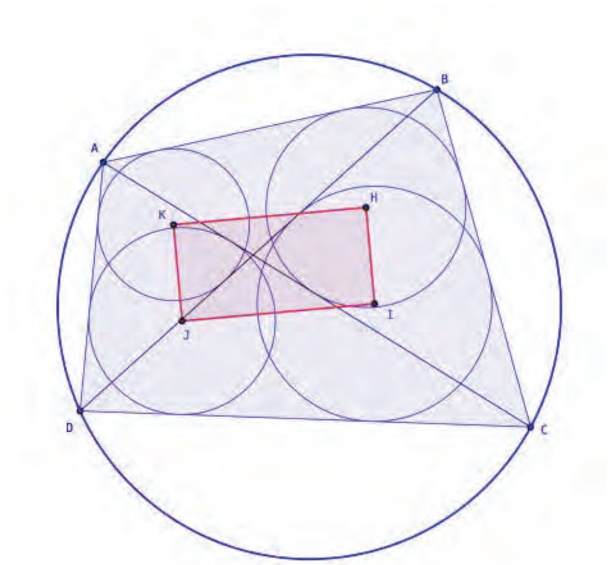


**Figure 5.** Cyclic quadrilateral - Construction in GeoGebra[4]

**Resolution:**
To solve this problem we will need the following definition:

[4] `https://www.geogebra.org/m/suuwu4wj`

**Definition 1:** A quadrilateral inscribed in a circle, that is, its four vertices are on the same circle, is called a cyclic quadrilateral.

Let the triangle be $[BAD]$.

$$B\hat{K}A = 90^{\text{o}} + \frac{B\hat{D}C}{2} \tag{10}$$

Likewise the triangle $[CBA]$

$$B\hat{K}A = 90^{\text{o}} + \frac{B\hat{C}A}{2} \tag{11}$$

However, as the quadrilateral is cyclic (Definition 1) the $B\hat{D}A = B\hat{C}A$. Thus, $B\hat{K}C = B\hat{H}C$.
The quadrilateral $[BHKA]$ is cyclic, like this

$$B\hat{A}K + B\hat{H}K = 180^{\text{o}} \tag{12}$$

Similarly, the quadrilateral $[CIHB]$ is cyclic, so

$$B\hat{C}I + B\hat{H}I = 180^{\text{o}} \tag{13}$$

(3) and (4) implies that

$$B\hat{H}K + B\hat{H}I = 360^{\text{o}} - \frac{B\hat{A}D}{2} - \frac{B\hat{C}D}{2}.$$

Therefore, $I\hat{H}K = 90^{\text{o}}$. The other angles are treated similarly.

**Problem 5** Another version of the previous problem is the following:

Let be the cyclic quadrilateral $[ABCD]$ (Figure 6.1). Let $r_A, r_B, r_C$ and $r_D$ be the radii of the circumferences of the circles inscribed in the triangles $[DAB]$, $[ABC]$, $[BCD]$ and $[CDA]$, respectively. Show that $r_A + r_C = r_B + r_D$ [1] .

For the resolution the following theorem and lemmas [4] will be needed:

**Ptolemy's Theorem:** In a cyclic quadrilateral, the sum of the products of the lengths of the opposite sides is equal to the product of the lengths of the diagonals.

**Lemma 1** The area of a triangle $[ABC]$ can be expressed as $S_{ABC} = rs = \dfrac{abc}{4R}$, where $a$,$b$ and $c$ are the sides of the triangle; $s = \dfrac{a+b+c}{2}$ is the semiperimeter of the triangle $[ABC]$; and $r$

[5] `https://www.wasan.jp/kanagawa/matubara1.html`

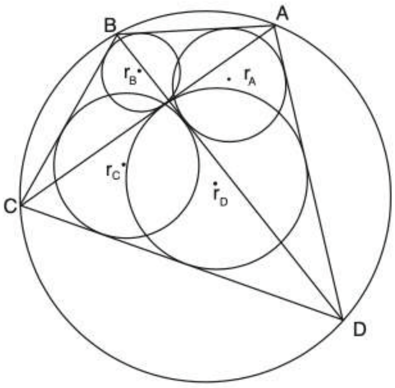


**Figure 6.1** Cyclic quadrilateral

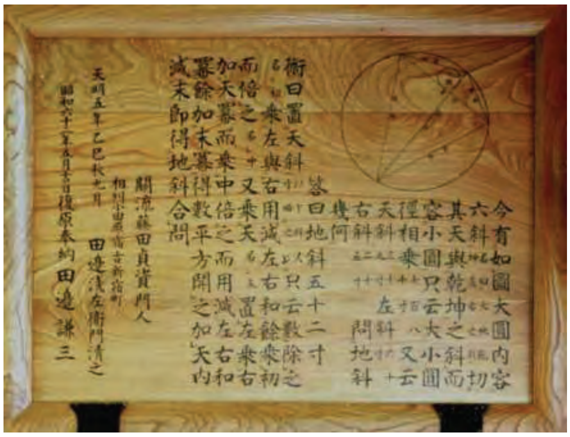

**Figure 6.2** Cyclic quadrilateral - Matsubara Shrine (Odawara) [5]

and $R$ are the radii of the inscribed and circumscribed circles of the triangle (see Figure 6.1). This lemma follows directly from two elementary theorems: 1) that the area of a triangle $[ABC]$ can be written as $S_{ABC} = rs$, where $r$ is the radius of the triangle's inner circle and is its semiperimeter; 2) the radius $R$ of the circumcircle is $R = abc/4S_{ABC}$.

**Lemma 2** $\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} = \overline{AC} \cdot \overline{BD}$ . This is known as Ptolemy's theorem. Note: From this point on we will use the following designations: $\overline{AB} = a$, $\overline{BC} = b$, $\overline{CD} = c$, $\overline{AD} = d$ , $\overline{AC} = e$, $\overline{BD} = f$. So Ptolemy's theorem for this problem becomes $ac + bd = ef$.

**Lemma 3** Let the designation be $abe + cde = bcf + adf$. From Figure 6.1 we see that if $S$ is the area of the quadrilateral, then $S = \frac{1}{2} \cdot a \cdot d \cdot sinA + \frac{1}{2} \cdot b \cdot c \cdot sinC$. But $\hat{A}$ and $\hat{C}$ are supplementary and so $S = \frac{1}{2}(ad+bc) \cdot sinA$. Likewise, $S = \frac{1}{2} \cdot a \cdot b \cdot sinB + \frac{1}{2} \cdot c \cdot d \cdot sinD = \frac{1}{2}(ab + cd) \cdot sinB$. On the other hand, the triangle $[ABD]$ is inscribed in the circle $R$, so by lemma 1, $S_{ABD} = adf/4R$. But also $S_{ABD} = \frac{1}{2} \cdot a \cdot d \cdot sinA$, giving $sinA = f/(2R)$. Likewise, $sinB = e/(2R)$. Eliminating $sinA$ and $sinB$ from the previous two expressions immediately results in $f(ad + bc) = e(ab + cd)$.

Remembering also the following theorem:

**Japanese Theorem:** Triangulate a cyclic polygon by lines drawn from any vertex. The sum of the radii of the triangles' inner circles is independent of the chosen vertex [2].

By lemma 1, $S_{ABD} = rs = \frac{adf}{4R} = r_A \frac{a+d+f}{2}$, with similar expressions for $S_{BCD}$, $S_{ACD}$ and $S_{ABC}$. Like this,

$$r_A = \frac{adf}{2R(a+d+f)}; \quad r_B = \frac{abe}{2R(a+b+e)}; \quad r_C = \frac{bcf}{2R(b+c+f)}; \quad r_D = \frac{cde}{2R(c+d+e)} \tag{14}$$

We have,

$$2R(r_A + r_C) = \frac{bcf(a+d+f) + adf(b+c+f)}{(b+c+f)(a+d+f)} \tag{15}$$

The numerator of the second member is $(a+d)bcf + (b+c)adf + f^2(bc+ad) = f[ab(c+d+e) + cd(a+b+e)]$, where,

$$f^2(bc+ad) = ef(ab+cd) \text{ by lemma 3.}$$

The denominator is $f^2 + f(a+b+c+d) + (b+c)(a+d)$, but by lemmas 2 and 3,

$$(b+c)(a+d) = (ac+bd) + (ab+cd) = ef + (f/e)(ad+bc),$$

and so,
the denominator is $\frac{f}{e}[ef + e(a+b+c+d) + e^2 + bc + ad] = \frac{f}{e}[(a+b+e)(c+d+e)]$, where lemma 2 was used again to obtain the second equality. Equation (12) then becomes

$$2R(r_A + r_C) = \frac{e[ab(c+d+e) + cd(a+b+e)]}{(a+b+c)(c+d+d)} = \frac{abe}{a+b+e} + \frac{cde}{c+d+e}$$

or by equation (11) the following result is obtained,

$$r_A + r_C = r_B + r_D$$

# 3 Sangaku as Activity

The proposed activity involves concepts of Euclidean geometry, were given to students from Basic Education, focusing on quadrilaterals, more specifically the cyclic quadrilaterals. In the class there is a student who is blind[6] and the challenge of Ptolemy's Theorem were given, while the other colleagues worked on another challenge, the Japanese Theorem.

Students constructed a cyclic quadrilateral with ruler and compass. We can verify some of these constructions obtained in Figure 7.

[6] The student with severe low vision has visual acuity less than 1/10 (scale of Wecker) and needs Braille to read

**Figure 7** Students' construction

After construction, students would have to answer two questions. The questions were as follows:

- Joins the points $K$, $H$, $I$ and $J$. What conclusion did you come to?
- Measures the length of the radii of the smallest circles. Does it check for the following: $r_H + r_J = r_I + r_K$?

Students would have to come to the conclusion that they were facing a rectangle and that equality was verified. We can verify by Figure 8 that the students reached the intended conclusions.

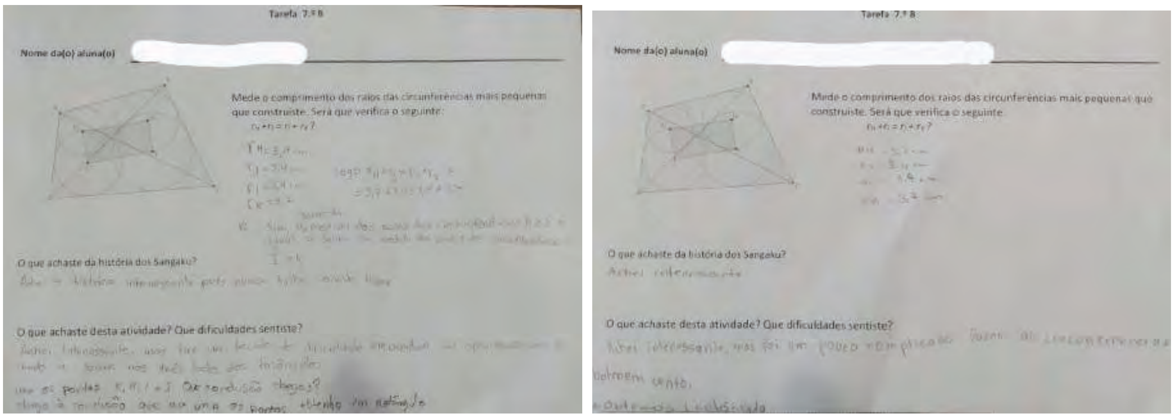


**Figure 8** Students' responses

The blind student performed an alternative activity using Ptolemy's Theorem because the compass she had available did not allow for "erasing" the auxiliary lines, and it could be very confusing to read on paper. She approached the task differently, using thick paper and a metallic compass adapted with accessories (toothed wheels) and a graduated ruler that was also adapted for her needs (Figure 9).

**Figure 9** Adapted material

With the adapted material, the student drew a circle and marked four points on it with a metallic pen. Then, with a pen with a toothed wheel, she traced the quadrilateral and its diagonals (Figure 10).

Finally, with the ruler she measured the lengths of the line segments to show the following: $\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} = \overline{AC} \cdot \overline{BD}$.

Verified that $\overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC} = 281.75$ and that $\overline{AC} \cdot \overline{BD} = 280.5$, despite not using materials with great precision the result is very close.

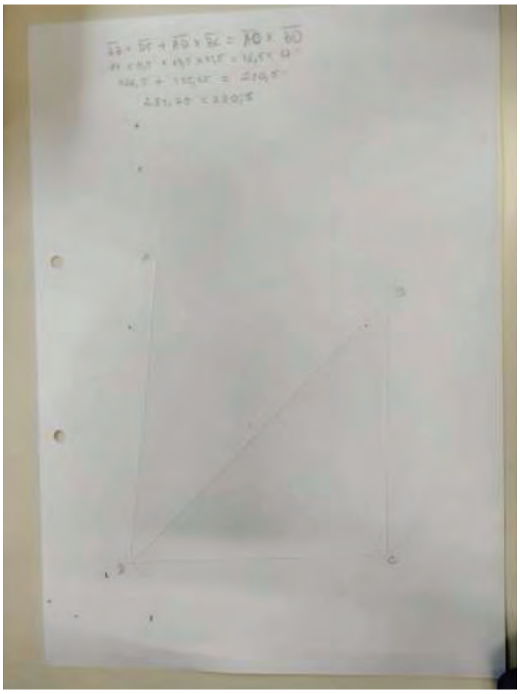

**Figure 10** Blind Students' construction

In the last part of the activity, the students were asked to paint their "work of art". The results of the drawings were quite diverse, resulting in a variety of creative and original images. In Figure 11, we can observe three students who chose to use different colors.

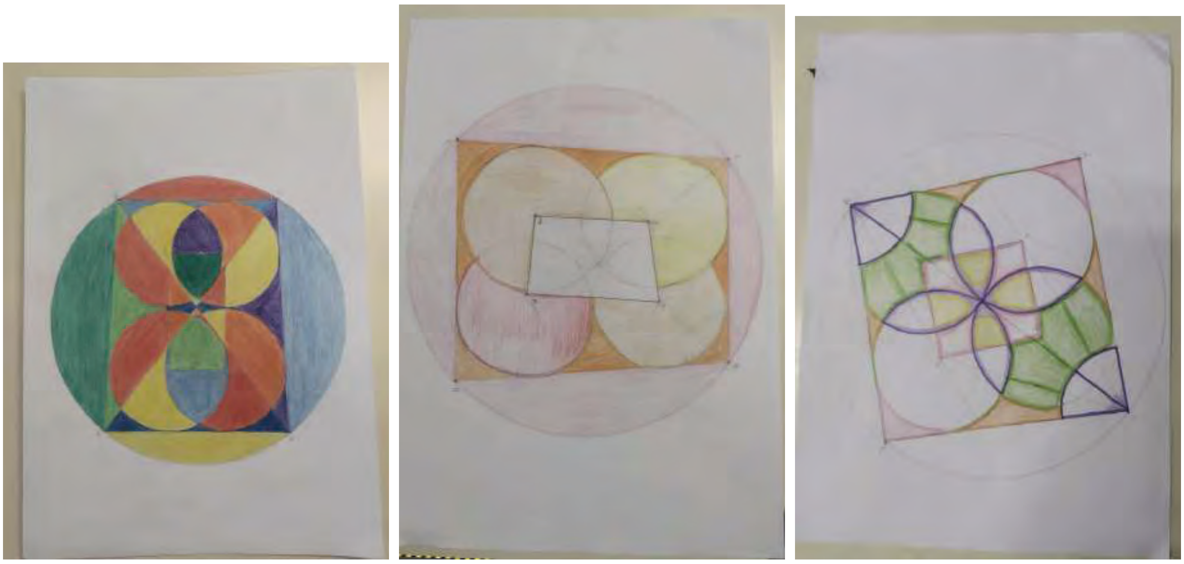

**Figure 11** Students' Sangaku

The blind student managed to paint (Figure 12), without help, her construction, the only help was the choice of colors, as she has difficulty distinguishing between dark colors and between light colors.

**Figure 12** Blind Student's Sangaku

# 4 Conclusion

Problems written in Sangaku demonstrate a surprisingly high standard in mathematics in Japan during the Edo period, especially in geometry and calculus. It is all the more surprising that such a high level was achieved independently of the West due to the strict isolation policy. Some interesting problems have been posed by many people from various socio-economic backgrounds. This fact may indicate that mathematics is attractive to many people, regardless of class, economic, ethnic, age and gender boundaries.

When comparing Euclid's work with a Sangaku, the two styles are very different, and therefore terms like 'proof' and 'theorem' should be used with caution when describing the Japanese tradition. The Sangaku authors did not employ proof by diagrams or axiomatics in the sense of Greek and Western mathematicians. Sangaku authors did not seem to enunciate theorems that were then proved with reasoning or logical argument. The tablets only contain a problem, an answer and a formula, which provide little information about how it was obtained from the figure and no reasoning about why it worked [5].

Integrating Sangaku into contemporary education offers a holistic and dynamic learning experience that enhances students' critical thinking, problem-solving, and creative skills while connecting them with historical and cultural dimensions.

Two geometric activities were presented: the Japanese Theorem and Ptolemy's Theorem for cyclic quadrilaterals. The challenges found in the wooden tablets, known as Sangakus, consisted of one problem, one answer, and one formula. However, these solutions offered little information about how they were obtained from the figures, lacking reasoning about their underlying principles [5].

The students were then challenged to work on a practical project, exploring different forms of expression and perceiving Mathematics as a tool for artistic creation and symbolic communication. Throughout the project, the students demonstrated autonomy and responsibility in its execution. The activity not only promoted critical thinking but also fostered creativity through interdisciplinary collaboration.

Considering the diverse range of possible activities from wooden tablets to Sangakus, the proposals were carefully selected to match the students' educational level. This particular work stands out for its originality in involving basic school students. Furthermore, the presence of a blind student in the class posed an additional challenge. However, through inclusive design, the objective of involving all students in the activity was successfully achieved. This inclusion of all students, including the blind student, highlights the significance of accessibility and equal opportunities in the educational process.

Employing an interdisciplinary approach, the activity stimulated students' creativity, motivating them to create visual depictions of the problem and interact with artistic materials. This process concurrently simultaneously their abilities in visual communication and mathematics.

Incorporating creativity and innovation into STEM education, whether through technology-driven approaches [7] or non-technological means [10], creates a dynamic learning environment that prepares students for the challenges and opportunities of the modern world. By fostering creativity, encouraging innovation, and promoting computational thinking, we empower the next generation with the essential skills to shape a future driven by technology [8].

# 5 Acknowledgements

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# Arts and Maths: a STEAM introduction to envelopes with automated methods*

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**Abstract**

Starting from a piece of string art, we propose a STEAM approach to motivate activities about envelopes of parametric families of surfaces. The examples are provided by 1-parameter families of planes and yield ruled surfaces with cuspidal edges. The topology of the surface obtained by geometric and algebraic work can be compared to the shape of well-known monuments (whence an incitement to outdoor mathematics). Later, we discuss the transition from 2D to 3D: regarding the automated methods, it is non trivial as commands available in a 2D setting may not be available for working in 3D. Nevertheless, the algebraic manipulations are similar in 2D and 3D, based on the same packages. Here, the main example is offered by an astroid in the plane, and its 3D generalization as an astroidal surface. Finally we discuss the examples according to the STEAM approach, and to Balacheff's computer transition.

# 1 Introduction

## 1.1 String art and beyond

A popular activity consists in creating pieces of art using a wooden plate, nails and threads, what is called string art. Figure 1(a) shows an astroid created with string art. Note that actually an astroid is a plane curve, but the concrete creation provides a kind of surface in 3-dimensional space, because of the thickness of the threads. This will be an incitement to extend the activity to 3D.

This piece of string art can be modelled using a Dynamic Geometry System (DGS). Figure 1(b) shows a screen snapshot of a GeoGebra applet which creates such an astroid dynamically. The concrete basis of the applet is the envelope of positions of a given ladder whose foot is on the $x-$axis and whose head is on the $y-$axis; remember that the ladder's length is constant. Various methods can be used for the construction, depending on the mathematical background and the

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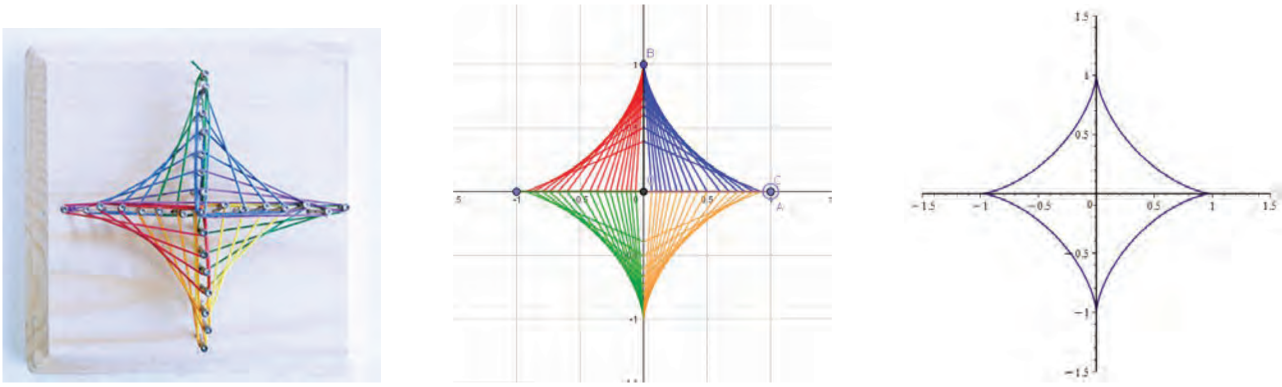


(a) Concrete realization (b) Model with a DGS (c) A plot with CAS

Figure 1: Astroid as string art and modelling

software literacy of the user. For example, the applet by Z. Kovács[1] shows a construction and provides an implicit equation for an astroid, using the automated command **Envelope**. Another applet provides a full construction of the astroid (it is analyzed in Section 3). Here, in Figure 1, the plots have been created in the first quadrant and reflected into the other quadrants. Then colors have been changed. This provides the dynamic effect of simultaneous creation in all the quadrants, moving one point with the mouse. The slower the moves, the higher the number of plotted segments. We may mention that sometimes, an illusion of 3D may be experienced, but not more than an illusion.

On a sunny day, the surface of a cup of coffee (Figure 2(a)) or a table where you forgot a brass ring (Figure 2(b) offer other examples. The curve appearing on the coffee is called a cardioid; it is the envelope of the solar rays reflected by the cup. See also Appendix A.3. Figure 2(c) shows a model with string art.

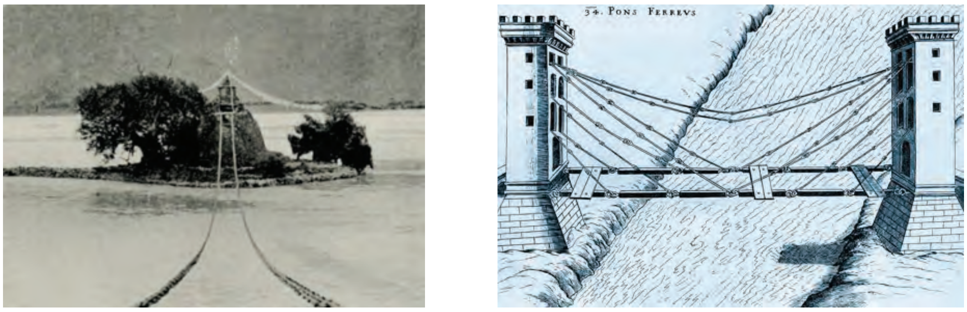
(a) Sun on a cup of coffee (b) Reflection on a copper ring (c) A model with threads

Figure 2: Playing with light

String art is not limited, neither to plane constructions nor to the classroom. Outdoor mathematics give access to numerous string constructs. A long time ago, engineers conceived hanging bridges for utility reasons, among them to be able to cross very wide rivers without pillars in the middle. Figure 3 shows (a) an old chain bridge at Chaksam (Tibet), built in 1430,

[1]There are two mathematicians with this name, working in closely related areas. In this paper,we relate to the colleague from Linz, Austria, `https://matek.hu/zoltan`.

and (b) a drawing by Fausto Veranzio in 16th century[2]. In recent years, this kind of bridges are



(a) An old bridge in Tibet (b) A Renaissance drawing

Figure 3: Old hanging bridges

also conceived as pieces of art, actual realizations of ruled surfaces. Figure 4 shows two views of hanging bridges: (a) a bridge over the sea in Greece, and (b) a bridge for the light train in Jerusalem, over one of the busiest road crossings. The needed curvature of the tracks provided an opportunity for a wing design.

(a) The Rion-Antirion bridge, Greece (b) The chords bridge in Jerusalem, Israel

Figure 4: Modern hanging bridges

## 1.2 STEAM Education

STEAM is an acrostics for Science, Mathematics, Technology, Engineering, Arts and Mathematics. It represents an educational approach that aims to spark an interest and lifelong love of the arts and sciences in children from an early age, but not only. The first example above is doable by young children, the next one probably not. What we will show later is suitable for undergraduates, or at least for gifted High-School students having access to an extension of the curriculum. Until quite recently, a STEM approach was defined and analyzed, the A for Arts joined in the last years. STEM literacy has been defined as the ability to identify, apply, and integrate science, technology, and mathematical concepts to understand complex problems and innovate in their solution" [6]. The inclusion of the artistic disciplines is not

[2]Both images are available in Wikipedia.

arbitrary. It broadens the interdisciplinary perspective by incorporating an external area into the scientific field. This is true for various artistic domains; here we deal with some visual arts, including architecture (what doubles the meaning of A). In fact, this A incites to use the cultural background of the students in order for them to develop new knowledge and new skills [20, 42].

Actually, the influence is multidirectional. Science, Technology, Engineering, the Arts and Mathematics are similar fields of study; they involve creative processes and none uses just one method for inquiry and investigation. Moreover, "since the European Community has recognized the role of life-long learning and its impact on the education of aware citizens , non-formal contexts started deserving the attention of scientists dealing with education and in particular science education" [43].

In all the components of STEAM, the name of the game is given by the 4 C's of 21st education: Collaboration, Communication, Critical Thinking and Creativity [39]. Important skills are included in these 4 C's: creativity and productivity are a must for innovation, information and media literacy are part of the skills for communication and collaboration, together with leadership and responsibility, initiative. In previous works, we mention the rich influence of using the cultural background of the student, to attract them to maths and other scientific domains [20]. Actually a 5th "C" should be added, namely Curiosity, without which Creativity may be limited. Curiosity will lead to and will be nurtured by exploration of general media, and to developing cross-cultural skills, which may be an important component of Communication and Collaboration..

For the examples shown in this paper, we use GeoGebra, and the Maple 2023 Computer Algebra System, both for its 3D plotting features and its algebraic abilities, working with polynomial rings.

## 1.3 New knowledge developed in a technology-rich environment

Artigue [4] notes that "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations". One of the most famous events is the transition from writing with a feather to writing with an iron quill. This has been noted by P. Lavoie [36], following whom Guin and Trouche write [32] "Lavoie, for instance, shows the consequences of the introduction of the iron quill (instead of the goose quill) for the learning of arithmetic in the 19th century (writing more easily allowed pupils to do longer computations by hand and this lead to an earlier introduction of arithmetic in curricula)."

More recently, in the 2nd half of the 20th century, hand held numerical calculators made the usage of numerical tables obsolete. Then new algorithms have been written, enabling the development of symbolic calculators and packages for personal computers. The era of Computer algebra Systems (CAS) and of Dynamic Geometry Systems (DGS) began. In some cases, these CAS were devoted to a specific mathematical domain, but general purposes CAS have also been developed; see [18]. An interesting comparison of the effects of different kinds of software for the same kind of problems is given in [33]. Nowadays, general purpose systems are ubiquitous. They provide an environment including Geometry, Algebra, one-variable and multi-variable Calculus, complex variables, etc. In what follows, we explore algebraic curves and surfaces. Therefore, we use mostly the geometric features and the algebraic packages, mostly the Polynomial Ideals and Gröbner packages.

The geometric objects of our study, envelopes of 1-parameter families of curves or of surfaces, appear first via a parametric presentation, obtained as the solution of a system of equations (see Definition 3 in next Section). This may be enough to explore their topology, but implicitization may be useful. A drawback is that implicitization can be heavy, as we will see in Section 3. Most CAS have a command for elimination, but computations may be very long (see [25]). Only a more profound insight into the algorithm may provide a better understanding, in particular an answer to the question "why do superfluous components appear?". This is due to the fact that elimination of variables actually computes the projection of a certain variety onto a space of smaller dimension and provides the closure of the desired variety, in Zariski topology [13, 38]. We experienced this phenomenon, not only for the determination of envelopes, but also for the determination of isoptics; see [22].

# 2 Envelopes of families of surfaces: mathematical treatment and automated methods

We consider a 1-parameter family of surfaces $\mathcal{F} = \{\mathcal{S}_u\}$ in 3-dimensional space, given by the equation $F(x, y, z, u) = 0$, where $u$ is a real parameter. We use the following definition of an envelope:

**Definition 1** *A family of surfaces is given by an equation of teh form $F(x, y, z, u) = 0$, where $u$ is a real parameter. An envelope of the family $\mathcal{S}_u$ is the image in 3-dimensional space of the solution set of the following system of equations:*

$$\begin{cases} F(x, y, z, u) = 0 \\ \frac{\partial}{\partial u} F(x, y, z, u) = 0 \end{cases}$$

This is one of the 4 definitions given in [12], and the only one given in [7].

We consider the family of planes whose equation is given by $F(x, y, z, u) = 0$, where $F(x, y, z, u) = x + uy - 2uz^2 - 3u^3$ and $u$ is a real parameter. According to Definition 3, we can compute a parametric presentation of an envelope of this family. We obtain:

$$(x, y, z) = (-6u^3 + 2u^2 v, 9u^2 - 4uv, v). \tag{1}$$

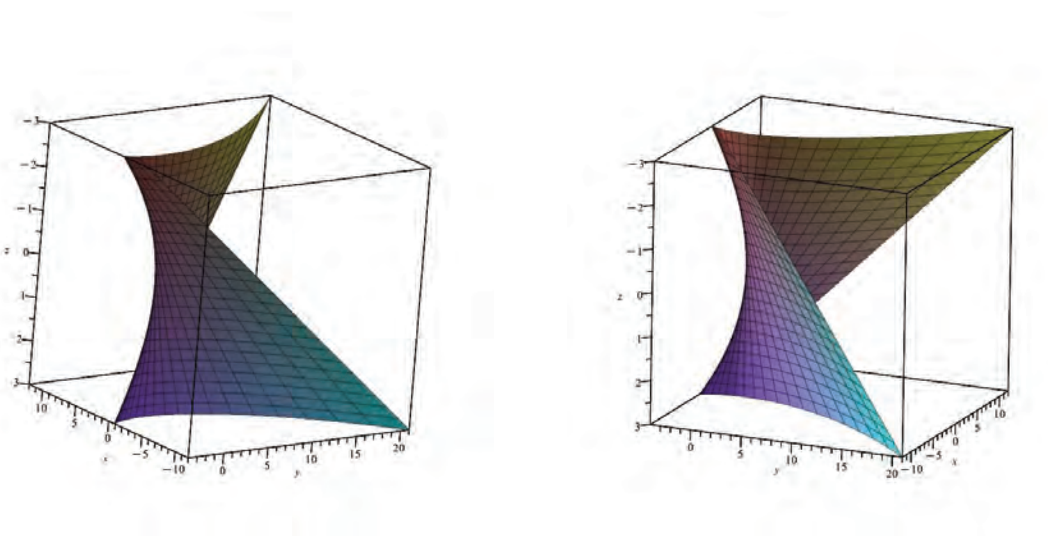
Before we proceed further, we wish to mention that the output on the screen is actually $(x, y, z) = (-6u^3 + 2u^2 z, 9u^2 - 4uz, z)$ and it is necessary to substitute another variable instead of $z$ (here we chose $v$). Otherwise, what follows is meaningless. Now, using elimination methods, as in [29], an implicit equation is obtained:

$$32xz^3 - 4y^2 z^2 + 108xyz - 12y^3 + 243x^2 = 0. \tag{2}$$

Figure 5 shows two views of this envelope.

When a 1-parameter families of planes has an envelope $\mathcal{E}$, this has interesting general properties[3] :

[3] A family of planes having a common line has no envelope, but in general the family has an envelope, and it verifies the properties mentioned here; see the Mathcurve website.



(a) (b)

Figure 5: Two views of the envelope of a 1-parameter family of planes

(i) The envelope $\mathcal{E}$ is a ruled surface;

(ii) It has a cuspidal edge $\mathcal{C}$;

(iii) The tangent to $\mathcal{C}$ are generators of $\mathcal{E}$.

The first two properties are apparent on Figure 5. The 3rd property is less visible. All the properties can be proven using suitable commands of a CAS.

In order to determine the cuspidal edge $\mathcal{C}$, two methods can be applied:

(i) Looking for singular points of the surface determined by the parametric presentation (1)

(ii) Looking for these points using the implicit equation (2) and computing a gradient.

Maple offers two packages to compute a gradient. We used the *Student[MultivariateCalculus]* package, the second one seems to interfere with the *PolynomialIdeals* package and to make the algebraic computations unreachable. We display here the Maple session:

```
restart:with(plots):
with(PolynomialIdeals):with(Student[MultivariateCalculus]):
F := -3*t^3 - 2*t^2*z + t*y + x:
derF := diff(F, t):
env := solve({F = 0, derF = 0}, {x, y, z}):
envplot1 := plot3d([rhs(env[1]), rhs(env[2]), rhs(env[3])], t = -1 .. 1, z = -3 .. 3,
            labels = [x, y, z]):
envplot2 := plot3d([rhs(env[1]), rhs(env[2]), rhs(env[3])], t = -1 .. 1, z = -3 .. 3,
            transparency = 0.4, labels = [x, y, z]):
p1 := x - subs(z = u, rhs(env[1]));
p2 := y - subs(z = u, rhs(env[2]));
p3 := z - subs(z = u, rhs(env[3]));
```

```
K := <p1, p2, p3>:
KE := EliminationIdeal(K, {x, y, z})
FF := Generators(KE)[1]:
G := Gradient(FF, [x, y, z]):
cuspedge := solve({G[1] = 0, G[2] = 0, G[3] = 0}, {x, y, z}):
cuspedgeplot := plot3d([subs(z = u, rhs(cuspedge[1])), subs(z = u, rhs(cuspedge[2])),
                subs(z = u, rhs(cuspedge[3]))], u = -3 .. 3, thickness = 5):
display(envplot2, cuspedgeplot):
```

The variable $FF$ denotes the polynomial displayed in Equation (2). The output of the **Gradient** command is in matricial form, namely

$$G = \begin{bmatrix} 32z^3 + 108yz - 486x \\ 8yz^2 + 108xz + 36y^2 \\ 96xz^2 + 8y^2z + 108xy \end{bmatrix} \tag{3}$$

whence the format of the **solve** command to determine singular points. The solution set of this command determines a curve with the following parametrization:

$$(x, y, z) = \left(-\frac{8}{243}t^3,\ -\frac{4}{9}t^2,\ t\right). \tag{4}$$

The previous remark about the need to substitute a new variable instead of $z$ is still valid here. Figure 6 shows the envelope with the cuspidal edge enhanced. For this purpose, the envelope has been plotted with some transparency (defined as *envplot2* in the above session).

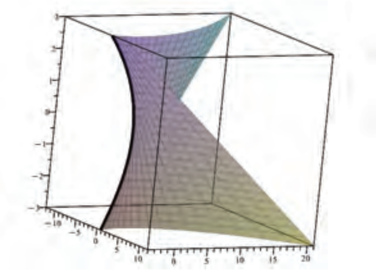


Figure 6: A view of the envelope with its cuspidal edge

# 3 A 3D version of an astroid

WLOG we can define an astroid as the envelope of a segment of length 1 whose end points are on the $x-$axis and on the $y-$axis respectively. It has the following parametric equation:

$$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \tag{5}$$

where $a$ is a positive parameter and $t \in [0, 2\pi]$. This astroid is also defined by the implicit equation

$$x^{2/3} + y^{2/3} = a^{2/3}. \tag{6}$$

Figure 1(c) shows the astroid for $a = 1$. Note that, on the one hand, this does not define an astroid as an algebraic curve, and on the other hand, that it request to be careful with powers of negative reals. In the present case, problems will be avoided if Equation (6) is written in the following form:

$$(x^2)^{1/3} + (y^2)^{1/3} = (a^2)^{1/3}. \tag{7}$$

Algebraic manipulation leads to the equivalent equation of degree 6:

$$(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0, \tag{8}$$

showing that an astroid is actually an algebraic curve. For a concrete construction as an envelope, we refer to the applets mentioned previously.

An astroid can be studied with automated methods. Here we use GeoGebra's **Envelope** command. Figure 7 shows a snapshot of a devoted applet. The syntax of the command is **Envelope**(<path>,<point>), therefore the applet has been built as follows:

i. A segment $AB$ is defined on the $x-$axis. It is denoted by $f$, which in GeoGebra denotes also its length.

ii. A point $C$ is attached to the segment. This will be the variable point.

iii. Denote by $E$ the point of intersection of the $y-$axis with the circle centered at $C$ with radius $f$.

iv. The segment $CE$ is denoted by $g$.

v. Apply the command **Envelope**($g$,$C$).

The construction refers to objects in the first quadrant only, but as in Figure 7:

- The output on the graphical window is an (almost) entire astroid. The plot is inaccurate in a small neighborhood of each vertex, as the vertices are singular points. A better plot could be afforded by a parametric representation of the astroid, which is not provided by the command. The inaccuracy of plots around singular points is discussed in [44] for any CAS.

- The astroid's equation, given by a quartic polynomial, is displayed in the algebraic window.

- This polynomial equation defines the entire astroid, not only an arc in 1st quadrant. The reason is that the computation, based on Gröbner bases and elimination, computes a Zariski closure.

We wish now to make a transition towards 3D, i.e., to define a surface generalizing the astroid to 3D. As the current version of GeoGebra does not offer a 3D version of the **Envelope**

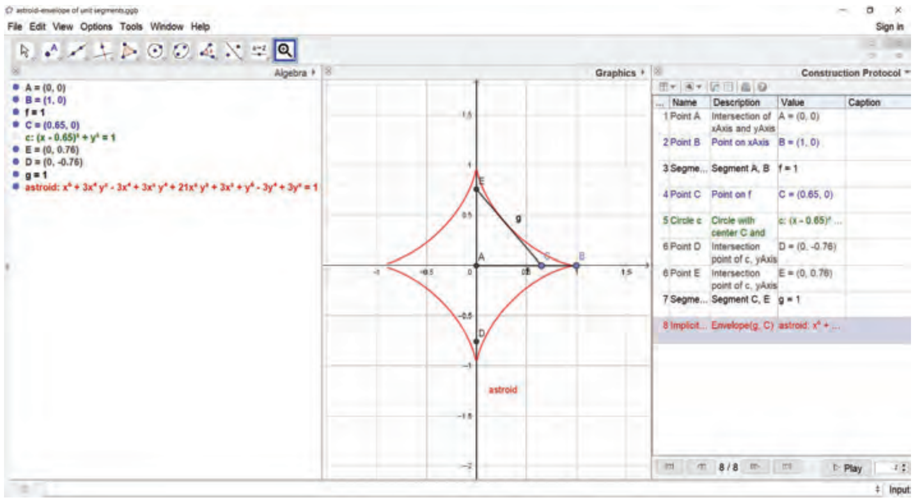


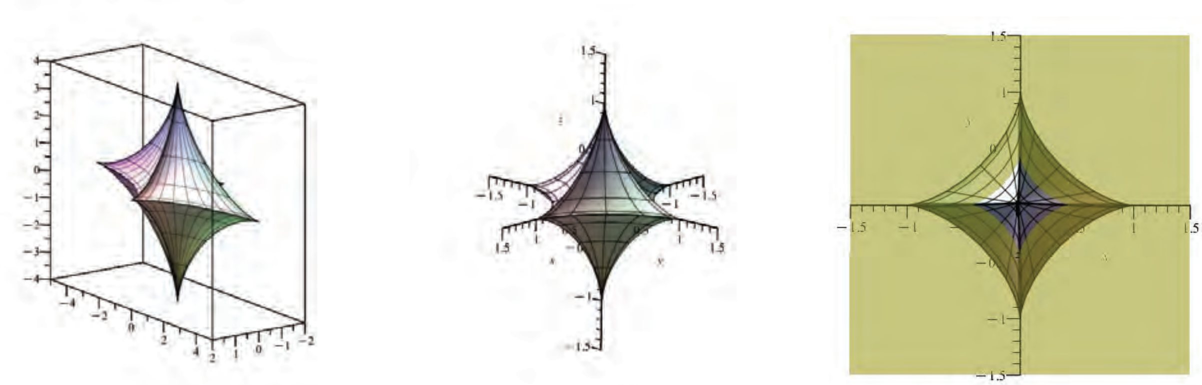

Figure 7: Snapshot of the construction of an astroid with the Envelope command

command (neither does Maple anyway), we adopt another method. Following the Mathcurve page, we define an *astroidal ellipsoid* $\mathcal{A}$ by the parametric presentation

$$\begin{cases} x = a\ \cos^3 u \cdot \cos^3 v \\ y = b\ \cos^3 u \cdot \sin^3 v \\ z = c\ \sin^3 v \end{cases} \qquad u \in [0, 2\pi],\ v \in [0, \pi]. \tag{9}$$

where $a > 0, b > 0, c > 0$. If $a = b = c$, the surface is called a called an *astroidal surface*; it can be also defined by the equation:

$$(x^2)^{1/3} + (y^2)^{1/3} + (z^2)^{1/3} = (a^2)^{1/3}. \tag{10}$$



(a) $(a, b, c) = (2, 5, 4)$ (b) $a = b = c = 1$, general view (c) $a = b = c = 1$, from above

Figure 8: Astroidal ellipsoids

In the present paper, we deal with the astroidal surface $\mathcal{A}$ given by $a = 1$. The intersections of the surface $\mathcal{A}$ with the coordinate planes are astroids (see Figure 8(c)). In the literature, this surface is also called *hyperbolic octahedron* or *astroidoctahedron*.

From Equation (10), by algebraic manipulations (i.e. moving terms from side to side, raising both sides to 3rd power, and iterating the process several times, it can be shown that an astroidal surface is an algebraic surface given by a polynomial of degree 18 in 3 variables[4]. Another way to derive an implicit equation for $\mathcal{A}$ is to use the same method as in [23, 24], substituting cosines and sines with rational expressions in Equations (9). We only sketch the method:

$$\begin{cases} \cos u = \frac{1-r^2}{1+r^2} \\ \sin u = \frac{2r}{1+r^2} \end{cases} \quad \text{and} \quad \begin{cases} \cos v = \frac{1-s^2}{1+s^2} \\ \sin v = \frac{2s}{1+s^2} \end{cases}.$$

and from there defining 3 polynomials in 5 variables $x, y, z, r, s$. They generate an ideal $K$ in $\mathbb{R}[x, y, z, r, s]$. Elimination of the parameters $r, s$ provides a polynomial $F(x, y, z) \in \mathbb{R}[x, y, z]$ whose vanishing set is the Zariski closure of the required envelope.

An astroidal surface is also an envelope of a family of planes. As this is a really non-trivial statement, we emphasize it:

**Proposition 2** *An astroidal surface is the envelope of the planes intersecting the 3 axes at the 3 vertices of a triangle, for which the distance between the center of gravity and the origin $O$ is constant, equal to $a/3$.*

We will only sketch the proof. Consider 3 points $A(\alpha, 0, 0), B(0, \beta, 0), C(0, 0, \gamma)$, on the axes in the 1st octant, i.e. $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. The center of gravity $G$ of the triangle $ABC$ has coordinates $\frac{1}{3}(\alpha, \beta, \gamma)$, thus we have $OG = \frac{1}{3}\sqrt{\alpha^2 + \beta^2 + \gamma^2}$. We have:

$$OG = \frac{a}{3} \qquad \Leftrightarrow \gamma = \sqrt{a^2 - 9\alpha^2 - 9\beta^2}. \tag{11}$$

The plane of the triangle $ABC$ has equation

$$-\gamma\beta x - \alpha\gamma y \alpha\beta z + \alpha\beta\gamma = 0. \tag{12}$$

Substitution of the formula (11) into Equation (12) yields an equation of the plane $ABC$ depending on 2 parameters only. The envelope of the 2-parameter family of planes is computed by standard methods, as in [21, 17], according to the following definition:

**Definition 3** *A family of surfaces $\mathcal{S}_{u,v}$ is given by an equation $F(x, y, z, u, v) = 0$, where $u, v$ are real parameters. An envelope of the family $\mathcal{S}_{u,v}$ is the image in 3-dimensional space of the solution set of the following system of equations:*

$$\begin{cases} F(x, y, z, u, v) = 0 \\ \frac{\partial}{\partial u} F(x, y, z, u, v) = 0 \\ \frac{\partial}{\partial v} F(x, y, z, u, v) = 0 \end{cases}$$

The geometric construction that we performed is valid in the first octant. What happens in the seven other octants is obtained by reflections about the coordinate planes.

As mentioned in the previous section, the envelope of a 1-parameter family of planes has a cuspidal edge. In [21], an example of a 2-parameter family with the same property has been given. We can prove this here too. In the previous section, we could use the implicit

[4]The computation takes a very long time.

equation and gradients to find the cuspidal edge. Here, the implicit equation is too complicated (therefore, we did not display it - the interested reader can derive it using a CAS). We can show that the same situation appears here, using partial derivatives of Equations (9). We have:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} -3\cos^2 u \cdot \cos^3 v \cdot \sin u & -3\cos^3 u \cdot \cos^2 v \cdot \sin v \\ -3\cos^2 u \cdot \sin^3 v \cdot \sin u & 3\cos^3 u \cdot \sin^2 v \cdot \cos(v) \\ 3\sin^2 u \cdot \cos u & 0 \end{pmatrix} \tag{13}$$

Singular points correspond to values of the parameters $u$ and $v$ for which the matrix vanishes. These points are on the coordinate planes. The corresponding sets are enhanced in Figure 9.

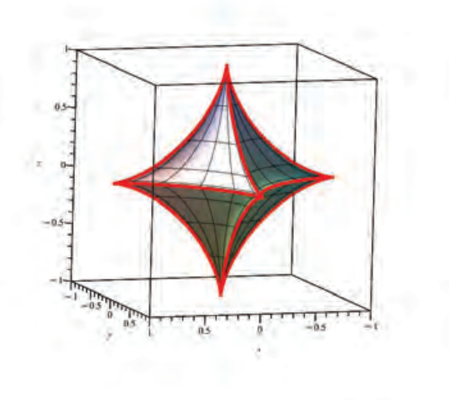


Figure 9: An astroidal surface with enhanced cuspidal edges

As visualization of a 3D object on a screen may be difficult to understand (it has to be rotated in several ways to have enough points of view), we can use a quite new technology[5],we mean 3D printing. Figure 10 shows such a 3D printed astroidal surface. Note that for technical reasons, it has been 3D-printed in 3 steps: the first 2 steps were identical, namely 3D-printing twice half of the object, let's say the part of the surface above the $xy-$plane, then gluing them together (not really high-tech).

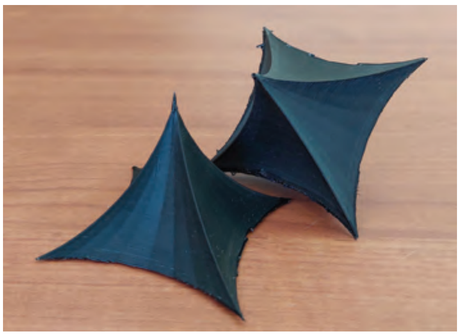


Figure 10: A 3D printed astroidal surface

[5]At least, new in educative environments.

# 4 Discussion

## 4.1 The needed dialog between technologies

The different affordances of a CAS and a DGS complete each other. Generally one is stronger for interactive display of curves and surfaces, and the other for algebraic computations. Actually most CAS contain plotting features, but require a priori programming. Then some intervention is possible during the animation, but no interactive "essential" changes. On the other hand, a software such as GeoGebra, began as a DGS, but has been supplemented with several other components, in particular a CAS (called Giac [35]). A new version of this CAS has been released recently. The need of an automated CAS-DGS dialog has been explained a long time ago (see [41]) and has still to be developed. In many cases, the dialog is performed by Copy-Paste only; see [23].

Development of automated dialog between CAS and DGS will strongly ease the exploration and discovery of new properties. Among the 4 C's of 21st Century Education, are mentioned Collaboration and Communication. If, in the original OECD documents [39], they were meant between humans, we advocate for these two C's also between man and machine, a domain which has already seen numerous studies, but also now between machines.

There exist cases where communication between technologies exist, such as export images as jpg or other format, export a session as a LaTeX file, etc. This is useful, but does not fit what we mean by a dialog. Here, we can 3D print some objects by exporting the output of Maple plot commands to STL, the language used by the printer (without a human having to translate). A similar situation prevails with GeoGebra.

This dialog is not intended only for easing the technical work. It enables to concentrate on understanding the situation and the proposed task. A DGS and a CAS provide several registers of representation (in the sense of Duval [30, 31]): numerical, graphical, symbolic, etc. A central issue is switching between registers, most CAS enabling to switch from symbolic to graphical (generally via the numerical, after some discretization of the data because of hardware constraints, i.e. the pixel definition - see [5], but this may be transparent to the user), sometimes in reversed direction, from graphical to numerical, more rarely from graphical to symbolic. This last switch is proposed by GeoGebra Discovery[6], in particular with the commands used in the present paper. 3D printing adds a new register, making mathematical objects more present, concrete. This fills a gap, mentioned by Duval [30, 31]: unlike Physics, Chemistry and other scientific domains, the objects under study in mathematics are abstract, and not graspable with hands. Therefore, they need these registers of representation. 3D printing offers a new register, more concrete.

## 4.2 3D printing

Visualization of a 3D object on a screen is not easy, sometimes an obstacle. New technologies may help, in particular 3D printing. Figure 12 enables a comparison between visualization on the screen and concrete realization. Of course, 3D printing is in itself a collaboration between CAS-DGS and 3D printer, the last one having its own language. For this kind of

[6]GeoGebra Discovery is an experimental version of GeoGebra. It offers both extended versions of existing commands and new commands. For existing commands, the improvement is to offer a symbolic proof, when the orginal one is numerical.

communication, we need to make the following remark, already addressed in [17]: the data transfer from the DGS to the 3D printer requires the object to be recognized by the DGS as a geometric construct. For a reader who did not experience this issue, we explain it on a 2D example in Appendix A.1.

All this discussion does not apply to a 3D problem. In GeoGebra, no equivalent to the 2D-automated commands **Locus**, **LocusEquation** and **Envelope** exist (at least to the day this paper is written). Therefore, other tools have to be used. Algebraic methods of a CAS, namely commands such as **Solve** exist no matter the dimension, and when Gröbner methods have to be used, they work. It should be noted that the algebraic engine at work for these commands (via Giac in GeoGebra's background, or visible when working with Maple, is mostly Gröbner packages; see for example [24, 18]). The differences are transparent to the user. An important difference lies in visualization, a topic which has been addressed widely in the past. This issue is partly answered by 3D printing: the objects are not represented on a screen (a 2D representation of a 3D object, but can be manipulated by hand, offering all the possible points of view.

This has already been illustrated by the author regarding canal surfaces [17]. Figure 11 shows a model of a twisted lemniscate, defined by parametric equations[7]. It had to be thickened and the equations to be prepared to be understood by the software as a geometric object. The original data, written with Maple, defined a curve. The thickness option of the **spacecurve** command was not enough for that.

Moreover, a mathematical activity involving 3D printing enables a strong communication between participants. See [2, 3].

Figure 11: A twisted lemniscate

Despite the astroidal surface $\mathcal{A}$ seeming simpler than the canal surface evoked above, this example reveals another issue: in order to 3D print $\mathcal{A}$, it is necessary to understand its symmetries. To achieve this understanding, the teacher and the students may have a dialog[8] around the properties of the defining functions of the surface, such as parity, and what are their consequences. Figure 8 illustrates the nonintuitive situation, whence the necessity to deal with general properties of the functions.This enables to decompose the printing into 2 parts, which is compulsory to print on the machine table; see Figure 12. Afterwards, a low level task has still to be done: to glue the two parts together.

[7]This, and what is shown in Figure 12, have been 3D printed by E. Ulbrich, at Johannes Kepler University (Linz, Austria).

[8]Students, to whom the study of such surfaces is proposed, are beyond the need of a strong scaffolding around functions.

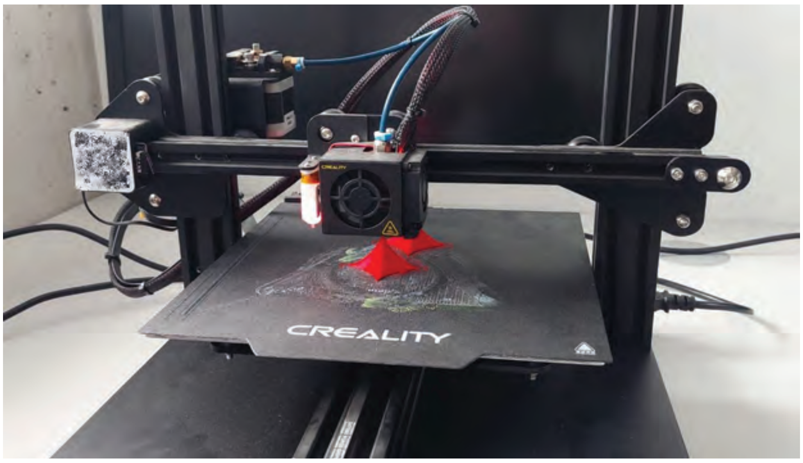
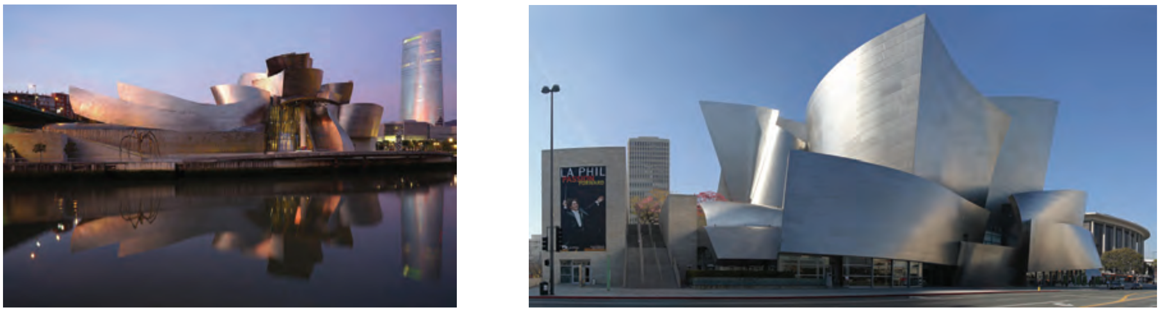



Figure 12: Printing in two parts

### 4.3 Back to Arts

As mentioned above, the A of STEAM is intended for Arts, but sometimes it can be reduced to Architecture. In subsection 1.1, we wrote that the exploration of envelopes of 1-parameter families of planes provides ruled surfaces, and we could compare this with the construction of hanging bridges. We may have another connection with architecture. Compare the topology of the envelope described in Section 2 with the monuments (to call them buildings does not seem to me enough) shown in Figure 13.




(a) Guggenheim Museum, Bilbao, Spain

(b) Walt Disney Hall, Los Angeles, USA

Figure 13: Two buildings whose geometry evokes the envelopes under study.



We mentioned visual arts and architecture, but many other examples can be used, coming from the everyday life of the students, or from their centers of interest. Figure 14 show two such examples. The boat bow shows a cuspidal edge which is itself singular (ask civil engineers why this shape has been chosen); actually the upper part (the blue one) and the lower part (the red one) show different examples of the same remark. The snow plow shows a truncated envelope with a cuspidal edge; the surface has been cut by a plane, probably to allow the plow driver to see where the train is going. Here too, we have examples for outdoor mathematical activities, and an incitement to learn mathematics based on everyday life and the cultural background of students (of course, not in every location on Earth, students may have the same experiences, such as a train with a snow plow and a big cargo boat).

(a) A boat bow

(b) A snow plow

Figure 14: Other concrete realizations of such envelopes

STEAM combines "arts" with STEM. This learning is aimed at improving student engagement, creativity, innovation, problem-solving skills, and other cognitive benefits [37]. With a STEAM approach involving the 4C's, students' motivation increases and their learning may be more effective; see [42]. We wrote already that Curiosity should be added, and is enhanced with ipout from the so-called "real world".

## 4.4 Modelling and Computer transposition.

Let us look again at Figure 1. An astroid is a plane curve. It has been modelled using string art, whose constraints were visible: instead of a plane curve, the output is a surface dependent on the threads' thickness. Then a computerized model has been constructed with a DGS. Jablonski [34] says that "Mathematical modelling is characterized through its interplay of reality and mathematics. It offers a way to integrate references to reality into the classroom and shows students where in everyday life their mathematical knowledge can be applied." Here we see that interplay at work. The process is structured as follows:

i. A real-world problem is given and analyzed.

ii. It is translated into a mathematical setting (Descartes claimed that every problem has to be transformed into a system of equations).

iii. The mathematical problem is solved.

iv. The solutions have to be interpreted and validated with respect to reality.

The first 3 points have been processed using software, in particular automated methods. Then, we replaced the last point by an extension in another direction, namely generalizing to 3D. The possible paths for the usage of modelling are described in Figure 15: back to reality with a more profound understanding, or extending the reality towards other creations. The paths

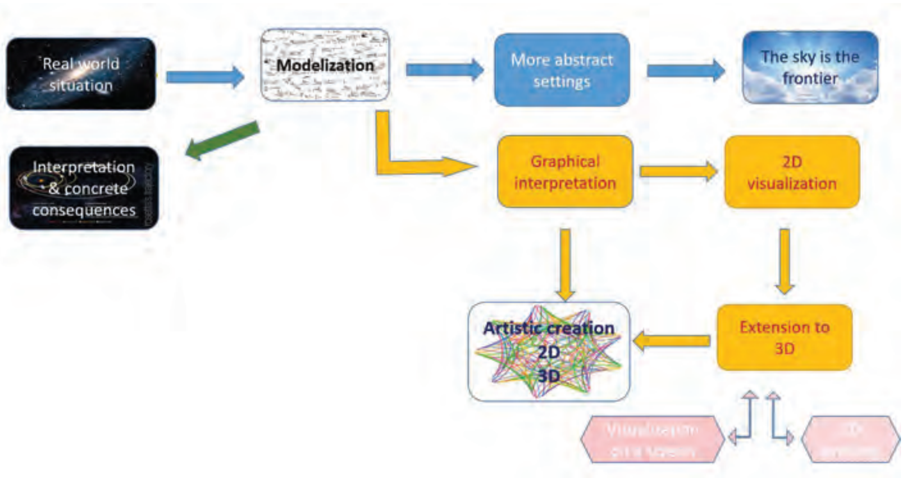



Figure 15: Different paths from modelling onwards

leading to "Artistic Creation" starting from parametric curves (modelling trajectories in space) are the topic of an ongoing work [28].

A computable model (in the sense of Balacheff [5]) was easy to obtain from the literature, a CAS or a DGS provided a good representation on the screen. And then, back to reality with a 3D printing, revealing once again that "the choice of representations implies a structuring of the data which determines processing constraints" [5]. Constraints, either of the hardware or of the software, when working with a CAS, have been analyzed by Guin and Trouche [32]. An additional constraint, called motivating constraint, is mentioned in [14]. Such a constraint provides a motivation to achieve a more profound understanding. We met such a constraint when analyzing the symmetries of an astroidal ellipsoid. In our work, we see two new appearances of constraints:

i. The recognition by the software of and object as a geometric object (v.i. Appendix A.1)

ii. More concrete: the impossibility to 3D print the astroidal surface in one shot, and the need to 3D print it in two parts, and then glue them.

Note that the example of an astroidal surface is quite easy to analyze prior the 3D printing. The symmetries of the surface are a direct consequence of parity of the functions involved in the parametric presentation of the surface. Other examples may be more difficult to analyze for symmetry: the difficulties of the computer transition are met early. The 3D printed object may provide confirmation a posteriori that the choices were accurate. Hopefully, the 3D printed object will enable discovery of new properties, maybe even a shorter way to have the object 3D printed.

# A Appendix

## A.1 Recognition of an object as a geometric construct.

We mentioned above that for certain GeoGebra commands, the object appearing on the screen has to be recognized by the software as a geometric object. We illustrate this claim in different situations, namely the usage of the **Trace On** feature, the usage of the **LocusEquation** command, and of the **Envelope** command, all relying on examples.

## A.2 Determination of a envelope and of a geometric locus

The automated determination of an envelope and of a geometric locus rely on the same algebraic machinery, no matter which software is used. Pech [40] works mostly with CoCoA[10], and in this paper we work with GeoGebra and Maple. The representations of the results may be different, but the mathematical meaning is the same. In this paper, we deal more with envelopes than with geometric loci; as, in the literature, there are more occurrences of computations of geometric loci, and also as GeoGebra has more commands for this determination than for envelopes, we discuss here the case of loci, whose automated methods have seen important developments along the years; see for example [10, 1, 9, 27]. Envelopes encounter similar issues.

We illustrate this remark with the following example (for the sake of simplicity, in 2D). Recall the bifocal definition of a Cassini oval:

**Definition 4** *Let be given two points $F_1$ and $F_2$ in the plane, and a positive real number $f$. The geometric locus of points $M$ in the plane such that $MF_1 \cdot MF_2 = f$ is called a Cassini oval. The points $F_1$ and $F_2$ are the foci of the oval.*

A plot and an equation of the curve can be obtained using GeoGebra's **LocusEquation** command, as shown in Figure 16. This is the equation appearing in the algebraic window labelled as *eq1*. The curve plotted on the screen is not a geometric construct recognized by the software; as a consequence, it is impossible to define a **Point on Object** (clicking with the mouse on the curve does not provide anything). Therefore, the equation has been copied (using the sequence Ctrl-C Ctrl-V). The new version is labelled by *eq2*. The new plotted curve looks identical to the previous one, but now it is a geometric object recognized by the software. It is now possible to define a Point on Object, and to continue with geometric constructs, such as the tangent and the normal to the curve at this point.

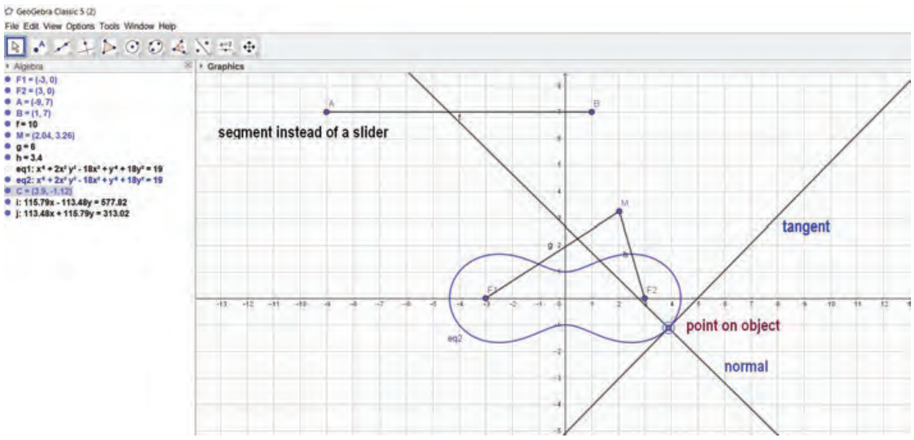



Figure 16: A Cassini oval as a geometric loci obtained with automated commands

At this point, it is important to mention that, instead of a slider for the parameter $f$, we used a segment $AB$, this in order to define every object as a geometric construct. Otherwise, the

[10]CoCoA stands for Computations in Commutative Algebra.

command **LocusEquation** may not work. Modifying the segment modifies the curve labelled *eq1*, but not the curve labelled *eq2*. This has still to be done by hand. Such issues have been met when studying offsets of Cassini ovals [24],and are under study in a subsequent ongoing work.

## A.3 Envelope

The same remark is generally valid for the output of the **Envelope** command, and nothing ensures that the **Point on Object** will provide the expected result. Actually, using the already mentioned GeoGebra applet for constructing an astroid as an envelope of unit segments, we could attach a point to the astroid. This is due to the fact that the construct is purely geometric.

We must mention a phenomenon which can often occur: the output on the screen may contain irrelevant components. This is the case with the following applet for the automated construction of a nephroid; see Figure 17, which displays a screenshot of the online applet. In this example, we obtain the nephroid as an envelope of the family of circle centered on the unit circle and tangent to the $y-$ axis. The construction follows steps similar to what is explained in

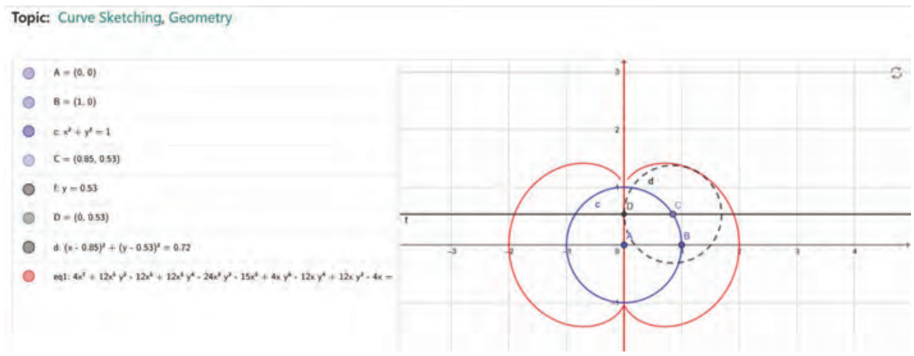



Figure 17: Automated determination of a nephroid

Section 3; in particular, the construction is fully geometric, i.e., does not use equations of lines and circles. The last command, namely **Envelope**($d$,$C$) (with the notations of the figure) yields both the plot (in red) and the corresponding equation (we reordered according to decreasing degree):

$$4x^7 + 12x^5y^2 + 4xy^6 + 12x^3y^4 - 12x^5 - 12xy^4 - 24x^3y^2 - 15x^3 + 12xy^2 - 4x = 0. \qquad (14)$$

A factor $x$ appears clearly, corresponding to he $y-$axis, the other factor (of degree 6) can be easily proven to be irreducible. It determines the nephroid. For geometric reasons, it is understood that only the segment of the $y-$axis inside the unit circle can be part of an envelope, but this cannot be determined only with the algebraic methods used by the **Envelope** command. These are based on computations of Gröbner bases and elimination, whence actually providing the Zariski closure of the requested envelope; see [13, 38]. More exotic examples can be found, they are beyond the scope of the present paper.

It is now time to recall the sunny cup of coffee from Figure 2. An automated construction is available as an applet by Z. Kovács. Other constructions of a cardioid as an envelope exist, but they cannot be treated with the automated tool used here. We could now be able to reproduce part of the work in [15] for the determination of isoptics of an astroid. This will be done in a subsequent paper.

Another situation where recognition of a construct as a geometrical object is important is the determination of offsets of curves. This is often included in the general framework of envelopes (see [12]), but in fact requires different methods, belonging to the determination of a geometric locus; for example, see [16]. An ongoing project tries to bypass the problems posed by the lack of recognition by the software of a locus as a geometric object (in which case the involved polynomials are of high degree).

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# Sangaku Mathematics Puzzles: A Catalyst for Cultivating Creative Thinking and Problem-Solving Abilities using The Geometer's Sketchpad

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***Abstract:*** *The purposes of this study were to examine how Sangaku Mathematics Puzzle serves as a catalyst for cultivating students' creative thinking and problem-solving abilities, aided by the dynamic software: The Geometer's Sketchpad. Action research was conducted in the College of Hospitality Industry Management, Suan Sunandha Rajabhat University, Bangkok, Thailand, in the year 2022. A total of 19 students studying in the second year of their bachelor's degree in education, majoring in mathematics, participated in this study. The duration of the action research project was about two months. A flipped classroom model incorporating cooperative learning and the Geometer's Sketchpad was employed in this study. These methods were used in line with the policy of the Ministry of Education in Thailand during the COVID-19 pandemic situation in the years 2021–2022. The research findings showed that with the combination of Sangaku puzzles and The Geometer's Sketchpad, students are encouraged to think outside the box and approach mathematical problems from different perspectives. In addition, the flipped classroom and cooperative learning model encourages teamwork and communication among students, promoting a deeper understanding of mathematical principles.*

## 1. Introduction

The Ministry of Education announced the policy of education in Thailand during the COVID-19 pandemic situation in 2021-2022: teaching and learning must carry on to ensure continuity of learning. The Secretary-General of the Office of the Basic Education Commission (OBEC), Mr.Amporn Pinasa, revealed that it is not necessary for the school to be closed and that teachers will teach online only. OBEC gave a policy to all schools to prepare for the five teaching methods [1]. The schools must provide teaching methods based on the context and appropriateness of each school.

The details of the five methods are as follows:

1) On-air refers to teaching via TV, satellite systems, and cable TV systems.
2) Online refers to teaching via the Internet, internet video conference, school, and other systems. Students learn through their computers, smartphone, or tablet.
3) On-hand method refers to teaching without viewing devices. Teachers will prepare learning materials, books, exercises, and worksheets for students to study at home with the help of their parents.
4) On-demand learning refers to teaching and learning through mixed media in electronic formats, such as watching videos, Live teaching notes to review lesson content after live instruction or studying the worksheets through Google Classroom or other applications designed by the teachers. The On-demand method allows students to study at their own pace whenever and wherever they want to, and
5) On-site method refers to in-person face-to-face learning lessons and/or enrichment activities at school or any place.

The students have to adjust the way they learn, and teachers have to change the way they teach, and they must be prepared for any situation that may arise. The Ministry of Education announced that on-demand learning materials are being increased, along with the production and distribution of workbooks that students can use at home. In addition, Dr. Ekachai Kisupan, the chairman of the Basic Education Committee, stated that we need to improve the knowledge and skills of mathematics teachers to deliver integrated mathematics instruction that emphasizes critical thinking. Teachers' competencies must meet the professional standards of teachers and can help students develop their critical thinking and problem-solving abilities.

Most importantly, the student-teacher must increase their skills and competencies in using creativity and modern information technology resources [1]. This is because the student-teacher will become a teacher in a few years. Based on the case study of Vanpetch, Y., and Sattayathamrongthian, M. [2], they found that the education obstacles due to the Covid-19 pandemic impact all sectors, however with every crisis, there is always an opportunity for a pedagogical review and learning something new. There is evidence that learning online can be effective. With the experience gained during the COVID-19 crisis, new digital learning possibilities could be implemented by educational institutions to stimulate the productivity of the lessons. Potential innovations include educational applications, platforms, and resources.

## 2. Flipped Classroom Model in Mathematics

The flipped classroom model employed in this study is based on Bergmann and Sams' research findings [3], the flipped classroom is a pedagogical model in which the typical lecture and homework elements of a course are reversed. Students watched a short video lecture at home before the class session, while during the in-class time, the students did the exercises, discussions, or projects. The flipped classroom model is a type of blended learning that increases students' engagement in learning mathematics. In the traditional teaching method, students learn mathematics content in the classroom and practice it at home. Mathematics knowledge can be divided into two parts: conceptual and procedural. Learning mathematics is more than computation, it is more than memorizing rules and facts. It is an investigation, exploring, experimenting, posing problems, and solving problems. To understand mathematics, students should be cognitively active while they learn.

Skemp [4] described understanding as the mental experience of a subject by a student relating an object to another object. A personal understanding of a concept involves grasping or acquiring the meaning of the object. According to Johnson and Johnson [5], small group work such as cooperative learning must be used in place of the traditional textbook lecture, whole-class discussion, and individual worksheets method of teaching in order to help students understand mathematics, communicate mathematically, develop confidence in their own mathematical ability, and be able to solve problems solving in mathematics.

## 3. Cooperative Learning: Group Investigation

Cooperative learning is one of the suggested teaching and learning approaches in student-centered classes and its use is consistent with the theories of learning and how children learn mathematics. There are a variety of cooperative learning methods based on the social psychological principles of cooperative learning. Cooperative learning methods have been adapted from different methods to meet the practical requirements of classrooms. Cooperative learning is a group-learning process built on the belief that students learn better when they talk and work together. Cooperative learning encourages group interaction using assigned roles, with each member sharing responsibility for the group and the work produced. Johnson and Johnson [6] described that the basic principles of cooperative learning must include four principles: positive interdependence, individual accountability, equal participation, and simultaneous interaction. Slavin [7] defined cooperative learning as a teaching method in which students work together in mixed-ability groups toward a common goal. Sharan [8] supported the idea that cooperative learning is a motivational strategy in which students collaborate in small groups to complete specific learning goals. Slavin explained that the essential components of cooperative learning consist of three concepts: team rewards, individual accountability, and equal opportunities for success. Slavin [9] explained in his research findings that cooperative learning can promote intrinsic motivation, which is a key element in successful teaching and learning. Cooperative learning can help students develop positive attitudes, which can generate higher self-esteem, and increased achievement. Through cooperative learning, students can increase their communication skills by interacting with team members. They can become actively involved in the learning process and therefore interested in what they are expected to learn. Research findings by Khairiree [10] show that cooperative learning methods like Math-Jigsaw do motivate learning-resistant students to want to learn and generate higher performance than would have been achieved in traditional classes.

### Cooperative Learning: Group Investigation

Cooperative Learning Method: Group-Investigation used in this study was based on the cooperative learning method developed by Sharan and Sharan [11]. Sharan described that the Group-Investigation model involved students working together in small groups with four basic features: interaction, investigation, interpretation, and intrinsic motivation. All these features are incorporated into the six steps of the Group-Investigation model as follows:

1) Class determines subtopics and learning procedures.
2) Students selected their own groups of 4-5 members according to their interesting subtopic. The group plans its investigations.
3) Groups work according to their plans.
4) Groups make their presentations, and
5) Teacher and students evaluate their projects.

Students work together in groups and carry out their plans. The teacher acts as a facilitator follows up and discusses with students in every step until they complete their projects.

## 4. Sangaku Mathematics Puzzles and the Geometer's Sketchpad

### Sangaku Mathematics Puzzles

The Japanese mathematical puzzle known as Sangaku first appeared in the Edo period (1603-1867). The Japanese word "Sangaku" means "mathematical tablet". These tablets presented numerous mathematical issues and geometric theorems in gorgeous calligraphy and elaborate pictures. Japanese people during the Edo period wrote mathematics problems on wooden boards or tablets and hung them under the roofs of Buddhist temples and Shinto shrines [12].

Sangaku looks like a work of art, and the figure is beautifully drawn in color. Each wooden tablet typically contains several illustrated puzzles or theorems, many Sangaku puzzles were based on Euclidian geometry. Sangaku puzzles were not only intricate works of art but also difficult mathematical challenges. They aimed to demonstrate creative problem-solving and comprehension of complex mathematical ideas. The uniqueness of Sangaku is the content of the problems. Ordinary triangle geometry mainly concerns the properties of a triangle. However, Sangaku's problems are concerned with some relationships arising from several mixed geometric figures like circles, triangles, and squares.

Sangaku usually presented their problems in three different sections:

1) *Problem text*: An introductory section describing the diagram and indicating which figure from the diagram the observer needs to find the value.
2) *Answer*: Section, either providing the numerical value of the sought figure or advising the reader to the following section.
3) *Formula*: Section containing a formula for calculating the solution.

Some Sangaku tablets were preserved from the many more produced in the country at the beginning of the Edo period. Today, these Sangaku tablets are preserved in museums and temples in Japan [13]. Geometry is one of the core contents in the syllabus of mathematics in Thailand. The contents and topics are similar to Sangaku puzzles, such as circles, triangles, squares, and Pythagoras's theorem. Teachers can use the Sangaku puzzle as a mathematics problem to design challenge worksheets for their students to solve the Sangaku puzzles. However, some Sangaku puzzles are not easy to draw. Majewski, Chuan, and Hitoshi [14] supported the idea that many Sangaku puzzles are not as simple to construct as they appear in the figures. Especially when circles are inscribed into something more complicated than a triangle, this can lead to very elaborate constructions. They explained that we can examine Sangaku mathematics puzzles using dynamic geometry software, try to construct some of them, and validate them using a computer software program.

**The Geometer's Sketchpad and Sangaku Puzzle Construction**

The Geometer's Sketchpad (GSP) is one of the dynamic mathematics software programs that provide opportunities for students to investigate and discover mathematics concepts, in particular geometric patterns [15]. Since 2004, the Ministry of Education Thailand has purchased a National license for the Geometer's Sketchpad and distributed it to schools and universities. GSP was translated into Thai language and widely used in Thailand. More than 1,000 mathematics teachers were trained to use GSP as a tool in their mathematics classes. After attending the workshops, many teachers consequently conducted action research in mathematics and incorporated the use of GSP as a tool in their classes. In using GSP, students learn by exploring, investigating, and discovering. GSP helps students visualize abstract mathematical relationships and various problem structures using pictorial representations [16]. The Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education, Thailand announced in the mathematics curriculum that the Geometer's Sketchpad is one of the indicators of software in mathematics. The teachers have to use GSP as a tool to teach mathematics subjects [17]. The teacher should enable students' abilities to use GSP to construct mathematics puzzle problems. These will enhance students' learning of mathematics with creative thinking skills, thinking carefully, and having fun. The teachers can include Sangaku puzzle topics to teach in any one of the five delivery formats that the Ministry of Education, Thailand has designated: on-air, online, on-hand, on-demand, or on-site.

## 5. Action Research in Mathematics Class

Action research was conducted in the College of Hospitality Industry Management, Suan Sunandha Rajabhat University, Bangkok, Thailand, in June-July 2022. A total of 19 students studying in the second year of their Bachelor of Education, majoring in mathematics, participated in this study. The students enrolled in the course BMA 3301: Blended Learning in Mathematics. The duration of the action research project was about two months.

A flipped classroom model incorporating cooperative learning: Group Investigation and the Geometer's Sketchpad were employed in this study. These methods were used in line with the policy of the Ministry of Education in Thailand during the COVID-19 pandemic situation in the years 2021-2022. The action research questions were:

1) How can a flipped classroom model be implemented using Group-Investigations and the Geometer's Sketchpad on Sangaku mathematics puzzles?
2) How does the Geometer's Sketchpad impact the students' creative thinking of mathematics and assist them in constructing Sangaku puzzles?

**Research Question 1:**
How can a flipped classroom model be implemented using Group-Investigation and the Geometer's Sketchpad on Sangaku mathematics puzzles?

The following activities showed how the author who acts as a teacher implemented the Cooperative learning method: Group-Investigation and the Geometer's Sketchpad on Sangaku mathematics puzzles in this research study. The steps of the cooperative learning method: Group-Investigation based on Sharan [15] were as follows:

Step 1: The class determines subtopics and learning procedures. The Sangaku mathematics puzzle was identified into four Sangaku puzzle subtopics, which are as follows:

**Sangaku Puzzle 1:** Satimiya Three Tangent Circles.

Given three circles tangent to each other and to a straight line as shown in the diagram. The radius of the large circle is $\mathbf{r_1}$, and the radius of the medium circle is $\mathbf{r_2}$.

What is the radius (**r**) of the small circle?

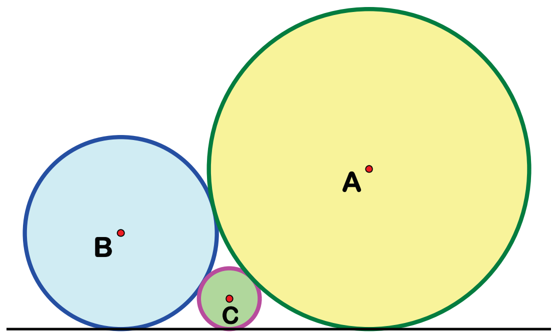


Figure 1: Three Tangent Circles

**Sangaku Puzzle 2:** Yoshifuji Mishima Problem

As shown in the diagram, there is a right-angle triangle which contains a circle, and an equilateral triangle.
What is the side length of the equilateral triangle in terms of $x$ and **y**?

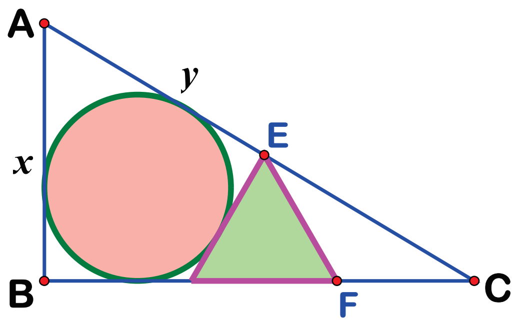


Figure 2: Sangaku Yoshifuji Mishima Problem

**Sangaku Puzzle 3:** Sangaku Problem 47

As shown in the diagram: Maximize **y** as a function of $x$ assuming BC = $a$ is constant.

(Sangaku Problem 47: Sacred Mathematics: Japanese Temple Geometry/Fukagawa Hidetoshi, Tony Rothman)

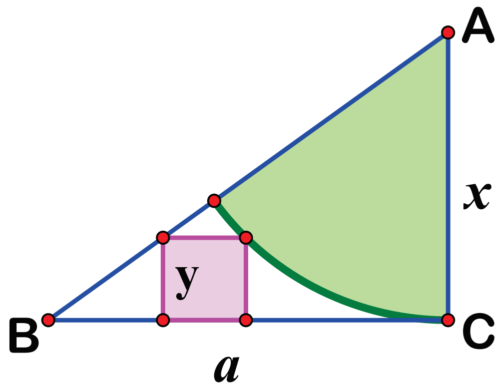


Figure 3: Sangaku Problem 47

**Sangaku Puzzle 4:** A Square-Three Arcs and a Circle

As shown in the diagram, Sangaku in a square-Three arcs and a circle.

Find a relationship between the radius of the circle and the side of a square.

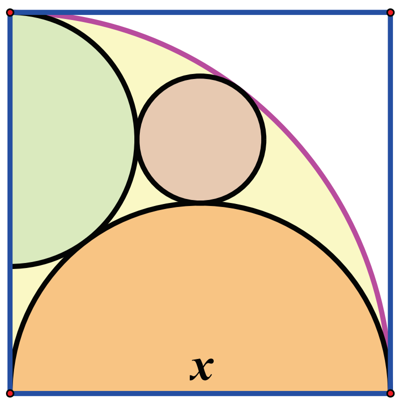


Figure 4: Sangaku in a square-3 arcs and a circle

Step 2: Students selected their own groups of 4-5 members according to their interest in the Sangaku puzzle. Each group is a mixed-ability group.

Step 3: Students plan their investigation and discuss it in their own groups. They plan to use the Geometer's Sketchpad to construct the Sangaku puzzle on their selected puzzle.

Step 4: Using a flipped classroom model. Students work in groups according to their plans inside the classroom and through an online platform. They constructed a Sangaku puzzle using the Geometer's Sketchpad. The students had a whole class discussion, and the teacher was a facilitator, helping them when needed.

Step 5: Each group made the presentation. Students explained to the whole class how they came up with their solutions and how to use the Geometer's Sketchpad to construct the Sangaku puzzle.

Step 6: The teacher and students evaluate the group projects based on the preset rubric.

**Research Question 2:**

How does the Geometer's Sketchpad impact the student's creative thinking of mathematics and assist them in constructing Sangaku puzzles?

The example of constructing the Sangaku Problem that impacts students' creative thinking in this paper is the Sangaku Puzzle 1: Samiya Three Tangent Circles.

*Sangaku puzzle-1*: Satimiya Three Tangent Circles

Given three circles tangent to each other and to a straight line as shown in the diagram. The radius of the large circle is $r_1$, and the radius of the medium circle is $r_2$.

a) What is the radius of the small circle?
b) How to construct the Three Tangent Circles.

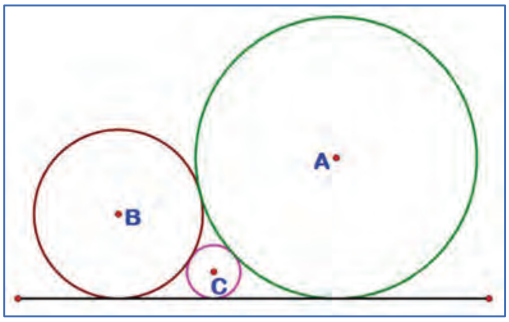


Figure 5 Sangaku Puzzle 1

a) What is the radius of the small circle?

*(i) Find FE:*

The right triangle ABH. From Pythagoras Theorem:

$AH^2 + BH^2 = AB^2$

$BH = FE$

$(r_1 - r_2)^2 + FE^2 = r_1^2 + r_2^2$

After a few calculation steps:

$FE = \sqrt{4r_1r_2}$

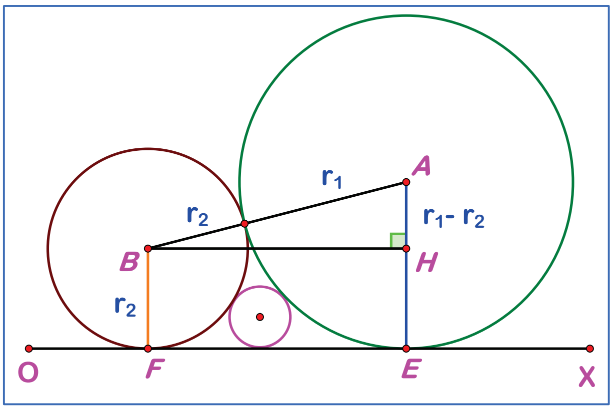


Figure 6 Find Distance FE

*(ii) Find the radius of the small circle?*

From the diagram: BCM and ACN are right angles triangles:

From Pythagoras Theorem:

$MC^2 = BC^2 - BM^2$; $MC = \sqrt{BC^2 - BM^2}$

$CN^2 = AC^2 - AN^2$; $CN = \sqrt{AC^2 - AN^2}$

$MC + CN = \sqrt{BC^2 - BM^2} + \sqrt{AC^2 - AN^2}$

From (i): $MC + CN = MN = FE = \sqrt{4r_1r_2}$

From the diagram:

$BC = r_2 + r$; $BM = r_2 - r$; $AC = r_1 + r$; $AN = r_1 - r$

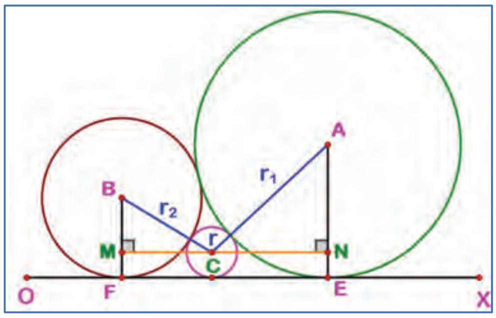


Figure 7 Find Radius of Small Circle C

$$\sqrt{(r_2+r)^2-(r_2-r)^2} + \sqrt{(r_1+r)^2-(r_1-r)^2} = MN = FE = \sqrt{4r_1r_2}$$

We have:

$[(r_2 + r)^2 - (r_2 - r)^2] + [(r_1 + r)^2 - (r_1 - r)^2]$

$= [\{r_2^2 + 2rr_2 + r^2\} - \{r_2^2 - 2rr_2 + r^2\}] + [\{r_1^2 + 2rr_1 + r^2\} - \{r_1^2 - 2rr_1 + r^2\}]$

$= [r_2^2 + 2rr_2 + r^2 - r_2^2 + 2rr_2 - r^2] + [r_1^2 + 2rr_1 + r^2 - r_1^2 + 2rr_1 - r^2]$

$= [4\,rr_2] + [4\,rr_1]$

Therefore:

$$\sqrt{(r_2+r)^2-(r_2-r)^2} + \sqrt{(r_1+r)^2-(r_1-r)^2} = \sqrt{4r_1r_2}$$

$$\sqrt{4rr_2} + \sqrt{4rr_1} = \sqrt{4r_1r_2}$$

$$\sqrt{rr_2} + \sqrt{rr_1} = \sqrt{r_1r_2}$$

After a few calculation steps, we will obtain:

⇨ $$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r}}$$

and; $$r = \frac{r_1r_2}{(\sqrt{r_1} + \sqrt{r_2})^2}$$

The example of students' group work on constructing *Sangaku puzzle-1*: Satimiya Three Tangent Circles using the Geometer's Sketchpad is as follows.

- Open the Geometer's Sketchpad, in the File Menu, choose New Sketch, and follow the instruction step-by-step.
- Used the *Number menu* to construct parameters: $r_1 = 8$ cm, and $r_2 = 5$ cm.
- Construct a circle A of radius 8 cm, tangent to a straight line OX at point E;
- AP is a diameter length = $r_1 + r_2$
- The circle with diameter AP intercepts line OX at point D.

We have the length $ED = \sqrt{r_1r_2}$

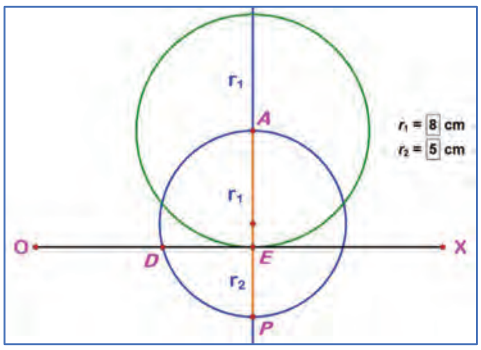


Figure 8: Construct a Large Circle A

- Construct a circle at D with a radius DE, intersect OX at point F
- Construct a line PD;
- Construct a perpendicular at point F, and intersect a line PD at a point B
- BF is a radius of circle B,

  Therefore B is the medium circle with a radius $r_2$.

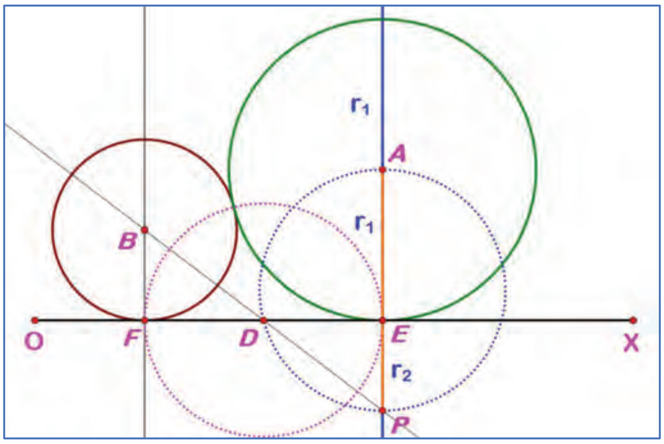


Figure 9: Construct a Medium Circle B

Construction a circle C with radius r = $\frac{r_1 r_2}{(\sqrt{r_1}+\sqrt{r_2})^2}$

- Construct a circle at point A with a radius $r_1 + r$ or $r_1 + \frac{r_1 r_2}{(\sqrt{r_1}+\sqrt{r_2})^2}$ intersect AP at point K
- Construct a circle at E with a radius EK, intersect AP at point T
- Construct a perpendicular at point T, and intersect a circle at point C

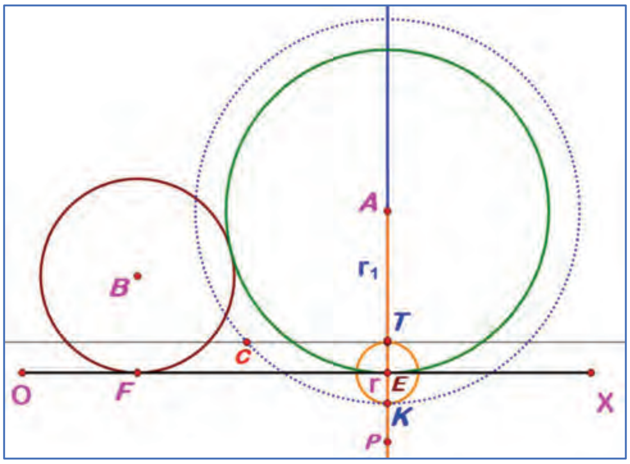


Figure 10: Find Center of Small Circle C

- Construct a perpendicular at point C, and intersect OX at point Q
- CQ is a radius of circle C,
- Therefore, C is the small circle with a radius ***r*** or

  $\frac{r_1 r_2}{(\sqrt{r_1}+\sqrt{r_2})^2}$.

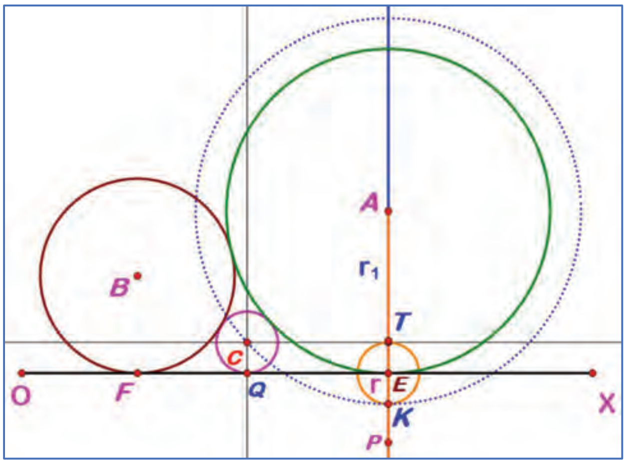


Figure 11; Construct a Small Circle C

- Construct circle interior of the three circles, select color as needed.
- Hide the circles and lines, we do not want to show.
- Completed Sangaku Puzzle 1 Satimiya Sangaku:Three Tangent Circles was shown on the right.

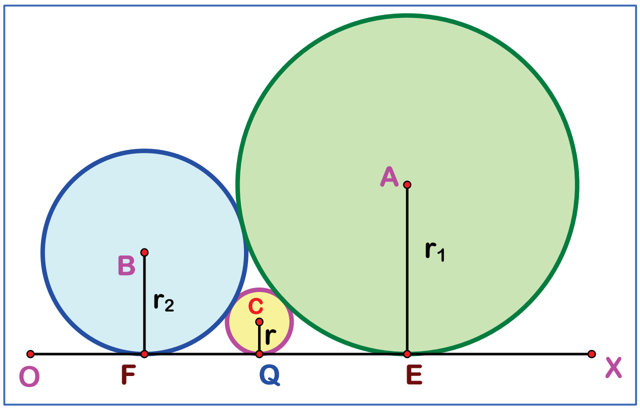

Figure 12: Satimiya Sangaku:Three Tangent Circles

**Students' Investigation**
With the use of the Geometer's Sketchpad, students animated the value of parameter $r_1$ and $r_2$ to investigate the relation of the "Sangaku: Three Tangent Circles". They found that no matter what the size of the two radii $r_1$, and $r_2$, the radius (r) of the small circle is always equal to $\frac{r_1 r_2}{(\sqrt{r_1}+\sqrt{r_2})^2}$ and the three circles tangent to each other and to a straight line.

## 6. Research findings

The research findings revealed that at the beginning of the research study during the Covid-9 pandemic the students had problems with communication within their group through online platforms. Some group members did not know how to solve the problem and could not use GSP to develop their Sangaku puzzles. Sangaku takes up a lot of problems about the figure, such as the properties of circles, and it is not easy for students to solve such problems. The students learned through a combination of pedagogical methods, using dynamic technology in mathematics, traditional face-to-face in-class, and online learning platforms. That was used in cooperative learning: Group Investigation, the Geometer's Sketchpad in constructing Sangaku mathematics puzzles. This knowledge and experience impact students' confidence in learning and teaching mathematics in the future.

## 7. Conclusions

In conclusion, the integration of Sangaku Mathematics Puzzle with The Geometer's Sketchpad presents an exciting and effective way to foster creative thinking and problem-solving skills in students. It empowers them to tackle intricate mathematical challenges, encourages collaborative learning, and ultimately enhances their overall mathematical proficiency. Overall, this integration ignites a passion for mathematics, making learning a dynamic and enjoyable process. Students build confidence in their abilities and become adept at tackling mathematical problems, not merely as exercises, but as opportunities for growth and enrichment. The combined power of the Sangaku Mathematics Puzzle and the Geometer's Sketchpad paves the way for a new generation of innovative thinkers, equipped with the essential skills needed to excel in various academic and professional endeavors. The "Sangaku Mathematics Puzzle" serves as a catalyst for cultivating creative thinking and problem-solving abilities, aided by the innovative tool, "The Geometer's Sketchpad."

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# Interactive visualization of curvature flows

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**Abstract**

*After a short motivation, we introduce several different curvature flows: A naive flow on the curvature under the heat equation, the curve-shortening flow, the mean curvature flow for imbedded surfaces, and the Ricci flow on surfaces of revolution and for abstract 2-manifolds. The main focus is on interactive visualizations using animations of curves, surfaces, and in the case of the Ricci also using flow on a metric field similar to Tissot's indicatrix. We refer to and briefly demonstrate an existing applet for the curve-shortening flow, and present our own code written in SageMath / Python (and MAPLE) for other flows, including for the Ricci flow on surfaces of revolution, thus recreating animations first presented by Rubinstein and Sinclair. Our code is publicly available on GitHub and invites for further experimentation, especially with different initial shapes.*

## 1 Introduction

Curvature flows are a beautiful mathematical topic and are under intense study in much cutting edge current research, while they are also very tangible and accessible to even school students in the form of science experiments with soap bubbles and films [14]. Calculus students are familiar with problems of minimizing area given a perimeter constraint. The classical version that does not limit itself to rectangles is the isoperimetric problem or Dido's problem which has had a huge impact on mathematics, and is a first example in the study of calculus of variations. [3]. Rather than just looking at static minimal (networks of) curves and surfaces, it is natural to try to continuously deform arbitrary such into minimal ones. Great models for the way nature does this are various notions of *curvature flows*. The main idea is that curvature should spread out, just like a giant soap bubble evolves towards having everywhere constant curvature, eventually becoming a sphere. Similarly to temperature evolving on a metal plate, with hot spots becoming cooler, and cold spots warming up. Eventually, the temperature variations on the plate should *diffuse* and eventually yield a constant temperature. Indeed, many curvature flows are solutions of partial differential equations that somewhat look like the heat, or diffusion

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equation $u_t = u_{xx} + u_{yy}$ – but with the second partials replaced by nonlinear functionals of $u = u(t, x, y)$ that involve various notions of curvature of the graph or surface.

Curvature flows have been highly publicized in the past 25 years, especially Hamilton's Ricci flow that Perelman utilized to prove the Poincaré conjecture [21]. However, the Ricci flow is a highly abstract flow lives on abstract manifolds, and is is very inaccessible to non-experts. Our objective is to present several notions of curvature flows that share fundamental underlying properties, yet are much more accessible and tangible than the general Ricci flow.

In general, differential geometry distinguishes between extrinsic and intrinsic properties of curves, surfaces, and manifolds. Roughly speaking, the curvature of a space curve and the mean curvature of a smooth surface in 3-space are extrinsic because they depend on how the curve or surface is imbedded in the ambient (surrounding) 3-space. On the other hand, the Gauss and Ricci curvature only depend on the metric, that is, the infinitesimal notion of distance. For example, the Gauss curvature, and geodesics (shortest curves) on a sheet of paper do not change no matter whether the paper is rolled into a cylinder, or cone, or lies flat on a table. The next more intriguing examples are flat tori and Klein bottles that are *constructed* (in our minds) by identifying (gluing together) opposite sides of a square in the plane (without or with reversing orientation). Nonetheless, given the original notion of distance in the plane (accommodating crossing the glued together edges like in a Pac-Man game), one still has well-defined notions of geodesics, namely straight lines in the rectangle picture. With their zero Gauss curvature, these abstract surfaces that cannot be imbedded in 3-space while preserving the original metric, or notion of distance.

We start with simple experiments with plane curves evolving under different kinds of curvature flows. Then we give some brief remarks about the *extrinsic* well-known curve-shortening and mean-curvature flow for curves and surfaces, respectively. Continuing our main focus is on *intrinsic* geometric flows, we experiment with the Ricci flow for two-dimensional surfaces in the very special case of surfaces of revolution. Throughout, the focus is presenting tools, and discussing their use for interactive visualization of the ensuing deformations. Data are generally assumed to be smooth, i.e., having continuous (partial) derivatives of all orders, unless specified otherwise.

# 2 A simple model at the level of multi-variable calculus

According to the fundamental theorem of curves every smooth immersed (i.e., everywhere nonzero derivative) curve $\gamma\colon I \mapsto \mathbb{R}^2$ in the plane is, up to Euclidean motions (translations and rotations) completely determined by its curvature.

We illustrate how to recover the curve from its curvature, and then animate many examples, especially closed curves that straighten out to become circles. [In the literature, length constrained curvature flows are discussed primarily as a modification of the curve-shortening flow (see the next sections), e.g., [18].]

Throughout $I$ stands for an interval, ray or $\mathbb{R}$ or the circle $I = S^1$. Without loss of generality, we assume that $\gamma$ is parameterized by arclength, customarily denoted by $s$, and thus $|\gamma'| = 1$. Denoting the (unit) tangent vector $T = \gamma'\colon I \mapsto \mathbb{R}^2$, for plane curves it is customary to define a unit normal vector field $N\colon I \mapsto \mathbb{R}^2$ along $\gamma$ such that $\{T, N\}$ is positively oriented, i.e., with common abuse of notation $\det(T|N) = +1$. With this, the *signed* curvature of plane curves is defined as $\kappa\colon I \mapsto \mathbb{R}$ by $\kappa = \langle \gamma'', N\rangle$ where $\langle \cdot, \cdot rangle$ is the standard inner product on $\mathbb{R}^2$.

Given any smooth function $\kappa\colon I \mapsto \mathbb{R}$ and initial data $\gamma(0) = p \in \mathbb{R}^2$ and $\gamma'(0) = (\cos\theta_0, \sin\theta_0)$ for some $\theta_0 \in \mathbb{R}$, the first antiderivative yields for $s \in I$ and $|s|$ sufficiently small $\theta(s) = \int_0^s \kappa(\sigma)\,d\sigma$. Note, that $\kappa\colon S^1 \mapsto \mathbb{R}$ *periodic* does not imply that its antiderivative $\theta$ is periodic. A further integration yields a parameterized curve $\gamma\colon I \mapsto \mathbb{R}^2$ (with the same caveat about periodicity as above) that is the unique curve (parameterized by arclength) whose curvature function agrees with $\kappa$.

A few suggestions for interactive classrooms:

- Start with the traditional *straightforward* direction: Given a formula for a curve, plot it and propose what the graph of $\kappa$ may look like. Then use basic differential calculus (using the usual formulas for curves not necessarily parameterized by arclength) to calculate its curvature, and plot it to confirm the guess – or elaborate what mistakes were made.
- Given a sketch of the graph of $\kappa$ try to sketch a curve whose curvature looks like $\kappa$. Confirm using computing technology such as our Sage-applet or MAPLE-worksheet. Typical examples for the curvature $\kappa(s)$ should/may include (for constants $a$, $b$, $c \in \mathbb{R}$ with various signs) $\kappa = c$, $\kappa\colon s \to c + a\cos bs$, $\kappa\colon s \to a/(b+s)$ for $s > -b$, and $\kappa\colon s \to a/(1+bs^2)$. In the latter, pay special attention to the graphical meaning of $\int_{-\infty}^{\infty} \kappa(\sigma)\,d\sigma$.

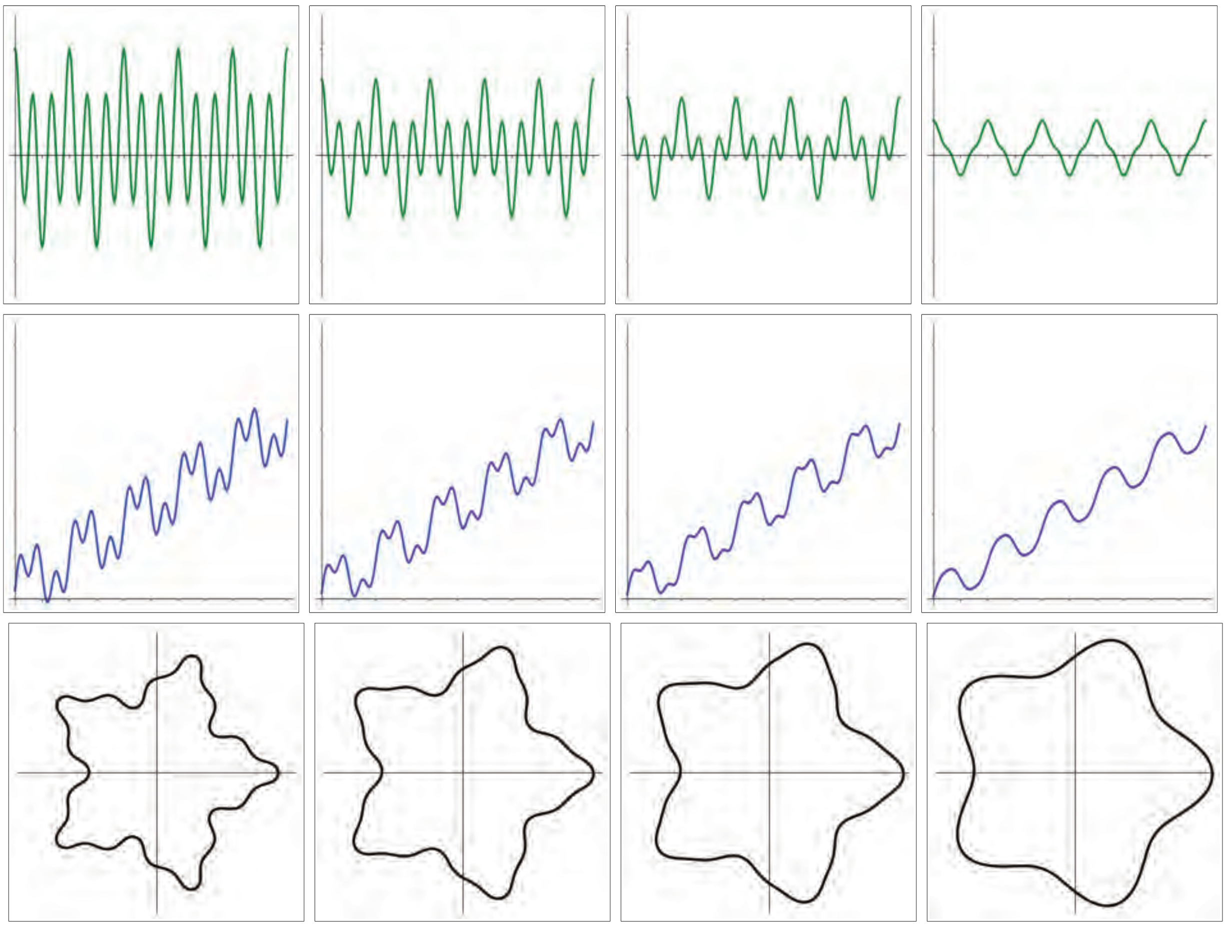

Figure 1: Curvature flow on closed curves with fixed length. Time increases left to right.

Calculations involving arc-length are notoriously hard to complete within the traditional set of elementary functions. This is a good place to explore what computer algebra can do, and else explore the trustworthiness of naive numerical approximations using popular computing packages or applets like the one we provide.

After this *warm-up* let us get started with curvature flows. Instead of considering individual curves, in general much more insight is gained by considering parameterized families of curves. The above with $a$, $b$, $c \in \mathbb{R}$ are typical algebraic examples. Here and in the sequel we want to consider a single parameter $t \in J \subseteq \mathbb{R}$ which we identify with the tangible notion of time. Technically for some $T \in (0, \infty]$ consider smooth functions $\gamma\colon [0,T) \times I \mapsto \mathbb{R}^2$ and the corresponding $\kappa\colon [0,T) \times I \mapsto \mathbb{R}$.

Similar to the much better known and well-studied case of the curve-shortening flow (which we briefly address in the next section), one simple idea is to ask that curvature *spreads out*, *tries to equalize*: Short, highly curved segments try to straighten out by pushing some of their curvature to neighboring segments. For curves with a *fixed total length* (i.e. fixed domain $I$) this suggests posing the diffusion (or heat) equation: The rate of change of curvature with time is a positive multiple of the second derivative of curvature with respect to its natural parameter arclength $s \in I$ (using subscripts to denote partial derivatives)

$$\kappa_t = c\,\kappa_{ss} \tag{1}$$

This very simple example is accessible to multivariable calculus students. Indeed, in the second author's multivariable classes, students routinely use EXCEL worksheets to estimate numerical derivatives and explain their meaning – and usually culminating in numerically solving the heat equation and graphically observing *diffusion* of hot and cold spots.]

Unlike the much more involved curve-shortening flow addressed in the next section, this is just the elementary *linear* heat equation.

The first examples suggested in the above *warm-up* provide good starting conditions – make sure that before doing any computations to test your (and your students) intuition and sketch the graphs of the evolving curvature functions and their corresponding curves.

We find that some of the most attractive introductory examples are given by initial curves that are (low order) trigonometric polynomials (or finite Fourier series) such as

$$\kappa(0,s) = a_0 + a_1 \cos s + a_3 \cos 3s. \tag{2}$$

It is a simple exercise to find conditions on the parameters such that the corresponding curve is closed and simple, and indeed attractively look like a flower with many petals. A special advantage of such curves or curvature functions is that the solution of the diffusion equation (1) can easily be written out in closed form using the routine separation of variables technique (for suitable boundary data)

$$\kappa(t,s) = a_0 + a_1 e^{-ct} \cos s + a_3 e^{-9ct} \cos 3s. \tag{3}$$

The corresponding computations are very easy to implement using computer algebra, or *sympy*, or numerical integration (e.g., using *numpy*). Typical results are illustrated in figure 1.

**Extension:** For fun, replace the diffusion equation (1) by the (linear) wave equation to obtain some truly inspiring animations. Note that in this case, the solution can be written down in closed form just as easily as for the diffusion equation. It is immediate to create similar animation with the curvature evolving according to the wave equation, reminiscing the motions undergone by giant soap bubbles mentioned in the discussion.

# 3 Curve Shortening Flow

A common basic question is: "How do physical curves and surfaces and surfaces evolve with time if for example the driving force is given by *surface tension*?" In the case of soap bubbles, modeling surface tension as proportional to surface area, with an the apparent objective is minimizing surface area, (or total length) given a fixed volume (or area). The latter objective is well known as Dido's problem (see also: *isoperimetric inequality*).

Intuitively, every point on the image of the curve $\gamma(t\,\cdot\,)$ shall move in the direction normal to the curve (in the convex direction) at a rate proportional to the (signed) curvature at this point, often written as $\frac{\partial}{\partial t}\gamma(t,s) + \kappa(t,s)\,\mathbf{N}(t,s)$. Using standard subscript notation for derivatives, the flow is again generated by the deceptively innocent looking differential equation

$$\gamma_t = c\,\gamma_{ss}. \tag{4}$$

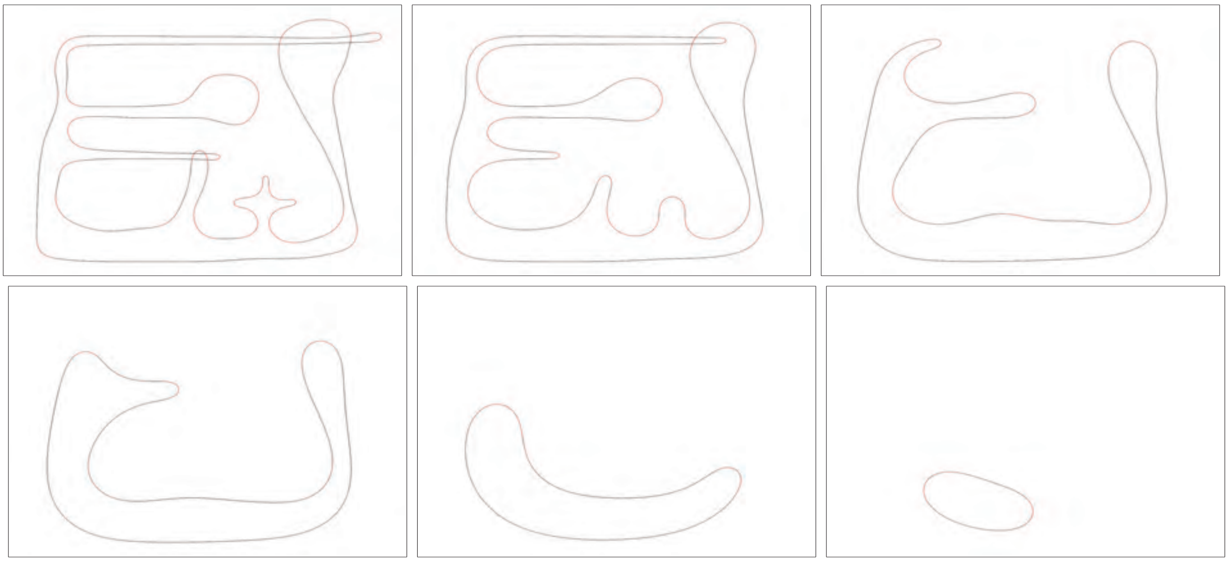

Figure 2: Curve shortening flow: Carapetis' applet

The key observation is that the arclength $s$ of the curve $\gamma(t\,\cdot\,)\colon [0, L(t)] \mapsto \mathbb{R}^2$ depends on the time $t \geq 0$, and thus the partial differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ no longer commute.

We recommend the recent monograph [1] for both a detailed introduction and also a meticulous technical development of properties of the *extrinsic* geometric flows addressed in this and the next section. Most notable is that this flow is the (negative) gradient flow on a space of smooth curves for the length functional (lemma 2.4. in [1]). A central result for the curve shortening flow is the Gage–Hamilton–Grayson theorem, which roughly says that every smooth simple closed curve will remain without self-intersections, become convex, and eventually converge to a *round point*.

While we have no new visualization tools for this flow to present, we highly recommend the wonderful free applet (similar to JavaScript) [8] by A. Carapetis. The user may draw any curve in the plane, and then study the evolution of the curve under the curve shortening flow. Figure 2 provides a simple sequence of screenshots for a simple curve that initially has multiple self-intersections.

# 4 Mean curvature flow for surfaces

The mean curvature flow on smooth surfaces $\subseteq \mathbb{R}^3$ imbedded in Euclidean space is an *extrinsic* flow that is an intuitive analog of the curve shortening flow discussed in the previous section. From a calculus of variations perspective, it is the (negative) gradient flow on a space of smooth closed surfaces for the volume functional.

Heuristically, every point of the surface moves in the direction of the normal vector (with consistently chosen orientation) at the rate of the mean curvature, usually denoted by $H$. It is easiest defined via the *extrinsic* Gauss map $\mathbf{N}\colon S \mapsto S^2 \subseteq \mathbb{R}^3$ that assigns to every point $p$ on the surface $S$ a unit normal vector $\mathbf{N}(p)$ (with consistent orientation). The mean curvature is one half the trace of the differential $d\mathbf{N}$, i.e., the average of its eigenvalues $\kappa_i$

$$H = \frac{1}{2}\operatorname{tr} d\mathbf{N} = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{5}$$

These are known as the principal curvatures, and they are the minimal and maximal *sectional* curvatures of the curve intersections of the surface with planes that contain the normal line at the given point.

The normalized mean curvature flow that preserves volume is familiar as the *surface tension flow* that models, e.g., the motion of soap bubbles, or more general surfaces of drops of liquids.

The literature on the mean curvature is vast, and we again refer to the monograph [1] for both a detailed introduction and also a meticulous technical development of its properties. Most notable are its close connection with the Laplacian, harmonic analysis, and minimal surfaces defined by the vanishing of $H = 0$.

While smooth closed (uniformly) convex surfaces also shrink into a point, they do so in finite time. However, flows initialized by nonconvex surfaces may develop singularities with the curvature becoming unbounded in finite time, see, e.g., [9]. One of the best known examples is a surface in the shape of a smooth dumbbell whose neck's diameter will shrink much faster than the *radii* of the *sphere like* ends, then *pinch off* in finite time and become two disconnected surfaces. Pictorially, such behavior is reminiscent of the Ricci flow in higher dimensions, and is the opposite of the Ricci flow in two dimensions as illustrated in figure 6. Collapsing solutions of ancient flows (i.e., that exist backwards in time to negative infinity) are investigated in [6], while positive results on the flow through singularities are presented in [2].

# 5 Riemannian manifolds and the Tissot indicatrix

This section provides a gentle first step beyond multi-variable calculus towards Riemannian manifolds, that will be utilized in the last two sections in the context of the Ricci flow. Multivariable calculus and first differential geometry courses usually only consider smooth two-dimensional surfaces that are imbedded in three-dimensional Euclidean space.

For simplicity, consider the graphs of the functions $f_\pm\colon (x, y) \mapsto x^2 \pm y^2$, illustrated in the images in the top row of figure 3. This computer illustration uses a very coarse $6 \times 6$ grid of points $(x_i, y_j)$ in the domain and connects the points $(x_i, y_j, f_\pm(x_i, y_j))$ by line segments whose projections onto the $xy$-plane from a regular square grid. Contrary to what the filled-in patches in the computer illustration suggest, in general, the four corners of each patch do *not* lie in any plane. But as the number of grid points increases, the patches are better and

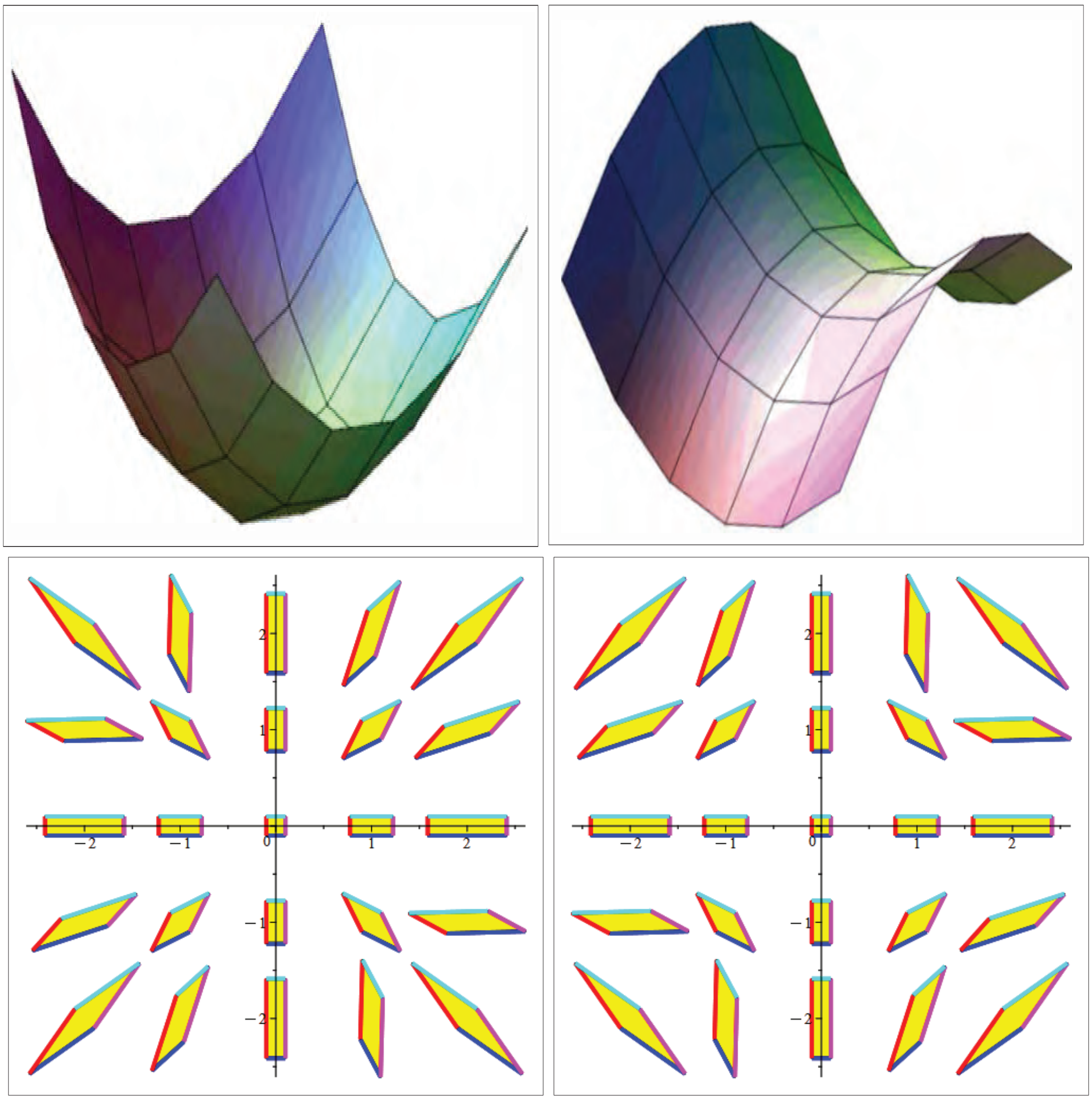

Figure 3: Chopping up surfaces into almost parallelograms

better approximated by (planar!) parallelograms – provided this is the graph of a continuously differentiable function.

Next think of cutting up the surface along these grid lines, and drop the near parallelograms into the plane, as suggested in the images in the bottom row of figure 3. Note that each patch is necessarily at least as large as the $(\Delta x, \Delta y)$ rectangle beneath it, and thus we uniformly rescaled the patches, and chose the orientation to most closely reflect the orientation of the path above in 3-space. This picture shows parallelograms whose sides have lengths identical to the lengths of the averaged opposite sides of the original patch

$$\frac{(x_{i+1}, y_j, f_\pm(x_{i+1}, y_j)) - (x_i, y_j, f_\pm(x_i, y_j)) + (x_{i+1}, y_{j+1}, f_\pm(x_{i+1}, y_{j+1})) - (x_i, y_{j+1}, f_\pm(x_i, y_{j+1}))}{2},$$

$$\frac{(x_i, y_{j+1}, f_\pm(x_{i+1}, y_j)) - (x_i, y_j, f_\pm(x_i, y_j)) + (x_{i+1}, y_{j+1}, f_\pm(x_{i+1}, y_{j+1})) - (x_{i+1}, y_j, f_\pm(x_{i+1}, y_j))}{2}.$$

The key mental exercise is to try to glue (tape) together the respective edges, blue to cyan, and red to magenta, to recover the original surface in 3-space.

Flattening out the patches into plane parallelograms made the curvature inside each surface patch disappear, or, in fancy terms, *move and concentrate* it along the edges and at the vertices. However, on the larger scale the curvature is still highly visible: Think of the geodesic circles of radius $r > 0$ about the center $(0, 0, 0)$ in either graph, that is, the locus of all points that have the same distance $r$ from the origin. In the case of the elliptic/circular paraboloid $z = x^2 + y^2$ this again a circle. Indeed, for geodesic radius $r = 1$, by calculating the arclength along the arc $(t, t^2)$ we find that the geodesic circle with radius $r = 1$ inside the surface is a circle of radius $\rho \approx 0.7639$ in the plane $z \approx 0.5835$, with circumference less than $2\pi$. Conversely, in the case of the hyperbolic paraboloid $z = x^2 - y^2$ one obtains an undulating curve that indeed has a circumference exceeding $2\pi r$. Indeed, these reflect that the elliptic and hyperbolic paraboloids have strictly positive and strictly negative Gauss curvature, respectively.

What matters for us is that by visually patching/gluing together families of parallelograms, it appears quite clearly that in the first case the resulting surface curves to form a bowl shape (possibly upside down), while in the case there is extra length to be used up, which results in an undulating surface such as a saddle.

We continue with a few introductory remarks about abstract manifolds and Riemannian metrics. As the first step, smooth manifolds may be thought of as generalizations of smoothly parameterized curves and surfaces in the plane or in 3-space, but now parameterized by $n$ parameters and being *surface-like* objects in some high dimensional space $\mathbb{R}^N$. Indeed, due to Whitney's and Nash's imbedding theorems (compare the next section), this is a totally legitimate view. However, the modern view considers abstract (differentiable) manifolds as general (topological) spaces. The principal requirements are that every point is contained in a *local coordinate chart* which is an open neighborhood together with a local diffeomorphism onto (an open subset) of a Euclidean space, together with some compatibility condition of overlapping charts. The local diffeomorphism is called a system of local coordinates, and may be thought of as the inverse of a *local parameterization*, as in multivariable calculus.

Whereas traditional surfaces inherit a notion of distance from the ambient (surrounding) space, one endows abstract manifolds with abstract Riemannian metrics (typically, in physical settings, interpreted as energy or momentum *tensors*) which in turn provide inner products on the tangent spaces of the manifold. For a two-dimensional manifold, such a Riemannian metric may be considered (in local coordinates) as a smooth positive definite two-by-two matrix valued functions on the manifold.

As simple examples for Riemannian metrics (using notation similar to [11]) consider a sphere and a hyperbolic paraboloid imbedded in $\mathbb{R}^3$ with (local and global) parameterizations defined by $\mathbf{x}\colon (-\pi, \pi) \times (0, \pi) \mapsto \mathbb{R}^3$ and $\mathbf{y}\colon \mathbb{R}^2 \mapsto \mathbb{R}^3$,

$$\mathbf{x}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \quad \text{and} \quad \mathbf{y}(u, v) = (u, v, u^2 - v^2). \tag{6}$$

In these cases the induced Riemannian metrics (or first fundamental forms) are represented by $2 \times 2$ matrices, in various notations

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = J^T J, \tag{7}$$

where $J$ is either of the Jacobian matrices of partial derivatives

$$J_1 = \left(\frac{\partial \mathbf{x}(\theta,\phi)}{\partial(\theta,\phi)}\right) = \begin{pmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi \end{pmatrix} \quad \text{and} \quad J_2 = \left(\frac{\partial \mathbf{y}(u,v)}{\partial \mathbf{y}(u,v)}\right). = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & -2v \end{pmatrix} \tag{8}$$

$$g_{\mathbf{x}} = J_1^T J_1 = \begin{pmatrix} \sin^2\phi & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{\mathbf{y}} = J_2^T J_2 = \begin{pmatrix} 1+4u^2 & -4uv \\ -4uv & 1+4v^2 \end{pmatrix} \tag{9}$$

The columns of the Jacobians $J$ visually correspond to velocity or tangent vectors along the grid lines (coordinate curves) of the parameterized surface. Their inner products in $J^T J$ and entries $g_{ij}$ of the Riemannian metric correspond to their lengths (squared) and the angle between the velocity vectors.

We would like to visualize the metric *field* $g$ (a matrix valued functions of the parameters $(u, v)$). Instead of drawing two arrows at each point $(u, v)$ corresponding to the column vectors of $g(u, v)$, our eye find it easier to consider the parallelogram at each point $(u, v)$ spanned by these column vectors.

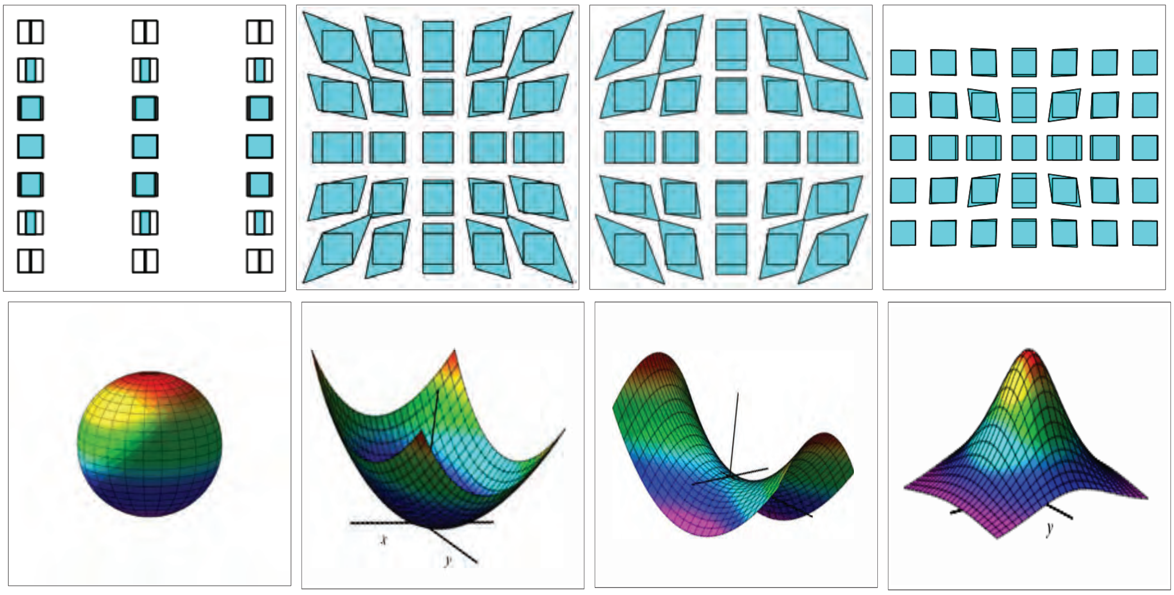

Figure 4: Squint your eyes: Riemannian metrics and surfaces

Indeed, when replacing $g$ in the above discussion by its positive definite square root, then these parallelograms most accurately depict the deformation of a small $(\Delta u, \Delta v)$ rectangle in the parameter plane under the parameterization $\mathbf{x}$ as, e.g., in (6). This qualitatively recovers the discussion about dropped patches, as illustrated in figure 3.

Figure 4 provides a few more examples for visualizing the Riemannian metric for familiar surfaces. Try to squint your eyes and recover the original surface from mentally *patching* together the parallelograms.

This way of visualizing Riemannian metrics is not a totally new idea, but indeed very closely related to the classic *Tissot indicatrix*: This tool has been used extensively in the mid 19th century in cartography and geography, to communicate geometric properties of various projections, such as the Mercator map. For an in-depth discussion of the history and the mathematics of Tissot's indicatrix see, e.g., [20]. It clearly demonstrates the preservation, or lack thereof, of angles or areas, and showing where the biggest distortions happen, e.g. near the polar regions. Whereas figure 4 employed parallelograms – which indicate the choice of coordinates and bases of the tangent spaces, the more geometric point of view considers distortions of circles (rather than of squares) into ellipses (rather than parallelograms). The next section will adopt this classical choice – at the cost of making it harder *to see* the surface by *squinting*, but more useful for see changes with time as the metric changes according to the Ricci flow.

# 6 2d Ricci-flow and visualization via Tissot's indicatrix

Whereas the Gauss map $\mathbf{N}\colon S \mapsto S^2 \subseteq \mathbb{R}^3$ defined in section 4 depends on the extrinsic notion of normal vector fields, Gauss showed in his celebrated *Theorema Egregium* in 1827 that the Gauss curvature $K = \det d\mathbf{N}$, the determinant of its differential is *intrinsic*: it only depends on the Riemannian metric, not on the imbedding of the surface into any ambient space. In particular, planes, cones, and cylinders have the same zero Gauss curvature as the latter two may be obtained by rolling up a (part) of the plane.

In higher dimensions, a general notion of curvature for Riemannian manifolds is given by the Riemann curvature tensor Riem, but for many purposes adequate information is provided by the Ricci curvature *tensor* Ricc – which is simply a $2 \times 2$ matrix in our context. Starting in the 1980s, foremost Hamilton [13] introduced and analyzed the *Ricci flow*, which is defined as a solution of the partial differential equation

$$\frac{\partial}{\partial t}\, g = -2 \text{ Ricc} \tag{10}$$

In our context $g$ and Ricc are time-varying $2 \times 2$ matrices, or rather matrix-valued functions of the parameters/coordinates (e.g., $(u, v)$ or $(\theta, \phi)$) defining the surfaces/manifolds). This flow was instrumental to Perelman's celebrated proof [21] of the Poincaré Conjecture. It has spawned a huge field of scintillating academic research of analysis of geometric flows.

Since these partial differential equations generally live on higher dimensional abstract manifolds, it is notoriously difficult for nonexperts to visualize such flows. Indeed, technically the flow takes place on the space of Riemannian manifolds, practically on a space of Riemannian metrics, rather than moving points of surfaces in some ambient space. One naturally asks: *"(When) is there hope to* see *this flow as families of surfaces in 3-space evolving with time?"*

Whitney's strong imbedding theorem from 1944 asserts that every smooth manifold M of dimension $n \geq 2$ can be imbedded into $\mathbb{R}^{2n}$ (and can be immersed into $\mathbb{R}^{2n-1}$). [The usual depiction of a Klein bottle in Euclidean 3-space is an immersion, but, due to its self-intersections, not an imbedding.] In our context the most fundamental results are Nash's imbedding theorems, starting with [19], that guarantee that every Riemannian manifold can be isometrically imbedded into some Euclidean space. [Isometric means that the metric the imbedded surface inherits from its ambient space agrees with the Riemannian metric of the original Riemannian manifold.]

Among the vast literature of isometric imbedding theorems, we briefly comment on some that address the case of two-dimensional Riemannian manifolds. Highly technical early work [22, 23] presented theorems of possible imbeddings and counterexamples depending on the degrees of smoothness. The case when the Gaussian curvature changes sign is addressed in

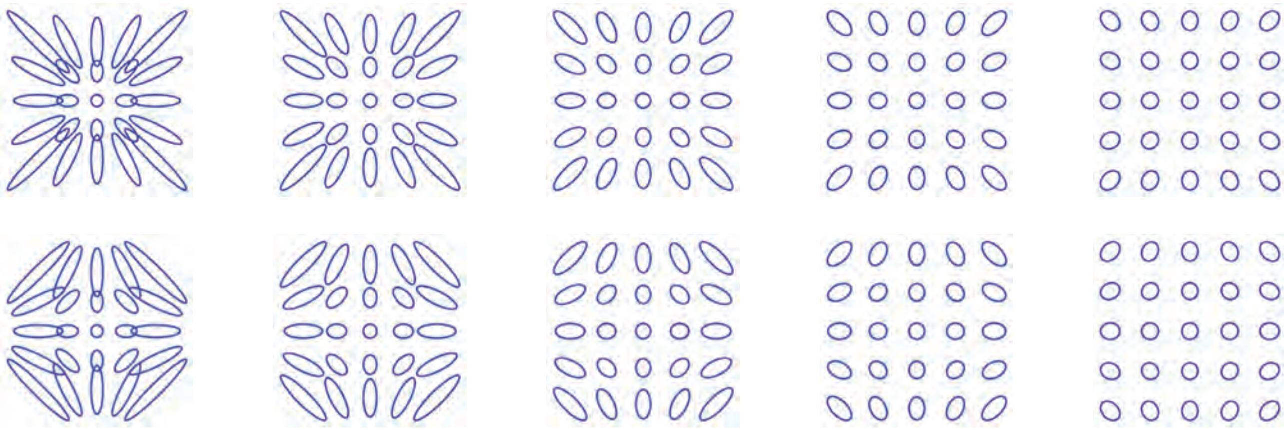

Figure 5: Visualizing of 2d-Ricci flows for initial surfaces $z = x^2 \pm y^2$

[17]. For detailed introductions to the Ricci flow in two dimension see [7] and [15] Some efforts to visualize the two-dimensional Ricci flow are presented in [10]. But we would like to warn the reader about highly counterintuitive results, for example in 2013 it was shown in [5] that the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ or $[0,1]^2$ with edges appropriately identified with the flat metric $g$ being the identity can be isometrically imbedded in Euclidean 3-space by a $C^1$ map.

But in spite of all these positive imbedding theorems one should not expect that in general, even if the *initial value* is an analytic surface in $R^3$, that the Riemannian manifolds evolving under the Ricci flow have any nice isometric imbedding in $R^3$ at any time $t > 0$.

Thus instead of focusing on trying to animate the Ricci flow on imbedded surfaces (as in the next section) we have been experimenting with trying to directly visualize the Ricci flow on the space of Riemannian metrics, presented by fields like the Tissot indicatrix. Two typical examples, presented in figure 5, clearly demonstrate the flattening of some bounded parts of a circular and a hyperbolic paraboloid as initial conditions (initial surfaces). We continue to experiment with this new MAPLE worksheet (and will rewrite it in SageMath, too). This again is a small program that is ideally suited for in-class demonstrations, inviting suggestions from students for different initial conditions, and for them to use it for budding research projects. The sample images in figure 5 are proof of concept, but there are many open questions about well-posedness of geometric flows on noncompact surfaces such as these graphs of polynomial functions (there is hope, as they become flat near infinity). The next experiments may involve the standard torus as a surface of revolution, a *double torus* (e.g., a fattened up lemniscate) etc.

# 7 Ricci flow on surfaces of revolution

While in general it may be hard to predict when the Ricci flow will deform surfaces imbedded in $\mathbb{R}^3$ into surfaces that for every $t \geq 0$ again maybe imbedded in $R^3$, in 2003 Engman [12] gave a simple necessary and sufficient condition for surfaces of revolution that are homeomorphic to a sphere to have isometric $C^1$ imbeddings into $\mathbb{R}^3$: The integral of the Gauss curvature over every geodesic disc centered at either pole must be nonnegative. Complementing this, the recent thesis [25] provides a readable survey on intrinsic geometric flows on manifolds of revolution.

In 2005 Rubinstein and Sinclair [24] demonstrated that 2-dimensional surfaces of revolution homeomorphic to spheres that are initially isometrically imbedded in $\mathbf{R}^3$ remain isometrically imbedded in $\mathbb{R}^3$ for as long as a smooth Ricci flow solution exists. In the same article they presented a numerical simulation of the Ricci flow, with very careful analysis of singularities, which gave rise to captivating series of images of such surfaces under the Ricci flow, that now are ubiquitous on the WWW, even popular on T-shirts.

This section closely follows the numeric implementation of the Ricci flow in this article [24], with new code in SageMath/Python that we have made freely available at GitHub [4]. To give the reader a better idea what this entails, we review some of the construction and terminology used in [24], as well as similar notation.

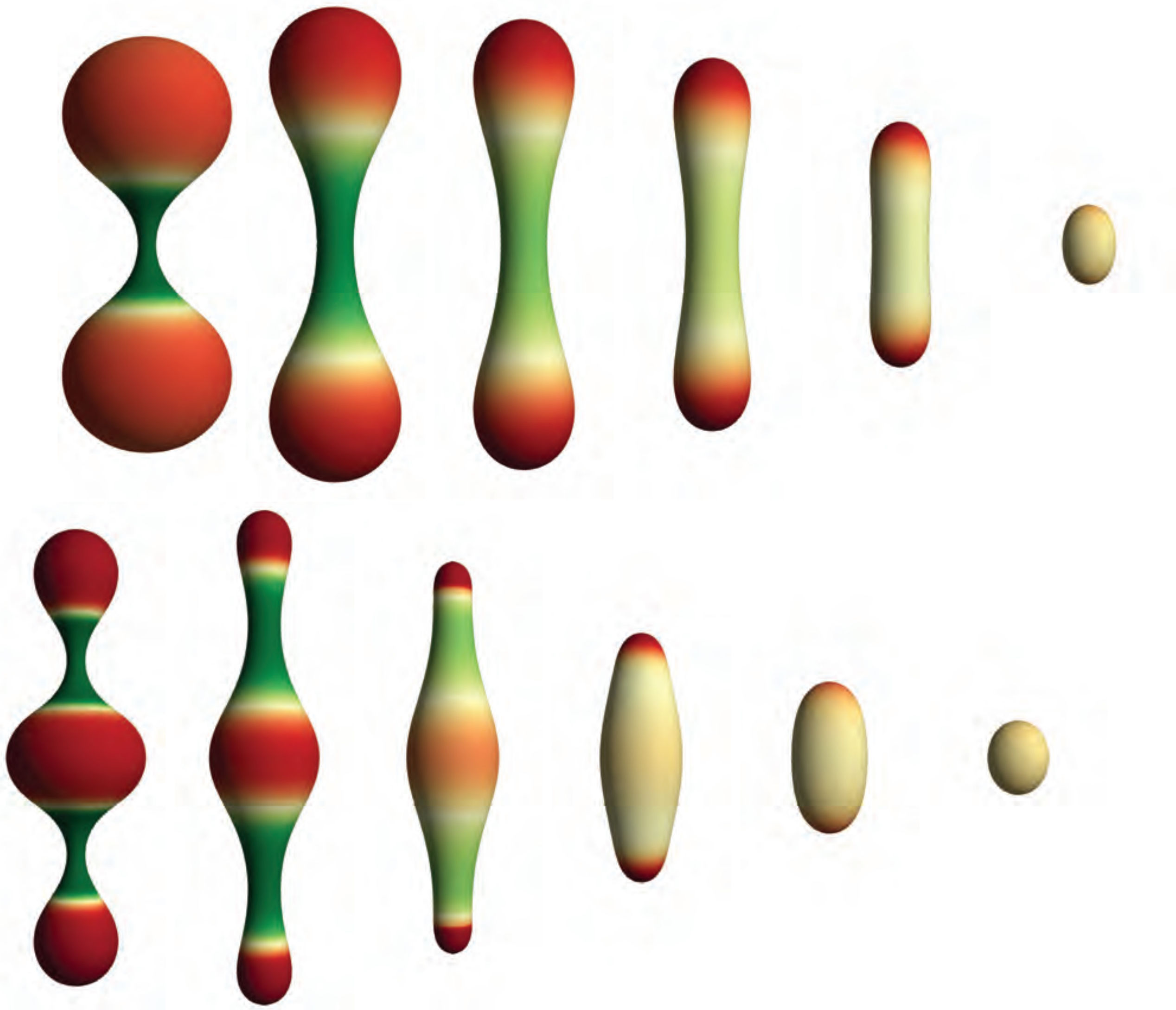

Figure 6: Ricci flow on surfaces of revolution, colored by Gauss curvature.

In the 2-dimensional case the Ricci curvature Ricc $= K\,g$ is just a scalar multiple of the metric $g$ where $K$ is the Gauss curvature (which n turn depends only, but in an intricate way, on $g$ and its partial derivatives). Closely following [24] the code `ricci_flow.sage` simulates the Ricci flow on surfaces of revolution, obtained by revolving a parameterized curve $\rho \mapsto (x(\rho), y(\rho))$ in Cartesian coordinates about the first coordinate axes. Think of $\rho$ initially, at time $t = 0$ parameterizing the curve by arclength, but as the curve changes, $\rho$ loses that interpretation.

Whereas the multi-variable calculus exercise asks to calculate the metric for a given parameterized surface, recall that the Ricci flow is takes place at the level of the metric $g$, and our task is to recover the surface (of revolution) from the metric. Thus we start with an initial metric

$$g_0 = \begin{pmatrix} h_0(\rho) & 0 \\ 0 & m_0(\rho) \end{pmatrix} \tag{11}$$

where for user specified parameters $c_i \geq 0$

$$h_0 \equiv 1, \quad \text{and} \quad m_0(\rho) = \left( \frac{\sin(\rho) + c_3 \sin(3\rho) + c_5 \sin(5\rho)}{1 + 3c_3 + 5c_5} \right)^2 . \tag{12}$$

For any metric $g$ of this diagonal form, i.e., for any $h, m \colon [0, \pi] \mapsto \mathbb{R}_{\geq 0}$ the corresponding curve of revolution as a parameterized curve in Cartesian coordinates is obtained as

$$x(\rho) = \int_0^\rho \sqrt{h(s) - \left( \frac{\partial \sqrt{m(s)}}{\partial s} \right)^2} \, \mathrm{d}s \, , \quad \text{and} \quad y(\rho) = \sqrt{m(\rho)}, \tag{13}$$

At any time $t \geq 0$, given a metric $g = g(t, \rho)$ for $\rho \notin \{0, \pi\}$ the Ricci curvature at that time $t$, as a function of $\rho$, may be expressed as

$$\mathrm{Ricc} = \left( -\frac{(\sqrt{m})''}{h\sqrt{m}} + \frac{m'h'}{4mh^2} \right) \begin{pmatrix} m & 0 \\ 0 & h \end{pmatrix} . \tag{14}$$

Because $m(0) = m(\pi) = 0$ and $m$ appears in the denominator of $K$, these formulae are not used near the singularities, i.e., for $\rho \in [0, \varepsilon)$ and $\rho \in (\pi - \varepsilon, \pi]$, but instead a cubic extrapolation is carried out to close the surface at the poles. To simulate Ricci flow, `ricci_flow.sage` implements a 4th-order Runge-Kutta (RK4) algorithm for the differential equation for Ricci flow (10).

Periodically, reparameterizations are performed in order to prevent numerical instabilities. due to the data points getting too close together as the Ricci flow stretches the surface. In the final frame of each animation in figure 6, the numerical instability near the poles starts to become apparent; ripples can be seen in the color coding by Gauss curvature.

# 8 `ricci_flow.sage`, SageMath, and classroom technology

The graphics in figure 6 were produced by `ricci_flow.sage`. Mathematically, these evolutions of surfaces closely reproduces the breathtaking results of [24] which are widely familiar, from Wikipedia to T-shirts. The surfaces in these graphics are colored by Gauss curvature: red shades represent positive, and green shades represent negative Gauss curvature.

The `ricci_flow.sage` code, and associated code, is available on GitHub [4]. The code contains numerous variables which control the initial surface shape, the Ricci flow simulation parameters, which plot animations get generated, plot parameters, and camera viewpoint (if saving surface animation frames to individual images). The code is documented, and most variables are documented by comments. The folder to which images are saved is determined by the `folder_name` variable in the code. The code can be downloaded either by downloading a zip file from its GitHub repository page, or from the command line using `git clone` (if Git is installed). To run the code, SageMath first needs to be installed. The SageMath installation directory can be found at `https://doc.sagemath.org/html/en/installation/index.html`. Once SageMath is installed, `ricci_flow.sage` can be run by opening a terminal and navigating to the `ricci_flow` folder within the cloned/downloaded repository, and running `sage ricci_flow.sage`.

Throughout, `ricci_flow.sage` makes frequent use of built-in SageMath and Python features, including cubic interpolation, numerical integration and differentiation, matrix/vector operations, trigonometric functions, list comprehension, and others. The wide range of available mathematical functions makes the source code for implementing Ricci flow in SageMath far more compact than a similar implementation in a language such as C. Additionally, the free and open-source nature of SageMath along with its Python foundation makes SageMath code far more accessible to students than code written in proprietary languages such as MATHEMATICA and MAPLE would be.

We conclude with some remarks about the choices of technology in mathematics classrooms at the undergraduate (and advanced high school level).

For well over 30 years the 2$^{nd}$ author has relied on and still continues to primarily utilize the commercial symbolic computer algebra systems MAPLE and Mathematics for research and in class for demonstrations and student assignments, from calculus classes to postdoctoral settings. He maintains a very large collection [16] of MAPLE worksheets that are freely available on the WWW, and are regularly used for in-class demonstrations and experimentation, and by students on their own. But the ubiquitous availability of simple graphing applets like DESMOS, as well as freeware such as the free online off-shoot `alpha` of MATHEMATICA, much reduce the need and interest by students to learn the details of the dominant computer algebra systems, and even of MATLAB. Instead, substantial experience in coding in especially the language Python, which is just as old as MAPLE and MATHEMATICA, is rapidly becoming a most valuable marketable skill expected on resumes of our graduates.

The recent substantial expansion and improvement of the Python library `sympy` for symbolic computer algebra led the 2$^{nd}$ author, while still presenting in-class demonstrations in MAPLE, to expect their students to instead try out and complete assignments using `sympy`, Python, and SageMath, starting with vector calculus and differential geometry with their very large algebraic expressions involving many partial derivatives and symbolic inverses of matrices that should always be carried out by machines.

Moreover, the advent of sharing platforms like github introduces students to the necessary insight that no-one can recreate code for all intermediate steps oneself. Students need to learn to make smart decisions on what to code oneself, and when to rely on available libraries, critically evaluating their trustworthiness, and eventually publicly sharing their own work.

# 9 Acknowledgments.

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# The mathematics of the "solera" system

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**Abstract**

*Fractional blending, also known as the "solera system", is a technique dating from the mid 19-th century, for the aging of liquids such as fortified wines, spirits, and balsamic vinegars. Such products require careful aging before they can be sold, and careful mixing of liquids from different ages is thus required. At each stage, every six months for example, or each year, a new un-aged liquid is added to the system, and a sequence of mixings is used to "filter", as it were, this new material through the system. The result at the end is a liquid carefully blended from different ages, with the oldest predominating. When properly done, this ensures a constant supply of an appropriately aged product. The mathematics can be described as a sequence of difference equations, or recurrence relations, which leads into some matrix algebra, and it turns out that this mathematics is more interesting than the simple explanation of the system might lead one to believe. This article explores this mathematics, using a computer algebra package for all the heavy lifting.*

## 1 Introduction

The *solera system* [7] originated in Spain in the 19th century. It consists of a selection of barrels—of sherry, for example—all of which together form the *criaderas*, or nursery.

A solera system is generally visualized as a pyramid, as shown in figure 1. The bottom barrels are in fact the solera, which means "floor" or "sole" in Spanish.

Although a pyramid is a standard representation of a solera system, in fact the different rows of barrels may be spread out in different cellars, just very carefully labelled.

At the end of each aging period; a year say, or maybe six months, one-quarter of the sherry in each of the bottom barrels is taken away for bottling. They are filled up from the third row; each barrel of which loses one-third of its contents. These barrels, in turn, are refilled from the second row, so each barrel here loses half its contents. And these barrels are filled from the top barrel, which thus becomes empty. This top barrel is then filled with the newest sherry for aging.[1]

The beauty of the solera system, is that if it is carefully managed, the bottom barrels will always contain an old mixture which can be sold. And this is constantly renewed. This system thus provides a continually renewed aged mixture.

Of course there's much more to a solera system than this. A great deal spends on the skill of the cellar master, first to ensure that all barrels in the criaderas are maintained at optimum temperature and humidity, and then to ensure that the transfer of sherry between barrels is done in such a way so as not to disturb the maturing sherry.

[1] A very good explanation, with solera animations beginning at 7:37, can be found at `https://bit.ly/3r7Nh4Z`.

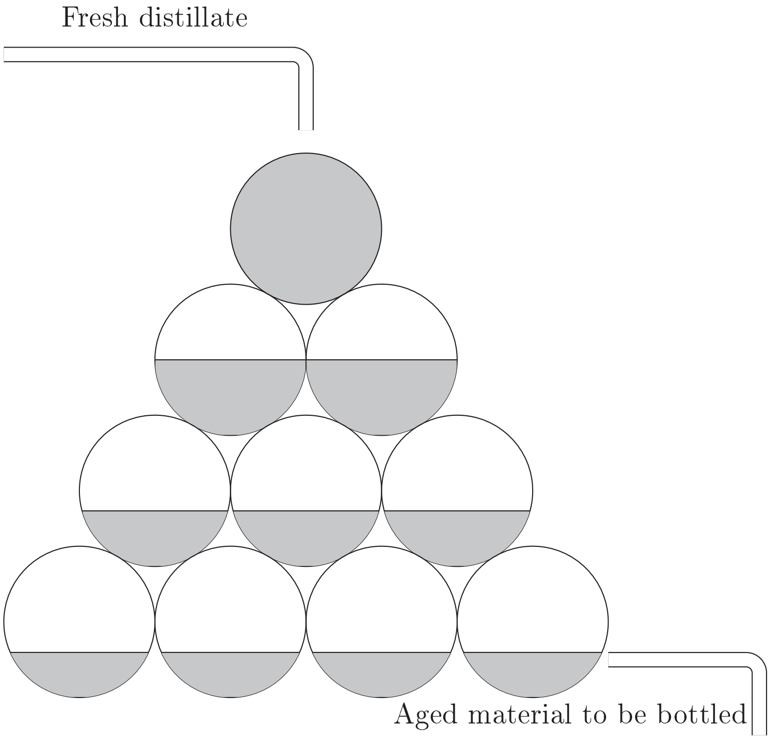


Figure 1: The solera system

This system is now in use world-wide to manufacture fortified wine and spirits of all sorts, as well as condiments such as balsamic vinegar.

## 2 Basic modelling

Before we start, note that there are several articles already about fractional blending; for example [1, 3, 6]. None of these adopt the approach we've taken, nor do they seem to use the same mathematical model to describe the blending. Moreover, none of them use a CAS (the first two were written long before and CAS was available), and as we shall see, the use of a CAS allows us to work with a simple model, and to manage with ease the expressions that arrive in the course of our use.

To develop a model, we shall consider four barrels of different volumes, which at aging time $n$ will have ages $a_n$, $b_n$, $c_n$, $d_n$. In the four layer system (as in figure 1), one quarter of $d_n$ is taken off and replaced from $c_n$. Thus immediately after this transfer, the age of the mixture in $d$ is

$$\frac{3}{4}d_n + \frac{1}{4}c_n$$

At the end of the next aging period, before the next transfer, this mixture will have aged by one period which produces

$$d_{n+1} = \frac{3}{4}d_n + \frac{1}{4}c_n + 1.$$

Likewise we have

$$c_{n+1} = \frac{2}{3}c_n + \frac{1}{3}b_n + 1$$
$$b_{n+1} = \frac{1}{2}b_n + \frac{1}{2}a_n + 1$$
$$a_{n+1} = 1.$$

The last equation is because the barrel on the top row is emptied and filled afresh. What we have now is a system of *difference equations* [5, 2] relating the ages at stage $n$ to the ages at the previous stage.

Suppose at the end of the first periods, before any transfer is done, we have

$$a(1) = 1, \quad b(1) = 2, \quad c(1) = 3, \quad d(1) = 4.$$

To avoid too much tangled algebra, our tool of choice will be Python's SymPy [4] module, which provides an `rsolve` command for solving difference equations. To do this, first rewrite using integers only:

$$2b_{n+1} = b_n + a_n + 2$$

and use $a_n = 1$ as shown in Listing **??**.

```
In[]: import sympy as sy
In[]: b = sy.Function('b')
In[]: c = sy.Function('b')
In[]: d = sy.Function('b')

In[]: bn = sy.rsolve(2*b(n+1)-b(n)-3,b(n),{b(1):2}); bn
```

$$3 - \frac{2}{2^n}$$

```
In[]: cn = sy.rsolve(3*c(n+1)-2*c(n)-bn-3,c(n),{c(1):3}); cn
```

$$-\frac{15}{2}\left(\frac{2}{3}\right)^n + \frac{4}{2^n} + 6$$

```
In[]: dn = sy.rsolve(4*d(n+1)-3*d(n)-cn-4,d(n),{d(1):4}); dn
```

$$\frac{45}{2}\left(\frac{2}{3}\right)^n - \frac{76}{3}\left(\frac{3}{4}\right)^n - \frac{4}{2^n} + 10$$

This means that the asymptotic age of the solera barrels will be 10 aging periods. If the transfer of material between the barrels takes place every six months, the age in the bottom barrels will approach 5 years.

We can work with the recurrence relations to obtain individual equations for each one. We start with

$$2b_{n+1} = b_n + 3 \quad \Rightarrow \quad 2b_{n+1} - b_n = 3 \tag{1}$$

and we write out the recurrence relation for $c_n$ twice:

$$3c_{n+1} = 2c_n + b_n + 3 \tag{2}$$

$$3c_{n+2} = 2c_{n+1} + b_{n+1} + 3 \tag{3}$$

Now suppose we multiply the equation (3) by 2 and subtract equation (2) from it:

$$6c_{n+2} - 3c_{n+1} = 4c_{n+1} - 2c_n + 2b_{n+1} - b_n + 2(3) - 3.$$

Bu equation 1 we can replace $2b_{n+1} - b_n$ with 3, thus producing

$$6c_{n+2} - 3c_{n+1} = 4c_{n+1} - 2c_n + 6$$

This can be cleaned up to produce

$$6c_{n+2} - 7c_{n+1} + 2c_n = 6. \tag{4}$$

The same thing can be done to produce a recurrence relation for $d_n$, writing it out three times and eliminating the $c_n$ terms by equation (4):

$$4d_{n+1} = 3d_n + c_n - 4 \tag{5}$$

$$4d_{n+2} = 3d_{n+1} + c_{n+1} - 4 \tag{6}$$

$$4d_{n+3} = 3d_{n+2} + c_{n+2} - 4 \tag{7}$$

In this case, to use equation (4) we compute

$$6 \times (7) - 7 \times (6) + 2 \times (5)$$

This will eliminate all the $c$ terms, replacing them with 6, and will produce:

$$24d_{n+3} - 28d_{n+2} + 8d_{n+1} = 18d_{n+2} - 21d_{n+1} + 6d_n + 6 + 4$$

This can be rewritten as

$$24d_{n+3} - 46d_{n+2} + 29d_{n+1} - 6d_n = 10 \tag{8}$$

The coefficients in equations (4) and (8) may look at first to be quite random, but this is not the case. We first notice that the characteristic equations for each recurrence relation are easily factorized into linear factors:

$$6\lambda^2 - 7\lambda + 2 = (2\lambda - 1)(3\lambda - 2)$$

$$24\lambda^3 - 28\lambda^2 + 29\lambda - 6 = (2\lambda - 1)(3\lambda - 2)(4\lambda - 3)$$

and this pattern can be continued.

Also, if we create an array $A$ of the coefficients of each relation:

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | | | | |
| 2 | 2 | −1 | | | |
| 3 | 6 | −7 | 2 | | |
| 4 | 24 | −46 | 29 | −6 | |
| 5 | 120 | −326 | 329 | −146 | 24 |

it is easy to see that (assuming all empty cells to have zero values):

$$A_{n,1} = n!, \quad A_{n,k} = nA_{n-1,k} - (n-1)A_{n-1,k-1} \quad \text{for} \quad k \geq 1.$$

# 3 Matrix formulation

The difference equations can be written in matrix form as

$$\begin{bmatrix} b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} b_n \\ c_n \\ d_n \end{bmatrix} + \begin{bmatrix} 3/2 \\ 1 \\ 1 \end{bmatrix}$$

We don't include an equation for $a_n$ as that is constant. Writing

$$\mathbf{b}_{n+1} = A\mathbf{b}_n + X$$

and with a starting vector $\mathbf{b}_1$, we have

$$\mathbf{b}_{n+1} = A^n\mathbf{b_1} + (A^{n-1} + A^{n-2} + \cdots + A^2 + A + I)X$$

and the sum of powers of $A$ can be written as

$$(A^n - I)(A - I)^{-1}.$$

and so the expression for $\mathbf{b}_{n+1}$ is:

$$\mathbf{b}_{n+1} = A^n\mathbf{b}_1 + (A^n - I)(A - I)^{-1}X. \tag{9}$$

The matrix $A$ is diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

Thus

$$A^n = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} (1/2)^n & 0 & 0 \\ 0 & (2/3)^n & 0 \\ 0 & 0 & (3/4)^n \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1/2)^n & 0 & 0 \\ -2(1/2)^n + 2(2/3)^n & (2/3)^n & 0 \\ 2(1/2)^n - 6(2/3)^n + 4(3/4)^n & -3(2/3)^n + 3(3/4)^n & (3/4)^n \end{bmatrix}$$

Given the matrix $A^n$ we can now develop the rest of the result, noting that

$$(A - I)^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -3 & 0 \\ -2 & -3 & -4 \end{bmatrix}$$

and using equation (9). This produces

$$\begin{bmatrix} b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} 3 - \left(\frac{1}{2}\right)^n \\ 6 + 2\left(\frac{1}{2}\right)^n - 5\left(\frac{2}{3}\right)^n \\ 10 - 2\left(\frac{1}{2}\right)^n + 15\left(\frac{2}{3}\right)^n - 19\left(\frac{3}{4}\right)^n \end{bmatrix}$$

To get the equations for $b_n$, $c_n$ and $d_n$ note that, for example, $(2/3)^{n-1} = (3/2)(2/3)^n$. Thus all powers can be scaled to produce powers one less. The result is:

$$
\begin{aligned}
b_n &= 3 - 2\left(\frac{1}{2}\right)^n \\
c_n &= 6 + 4\left(\frac{1}{2}\right)^n - \frac{15}{2}\left(\frac{2}{3}\right)^n \\
d_n &= 10 - 4\left(\frac{1}{2}\right)^n + \frac{45}{2}\left(\frac{2}{3}\right)^n - \frac{76}{3}\left(\frac{3}{4}\right)^n
\end{aligned}
$$

and it will be seen that these are the same equations as obtained earlier on using the `rsolve` command in Python's SymPy module.

All of the above can be simplified by writing the recurrence as:

$$
\begin{bmatrix} b_{n+1} \\ c_{n+1} \\ d_{n+1} \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 3/2 \\ 1/3 & 2/3 & 0 & 1 \\ 0 & 1/4 & 3/4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_n \\ c_n \\ d_n \\ 1 \end{bmatrix}
$$

The matrix $M$ can also be diagonalized as $M = ADA^{-1}$:

$$
\begin{bmatrix} 1/2 & 0 & 0 & 3/2 \\ 1/3 & 2/3 & 0 & 1 \\ 0 & 1/4 & 3/4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & -1 & 0 & 6 \\ 2 & 3 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & -1 & 0 & 6 \\ 2 & 3 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}
$$

Since the top three diagonal elements of $D$ are all less than one, the limiting value of $D^n$ will be a $4 \times 4$ matrix all zero except for the bottom right element, which is 1. This means that the limiting values of the ages are:

$$
\begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & -1 & 0 & 6 \\ 2 & 3 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ -2 & -1 & 0 & 6 \\ 2 & 3 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 10 \\ 1 \end{bmatrix}
$$

This approach also shows that the limiting value is independent of the starting ages, since

$$
\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ c_0 \\ d_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 10 \\ 1 \end{bmatrix}.
$$

## 4 Limiting values

As $n$ increases, the age in the barrels at each layer approaches a constant value. We diagonalize $A$ as $PDP^{-1}$ and since every diagonal element of $D$ is less than one, it follows that $A^n$ approaches zero. From equation (9) if $A^n$ is set to zero, we have, as a limiting vector:

$$
-(A-I)^{-1}X = -\begin{bmatrix} -2 & 0 & 0 \\ -2 & -3 & 0 \\ -2 & -3 & -4 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}
$$

To generalize this, suppose we have $k$ rows, with $A$ being the matrix whose diagonal elements will be 1/2, 2/3, 3/4, 4/5, ... and whose sub-diagonal elements will be 1/3, 1/4, 1/5, ..., like this:

$$A = \begin{bmatrix} 1/2 & & & & & \\ 1/3 & 2/3 & & & & \\ & 1/4 & 3/4 & & & \\ & & 1/5 & 4/5 & & \\ & & & \ddots & \ddots & \end{bmatrix}$$

Then:

$$A - I = \begin{bmatrix} 1/2 & & & & & \\ 1/3 & -1/3 & & & & \\ & 1/4 & -1/4 & & & \\ & & 1/5 & -1/5 & & \\ & & & \ddots & \ddots & \end{bmatrix}$$

Hand or computer-aided computation (for example, the Python command `(A-sy.eye(3)).inv()`) can be used to show that for increasing sizes, the matrices $(A - I)^{-1}$ are:

$$\begin{bmatrix} -2 & 0 & 0 \\ -2 & -3 & 0 \\ -2 & -3 & -4 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ -2 & -3 & -4 & 0 \\ -2 & -3 & -4 & -5 \end{bmatrix}, \quad \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ -2 & -3 & -4 & 0 & 0 \\ -2 & -3 & -4 & -5 & 0 \\ -2 & -3 & -4 & -5 & -6 \end{bmatrix}$$

This pattern is easily proved by induction. Suppose that $A_k$ is the $k \times k$ version of $A - I$, and $B_k$ is the inverse. Then we can define $A_{k+1}$ and $B_{k+1}$ with block matrices:

$$A_{k+1} = \left[\begin{array}{c|c} A_k & 0_k \\ \hline 0 \quad 0 \ldots 0 \quad \frac{1}{k+1} & -\frac{1}{k+1} \end{array}\right], \qquad B_{k+1} = \left[\begin{array}{c|c} B_k & 0_k \\ \hline -2\ -3\ -4 \ldots -k & -(k+1) \end{array}\right]$$

writing $u_{k+1}$ for the bottom left block of $A_{k+1}$, and $v_{k+1}$ for the bottom left block of $B_{k+1}$, their product is

$$A_{k+1}B_{k+1} = \left[\begin{array}{c|c} A_k B_k + 0_k v_{k+1} & A_k 0_k + 0_k[-(k+1)] \\ \hline u_k B_k + (-\frac{1}{k+1})v_k & u_k 0_k + (-\frac{1}{k+1})(-(k+1)) \end{array}\right]$$

All cells are easily computed, but the bottom left cell needs a little explanation. We note that $u_k$ consists of only one non-zero value $1/(k+1)$, in the last place, so that $u_k B_k$ essentially multiples the last row of $B_k$ by that non-zero value. But the last row of $B_k$ and $v_k$ are the same vector, so the bottom left cell is

$$\frac{1}{k+1} v_k + \left(-\frac{1}{k+1}\right) v_k = 0.$$

Thus

$$A_{k+1}B_{k+1} = \left[\begin{array}{c|c} I & 0_k \\ \hline 0_k^T & 1 \end{array}\right]$$

which is the $(k+1) \times (k+1)$ identity matrix.

Since the limiting age of the barrels in the solera row has been shown to be

$$-(A-I)^{-1}X$$

where $X$ is a column vector starting with 3/2 but all other values are 1's. This is then equal to

$$-\begin{bmatrix} -2 & & & & \\ -2 & -3 & & & \\ -2 & -3 & -4 & & \\ \vdots & & & & \\ -2 & -3 & -4 & \cdots & -k \end{bmatrix} \begin{bmatrix} 3/2 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3+3 \\ 3+3+4 \\ \vdots \\ 3+3+4+\ldots+k \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 10 \\ \vdots \\ \frac{k^2+k}{2} \end{bmatrix}$$

The final value shows that the limiting age of the solera barrels in a system with $k$ layers is $(k^2+k)/2$.

## 5 Other fractions

Up until now, we have been moving one compete barrel between rows, thus leaving the top barrel completely empty each time. But suppose we move some fraction $p$ of a barrel instead, where $0 < p \leq 1$.

As before, we'll start with a four-tier system. Emptying a total of $p$ barrels from the solera reduces each barrel by $p/4$. This amount needs to be replaced from the third row, thus reducing each of those barrels by $p/3$. Similarly, $p/2$ barrels will be taken from each of the barrels in the second row, and as described at the beginning of this section, $p$ from the top barrel.

Since the top barrel is not necessarily completely emptied, it will need to be considered. The recurrence relation relating ages in each barrel are then:

$$\begin{aligned} a_{n+1} &= (1-p)a_n + 1 \\ b_{n+1} &= (1-\frac{p}{2})b_n + \frac{p}{2}a_n + 1 \\ c_{n+1} &= (1-\frac{p}{3})c_n + \frac{p}{3}b_n + 1 \\ d_{n+1} &= (1-\frac{p}{4})d_n + \frac{p}{4}c_n + 1 \end{aligned}$$

If $p = 1$ these equations are equal to our original equations.

It is in fact easy to solve these using matrix methods, with

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} 1-p & 0 & 0 & 0 \\ \frac{p}{2} & 1-\frac{p}{2} & 0 & 0 \\ 0 & \frac{p}{3} & 1-\frac{p}{3} & 0 \\ 0 & 0 & \frac{p}{4} & 1-\frac{p}{4} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

If we write this as

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + \mathbf{b}$$

then, as before

$$\mathbf{x}_n = A^n\mathbf{x}_0 + (A^n - I)(A-I)^{-1}\mathbf{b} \tag{10}$$

and so the solution reduces to finding $A^n$. But again we have an easily diagonalizable matrix, with

$$\begin{bmatrix} 1-p & 0 & 0 & 0 \\ 1-\frac{p}{2} & \frac{p}{2} & 0 & 0 \\ 0 & 1-\frac{p}{3} & \frac{p}{3} & 0 \\ 0 & 0 & 1-\frac{p}{4} & \frac{p}{4} \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1-\frac{p}{2} & 0 & 0 \\ 0 & 0 & 1-\frac{p}{3} & 0 \\ 0 & 0 & 0 & 1-\frac{p}{4} \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -3 & -2 & -1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}^{-1}$$

Also,

$$(A-I)^{-1} = \begin{bmatrix} -\frac{1}{p} & 0 & 0 & 0 \\ -\frac{1}{p} & -\frac{2}{p} & 0 & 0 \\ -\frac{1}{p} & -\frac{2}{p} & -\frac{3}{p} & 0 \\ -\frac{1}{p} & -\frac{2}{p} & -\frac{3}{p} & -\frac{4}{p} \end{bmatrix}$$

Substituting into equation (10), and with $a_0 = 1$, $b_0 = c_0 = d_0 = 0$ produces

$$\begin{aligned} a_n &= \left(1-\frac{1}{p}\right)(1-p)^n + \frac{1}{p} \\ b_n &= \left(-1+\frac{1}{p}\right)(1-p)^n + \left(1-\frac{4}{p}\right)\left(1-\frac{p}{2}\right)^n + \frac{3}{p} \\ c_n &= \left(\frac{1}{2}-\frac{1}{2p}\right)(1-p)^n + \left(-2+\frac{8}{p}\right)\left(1-\frac{p}{2}\right)^n + \left(\frac{3}{2}-\frac{27}{2p}\right)\left(1-\frac{p}{3}\right)^n + \frac{6}{p} \\ d_n &= \left(-\frac{1}{6}+\frac{1}{6p}\right)(1-p)^n + \left(2-\frac{8}{p}\right)\left(1-\frac{p}{2}\right)^n + \left(-\frac{9}{2}+\frac{81}{2p}\right)\left(1-\frac{p}{3}\right)^n \\ &\quad + \left(\frac{8}{3}-\frac{128}{3p}\right)\left(1-\frac{p}{4}\right)^n + \frac{10}{p} \end{aligned}$$

We can see that the limiting values in the solera row, in a criederas with $k$ layers, will be

$$\frac{k^2+k}{2p}.$$

And this can be easily shown by noting the general form of $(A-I)^{-1}$, which can be established by induction. Let $A_k$ be the matrix $A-I$ of size $k \times k$. Then we can express $A_{k+1}$ as a block matrix:

$$A_{k+1} = \left[\begin{array}{ccccc|c} & & A_k & & & 0_k \\ \hline 0 & 0 & 0\ldots0 & & \frac{p}{k+1} & -\frac{p}{k+1} \end{array}\right]$$

If $B_k$ is the inverse of $A_k$, then

$$B_{k+1} = \left[\begin{array}{cccc|c} & & B_k & & 0_k \\ \hline -\frac{1}{p} & -\frac{2}{p} & -\frac{3}{p} & \dots -\frac{k}{p} & -\frac{k+1}{p} \end{array}\right]$$

Writing the lower left block of $A_{k+1}$ as $u_k$ and of $B_{k+1}$ as $v_k$, we have:

$$A_{k+1}B_{k+1} \left[\begin{array}{c|c} A_kB_k + 0_kv_k & A_k0_k + 0_k(-\frac{k+1}{p}) \\ \hline u_kB_k + (-\frac{p}{k+1})v_k & u_k0_k + (-\frac{p}{k+1})(-\frac{k+1}{p}) \end{array}\right]$$

By the induction hypothesis, $A_kB_k = I$. For the bottom left, we note that by construction, $v_k$ equals the bottom row of $B_k$. And since $u_k$ consists of zeros except for a final value of $p/(k+1)$, the results of the first product is simply this value multiplied into every element of the last row of $B_k$. Thus:

$$\begin{aligned} u_kB_k + \left(-\frac{p}{k+1}\right)v_k &= \left(\frac{p}{k+1}\right)\left[-\frac{1}{p} \quad -\frac{2}{p} \quad -\frac{3}{p}\dots -\frac{k}{p}\right] \\ &\quad + \left(\frac{p}{k+1}\right)\left[-\frac{1}{p} \quad -\frac{2}{p} \quad -\frac{3}{p}\dots -\frac{k}{p}\right] \\ &= 0 \end{aligned}$$

All other products are more straightforward; in the end we have

$$A_{k+1}B_{k+1} = \left[\begin{array}{ccc|c} & I & & 0_k \\ \hline 0 & 0 & 0\dots 0 & 1 \end{array}\right]$$

which is the identity, as required.

The limiting value of the age in the solera row is then

$$-(A-I)^{-1}\mathbf{b}$$

and since $\mathbf{b}$ consists entirely of 1's, this product is

$$\begin{bmatrix} \frac{1}{p} \\ \frac{1}{p}+\frac{2}{p} \\ \frac{1}{p}+\frac{2}{p}+\frac{3}{p} \\ \vdots \\ \frac{1}{p}+\frac{2}{p}+\frac{3}{p}+\dots+\frac{k}{p} \end{bmatrix} = \begin{bmatrix} \frac{1}{p} \\ \frac{3}{p} \\ \frac{6}{p} \\ \vdots \\ \frac{k^2+k}{2p} \end{bmatrix}$$

# 6 General arithmetic sequences

In the previous section, the number of barrels in row $k$ (starting with the top row numbered 1) is $k$. We now consider a more general system where the $k$-row contains $g + (k-1)h$ barrels, where $g, h \geq 1$ (and are integers). And at each stage $m$ barrels are moved into the top row and between other rows, with $m \leq g$.

In the first row, assuming all barrels to be full, $m$ barrels are moved to row 2, and $g - m$ barrels are left. We are thus moving a fraction of $m/g$ material between rows. But in this more general situation, we have $1 - m/g$ material left in each of the top barrels. This will then be aged between periods, so that:

$$a_{n+1} = \left(1 - \frac{m}{g}\right) a_n + 1. \tag{11}$$

Here the fraction $m/g$ corresponds to the fraction $p$ in the previous section. This means that we can re-purpose the equations given at the beginning of section 5 to obtain the other equations:

$$\begin{aligned} b_{n+1} &= \left(1 - \frac{m}{g+h}\right) b_n + \frac{m}{g+h} a_n + 1 \\ c_{n+1} &= \left(1 - \frac{m}{g+2h}\right) c_n + \frac{m}{g+2h} b_n + 1 \\ d_{n+1} &= \left(1 - \frac{m}{g+3h}\right) d_n + \frac{m}{g+3h} c_n + 1 \end{aligned} \tag{12}$$

Multiplying out to clear the fractions produces:

$$\begin{aligned} g a_{n+1} &= (g - m) a_n + m a_{n-1} + g \\ (g + h) b_{n+1} &= (g + h - m) b_n + m a_n + g + h \\ (g + 2h) c_{n+1} &= (g + 2h - m) c_n + m b_n + g + 2h \\ (g + 3h) d_{n+1} &= (g + 3h - m) d_n + m c_n + g - 3h \end{aligned}$$

The first two can be entered into SymPy as:

```
In[]: n,m,g,h = sy.var('n,m,g,h')
In[]: an = sy.rsolve(g*a(n+1)-(g-m)*a(n)-g,a(n),{a(1):1})
In[]: bn = sy.rsolve((g+h)*b(n+1)-(g+h-m)*b(n)-sy.simplify(m*an)-g-h,\
      b(n),{b(1):0})
```

These turn out to have the splendid solutions:

$$a_n = -\frac{g\left(\frac{g-m}{g}\right)^n}{m} + \frac{g}{m}$$

$$b_n = \frac{\left(\frac{g+h-m}{g+h}\right)^n \left(-g^3 - 3g^2 h + g^2 m - 3gh^2 + ghm - h^3\right)}{ghm + h^2 m - hm^2} + \frac{g^2 \left(\frac{g-m}{g}\right)^n + h\,(2g + h)}{hm}$$

With a little bit of algebra (helped by SymPy) this last can be written as:

$$\begin{aligned} b_n &= -\frac{(g+h)(g^2 + 2gh - gm + h^2)}{hm(g + h - m)} \left(\frac{g + h - m}{g + h}\right)^n + \frac{g^2}{hm} \left(\frac{g - m}{g}\right)^n + \frac{2g + h}{m} \\ &= \frac{g^2 + 2gh - gm + h^2}{hm} \left(\frac{g + h - m}{g + h}\right)^{n-1} + \frac{g^2}{hm} \left(\frac{g - m}{g}\right)^n + \frac{2g + h}{m} \end{aligned}$$

Already we see that we're obtaining expressions of considerable complexity. It turns out that the mechanisms of SymPy are unable to solve the comparable equation of $c_n$ directly, but we can give it some help.

First note that the form of the difference equation for $c_n$ is

$$c_{n+1} = \left(1 - \frac{m}{g+2h}\right) c_n - \frac{m}{g+2h}\left(A\left(1-\frac{m}{g}\right)^n + B\left(1-\frac{m}{g+h}\right)^n + C\right) - 1$$

where $A, B, C$ are the coefficients from $b_n$. We can write this more simply as:

$$c_{n+1} = tc_n - Ax^n - By^n - C - 1$$

where $A, B, C$ now include the multiplier $m/(g+h)$. This can be solved:

```
In[]: from sympy.abc import t,x,y,A,B,C
In[]: cn = sy.rsolve(c(n+1)-t*c(n)-A*x**n-B*y**n-1,c(n),{c(1):0})
```

The output is too long to be shown, but all we need to is to extract from the expression for $b_n$ the papers corresponding to $A, B, C, X, y$ and substitute them into the expression for $c_n$ just obtained, along with the definition for $t$.

We have seen that the expression for $b_n$ is reasonably hideous, but the coefficients can be extracted by setting various of the powers to zero with a little function:

```
In[]: def power_sub(i,j):
      ....temp = {((g-m)/g)**n:i,((g+h-m)/(g+h))**n:j})
      ....return(temp)

In[]: C1 = bn.subs(power_sub(0,0))
In[]: A1 = (bn - C1).subs(power_sub(1,0))
In[]: B1 = (bn - C1).subs(power_sub(0,1))
```

Here `C1` is the constant term of $b_n$, and `A1`, `B1` are the coefficients of

$$\left(\frac{g-m}{g}\right)^n, \quad \left(\frac{g+h-m}{g+h}\right)^n$$

respectively. Finally all of this can be put into the expression for $c_n$:

```
In[]: x1 = (g-m)/g
In[]: y1 = (g+h-m)/(g+h)
In[]: t1 = (g + 2*h - m)/(g + 2*h)
In[]: A1 *= m/(g+2*h)
In[]: B1 *= m/(g+2*h)
In[]: C1 *= m/(g+2*h)
In[]: cn1 = cn.subs({A:A1,B:B1,C:C1,x:x1,y:y1,t:t1})
```

This is still a very complicated expression, but we can get a sense of it by noting that it will have the form:

$$c_n = C_3\left(\frac{g+2h-m}{g+2h}\right)^n + C_2\left(\frac{g+h-m}{g+h}\right)^n + C_1\left(\frac{g-m}{g}\right)^n + C_0 \tag{13}$$

As for $b_n$ above, we can find the values of the coefficients $C_k$ by setting various of the powers to zero, again with a small function:

```
In[]: def power_sub3(i,j,k):
      ....temp = {((g-m)/g)**n:i,((g+h-m)/(g+h))**n:j,
                  ((g+2*h-m)/(g+2*h))**n:k}
      ....return(temp)

In[]: C0 = sy.simplify(cn2.subs(power_sub3(0,0,0)))
In[]: C1 = sy.simplify((cn2-C0).subs(power_sub3(1,0,0)))
In[]: C2 = sy.simplify((cn2-C0).subs(power_sub3(0,1,0)))
In[]: C3 = sy.simplify((cn2-C0).subs(power_sub3(0,0,1)))
```

These are:

$$C_0 = \frac{3(g+h)}{m}$$
$$C_1 = \frac{2g+h}{g+2h}$$
$$C_2 = \frac{g^4 + 4g^3h - g^3m + 6g^2h^2 - 2g^2hm + 4gh^3 - gh^2m + h^4}{h^2m\,(g+h-m)}$$
$$C_3 = \frac{-g^4 - 8g^3h + g^3m - 24g^2h^2 + 4g^2hm - 32gh^3 + 4gh^2m - 16h^4}{2h^2m\,(g+2h-m)}$$

Substituting these in equation 13 will provide the full solution for $c_n$. Note that since all the powers are of values less than 1, the limiting value as $n$ increases is the constant term.

We can now turn our attention to the fourth row, given as the solution to the difference equation for $d_n$. However, given the complexities in trying to solve the previous equation for $c_n$, we will not try (although we could, if we felt like giving ourselves a hard time), but go straight to determining the limit.

The equation is

$$d_n = \left(1 - \frac{m}{g+3h}\right) d_n + \frac{m}{g+3h} c_n + 1$$

and its solution will have the form

$$d_n = A_1 \left(\frac{g+3h-m}{g+3h}\right)^n + A_2 \left(\frac{g+2h-m}{g+2h}\right)^n + A_3 \left(\frac{g+h-m}{g+h}\right)^n + A_4 \left(\frac{g-m}{g}\right)^n + A_5$$

Our only concern is to find $A_5$. We shall begin as we did for $c_n$ above, but without aiming to determine any of the coefficients, simply set all the powers to zero.

```
In[]: A,B,C,D,x,y,z = sy.var('A,B,C,D,x,y,z')
In[]: dn = sy.rsolve((g+3*h)*d(n+1)-(g+3*h-m)*d(n)-\
      m*(A*x**n+B*y**n+C*z**n+D)-g-3*h,d(n),{d(0):0})
In[]: dn1 = sy.simplify(dn)
In[]: const_d = dn1.subs({((g+3*h-m)/(g+3*h))**n:0,x**n:0,y**n:0,z**n:0})
```

Here the expression $Ax^n + by^n + Cz^n + D$ stands for $c_n$; the values of $A$, $b$ and $C$ are irrelevant because each of $x$, $y$, $z$ ae less than 1. This produces:

$$\frac{Dm + g + 3h}{m}$$

But the $D$ is in fact the constant term from $c_n$, and substituting this for $D$ produces the limiting age for the material in the fourth row:

$$\frac{4g+6h}{m}.$$

Note that if we set $m = g = h = 1$ corresponding to the original setup, then this value is equal to 10 (as it should).

The limiting values in the first four rows are:

$$\begin{aligned} a_\infty &= \frac{g}{m} \\ b_\infty &= \frac{2g+h}{m} \\ c_\infty &= \frac{3g+3h}{m} \\ d_\infty &= \frac{4g+6h}{m}. \end{aligned}$$

A pattern is now apparent, and can be continued for increasing $n$. Note that if all variables are equal to 1, these limiting values are 1, 3, 6, 10, as previously.

We can demonstrate this by considering the general solution of a difference equation, for example

$$x_{n+1} = \lambda x_n + pz^n + qw^n + r.$$

Assuming that all of $\lambda, z, w$ are distinct, and using the method of undetermined coefficients, then the homogeneous solution will be of the form $x_n = A\lambda^n$ and the particular solution of the form $y_n = Pz^n + Qw^n + R$.

Substituting $y_n$ into the difference equation produces

$$Pz^{n+1} + Qw^{n+1} + R = \lambda(Pz^n + Qw^n + R) + pz^n + qw^n + r.$$

We can now equate the coefficients of $z^n$, $w^n$, and the constant term on both sides; these equations are, respectively:

$$\begin{aligned} Pz &= \lambda P + p \\ Qw &= \lambda Q + q \\ R &= \lambda R + r \end{aligned}$$

It is the last value which concerns us, and so

$$R = \frac{r}{1-\lambda}.$$

This is clearly generalizable to particular solutions of any length. In our context, all values $\lambda, z, w$ are less then 1, so that in the limit only the constant term matters.

Going back to the original set of difference equations 11 and 12 we have that the constant term for $a_n$ is $g/m$.

This means that the constant term in the equation for $b_n$ is

$$r = \frac{m}{g+h}\left(\frac{g}{m}\right) + 1 = \frac{g}{g+h} + 1.$$

For this equation,

$$\lambda = 1 - \frac{m}{g+h}$$

and so

$$\frac{r}{1-\lambda} = \frac{2g+h}{m}$$

as we found previously. For the next equation (for $c_n$), we have

$$r = \frac{m}{g+2h}\left(\frac{2g+h}{m}\right) + 1 = \frac{2g+h}{g+2h} + 1$$

Since here, $\lambda = 1 - m/(g+2h)$, we have

$$\frac{r}{1-\lambda} = \frac{3g+3h}{m}.$$

In general, suppose that $k_i$ is the constant term corresponding to the $i$-th row. We then have

$$\begin{aligned} k_i &= \left(\frac{m}{g+(i-i)h}k_{i-1} + 1\right) \Big/ \left(\frac{m}{g+(i-1)h}\right) \\ &= k_{i-1} + \frac{g+(i-1)h}{m}. \end{aligned}$$

Since $k_1 = g/m$, we have, in general,

$$k_i = \frac{ig + i(i-1)h/2}{m} = \frac{2ig + i(i-1)h}{2m} = \frac{i}{2m}(2g+(i-1)h).$$

## 7 A final generalization

Suppose now that the numbers of barrels in row $i$ is $g_i$, where these values form a non-decreasing sequence.

$$1 \le g_1 \le g_2 \le g_3 \le \dots$$

Again we use $m$ for the number of barrels-worth of liquid moved each time. We won't attempt to try and find solutions to the difference equations, but simply determine the constant values for each row. As before, the first constant value will be

$$k_1 = \frac{g_1}{m}.$$

A difference equation, say for row 4, will look like this:

$$d_{n+1} = \left(1 - \frac{m}{g_4}\right) d_n + \frac{m}{g^4} c_n + 1$$

so that in all equations, $g + (i-1)h$ will be replaced with $g_i$. Then for row $i > 1$, we will have

$$\begin{aligned} k_i &= \left(\frac{m}{g_i}k_{i-1} + 1\right) \Big/ \left(\frac{m}{g_i}\right) \\ &= k_{i-1} + \frac{g_i}{m}. \end{aligned}$$

This means, for example, that

$$k_4 = \frac{g_1 + g_2 + g_3 + g_4}{m}$$

and so in general,

$$k_n = \frac{1}{m} \sum_{i=1}^{n} g_i.$$

This generalizes the results of previous sections.

# 8 Final remarks

Although this work was inspired by a "real-life" situation (the visit of the author to a friend's winery and distillery), it is in fact a nice example of the use of a computer algebra system to solve systems of difference equations, both by standard techniques and also by the use of matrix algebra. Even when the complexities were such that individual solutions could not be (easily) obtained, we showed how to find the limiting values, which is in fact what counts in the world of fine wines and spirits. As a multi-billion dollar industry covering much of the world, this is clearly an industry worth considering, at least mathematically. We note that fractional blending is most often used for alcohol, and so might seem of no interest to people who don't or can't drink alcohol. However, it is also used for non-alcoholic purposes such as for fine balsamic vinegars. We thus note that fractional blending systems have a very wide usage, and the science of them may thus have a wide appeal.

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# Making Geometry Dynamic:
## Design Considerations in Mathematical Interactivity

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**Abstract**: *This paper surveys situations in the early development of Dynamic Geometry Software in which designers had to "invent" plausible mathematical behaviors for specific dynamic configurations. It offers both a case study in the design of mathematical software and a reflection on the potential contribution dynamism makes to the history of mathematical representation.*

## 1. Introduction

How do mathematical representations—mathematical media, notations, and conceptions—arise, and how do they evolve? Sixty years into a growing diffusion of computational activity and software technology across mathematical practice, and half as long into an era in which various Dynamic Geometry packages have arisen as the most widely used such tools in the teaching and learning of mathematics, is it surprising that not one of these packages feature the most characteristic—even emblematic—representation of school geometry across the century before these new tools' arrival, the two-column proof? The nature of activity changes as the tools change through which activity is pursued. Adopting the position that emerging tools and representations shape, as much as are shaped by, standing ideas and conceptions of a domain, this paper surveys the origins of Dynamic Geometry software for moments where the gestural and visual paradigm for mathematical interaction introduced by such software had to extend or refigure practices and implications of "static" geometry to enable its new and dynamic approach.

This account is personal and to some degree subjective. My colleague Steven Rasmussen and I first coined the phrase "Dynamic Geometry," and I write about its design from my experience creating one of its first incarnations: *The Geometer's Sketchpad* [1]. My account focuses primarily on *Sketchpad,* but attempts to portray other perspectives in the fertile design dialog that occurred across software packages beginning in the early 1990s and continuing, to some extent, to date. However, my broad interest behind this account—in, through design, creating and clarifying the role of gesturally-mediated dynamic manipulations in mathematics software—features not one, but two, topics notably difficult to engage in print. Dynamic gesture, as I use the term, refers to physical motions and actions (hand, or hand/eye/body movements) that almost by definition constitute non-verbal and non-symbolic acts of communication, whether between people or between a person and a manipulated machine. I can describe such actions in print, but I cannot *communicate* them here effectively. And dynamic software is, of course, software whose content is not only graphical in nature but moving, animating, and evolving continuously in time and in response to gestured directions. It is decidedly *not* software that presents a text of symbols conveniently transcribable in print. Scholarship in both domains struggles with capturing the essential physicality of the gesture, and the inherent dynamics of the software, in words and static pictures. This paper accompanies an invited lecture I shall give at ATCM, but where within a few minutes in a talk I can give a half dozen compelling examples of dynamic gestures and dynamic software responses to them, in text one labors for paragraphs to evoke the physicality or dynamics critical to a single example. Thus, the presentations shall necessarily differ, as the media are so different in regard to critical fault-lines of my topic. (Fortuitously, this very fact—that choice of representational media is consequential! —is a fundamental conclusion of both my presentations.) My talk covers examples drawn not just from *Sketchpad* but also from my lesser-known works and from my work-in-progress and aims to highlight design comparisons and their implications. In this paper instead, I focus on the single example of dragging within Dynamic Geometry, to detail a single design in context, while hoping the reader has familiarity with the actual *act* of dragging in Dynamic Geometry sufficient to compensate for the deficiencies of print description. Even before we take up that subject, however, we must first ask "Why consider dynamics at all?" and "Why consider design?"

## 2. Why Dynamism?

Why focus on dynamic motions and gestures—on the often non-verbal waving of our hands—when we more typically consider mathematics as a set of concepts, skills, and procedures? Broadly, dynamic gestures and motions are a placeholder for an entire spectrum of embodied actions, intuitions, and understandings we might consider when we replace a perspective of mathematics as a knowledge domain in favor of a perspective on it as a human activity. Neither perspective invalidates the other: the first simply considers a set of propositions, symbolically expressed and organized by principles of abstraction, whereas the second considers the construction, communication, comprehension, and use of those propositions. And yet, a student of mathematics history rapidly learns that one of the social conventions of mathematical practice is to pretend mathematical practice has no social conventions: neither animate people nor animate—moving—objects appear to be acceptable in the empyrean of formal mathematics, in which (in the famous words of Nicolas Balacheff [2]) knowledge appears only and always "detemporalized, depersonalized, and decontextualized." Since this postured independence from all human action or motion makes mathematics a decidedly inhospitable one for many human students (who themselves are irrevocably temporalized, personalized, and contextualized!), it is worth briefly touring arguments for the importance and contribution of the dynamism, motion, and gesture to mathematical thinking.

Such a tour could usefully begin at the dawn of mathematics. Where Western accounts often trace math's origin to the work of the ancient Greeks—Plato coins the term "mathematics"—of course at that time arithmetic and geometry were already well-developed practices, so we look earlier. Drawing on the anthropological literature and Indian, Chinese, Babylonian, and Egyptian documents as well as Greek ones, Seidenberg ([3], [4]) argues compellingly that both disciplines have their earliest origin in religious ritual, and that—specifically—ritualistic chanting and religious procession are the forebears of number sequence, of cardinality and ordinality. Thus the moving body, and the social concert of moving bodies in song and in parade, is the very font of mathematics. Jumping 5,000 years ahead to the opposite end of the timeline, Lakoff and Nunéz [5] trace an arc of influence to present-day mathematics, which they argue, from cognitive science perspectives, is constructed through a small but powerful set of embodied understandings in which mathematical abstractions are metaphors based on concrete physical actions and sense perceptions of the physical body. Inside these two historical bookends, the Platonic move *away* from embodied and enactive (motion-based) understandings seems almost contrarian. Plato arises in an intellectual milieu in which Zeno's "paradoxes" have only recently challenged arguments based on temporality and temporal continuity, so there is philosophical motivation to such a disposition. But present-day scholars like James Kaput or Brian Rotman might argue such a turn arises as much as a consequence of the available infrastructure for mathematical representation and communication in the time of the Greeks (a move from the diagrams of "sand reckoning" to the symbols of writing and "print") as from any explicit epistemological rejection of embodiment, temporality, or dynamism. (Recall that Archimedes—a dynamic geometer! —was more influential in his day than was his rough contemporary Euclid, whose eventually important *book* established the next 2,000 years' pursuit of geometry as a static enterprise.) In either case, the Platonic conceit is clearly situated within a larger narrative in which dynamics and motion-based reasoning are grounded, and continue to ground, mathematical experience and comprehension.

## 3. And Why Design?

And why talk about design? Is the art, mechanics, and practice of design in a particular area, say mathematics education technology, relevant to anyone besides the small community of practitioners working in that area? Yes. We are all users of designed objects; an awareness of, sensitivity to, and ability to think critically about design makes us better-educated consumers of the products or technologies we use. But more importantly, we are all also designers. In mathematics education, we are designers in our choices of approaches to problems. There, design influences how we assemble whatever physical or conceptual toolkit we bring to a given task, and how we orchestrate those elements (what scripts we follow, what existing skills we rehearse) in accomplishing that task. We are also designers of activities, lessons, and classes for our students, sometimes merely in the sensibilities we bring to selecting and sequencing existing curricular resources, other times more aggressively in creating our own new resources out of rawer ingredients. Finally, if or where we use mathematical technologies that are themselves authoring systems—systems where we can produce ideas rather than merely consume them—we are designers of interactive technologies ourselves. In environments like Computer Algebra Systems or Dynamic Geometry Systems, a digital object we create for ourselves, to answer our own private curiosity, is never far from one that explains that same curiosity to another, and a frequent turn in the development of one's work in an environment like *Sketchpad*, comes at the moment one asks "now how can I make this useful for someone else?" In all these cases, we are engaged in design.

And yet, effective design often resists "strategic" or "paradigmatic" thinking—solutions or design strategies that elegantly solve problems in one context often become either simplistic clichés or arcane prescriptions, when reapplied in an unrelated context. We have all seen "effective design principles" stripped of, and promoted outside, any specific application—"put the important thing first!", say, or "keep it simple, stupid!"—and we can easily summon specific problems in which these principles would be not just ineffective but even, perhaps, counterproductive. (Mathematics itself is ripe with such examples: in a deductive argument, we usually need to state the important thing *last!* And we cannot motivate introducing tools of mathematical simplification—factoring, or parentheses, or outsourcing a claim to a lemma or corollary—unless we are willing to tolerate the emergence of sufficient complexity to justify them!) So rather than spend too much effort considering abstract *principles* of design, we must consider design in application, design in specific instances and practices rather than in generalizations and abstractions. In this, getting good at design is like getting good at sport: a theory of volleyball only takes you so far. Eventually, you have to play the game.

## 4. Design and dynamics, in Dynamic Geometry

#### 4.1 The origin of dragging

Now let us explore in detail an actual instance of a designed and gestural form of mathematical interactivity, dragging, as it appears in Dynamic Geometry software such as *The Geometer's Sketchpad* or *Cabri* [6]. While these environments are mathematically rich and find diverse curricular applications, the act of dragging is at their heart and was widely celebrated at their inception as their most important contribution to the evolving corpus of geometry technologies ([7], [8]). Broadly, dragging allows you physically to move one or more elements of a graphical construction or diagram, while preserving all the stated definitions that relate that element to other elements of the figure. In the single gesture of moving your computer mouse across the screen, you tour dozens if not hundreds of continuously-related examples of your construction, its fundamental invariances intact while inessential properties melt away. A diagram is no longer an *example* of a construction, it instead appears almost as the *general case*. It is a powerful tool of mathematical generalization, and since you yourself are guiding the mouse dragging the vertex through all these limitless examples, you find yourself a tremendously empowered agent of mathematical causation and generalization. Thirty years after the invention of Dynamic Geometry dragging, the idea has spread so widely through software and mathematics that it is hard to imagine the sensation of its original impact. And even in that moment, many mathematicians felt an uncanny sense of familiarity upon first encountering Dynamic Geometry or a welcome inevitability ([9]): Dynamic Geometry Software had taken a fantasy of continuously evolving geometric configurations out of the closed garden of mathematicians' daydreams and put it into the literal hands of mathematical explorers everywhere. It is almost as if Dynamic Geometry was not a *designed* idea but rather an essential or pre-existing one that software simply made "real."

In one sense, this claim of inevitability is, in fact, correct. The same powerful idea—dynamic dragging of geometric elements with responsive diagrams adjusting to that motion—emerged simultaneously and completely independently in the United States with *Sketchpad* and in France with *Cabri Géomètre*. Both programs were already well under development by their respective research clusters when they first encountered each other at a NATO Advanced Research Workshop in Grenoble in 1989, and the initial similarities of their approaches to this novel idea was breathtaking. Since the name *Cabri* is a play on the French term for "electronic notebook," even their titles were similar! Yet this is perhaps less a case of raw serendipity than it first appears. The research projects separately giving rise to these programs trace back to the mid-1980s, shortly after the 1984 debut of the Apple Macintosh. The Macintosh introduced bitmap display screens, mouse-based interactions, and graphical user interfaces to the computing public, and these were technologies that individually and in concert disrupted many paradigms of software interaction that preceded them. The Mac's novel hardware and software capabilities forced software professionals in all domains to ask: "what are the implications of graphical visualization and direct manipulation on the objects and workflows of *my* work?" *Sketchpad* and *Cabri* both asked this question, and answered it similarly.

Yet it is by no means given that something as "new" as Dynamic Geometry would arise from such reflection. In many domains—including within the reigning geometry technologies of the day (that is; of *before* the advent of the fully graphical user interface)—the answer was some form of "we shall continue as before, but now use a mouse to choose from menus and icons directing our previous workflows," in a pattern designers have come to call *skeuomorphism* , wherein new objects or technologies are intentionally designed to resemble older technologies they may seek to replace. The ornamental triglyphs of Doric stone columns in classical Greek temple architecture resemble structural features of the wooden temples they replace. While some dismiss the skeuomorphic tendency in design as a form of nostalgia, I believe instead it reflects the difficulty of envisioning what one cannot already, in some sense, see. The skeuomorph signals less the present's longing for the past than its stranglehold on potential futures, in other words. Comically, in the case of *Sketchpad*, the very earliest design motivation for draggable configurations had nothing to do with mathematical exploration or geometric generalization and was instead itself an instance of skeuomorphism. I originally imagined the main purpose of the software was to enable mathematicians to produce print illustrations—perhaps for publication in print journals! In that context, the ability to rearrange their constructions on-screen to fit the available page space in print would be welcome compared to an alternative of rebuilding them, repeatedly, to meet the layout requirements. It was only through a process of considering consequences, and through experiment with some of the first actual prototypes of moveable—dynamic—geometry, that its profound mathematical novelty, and its vast potential impact on visualization and learning, became apparent.

In this preliminary account we gain insight into an essential characteristic of designs' relationship to the domains in which they appear. In the case of software design, the affordances of available hardware not only dictate the limits of what is *possible* in software (an obvious point), but also what is *conceivable* in software (a less obvious one). Thus at any moment within a domain where software finds application, the affordances of contemporary hardware assert an influence not just on our thinking about software but on our thinking about the domain itself. In the 1960s, pre-graphical computers acted entirely as symbol processors; hardware supported text-based input and text-based output. The quest of geometry software in that era—and therefore, of considerable work both in geometry and in computer science—was to automate deduction, a task which conceives of geometric reasoning as a symbol-processing activity. In the 1970s and early 1980s, hardware developed limited support for graphical output but remained symbol-based (keyboard driven) in their input register. The geometry software of that era—Logo ruled the day—involved students' authoring symbolic procedures that produced entertaining or attractive images. Here geometry was acknowledged as pertaining to the structure of space or the plane—the graphical output was relevant—but reasoning and problem-solving was still propositional and linguistic. With the debut of the Macintosh, hardware evolved to feature a ubiquitous bitmapped screen (an always-available 2D graphical output, which the metaphor of "scrolling" makes infinite) and a ubiquitous mouse (an always-available 2D graphical input, which the action of continuous "rolling" makes infinite). It is hard to conceive a better set of physical affordances on which to model the mathematical idea of an unbounded two-dimensional Euclidean plane! And thus in that moment Dynamic Geometry became possible, and the discourse of contemporary school geometry shifted again to emphasize construction and visualization of planar objects rather than their symbolic manipulation. Uncritically, we might like to think of mathematics as Platonically ideal, as impervious to transitory issues of representation or affordance; but in transitions such as these, we see it is constructed by them.

#### 4.2 Deterministic designs for dynamic indeterminacies

Even if the *possibility* of Dynamic Geometry is a direct implication of the emergent computer hardware of the 1980s, its inevitability or mathematical obviousness is, in fact, an illusion. On close inspection, the manipulation of even simple geometric diagrams in space and time opens numerous mathematically unresolvable questions. Thus the "inevitability" of the consequences of dragging owes as much to design as to deduction or to the certainties of mathematical definition—of *static* mathematical definition. Let us briefly consider several examples of situations where dynamics needed to be newly designed for, rather than was defined by pre-existing, geometric purposes, in order to survey the variety of considerations contributing to such design.

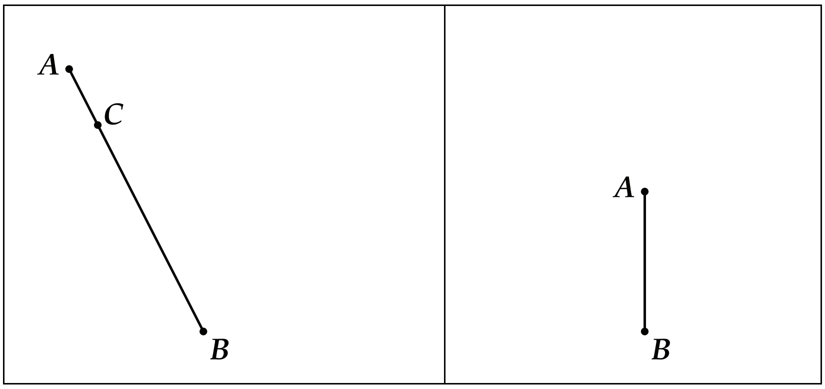



**Figure 1**. Membership indeterminacy: as *A* moves where is *C*?

In the simple example of Figure 1 (left), point *C* is defined as a given point on segment *AB*. The user drags *A* closer to *B*, resulting in *A*, *B*, and the segment *AB* as shown on the right. Where should *C* appear in this segment? Should it maintain its original distance from *A*, though the segment is now shorter? Or is its original distance from *B*? Since all we know is that *C* is defined *somewhere* on *AB*, does that mean *anywhere*? Should it move to a random location? Or perhaps should it stick as close as possible to its original planar position, i.e. to *C* (left), minimizing the distance it itself moves?

*Sketchpad*, *Cabri*, and every other Dynamic Geometry tool I know "answer" this question identically: under changes to *AB*, *C* maintains a constant ratio of division of *AB*. Thus since *C* (left) is roughly 30% of the distance from *A* to *B*, we can locate *C* (right) similarly. This design has the elegant effect of moving *C* in linear proportion to the overall motion of either endpoint and thus maintains a conceptual symmetry between endpoints while avoiding the stochastic chaos of *C* hopping about at random even when the segment is minimally perturbed. And yet, by preserving a constant ratio across dynamic manipulation, the solution introduces a mathematical artifact—a barycentric invariance—to the generalization that is absent from the construction's stated definition. We expect students to be staggered by facts like the concurrence of a triangle's medians in a single point, but in Dynamic Geometry this is unremarkable: dynamically preserved proportions guarantee *any* three cevians of a triangle that *ever* intersect in a point will *always* intersect in a point, no matter how we deform the triangle.

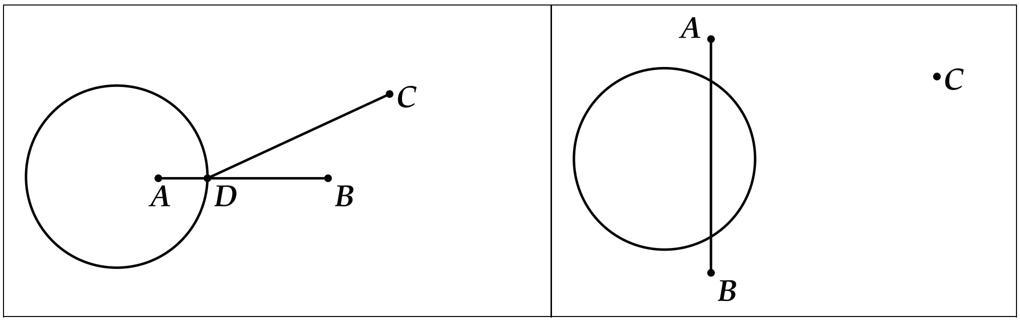



**Figure 2.** Quadratic indeterminacy: as *AB* (left) moves, where is *D* (right)?

In the static configuration of Figure 2 (left), a circle is intersected by a given segment *AB*; and a second segment, from *C*, connects to this intersection *D*. Now we drag points *A* and *B* into the configuration at right, leaving the circle and *C* unchanged. *AB* now intersects the circle twice rather than once. Where should *D* (and therefore segment *CD*) be located in this new configuration?
Mathematically a system of one solution has evolved into a system of two, with no clear mapping between the configurations in Euclidean geometry. To provide a definition of motion that allows *CD* to reappear in the righthand configuration, software developers must augment Euclidean geometry with a dynamic theory and behavior for quadratic intersections, and again balance aesthetic and functional considerations with concern for mathematical artifacts such designs will introduce in naïve exploration.

Here there is less agreement among software designers about what solution might be most fitting to the problem's constraints. Historically *Sketchpad* and *Cabri* both adopt solutions that begin by treating the segment *AB* as intrinsically directed, thus enabling a dynamic discrimination between the two potential intersections. However, they differ in their handling of many of the special cases that arise with finite segments, or with arcs that can effectively "invert" their orientations—and therefore the identity of their quadratic roots. Unhappily, in both programs, it is possible to cause a point to hop from one of two possible circle intersections to the opposite, under only a subtle motion of the configuration's independent points, a discontinuity of output despite a continuous transformation of input. The third major Dynamic Geometry software package, *Cinderella* [11], took issue with this design, and introduced a new solution based on a sophisticated internal mathematical architecture that guaranteed continuity in such transformations. Unfortunately, subsequent research [12] proved that this gain simultaneously eliminated determinism—the highly desirable property that the position of a construction's independent objects fully determines the position of its dependent ones. More recent Dynamic Geometry implementations (such as *GeoGebra* or *Desmos*) have returned to the designs first proposed for *Sketchpad* and *Cabri*, while sustaining numerous local exceptions and variations.

Numerous other situations arise in which the designer is forced to choose between two or more equally plausible mathematical outcomes of a certain dynamic interaction, including ones less narrowly divergent than the choice of "plus or minus a square root" arising between the roots of a quadratic. Consider two points *A* and *A′* related by geometric reflection through a mirror line *l.* In most if not all Dynamic Geometry software, dragging point *A* causes *A′* to move in mirror image. But an equally viable interpretation would move line *l* instead, maintaining a position bisecting the moving *A* and the fixed *A'*. Here a designer's choice (assuming a designer considers both options!) is more subtly pedagogical: should reflection dynamics emphasize the "equal but opposite" motions of mirroring—an emphasis on isometrics and chirality—or instead the "perpendicular bisector" role of the mirror, of potentially greater value to construction and proof?

Perhaps the broadest category of designed "inventions" of mathematical behavior in Dynamic Geometry involves situations where users attempt to drag some dependent object—an object that is constructed rather than independent or given. There is no precedent and even no allowance for such an operation in a geometry of strict deduction. To adhere narrowly to forward inferences reasoning only from givens to conclusions would outlaw such ideas categorically. The earliest versions of *Cabri* enforced such stricture. But in designing early *Sketchpad*, I remarked that in an environment in which different graphical objects of similar type (e.g. different segments) had similar visual identity and similar mouse-based accessibility, whether they were independent or dependent objects, users (including myself) would often *want* to drag such objects, and would often *try* to drag them. An important early design conception I held of *Sketchpad* was it should function as a tool abetting users' pursuit of their own goals rather than either as an authority such as a teacher establishing external goals or a censor didactically prohibiting them. An error message—"dragging such an object is outlawed by the very precepts of deduction itself!"—seemed ill-advised in this situation and unlike the behavior of any physical tool with which I was familiar. So from its earliest version *Sketchpad* has attempted to support users' desire to drag dependent objects, and to assert consistent and predictable mathematical behaviors to such actions.

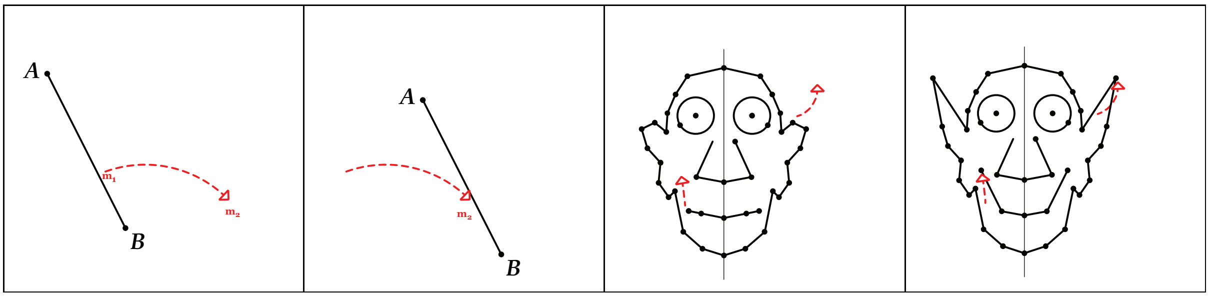
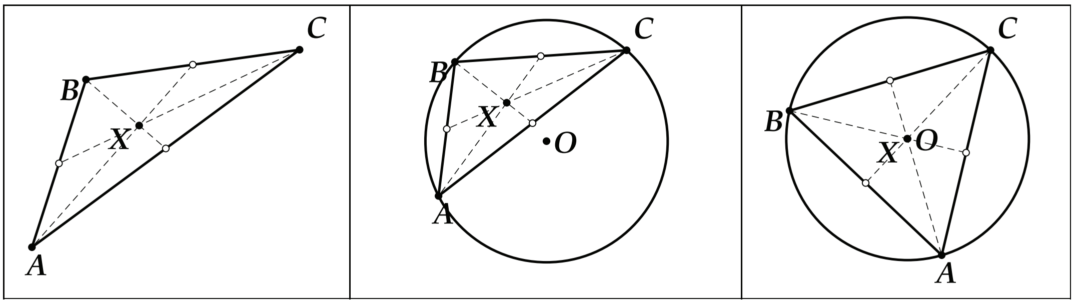



**Figure 3**. Base cases of "reverse dragging"

Figure 3 illustrates two extremely simple applications of reverse dragging. In the before/after sequence at left, a user drags a segment to the right. More precisely, a segment *AB*—an object that is itself dependent on the location of independent points *A* and *B*—is dragged by a mouse gesture from plane location *m1* to *m2*. (In this notation, *m1* and *m2* are not points defined in the construction; they represent only the extremes of a mouse motion.) In a deductively strict interpretation, the situation is under-constrained: an infinite number of potential segments pass through *m2*; the segment could reasonably evolve into any of these. Even introducing the same proportionate invariance as we introduced in Figure 1—with *m2* diving the new segment in the same ratio *m1* divided the old segment—fails to limit usefully the solution space. Instead, *Sketchpad* adopts the (I hope) obvious solution of attempting to preserve the length and orientation of the dragged segment, yielding the parallel translation of *AB* (shown in Figure 3 left, *after*). The important point here is that the justification for such a design draws from any of a number of sources—physical plausibility (in suggesting the segment maintains some "inertial" length or orientation, where possible), or from embracing a hypothetical "tool nature" (tools don't give error messages), or even from imagined user convenience (perhaps the user *wanted* to move the existing segment!)—rather than from any prescriptive mathematics preceding dynamic interactivity!

In the sequence in Figure 3 right, the image is symmetric with respect to a vertical mirror of reflection. Moving elements on either side of the mirror causes the opposite image to move opposite, regardless of which of the pair is technically an "independent" object. Even though one side necessarily preceded the other in the construction sequence behind the original configuration, the designed dynamics emphasize the symmetry of the present transformation over the dependency of its historical construction.

These cases of "reverse dragging" are no more arbitrary—or rather, no more dependent on new innovation and productive design—than the earlier discussed figurations, but they appear to be more controversial among practitioners. *Cinderella* joined *Cabri* in adopting a rigid, forward-only propagation of dependency. Then subsequent versions of *Cabri* relented, acknowledging the tremendous convenience users found in being able to reposition a segment or a circle by dragging that object directly, rather than by dragging its defining points. Then *GeoGebra* copied *Cabri*'s model. But both suffer from a somewhat capricious limitation: reverse dragging works only on objects that themselves depend immediately from independent points, yet other similar objects remain irreversible. Attempting to drag them has no effect. This gives rise to an unfortunate situation in which the very draggability of an object no longer depends on that object's fundamental definition (a segment defined through two points, say), but instead on non-obvious conditions imposed on the its ancestry. (To drag a segment through two points it is not sufficient, in *GeoGebra* for example, for its parental points to be draggable; they must instead be free.)




By contrast, *Sketchpad*—and more recently, *Desmos*—deploy reverse draggability as a generalized premise of Dynamic Geometry, as something consistently available rather than limited to special expedient circumstances. Dragging any constructed element attempts to relocate its determining geometry in such a way that the element remains under the mouse over the course of the motion. This is of course a heuristically-defined search, since—just as in the forward-dragging cases listed above—multiple and diverse solutions exist. From an analytic perspective we are now asking neither "what is the solution to this set of equations?" nor "which of the multiple solutions to this system of equations is preferrable?" but rather "which system of equations might produce this desired solution?" It is an entirely different approach to problem posing, and occasionally the system yields startling insight. In the example of Figure 4, left, a user begins with construction of medians of a triangle, that concurs in a point (the centroid *X*). Knowing the circumcenter of a triangle is a similar point of concurrence (of side bisectors), she might wonder "is it possible for the centroid and circumcenter to be the same point?" Drawing a circle and reinscribing her triangle inside it allows her to compare the triangle's centroid to its circumcenter *O* (Figure 4, middle). She then drags the centroid—a highly dependent object, a point of concurrence of three medians, which are themselves segments connecting points on the circle (the reinscribed triangle's vertices) to midpoints of the sides opposite those vertices—and moves it toward the center of the circle. The triangle wriggles and stretches into a position (Figure 4, right) where $X$ and $O$ coincide at just the moment that—ah ha!—the triangle becomes equilateral. Such provocations to insight cannot occur for the strict axiomatician, for whom reverse dragging is geometric heresy, but are very in keeping with the inductive and exploratory approaches that were encouraged as complements to strictly deductive enterprise by the mathematics reform movements of the 1980s.

## Reflection

Collectively, this case history of design situations demonstrates that when the surface of traditional mathematical practice is wrapped taut across a new representational medium, fissures or tears or holes open up in which prior mathematics is insufficient to describe the new shape. These can be viewed as inadequacies of the novel medium or—equally—as opportunities to ask and answer new mathematical questions. Thus every new representation both opens and closes doors to potential forms of activity. When we survey how new questions are answered and how such answers are ratified, we see a variety of mechanics at play. Sometimes even where a question has never been previously asked, a solution appears so obviously sufficient there is little interest in questioning it further (the point-on-segment example). In other situations, an issue appears ultimately resolved by reasoned consensus emerging from a community's grappling over time with various alternatives (the quadratic intersection example). Elsewhere, a variety of equivalent solutions seem to vary largely on sociocultural lines, reflecting pedagogic norms for instance that may vary place to place, and in these moments we may sustain multiple viable solutions although perhaps only one in one place at one time. (For example: should a constructed symmetry better appear as a construction or as a symmetry, in situations where both cannot be emphasized equally?). Even where a solution arises as the result of one individual's singular or quixotic vision, often such vision appears consonant with developments in surrounding socio-material and sociocultural milieux (as in the origin of Dynamic Geometry in the dawn of the modern graphical user interface; or in the relation of reverse dragging to the "inductive approaches" encouraged by the 1989 *NCTM Standards* [13]). These varied mechanics demonstrate not just how a novel tool arises within a social context but how a community of practice adopts and empowers a new representation.

It is hubris to suggest that a handful of software programs, or even an entire genre of them, form some sort of epochal advance in the long evolution of mathematical representation or practice. Software goes obsolete within the lifetime of its authors and compared to the durability of journal publication (to say nothing of the durability of *The Elements*!), is as fragile as Archimedes' drawings in the sand. And yet, the design and social ratification of Dynamic Geometry representations stands as a small example of an ongoing and much larger revolution in representation. The rise of propositional and symbolic mathematics, in the era of Plato and Euclid, corresponds historically to the rise of pervasive writing and the logocentric technologies of print. 2500 years later, print—with its relatively fixed symbol catalog and its physical archive in the bookshelves and libraries of the world—is being rapidly and systemically replaced by computational representations. While computers excel at symbol manipulation—and therefore will extend and further symbolic mathematics—it is perhaps a skeuomorphic error to think of them only as symbol processors; they are electricity processors. While we may code their currents and voltages into symbolic zeroes and ones, we then deploy those symbols to depict graphical diagrams and temporal motions, and to capture gesturing hands, probing fingers, and rolling mice. These non-symbolic infrastructures point the way to a post-symbolic mathematics, of which Dynamic Geometry is perhaps an early suggestion. As in biological evolution there is no goal or even clear direction, only a mechanism of change. Ultimately, what we mean by mathematics is a function of what we do, when we do mathematics, and of what we think, when we think mathematically. Both these in turn are constituted by the representations, technologies, infrastructure and affordances of the moment. When we look closely at any one of them, such representations or technologies reveal themselves to be caught, in that moment—as in every moment—between the nostalgia for a vanished past, and the uncertain vision of many potential futures.

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# How Many are Factorable?

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**Abstract**: *In this paper, an investigation in quadratic factoring is shared that begins with a straightforward problem that technology enables to evolve into other avenues of investigation of varying complexity and accessibility.*

## 1. Introduction

In 1995, the National Council of Teachers of Mathematics [1, pp. 1] called for a vision of mathematics (specifically algebra) that "shift emphasis away from symbolic manipulations toward conceptual understanding, symbol sense, and mathematical modeling." While one could argue that we are still far from that goal, the technology that is currently freely and readily available to both teachers and students is making this statement one that is certainly realizable. The use of technology to learn, teach, and do mathematics provides a chance to engage in classroom mathematics beyond what might be considered typical and what has been considered standard for years. Additionally, technology allows access to mathematics and mathematical ideas that might have been previously hidden or inaccessible. In this regard, it is most assuredly the ladder that Kennedy [2] argued for so many years ago as he saw the potential for technology to allow students to learn, play and ultimately discover and engage in meaningful mathematics.

The investigation shared in this paper is made possible because of the use of technology to serve as the ladder to allow students to access mathematics that could be a challenge due to the potential computational burden. The investigation (considering what quadratic polynomials are factorable) makes consistent use of technology as a way to make sense of the problem while attending to mathematical processes expected in any American high school algebra class. Technology further allows the opportunity to engage in study that envelopes many mathematical ideas while still staying grounded in basic processes.

This investigation engages students in several mathematical practices [3] and the process of doing mathematics [4]. Among the mathematical practices, the two most prevalent are 'making sense of mathematics and persevere in solving them' and 'reason abstractly and quantitatively'. As will be demonstrated, the main problem tackled in this work allows students to make sense of factoring while attending to both the process and theory of factoring. This is significant as many students learn the procedure of how to factor but are rarely asked to further consider the purpose and viability of that procedure. In order to accomplish this, abstracting some of the procedural aspects of factoring is required.

## 2. The Probability of a Quadratic Being Factorable

The process of factoring polynomials, specifically quadratic trinomials, is part of nearly every American high school mathematics curriculum. A significant amount of time is spent on learning techniques and related procedures to enable one to efficiently factor. Using technology tools such as spreadsheets, CAS, and graphing calculators we have the potential to deepen student understanding of factoring polynomials.

The initial problem investigated requires nothing more than the ability to know what it means to factor a basic one variable quadratic polynomial of the form $ax^2 + bx + c$ and to know what it means to be factorable. The initial problem investigated is modified from a high school textbook [5, pp. 738-740] to ask students to do the following-

> There are infinitely many quadratic expressions of the form $f(x) = ax^2 + bx + c$, but what percent are factorable over the rationals when $a$, $b$, and $c$ are each nonzero integers from -10 to 10?

For initial investigative purposes, students are first asked to generate at least 30 random polynomials that fit the problem criteria. While this is admittedly a small sample, part of the technology use here is having students think through the process of how to get technology to assist in the randomization and knowing that the results they are getting are correct. Several CAS (*Mathematica* specifically) have built in functions that can automate much of what we initially discuss here. However, most students are unfamiliar with even the most basic functions of CAS and so a spreadsheet is often the desired tool to use.

One way to generate the coefficients for the polynomials is to have a spreadsheet generate random numbers according to the criteria and eliminate those that do not fit. For example, the function RANDBETWEEN() in many spreadsheets will return a random integer in a specified range. With this command, if the range is specified as -10 to 10, then 0 will be a possible result, and so one needs to take care to account for this and not count those polynomials. While there are ways to work around this issue, such as using the CHOOSE() function combined with the RANDBETWEEN() function, most students initially working on the problem find it easiest to just throw out any polynomial not meeting the criteria and generate new examples as needed. If one uses a CAS like *GeoGebra* or *Mathematica*, there are built in commands to generate random polynomials, but those commands can have the same issues.

Regardless of the tool used, one will also need a test for factorability. Most students at this stage use a factor command from a CAS and manually type in each polynomial generated. Results from this initial work rarely result in more than 15% of the generated quadratics being factorable. While this will complete the problem, it far from solves the problem. Students are further challenged to make their work more robust and generalizable by being asked to consider what it means for a quadratic to be factorable. This is something most have never thought about before and so the spreadsheet can also provide one way to run this test.

To create a test for factorability, consider a typical quadratic polynomial, $ax^2 + bx + c$, with $a$, $b$, and $c$ having the criteria for the original given problem. We know that the discriminate is calculated as $b^2 - 4ac$ and must be a perfect square if the given quadratic is to be factorable. This is a key insight missed by many students and one not usually discussed or encountered explicitly in the high school setting. Usually, this criterion is implied but its meaning is lost and students do not understand that this is truly what it means to be factorable. This abstraction (consideration of the discriminant) is an essential realization for this investigation. To collect data with this in mind, the spreadsheet shown in Figure 2.1 provides one possible setup for the task. This approach uses a way to eliminate potential zero coefficients first and then randomly determine each polynomial coefficient. Columns A, B, and C are the polynomial coefficients as labeled, columns D and E calculate the discriminate value and tests that value to see if it is a perfect square. Finally, column F counts the number of polynomials that are factorable.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | a-coefficient | b-coefficient | c-coefficient | Discriminant | Factorable? | Total Factorable: |
| 2 | -4 | -2 | -2 | -28 | No | 3 |
| 3 | -9 | 3 | 8 | 297 | No | Percent Factorable: |
| 4 | -4 | -10 | 1 | 116 | No | 0.06 |
| 5 | 7 | -6 | 5 | -104 | No | |
| 6 | 10 | -5 | 10 | -375 | No | |
| 7 | -7 | 5 | -2 | -31 | No | |
| 8 | 4 | -4 | 6 | -80 | No | |
| 9 | -10 | 4 | -3 | -104 | No | |
| 10 | -6 | 2 | 7 | 172 | No | |
| 11 | 1 | 8 | -7 | 92 | No | |
| 12 | -3 | -5 | 10 | 145 | No | |

Figure 2.1

With this result in mind, an exhaustive investigation of the original problem can be completed. Given that there twenty possible coefficients for each term and the quadratics in question have three terms, there are (20)(20)(20) = $20^3$ = 8000 possible quadratic trinomials fitting the given problem. With this list, one can then test for positive perfect square discriminant values. In this problem the maximum value of the discriminate is calculated as $10^2 - 4(10)(-10) = 500$, thus only positive perfect squares less than or equal to 500 need be considered. So, for this class of polynomials, 892 are factorable meaning less than 12% (11.15%) of the total are factorable.

## 3. Further Investigation

These results provide a catalyst for a deeper investigation by expanding the allowable coefficients on the quadratics. Let $M \in$ Z and let [-$M$, $M$] represent the non-zero integer range for the coefficients of *f(x)*. The initial investigation relied heavily on brute force with a spreadsheet but with a CAS (specifically *Mathematica*) one can use a series of commands to efficiently count polynomials and collect needed information. Program 3.1 from *Mathematica* shows one possible approach.

Program 3.1

```
coefficients = Table[i, {i, 1, M}];
coefficients = Join[coefficients, -1 coefficients]
quadraticList =Flatten[Table[ax2+bx+c,{a,coefficients},{b,coefficients},{c,coefficients}]]

Length[quadraticList]
Discriminant[quadraticList,x]

result= Select[quadraticList,!IrreduciblePolynomialQ[#]&]
Length[result]
Sort[Tally[Discriminant[result,x]]]
```

Using this approach a table is first built to generate the allowable coefficients (the user specifies *M*) then **quadraticList** generates the possible quadratic functions. The length of the list can be found which specifies the number of quadratics generated. Next, the **Discriminate** function provides the list of discriminates for all the quadratics generated. In order to parse out those discriminates that are perfect squares, the built in function **IrreduciblePolynomial** is used. The user defined function **result** filters the list of generated quadratic functions to those that are factorable. The **Length** function returns the number of quadratics that are factorable. Finally, the **Sort** and **Tally** functions provide an ordered list of the perfect square discriminants along with how many times each one occurred.

These series of commands allow for efficient collection of data and more importantly, allow the determination of the percentage of quadratic polynomials that factor as *M* increases. Without technology, this avenue of exploration could prove nearly impossible to pursue. Table 3.1 shows results for different M values.

Table 3.1

| M | Total Quadratics | Number Factorable | Percent Factorable |
|---|---|---|---|
| 10 | 8000 | 892 | 11.15 |
| 11 | 10,648 | 1084 | 10.18 |
| 12 | 13,824 | 1404 | 10.16 |
| 13 | 17,576 | 1640 | 9.33 |
| 14 | 21,952 | 1972 | 8.98 |
| 15 | 27,000 | 2344 | 8.68 |
| 20 | 64,000 | 4628 | 7.23 |
| 25 | 125,000 | 7800 | 6.24 |
| 30 | 216,000 | 12,076 | 5.59 |

It may seem initially surprising or counter intuitive that the percentage of factorable polynomials decreases as the *M* value increase for the polynomial coefficients. Notice that the number of possible polynomials for each case can be calculated in general as $(2M)(2M)(2M) = 8M^3$ and the data show that the number of factorable polynomials is potentially modeled by a function that will always remain less than $8M^3$. The graph in Figure 3.1 shows how the total quadratic polynomials in each case compares to the number of factorable polynomials in each case. The curve represents the number of quadratics polynomials, while the individual data points are the number of factorable quadratic polynomials.

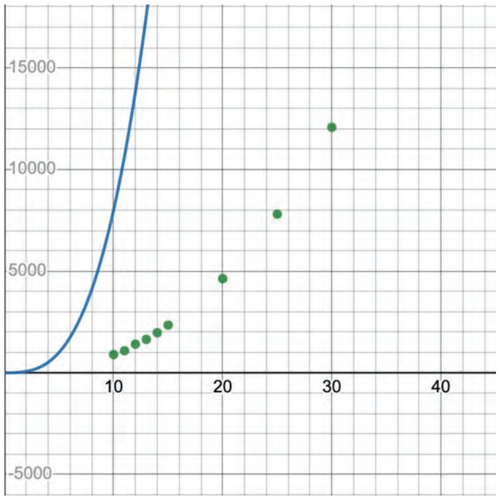


Figure 3.1

Using the discriminate one can calculate

$$M^2 - 4(-M)(M) = M^2 + 4M^2 = 5M^2$$

as the upper limit on the largest possible positive discriminate value. This also determines the entire range of discriminate values as $[1 - 4M^2, 5M^2]$. Thus, the largest number of perfect square candidates that can be obtained is bound by $(\sqrt{5})M$. If we wanted to be more precise, we can put the number $(\sqrt{5})M$ in the floor function. For example, when $M = 10$, the discriminant range is [-399,500] and there are 22 (bounded by $(\sqrt{5})10$ which is ~22.36) possible positive perfect square candidates. The exhaustive list in Table 3.2 shows that 441 (bounded by 500) is the largest square number that can appear in this range. There are no combination of numbers when $M = 10$ to yield a discriminate of $22^2 = 484$.

Table 3.2

| Discriminate | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of times occurred | 44 | 56 | 48 | 52 | 40 | 52 | 44 | 64 | 44 | 56 | 48 | 72 | 40 | 52 |

| Discriminate | 196 | 225 | 256 | 289 | 324 | 361 | 441 |
|---|---|---|---|---|---|---|---|
| Number of times occurred | 48 | 36 | 24 | 24 | 24 | 16 | 8 |

This upper limit increases to 24 possible perfect square candidates when $M = 11$, but like $M = 10$, the actual largest perfect square is 441. In this way, the range of the discriminate is growing faster than the upper limit on the actual perfect squares candidates. Graphically in Figures 3.2 and 3.3, one can see that the distribution of all discriminants is relatively stable and normal like and thus most likely will not significantly change for increasing values of $M$.

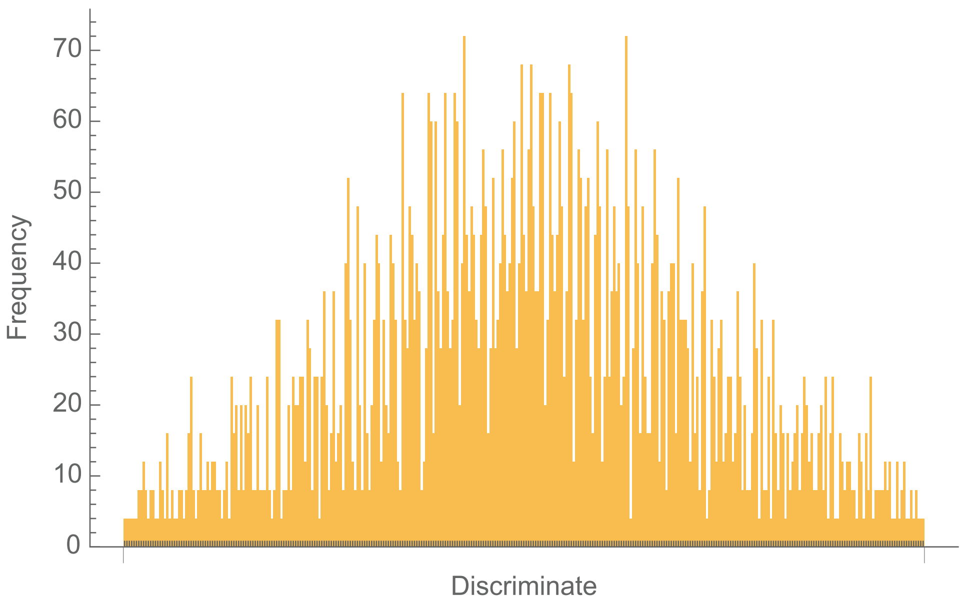


Figure 3.2 ($M = 10$)

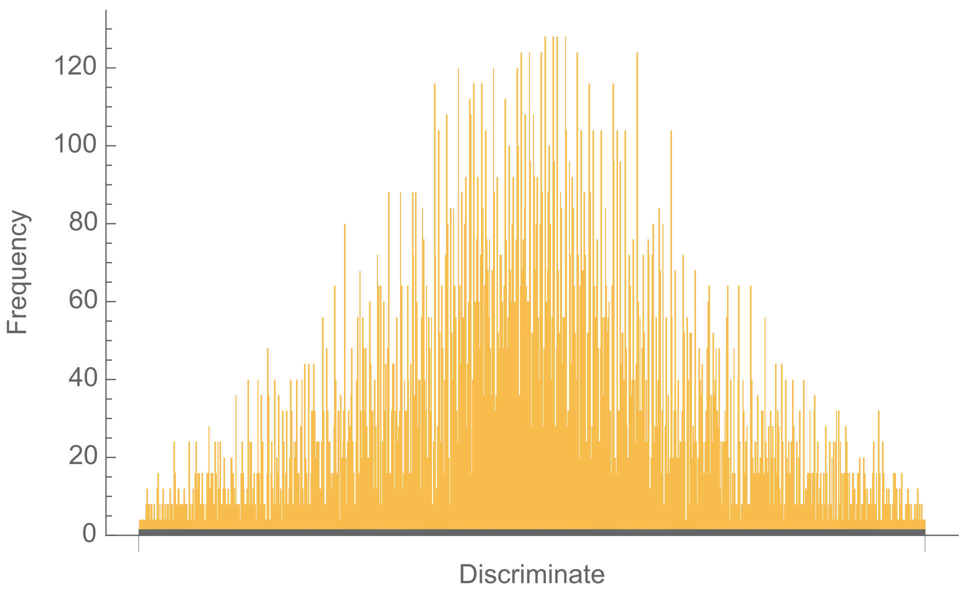


Figure 3.3 ($M = 15$)

There are still avenues of exploration and analysis left, but the work done here shows there is exploration and investigation in even the most routine of topics. The use of technology here was essential in gathering data, understanding the problem, and providing a playground for investigation. Technology provides the conduit for this investigation.

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# Applications of Lua for LaTeX Documents

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**Abstract**

*This article discusses various applications of Lua for LaTeX documents. It mainly focuses on enhancing the graphical aspect of LaTeX using Lua and creating Android applications from LaTeX documents. This is an extension of our work of creating computational packages for LaTeX using Lua. The computational packages we developed are available for LaTeX users on the CTAN repository and bundled with standard TeX distributions. In continuation of this work, the development and deployment of the luaplot package is discussed in this article. It also describes the outline for creating Android applications from LaTeX files using Lua and other resources. The Android applications created from LaTeX files do not need the internet, are static, and do not support calculations. The one purpose of using Lua for LaTeX documents is to reduce the dependence of LaTeX users on external software for computations and graphing. The other purpose is to provide techniques and methods to make mathematical content in LaTeX documents available to Android users. Some of the packages we developed can also be deployed for pedagogical purposes.*

## 1 Background and introduction

Lua [4] is a portable scripting language that we have used for different applications to LaTeX documents. The applications include computational and graphical aspects inside LaTeX documents and the creation of Android applications from LaTeX documents.

The article is organized in different sections. The application of Lua for performing basic mathematical computations inside LaTeX using Lua is described in the second section. The third section provides key features of the plotting tool that we developed and a few illustrations of it. The outline for the creation of Android applications from LaTeX files using Lua and other resources is given in the fourth section. The fifth section provides conclusions and the future plans and prospects of the research work are in the sixth section.

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# 2 Computations inside LaTeX documents using Lua

We have developed luamaths [9],luacomplex [6], luaset [14], luagcd [7], luatruthtable [15], luanumint [12], and luamodulartables [10] packages. The `luamaths` package is for basic arithmetic on real and complex numbers and evaluation of standard real-valued functions of a real variable. The `luacomplex` package is for computations on complex numbers. The `luagcd` package can be used to find the gcd of two or more integers and to verify Bezout's identity. The truth tables of boolean variables can be generated with the `luatruthtable` package. The numerical integration of real-valued functions of a real variable can be performed using the `luanumint` package. The `luamodularmaths` package generates modular addition and multiplication tables of positive integers. The linear algebra lualinalg tool [8] is a major highlight of the packages we developed. The standard computations on vectors and matrices can be performed with this tool inside LaTeX documents. The research article "Basic Mathematical Computations inside LaTeX using Lua" [22] describes some of these packages.

# 3 Plotting Graphs in LaTeX using Lua

We have developed the luaplot package [13] using Lua to plot graphs of real-valued functions of a real variable in LaTeX. It makes use of the Metapost system [17] and luamplib [11] and luacode [5] packages. It provides an easy way for plotting graphs of standard mathematical functions and their finite combinations. There is no particular environment in the package for plotting graphs. It also works inside floating environments of LaTeX like tables and figures. The compilation time to plot several graphs in LaTeX using the luaplot package is significantly less with LuaLaTeX engine.

The core idea is to load mathematical functions inside Lua and determining plot points using different methods available in Lua. After determining plot points in Lua, two different approaches are used:

- parse plot points to the MetaPost system via `luampblib`.
- parse plot points to the `tikz` package.

The MetaPost system is based on the Metafont to produce precise technical illustrations. Donald Knuth designed Metafont for TeX. John Hobby designed the MetaPost system to produce scalable PostScript or scalable vector graphics. The output from MetaPost can be directly included with LaTeX. The first approach thus offers a native way of plotting graphs inside LaTeX using Lua and MetaPost.

Tikz is designed by Till Tantau for producing vector graphics from different expressions. Drawing lines, arrows, paths, geometric shapes, etcetera is possible using Tikz [18]. Tikz commands can be considered as TeX macros, but Tikz itself is a language. LaTeX users widely use Tikz to produce different graphics. The second approach combines Lua and Tikz to plot graphs inside LaTeX.

## 3.1 Illustrations of the luaplot tool

Listing 1 illustrates the `luaplot` command. It generates graphs shown in Figure 1. For all plotting options, the Metapost package documentation [17] and guide [3] can be referred.

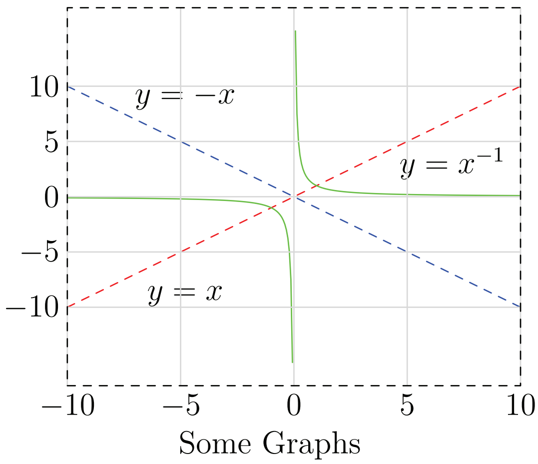


Figure 1: Plotting graphs with the luaplot package

Listing 1: Plotting with the luaplot command

```
\luaplot[
xmin=-10,
xmax=10,
plotpts=300,
hor='6cm',
ver='5cm',
clr={'red; blue; green'},
plotsty={"dashed evenly, dashed evenly"},
plotoptions={
[[
glabel(btex ${y=-x}$ etex,(-4.8,9));
glabel(btex ${y=x}$ etex,(-4.8,-9));
glabel(btex ${y=x^{-1}}$ etex,(7,3));
glabel.bot(btex Some Graphs etex, OUT);
autogrid(grid.bot,) withcolor .85white;
autogrid(grid.lft,) withcolor .85white;
frame.dashed evenly;
]]
}
]
{x,-x,1/x}
```

Listing 2 illustrates plotting graph of a function with the `luatikzpath` command. Multiple graphs can be plotted in a single picture environment.

Listing 2: Plotting with the luatikzpath command

```
\begin{tikzpicture}
\draw[thin,->] (-5,0)--(5,0)node[right]{$x$};
\draw[thin,->] (0,-3)--(0,2.5)node[above]{$y$};
\draw[red] \luatikzpath{sin(x^2)}{-4}{4}{100} node at (1.2,1.3)
```

```
   {$y=\cos(x^2)$};
\draw[blue] \luatikzpath{log(x)}{0.1}{4.9}{100} node at (3.9,1.8)
   {$y=\log(x)$};
\draw[blue] \luatikzpath{log(-x)}{-4.9}{-0.1}{100} node at (-3.5,1.8)
   {$y=\log(-x)$};
\end{tikzpicture}
```

Listing 2 generates graphs shown in Figure 2.

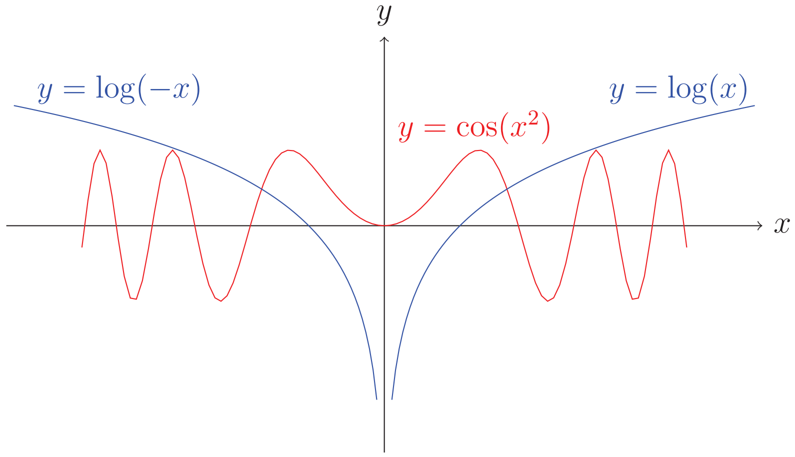


Figure 2: Plotting with the luatikzpath command

### 3.2 Known issues and limitations

The package does not use any external library supporting arbitrary precision arithmetic. The luaplot package can handle big and small numbers within the range of Lua that it supports. However, the MetaPost system does not support numbers in scientific notation. The coordinates of plot points produced in scientific notation are rounded off to 12 significant decimal places within the package. This may cause slight deviations from actual values. The same issue is not faced while parsing points to tikz as it supports input in scientific notation.

## 4 Android Applications from LaTeX files

This section describes the technique to create Android applications from LaTeX files containing mathematical content. We have created unique technique of producing Android Apps from LaTeX files. This technique has a tremendous advantage for displaying mathematical content. No online resources are required for displaying mathematical content as it is pre-compiled. This is a significant advantage. It reduces dependability on server-side scripts such as MathJax, which needs an internet connection. The mathematical content typed in LaTeX preserves its formatting. Further the mathematical content is mobile responsive. It is not easy to create mobile responsive mathematics content using other techniques. Another advantage is that one can use javascript and/or jquery. It reduces dependability on java in android applications for scripting purposes, and it can be preferred over other techniques for creating android applications containing mathematical content.

The following outlines the procedure for creating an Android app using a LaTeX file.

**I:** Pre-compilation of LaTeX file using Lua and make4ht.

**II:** Adding jquery library and javascript file(s).

**III:** Adding CSS file(s).

**IV:** Using the WebView Class to create Android app.

## 4.1 Resources required

The following resources are required for implementing the technique.

- A TeX distribution (such as TeXLive or MikTeX) with Make4ht package.
- A LaTex file with the article document class.
- Node.js, npm and Mathjax node.
- A small collection of Mathjax fonts.
- A custom filter `.lua` file
- A custom build `.mk4` file
- A custom configuration `.cfg` file
- The javascript and jquery files.
- A Cascading Style Sheets (CSS) file.
- Android Studio or other IDE.
- A java code for webview class.

There is a command line application provided by mathjax-node-page. It is `mjpage`. This processes HTML files and substitutes LaTeX or MathML code with plain HTML or SVG code. The mathjax node can be installed by using the following inline command.

```
npm install -g mathjax-node-page
```

There should be a woff subdirectory in the working directory containing MathJax fonts in the woff format for using this technique. It can be downloaded from MathJax resources.

## 4.2 Precompilation of LaTeX file using Lua and make4ht

This subsection provides the procedure to produce pre-compiled html document using Lua and the make4ht [16] package.

With all the resources in the working directory, a latex file containing mathematical content can be pre-compiled with mathjax to html file by executing the following inline command.

```
make4ht -uc custom.cfg -e mybuild.mk4 file.tex html5
```

The source code of the custom filter `mathnode.lua` file is given in listing 3. It is used for pre-compilation of mathematical content in html files.

Listing 3: Source Code mathnode lua file

```
 -- local mathnodepath = os.getenv "mathjaxnodepath"
-- print("mathnode", mathnodepath)
local mkutils = require "mkutils"
```

```
-- other possible value is page2svg
local mathnodepath = "mjpage"
-- options for MathJax command
local options = "--output CommonHTML"
-- math fonts position
-- don't alter fonts if not set
local fontdir = nil
-- if we copy fonts
local fontdest = nil
local fontformat = "otf"

local function compile(src)
  local tmpfile = os.tmpname()
  local filename = src
  print("Compile using MathJax")
  local command = mathnodepath .. " ".. options .. " < " .. filename .. " > "
      .. tmpfile
  print(command)
  local status = os.execute(command)
  print("Result written to: ".. tmpfile)
  mkutils.cp(tmpfile, src)
  os.remove(tmpfile)
end

-- save the css code from the html page generated by MathJax
local function extract_css(file)
  local f = io.open(file, "r")
  local contents = f:read("*all")
  f:close()
  local css = ""
  local filename = "mathjax-chtml.css"
  contents = contents:gsub('<style [^>]+>(.+)</style>', function(style)
    -- replace only the style for mathjax
    if style:match "%.mjx%-math" then
      css = style
      return '<link rel="stylesheet" type="text/css" href="'..filename ..'" />'
    end
  end)
  local x = assert(io.open(file, "w"))
  x:write(contents)
  x:close()
  return filename, css
end

-- Update the paths to fonts to use the local versions
```

```
local function use_fonts(css)
  local family_pattern = "font%-family:%s*(.-);.-%/([^%/]+)%.".. fontformat
  local family_build = "@font-face {font-family: %s; src: url('%s/%s.%s')
     format('%s')}"
  local fontdir = fontdir:gsub("/$","")
  css = css:gsub("(@font%-face%s*{.-})", function(face)
    if not face:match("url%(") then return face end
    -- print(face)
    local family, filename = face:match(family_pattern)
    print(family, filename)
    local newfile = string.format("%s/%s.%s", fontdir, filename, fontformat)
    Make:add_file(newfile)
    return family_build:format(family, fontdir, filename, fontformat,
       fontformat)
    -- return face
  end)
  return css
end

local function save_css(filename, css)
  local f = io.open(filename, "w")
  f:write(css)
  f:close()
end

return function(text, arguments)
  -- if arguments.prg then mathnodepath = arguments.prg end
  mathnodepath = arguments.prg or mathnodepath
  options    = arguments.options or options
  fontdir    = arguments.fontdir or fontdir
  fontdest   = arguments.fontdest or fontdest
  fontformat = arguments.fontformat or fontformat
  compile(text)
  filename, css = extract_css(text)
  -- use local font files if fontdir is present
  if fontdir then
    css = use_fonts(css)
  end
  save_css(filename, css)
  Make:add_file(filename)
  -- print(css)
  print(filename)
end
```

The source code of `mybuild.mk4` file is given in listing 4. It is a custom build mk4 file.

Listing 4: Source Code of mybuild mk4 file

```
local mathjax_node = require "mathnode"
local format = "woff"
Make:match("html$", mathjax_node, {fontdir = format, fontformat = format})
```

The custom configuration file `custom.cfg` can have the following code.

```
\Preamble{xhtml,mathml}
\begin{document}
\Css{body{font-family: MJXc-TeX-main-Rw, MJXc-TeX-main-Iw, MJXc-TeX-main-Bw,
   sans-serif;}}
\EndPreamble
```

These sets of codes are written by Michal Hoftich, a developer of make4ht package. It is available on the tex stack exchange website [19] and it can be freely used and shared under CC-BY-SA international license as per stack exchange website guidelines.

The MathJax node project is now archived. There is still better technique. The current version of make4ht now contains an extension for mjcli. It can be requested using the following command.

```
make4ht -f html5+mjcli filename.tex
```

## 4.3 Adding style and script files

After pre-compilation of LaTeX file into html file, style files (CSS files) and script files (javascript files and a jquery library) files are added to create a custom web app. This depends on the structure of the app that one wants to make. A user can design his style and script files to get the desired formatting and layout in the app.

## 4.4 Using the WebView Class to create Android app

There are many techniques that can be used to produce android applications from customized documents or web applications [2]. One technique to create android apps from latex files is to use `WebView class`. The `WebView class` [1] is an extension of android's `View class`. The WebView can display both online and offline web applications or even combination of online and offline applications. The advantage of this technique is that if online web applications are updated, then they get reflected in android applications without needing to incorporate updates in Android app. The user interface is preserved as well. The hierarchy of android webview in the java platform of android is given in Figure 3.

## 4.5 Examples based on the techniques

The following android applications are created based on developed technique.

1. Series of Real Numbers [21]

2. Riemann Integration [20]

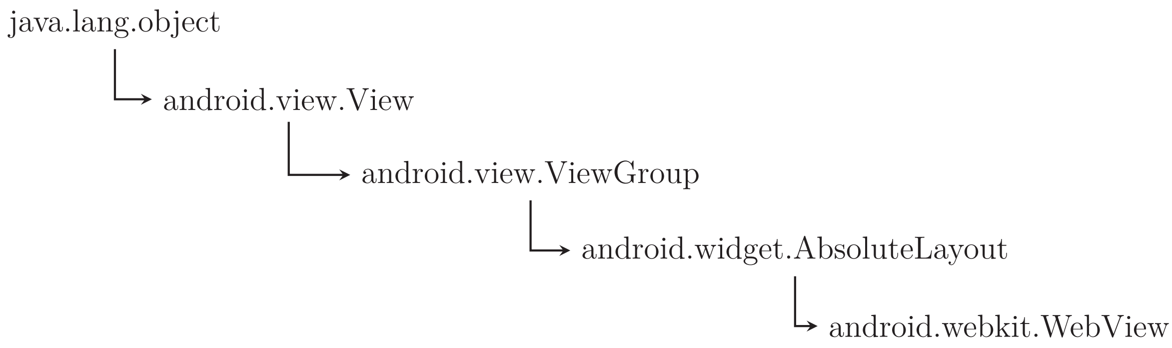


Figure 3: Android WebView Class

# 5 Conclusions

The following conclusions are drawn from the carried out work.

- Lua can effectively be used for the development of mathematical packages in LaTeX.
- Lua-based mathematical packages can reduce the dependence of LaTeX users on external software.
- Lua can be used to enhance graphical aspect of LaTeX.
- Lua-based mathematical packages can have pedagogical applications. Lua-based mathematical packages can be deployed to create interactive teaching modules in LaTeX. They can also be used to create different problem sets, assignments along with solution sets.
- Lua can be implemented in LaTeX to create customized documents, web applications and android applications.

# 6 Future work

The following ideas can be materialized in future.

- The techniques used in the creation of Android applications from latex files involve manual work of editing html and css files and scripts or programs. This part can easily be automated with Lua scripts. This automation will reduce the efforts required in the creation of Android applications.
- Lua-based mini Computer Algebra System can be developed and integrated into LaTeX. It will consist of Lua packages developed for performing symbolic mathematical computations.

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# Teaching to high school students the intimate relation between definite integrals, piecewise quadrature and areas

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**Abstract**

Numerical integration is an important topic for modern analysis and differential equations. From an educational point of view, in the high school mathematical curriculum, the definite integral is often understood as an application of indefinite integrals, and area computation as a further application. High school students enjoy calculating definite integral sound area by using the Fundamental Theorem of Calculus without understanding the essence of definite integrals as limits of Riemann sums, a natural idea leading to the more advanced notions of piecewise quadrature and measure.

We propose a unified teaching approach of definite integrals for high school students which allow not only a mathematical understanding of the notions of definite integrals and area computations through the notion of piecewise quadrature but also the relation with the different figures of area known already and prepare the ground for the notions of numerical integration and theory of measure to be learned at undergraduate level. Another important application is the calculus of limit of the sum of terms of an infinite sequence that can be represented as a finite area through piecewise quadrature.

To facilitate an authentic comprehension, the concepts are accompanied by GeoGebra scripts to visually depict their application in mathematical education, thus highlighting the direct integration of Information Technology within this domain.

## 1 Introduction

High school mathematical education is very important for acquiring essential knowledge and skills, for preparing students for university undergraduate level as well as later career, for making them able to live in a modern world packed with information.

In particular, we consider the limit to be a fundamental concept in calculus, an indispensable skill for any student interested in fields of pure or applied mathematics, physics, engineering, computer science, etc. Beside providing a rigorous way to define continuity of functions, derivation and so on, understanding limits is a necessary step for obtaining mathematical maturity, for building intuition and opening the way to modeling and solving phenomena of the real world.

Even though concrete problems related to limits can be found in the countless number of calculus textbooks, we point out that limits of the following type are challenging for many high school students.

$$\begin{aligned} &\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right]=\log 2\\ &\lim_{n\to\infty}\frac{n}{\sqrt[n]{n(n+1)(n+2)\ldots(n+n)}}=\frac{e}{4}. \end{aligned} \tag{1}$$

A related notion is the area which is introduced in elementary education to develop a sense of measuring and comparing shapes. Almost any teenager can tell you that the area of circle radius $r$ is $\pi r^2$, but how many of them understand why it is?

The unit of area is usually defined as the area of a circle with sides of unit length being equal to $1 \times 1$, and from here the area of any square or rectangle can be computed by decomposing it in one unit and simply counting the squares.

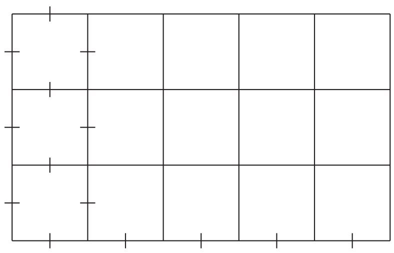

Figure 1: The notion of area.

The extension to triangle's area is quite intuitive.

However, when moving on to the circle area, things get more complicated. The irrational number $\pi$ can be defined as the length of a circle of unit diameter. To prove that $\pi = 3.14\ldots$ is firstly suggested by direct measurements, but to provide a rigorous mathematical proof of this fact, one needs to apply the squeeze theorem for limits to the perimeters double inequality

inscribed cyclic polygon $\leq$ circle length $\leq$ perimeter of the circumscribed.

The circle area formula is now proved by decomposing the circle into sectors and rearrange them.

Knowing circle length and rectangle area, it is trivial to obtain the area of the circle as $\pi r^2$.

The area mystery goes further into high school mathematics ([7], [8]). If $f : [a, b) \to \mathbb{R}$ is a continuous function, then after defining the primitive $F$, i.e. $F'(x) = f(x)$, the definite integral

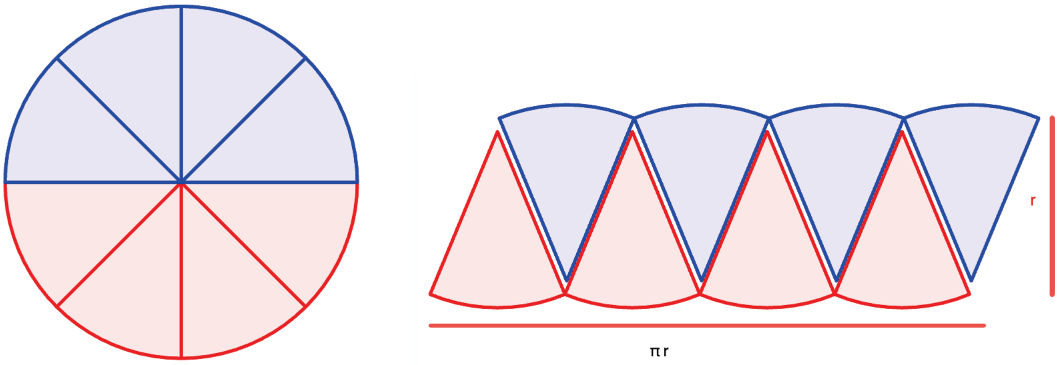


Figure 2: The area of the circle.

is defined by the formula

$$\int_a^b f(x)dx := F(b) - F(a) \tag{2}$$

and further, as application the area under the graph is defined by

$$S := \int_a^b f(x)dx \tag{3}$$

allowing in this way to students to easily to compute definite integrals and areas.

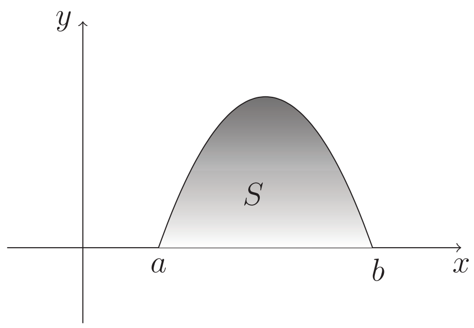


Figure 3: The area $Area\left(f\big|_{[a,b]}\right)$ under the graph of the function $y = f(x)$.

However, very few of them understand and can explain why the definite integral in (2) gives the area by formula (3).

In the present paper we purpose a unified teaching approach to definite integrals and areas in high school mathematics through the notion of limit that bring not only a deeper understanding of these notions, but also relation with circle area known already, computation of limits like the area in (1). In fact, formulas (2) and (3) are not definitions, but consequence of the piecewise quadrature definition (see for instance [1], [3], [4], [5], [6] and other similar textbooks).

Moreover, this kind of teaching prepare the background for more advanced topics like measure theory and numerical integration to be learned at the university level.

# 2 The piecewise quadrature

The piecewise quadrature, or approximating the area under the curve, and then passing to the limit, is the solution to the problems presented in the Introduction.

Indeed, for a function $f : [a, b] \to \mathbb{R}$, a partition $\Delta = (a = x_0, x_1, \ldots, x_n = b)$ of the interval $[a, b]$ and a system of $n$ points $t_i \in [x_{i-1}, x_i]$, $i \in \{1, \ldots, n\}$, we define the real number

$$\sigma_\Delta(f, t) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

called the **Riemann sum** associated to $f$, $\Delta$ and $\{t_i\}$.

Obviously, $t_i = x_{i-1}$, $t_i = x_i$ or $t_i = \frac{1}{2}(x_{i-1} + x_i)$ are obvious choices for $t_i$.

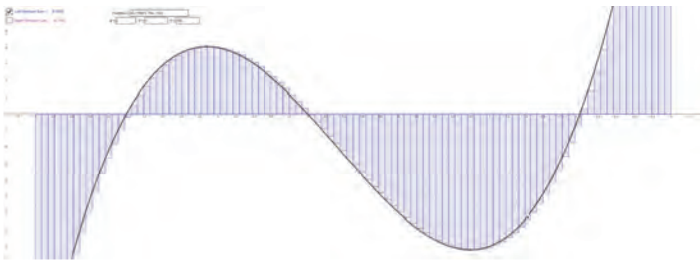

Figure 4: The Riemann sum.

Then the correct definition of a **Riemann integrable function** $f : [a, b] \to \mathbb{R}$ is that there exists a real number $I_f$ such that for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ such that for any partition $\Delta = (x_0, x_1 \ldots, x_n)$ of $[a, b]$ with $\|\Delta\| < \eta_\varepsilon$ and any points $x_{i-1} \leq t_i \leq x_i$, $1 \leq i \leq n$, the inequality

$$|\sigma_\Delta(f, \xi) - I_f| < \varepsilon$$

holds good. The number $I_f$ will call the **definite integral** of $f$ on $[a, b]$ and denoted by

$$\int_a^b f(x)dx.$$

In other words, if $\Delta_n$ is a sequence of partitions such that $\lim_{n\to\infty} \|\Delta_n\| = 0$ for any points system sequence $x_{i-1}^n \leqq t_i^n \leqq x_i^n$, then we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sigma_{\Delta_n}(f, \xi^n). \tag{4}$$

Using this definition, it is easy to check the celebrated **Leibniz-Newton formula**

$$\int_a^b f(x)dx = F(b) - F(a), \tag{5}$$

where $F$ is a primitive function of $f$, as well as to understand why the definite integral gives the area under the graph.

Indeed, if we consider the partitions sequence

$$\Delta_n = (x_0^n, x_1^n, \ldots, x_{k_n}^n) \text{ of the interval } [a,b],$$

such that $\lim_{n\to\infty} \|\Delta_n\| = 0$, then by the mean theorem applied to $F$ on $[x_{i-1}^n, x_i^n]$, it results that there exists $t_i^n \in (x_{i-1}^n, x_i^n)$ having the property

$$F(x_i^n) - F(x_{i-1}^n) = F'(t_i^n)(x_i^n - x_{i-1}^n).$$

However, since $F'(x) = f(x)$ on $[a,b]$, we get

$$F(x_i^n) - F(x_{i-1}^n) = f(t_i^n)(x_i^n - x_{i-1}^n)$$

and by summing up we obtain

$$\sigma_{\Delta_n}(f, t^n) = \sum_{i=1}^{k_n} f(t_i^n)(x_i^n - x_{i-1}^n) = \sum_{i=1}^{k_n} [F(x_i^n) - F(x_{i-1}^n)] = F(b) - F(a)$$

for any positive integer $n$. This gives the Leibniz-Newton formula (5).

The intuitive idea behind formula (5) is therefore quite easy to understand. The mean theorem for the primitive $F(x)$ provides the sequence of intermediate points $\{t_i^n\}$ where the slope of the tangent to the graph of $F(x)$ is parallel to the chord connecting the end points of the interval. By using these intermediate points for $f(x)$, then it results that the area $f(t_i^n)(x_i^n - x_{i-1}^n)$ coincide with the difference $F(x_i^n) - F(x_{i-1}^n)$, hence the conclusion.

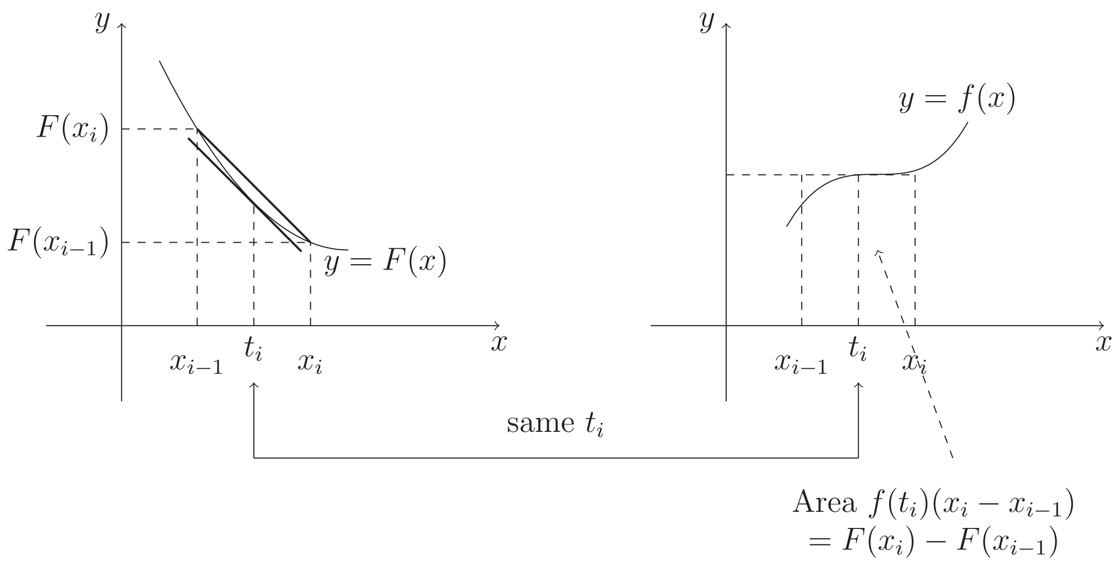


Figure 5: The relation of $F(x_i) - F(x_{i-1})$ and area of $f(t_i)(x_i - x_{i-1})$.

The correct statement about the area is as follows.

If $f : [a,b] \to \mathbb{R}$ is a continuous function, then the domain under the graph has finite area and this area is given by $\int_a^b f(x)dx$.

The proof is now easy.

Let us consider again the partitions sequence

$$\Delta_n = (a = x_0^n < x_1^n < \ldots < x_{p_n}^n = b)$$

such that $\lim_{n\to\infty} \|\Delta_n\| = 0$, and let us denote by $m_i^n$ and $M_i^n$ the lower and upper bounded of $f$ on the closed, bounded interval $[x_{i-1}^n, x_i^n]$, respectively.

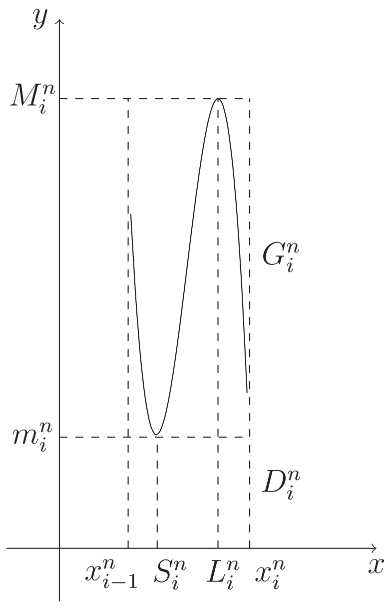


Figure 6: Computing area by piecewise quadrature.

The students should know (or it can be recalled) that any continuous function on a closed, bounded interval attains its bounds, i.e. in our case, there exists $S_i^n \in [x_{i-1}^n, x_i^n]$ and $L_i^n \in [x_{i-1}^n, x_i^n]$ ($S$ for small and $L$ for Large) such that

$$f(S_i^n) = m_i^n \text{ and } f(L_i^n) = M_i^n.$$

Let us denote by $D_i^n$ and $G_i^n$ the rectangles $[x_{i-1}^n, x_i^n] \times [0, m_i^n]$ and $[x_{i-1}^n, x_i^n] \times [0, M_i^n]$, respectively. Their union gives

$$E_n := \bigcup_{i=1}^{p_n} D_i^n,\ F_n : L = \bigcup_{i=1}^{p_n} G_i^n$$

with area

$$Area(E_n) = \sum_{i=1}^{p_n} m_i^n (x_i^n - x_{i-1}^n) = \sum_{i=1}^{p_n} f(S_i^n)(x_i^n - x_{i-1}^n) = \sigma_{\Delta_n}(f, S_i^n)$$

and

$$Area(F_n) = \sigma_{\Delta_n}(f, L_i^n).$$

The function f is continuous, hence integrable, and its definite integral is given by (4). If we choose any sequence points $x_{i-1}^n \le t_i^n \le x_i^n$, then clearly

$$Area(E_n) = \sigma_{\Delta_n}(f, S_i^n) \le \sigma_{\Delta_n}(f, t_i^n) \le \sigma_{\Delta_n}(f, L_i^n) = Area(F_n). \tag{6}$$

Taking now into account that

$$\lim_{n\to\infty} Area(E_n) = \lim_{n\to\infty} Area(F_n) = Area\left(f\big|_{[a,b]}\right)$$

by passing to limit in (6) it follows

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sigma_{\Delta_n}(f, S_i^n) = \lim_{n\to\infty} Area(E_n) = \lim_{n\to\infty} \sigma_{\Delta_n}(f, L_i^n) = Area(F_n),$$

hence

$$Area\left(f\big|_{[a,b]}\right) = \int_a^b f(x)dx.$$

where $Area\left(f\big|_{[a,b]}\right)$ is the area of the domain under the graph of $f$ on $[a, b]$, see Figure 3.

Finally, we consider the limits of the type (1). It can be easily computed if regarded as the limit of a Riemann sum. Indeed, the sum under the limit can be transformed as follows

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \ldots + \frac{1}{1+\frac{n}{n}}\right].$$

If we consider the function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x}$ and for any $n \ge 1$, we consider the equidistant partition

$$\Delta_n = \left(x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \ldots < x_n = \frac{n}{n} = 1\right)$$

with norm $\|\Delta_n\| = \frac{1}{n}$, and points system

$$t_i = x_i = \frac{i}{n},$$

then

$$\begin{aligned}\sigma_{\Delta_n} &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}}\left(\frac{i}{n} - \frac{i-1}{n}\right)\\ &= \sum_{i=1}^{n} \frac{1}{i+n} = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right] = \lim_{n\to\infty} \sigma_{\Delta_n}(f, t)\\ &= \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2.\end{aligned}$$

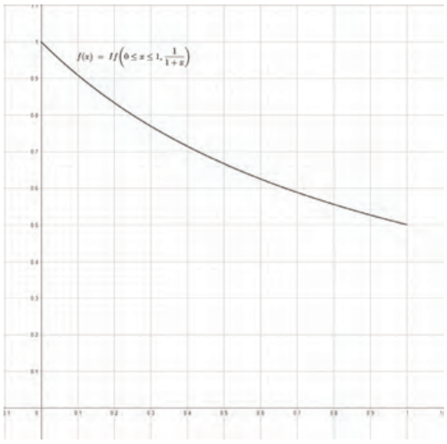


Figure 7: Computing limits by piecewise quadrature.

A further problem is the second limit (1). Indeed, we need to transform the product in sum by using logarithm.

We have

$$\log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = -\log n + \frac{1}{n}\left[\log n + \log(n+1) + \dots + \log(n+n)\right]$$

and, denoting

$$\log(n+1) = \log n\left(1+\frac{1}{n}\right) = \log n + \log\left(1+\frac{1}{n}\right)$$

we obtain

$$\log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \dots + \log\left(1+\frac{n}{n}\right)\right].$$

Therefore,

$$\lim_{n\to\infty} \log \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = \int_0^1 \log(1+x)dx = \log 4 - 1$$
$$\lim_{n\to\infty} \frac{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}}{n} = e^{\log 4 - 1} = \frac{4}{e},$$

hence

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n(n+1)(n+2)\dots(n+n)}} = \frac{e}{4}.$$

This is an important technique that high school student should master.

# 3 Materials

Computer technology plays a significant role in mathematical education, visually enhancing and adding new value to the traditional mathematical learning.

In the present project, we have used the computer in order to have the students visualizing mathematical concepts as area and piecewise quadrature and explore them through interactive scripts to better understand geometrical shapes and functions.

By integrating computer science into classical mathematical learning, we were able to create a more engaging, relevant, and enriching educational experience for our students.

In particular, we have used GeoGebra ([2]), developed by Markus Hohenwarter in 2001, which is a dynamic mathematics software that combines geometry, calculus and other fields of mathematics in an interactive and visual environment. It is a powerful tool for teaching and learning mathematics, as well as for exploring mathematical concepts and conducting research. It is available on various platforms, including Windows, macOS, Linux, and as mobile apps for Android and iOS.

Even though we are using it here merely for visual integration, GeoGebra includes all necessary tools for calculus, such as finding derivatives, integrals, and tangents to curves. It can plot a large variety of functions graphs and enhance the study of their behavior and limits.

## 3.1 The area of the circle

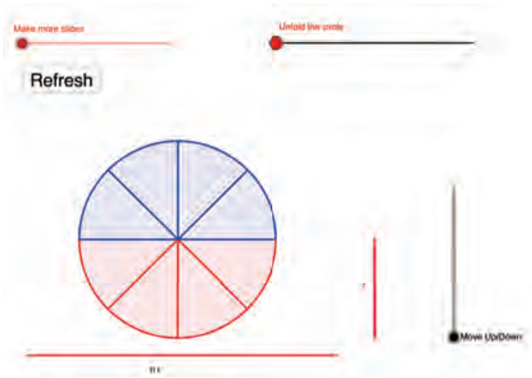


Figure 8: https://www.geogebra.org/classic/jzsxxnhq

### 3.2 The Riemann sum

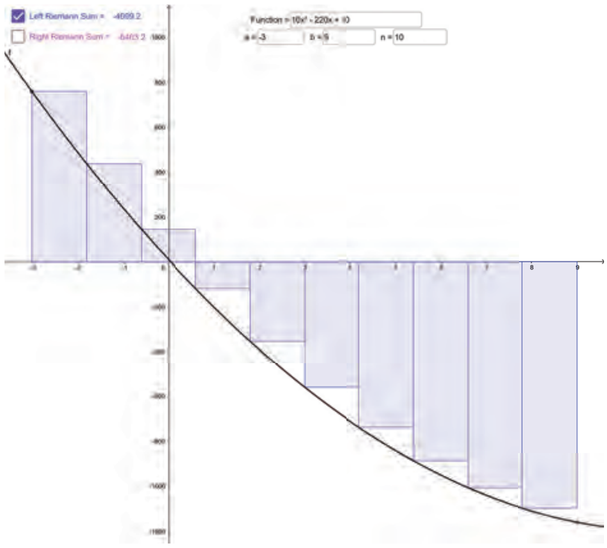


Figure 9: https://www.geogebra.org/classic/h4P4cjsT

# 4 Acknowledgements

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# 5 Conclusion

The concept of piecewise quadrature holds significant importance in high school mathematics, as it allows students to establish connections between the notions of limits and area.

Introducing piecewise quadrature prior to covering topics such as area and definite integrals can prove to be highly beneficial. It serves as a foundational motivation and model for these advanced concepts, enabling students to grasp the fundamental principles that underlie calculus. By employing this teaching model, students not only gain proficiency in numerical integration but also develop the ability to apply these mathematical concepts to real-world problems. Additionally, they learn to evaluate the accuracy and precision of approximations, which is a critical skill in various scientific disciplines.

By nurturing critical thinking and problem-solving skills through the study of piecewise quadrature, students are better prepared to become successful scientists. These skills become particularly valuable when they later encounter advanced numerical integration methods used extensively in engineering, physics, computer science, and other fields.

Through a comprehensive understanding of piecewise quadrature, students are empowered to tackle complex problems, analyze real-world scenarios, and make informed decisions. This knowledge lays a strong foundation for their academic and professional pursuits, equipping them with the necessary tools to excel in the STEM fields.

Moreover, the introduction of piecewise quadrature provides a bridge that connects seemingly abstract mathematical concepts to practical applications. Students can explore how these mathematical tools are utilized in diverse industries to solve problems, design solutions, and optimize processes. This real-world relevance fosters a deeper appreciation for mathematics and its practical implications.

In conclusion, integrating piecewise quadrature into the high school mathematics curriculum is a strategic move to enhance students' learning experience. By introducing this concept early on, students can build a solid understanding of numerical integration, critical thinking, and problem-solving skills. This preparation will undoubtedly pave the way for their future success as they pursue further education and careers in scientific and technical fields.

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# Enhancing Mathematics Instruction for Students with Visual Impairment: A Teacher Training Program on Accessible Online Math

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**Abstract**: *This research article presents a comprehensive training program designed to enhance the knowledge and skills of lower secondary math teachers in Thailand regarding producing online mathematics materials that cater to visually impaired students. The training program encompassed various topics, including utilizing LaTeX for writing math expressions and equations to facilitate screen readers and adopting vector graphics to ensure scalability for students with low vision. Additionally, the program trained math teachers to create accessible documents, illustrations, videos, web pages, and online tests. A total of 94 lower-secondary math teachers participated in a one-day training session conducted online via Zoom. Pre- and post-training evaluations were conducted to measure participants' knowledge of teaching mathematics to visually impaired students and their attitudes toward accessible mathematics. The evaluations also assessed participants' satisfaction levels and skills acquired from the training program. Statistical analysis revealed a significant improvement in participants' knowledge and attitudes toward accessible math ($p < 0.001$, paired-sample t-test). Furthermore, the findings indicated a high level of participant satisfaction, with an average rating of 4.66 out of 5.00, demonstrating the effectiveness of the training program. Participants strongly agreed that they obtained valuable skills in creating accessible math lessons, as indicated by an average rating of 4.47 out of 5.00. The results highlight the positive impact of taking accessibility factors into account in teaching mathematics online, ultimately fostering an inclusive learning environment for all students.*

## 1. Introduction

In recent years, the integration of technology in education has opened up new opportunities for online learning environments. Online lessons have been streamlined to be more interactive and engaging with tools that teachers and students can easily access and use. However, while online learning resources have improved for the general student population, there is still a significant gap in accommodating students with visual impairments.

Improving accessibility in other subjects may not be very complicated. Teachers, staff, or volunteers can create audiobooks, braille textbooks, or Word document files that support screen readers. Mathematics, on the other hand, is challenging to enhance accessibility because the subject relies heavily on communicating with expressions, equations, charts, diagrams, and images. In order to enhance the accessibility of online mathematics instruction, it is imperative for educators and media producers to gain a comprehensive understanding of the learning processes employed by visually impaired students, both in offline and online contexts. This entails recognizing the distinctive attributes of accessible mathematical online media and acquiring the necessary knowledge and skills to improve the production of existing and forthcoming online math resources to cater to the needs of visually impaired individuals. Moreover, fostering a positive attitude towards accessible mathematics is of utmost importance, as it serves as a driving force for consistently incorporating accessibility considerations into the development of online math content.

The project aims to increase the knowledge and skills of mathematics teachers in Thailand regarding the production of online mathematics materials that support visually impaired students. Additionally, the project focuses on fostering a positive attitude toward accessible online mathematics that supports all groups of learners, regardless of their visual abilities.

### 1.1 Visually impaired students and their special needs

The education system in Thailand classifies visually impaired students as blind and low vision [1]. The classification was announced by the Special Education Bureau, Office of the Basic Education Commission to help in manpower and resource planning, as the two groups of learners have different special needs as follows:

*Blind students in offline learning contexts* – Students use textbooks in braille with symbols to represent expressions and equations [2]. Illustrations, such as graphs, charts, and geometric figures, must be tactile. In addition, teachers must be specially trained to teach blind students.

*Blind students in online learning contexts* – Students learn from audiobooks and use screen-reading programs, such as NVDA and JAWS, that can read text on-screen aloud (primarily support English). A screen reader can read simple expressions and equations properly if they are in a compatible format, such as LaTeX, MathType, and Microsoft Equation [3]. However, Thai screen readers, such as PPA Tatip, currently do not support reading expressions and equations. Also, braille is inevitable for complex expressions and equations. Illustrations must include alternative text, captions, or descriptions to support screen readers. Videos and demonstrations must have voice narration, or the blind will not be able to learn.

*Low-vision students in offline learning contexts* – Students use large print [4] (materials with larger text) with high-contrast colors. With normal print, they need to look closer than usual and may need a magnifying glass. Low-vision students are generally able to study with normal students in classes.

*Low-vision students in an online learning context* – Students use screen magnification to increase their vision. Today's computers and mobile devices often have these functions built in. However, the content must be vectorized or have reasonable resolutions so that it can be enlarged to a certain extent without being pixelated.

### 1.2 Accessible online math

The blind students relied on lessons they can perceive using other non-visual senses, such as braille, which can be perceived by touch, and screen readers, which can be perceived by hearing [5]. Accessible math designs can be summarized by media type as follows:

#### Images

Math lessons inevitably require images, be they geometric figures, graphs, or charts. Improving the accessibility of digital images can be done as follows:

Images taken from digital cameras, mobile phones, screenshots, and scanner images are stored as raster graphics. They will be pixelated if enlarged too much. Raster graphics must be taken at a high resolution to be scaled up to a level that people with low vision can see. However, high-resolution images have large file sizes, which can take a long time to download and consume a lot of computational resources. Additionally, text embedded in photos cannot be selected or read by a screen reader.

Math illustrations should be created as vector graphics (such as SVG) so they can be scaled up without pixelation. Plus, they have much smaller file sizes than photos and screenshots. Screen readers can select and read text displayed on vector graphics (if created properly). Current display devices can display vector graphics without problems. Programs and applications like Illustrator and draw.io can provide vector graphics output. Applications like GeoGebra, besides being able to provide vector graphics output, allow teachers to share interactive content for low-vision learners to interact with. Some applications can output 2D and 3D models (such as OBJ and STL) that can be used to create tactile materials for blind learners using UV and 3D printers [6].

No matter how illustrations are created, content publishers need to provide descriptions of those images so that blind people can understand. Adding alternative text (alt text or alt attribute) is one of the most common methods. A screen reader will read alternative text when it reaches where the image is displayed. Additionally, the alternative text will appear if the image fails to load. Various online platforms, especially those used in education, provide alternative text input tools. Some educational platforms do not allow users to publish images without alternative text. For platforms that don't have an alt text tool, designers may add captions or describe the image in body text [7], which is another good practice.

### Expressions and equations

Most screen readers can read math expressions and equations properly if they are in a math environment:

- Within Microsoft Equation or MathType blocks, or
- Within TeX or LaTeX delimiters: {math}, {equation}, $...$, $$...$$, \(…\), or \[…\].

Screen readers do not recognize expressions and equations typed using subscripts, superscripts, symbols, or floating text boxes. For example, $x^2$+2x-5=0 will be read as "x two plus two x, five equals zero" with the minus sign being interpreted as a hyphen. Expressions and equations saved as images require alternative text. Using a suitable mathematical environment, expressions, and equations will be rendered in vector graphics, ensuring scalability for students with low vision.

### Documents and web content

A screen reader will recognize images, expressions, and equations in a document file (such as DOCX) and on a web page if they are in appropriate formats, as mentioned above. However, there are other requirements for accessible documents and web content. First, documents and webpages must be outlined using headings so that screen reader users can easily jump to different parts [8]. Numbered and bulleted lists must be created using the appropriate tools or tags so that screen readers recognize them as a sequential list. Most users organize topics using font sizes, colors, line spacing, and manual numbering, which reduces overall accessibility.

Tables in documents and web pages should be used for data presentation only. They should have a simple structure – the first row is a header specifying data in each column, and subsequent rows are corresponding datasets. Sales and transaction records are good examples. Cells should not be merged, and tables should not be used to arrange content. Avoid adding paragraph text to a table.

Blind people typically use the tab key as a shortcut to move to the next content both in documents and on the web. Content designers should ensure that content is presented in the correct order when pressing the tab key [8], especially for images and tables. All images and tables must be "in line with text" (no floating element) to allow the tab key and alt text to work properly.

Converting Word to PDF preserves page formats and most accessibility features, including headings and alt text. Unfortunately, expressions and equations lose screen reader compatibility upon PDF conversion. Getting equations in PDFs back to work with screen readers requires PDF tagging [9], which is a complex process. Therefore, teaching online mathematics to blind students should avoid PDF files.

Web designers and content management systems (CMS) like WordPress typically use MathJax or MathML to render expressions and equations typed in TeX and LaTeX format. These two plugins provide accessibility since they are compatible with screen readers. However, some plugins, such as Elementor with KaTeX, have accessibility issues. In addition, the content layout may differ depending on the device as a result of responsive templates provided by the CMS. Therefore, messages like "from the image shown on the left" should be avoided. Also, hyperlinks should have meaningful display texts, such as "Click to open the image on a new window," to prevent a screen reader from reading the entire URL.

**Video content**

Video media can present large amounts of information in a short period of time with a combination of motion pictures and sounds. Video materials for the blind must ensure that content is understood by viewers using audio alone. The narrator must explain what is depicted. Saying "from the image shown" or "from the following equation" must be avoided without explaining the image or reading the equation aloud. Video materials with only animations or demonstrations accompanied by background music without audio narration are not available for blind learners.

**Web and mobile applications**

Web and mobile applications used in education often convert original content into raster graphics and overlay interactive content. Presenting information in that way loses its accessibility properties and causes learning difficulties for visually impaired students. Investigating whether applications are accessible to the visually impaired can be done initially by selecting text and resizing content. If the text is selectable, then the screen reader would work. If the application does not provide a content resize function or the image becomes pixelated when enlarged, it will not support low-vision students.

### 1.3 Accessibility in the Thai online learning context

Visually impaired students in Thailand do not have much support [10]. Devices like braille displays, braille tablets, UV, and 3D printers are a rarity even in large schools. During the COVID-19 pandemic, a large number of blind students were dropped from the education system as most educational online content does not support screen readers [11]. Therefore, this project emphasizes that the trainee teachers use equipment and free applications both teachers and students can provide. The project organizers were trying to offer a way that doesn't interfere with the original teacher workflow but instead allows teachers to incorporate accessibility features as part of their online math instruction.

## 2. Related studies

Several studies have explored the development and effectiveness of accessible math learning materials for visually impaired students in online settings. The University of Arizona [12] has created an accessible word problem-solving application for visually impaired students as an iPad app with printed and braille materials. The app contains audio descriptions of images, options to adjust color and contrast, hints, and videos showing sample solutions using Apple's built-in accessibility features, such as Voiceover and Zoom. A study with 29 visually impaired students and their teachers revealed positive results. However, both students and teachers reported different quality of hints and found the video was not very helpful. It has been fixed by adding a video description that shows step-by-step troubleshooting.

It can be seen that text-to-speech is one of the key assistive technologies that support visually impaired students. Online learners typically have access to NVDA and JAWS, famous screen readers. However, one of the issues with screen readers is that they primarily read math expressions and equations in English. Screen readers are often not developed to read mathematical expressions and equations in other languages with distinct pronunciation structures. Screen readers and add-ons that can read math in other languages are in the research and development phase and are not yet widely available. Mahidol University [13] developed an i-Math application that can read expressions and equations in Thai. It was tested on 78 Thai secondary school students with visual impairment (52 blind and 26 people with low vision) and their teachers. The results showed that the developed application was able to pronounce expressions and equations in Thai correctly. Participants found the application useful, especially for visually impaired students who had not learned to write mathematics in braille. However, it has been found that reading aloud may not be effective for complex expressions or equations. A similar application [14] was developed for reading mathematics in Chinese.

Teachers are one of the most important factors in promoting inclusive education. The teacher's role is to understand how visually impaired people learn, especially in math and science, despite their small proportion in class. A case study [15] that interviewed a blind technician was conducted to analyze the factors affecting the learning of visually impaired students. The study found that educational institutions had enough assistive technology for the visually impaired, but teachers lacked the skills needed in special education. Many teachers did not believe in the potential of those learners and did not incorporate accessible media in teaching.

These literature reviews highlight the significance of developing accessible math learning materials, providing comprehensive teacher training, and utilizing appropriate assistive technologies to ensure an inclusive and effective online mathematics education for visually impaired students. Further research in these areas is necessary to continue advancing accessibility practices in mathematics instruction.

## 3. Methods

This project presented an online training program on creating accessible math materials for visually impaired students in the online learning context. Participants consisted of 94 lower-secondary mathematics teachers in Thailand sampled from 1,481 mathematics teachers in inclusive schools (where normal and visually impaired students study together) listed in the database of the Special Education Bureau, Office of the Basic Education Commission. The number of samples was calculated using the Taro Yamane formula [16] with an error of 10%. The sample was selected according to the proportion of inclusive schools in each region of Thailand.

During the introduction, participants responded to questions regarding knowledge and attitude toward accessible online math in a 5-scale format, in addition to general demographics questions. Materials and instruments have been reviewed by experts in teaching mathematics for the blind.

The instruction phase began by introducing how visually impaired students learn mathematics and the features of online mathematics materials they can learn from. The next part was a guide on how to create teaching materials using online tools that teachers and students can access, as follow:

- Adding alternative text to an image on Word documents, Content Management Systems (e.g., WordPress and Google Sites), and Learning Management Systems (e.g., Moodle, Google Classroom, and Open edX).
- Creating illustrations for math problems in a scalable vector graphics (SVG) format using GeoGebra and Draw.io.
- Introducing the use of colors in diagrams and charts to accommodate the visually impaired.
- Learning how screen readers interact with online math expressions and equations.
- Writing simple math expressions and equations in LaTeX format using CodeCogs.
- Utilizing LaTeX commands on websites and learning management systems, as well as providing alternatives for unsupported platforms.
- Recognizing other elements of accessible documents and web pages, such as headings and tab key navigation.
- Introducing how to narrate in video media to accommodate the visually impaired.
- Practicing creating mathematic quizzes using Microsoft Forms with built-in features for the visually impaired.

The training program was organized online through the Zoom application and lasted for a day. The training material was delivered in web format, which itself acts as an example of accessible online mathematics materials. At the end of the training, participants completed a satisfaction survey and questions about their acquired knowledge and skills, as well as their attitude toward accessible online mathematics. The questions are on a 5-level scale, as shown in Tables 4.1-4.3. Data were analyzed using mean and standard deviation. A paired t-test was used to compare pre- and post-training knowledge and attitude levels.

## 4. Results

A comparison of the trainees' knowledge before and after the training is shown in Table 4.1. It was found that the trainees didn't have much information about accessible online mathematics materials prior to the training (average at 2.50/5.00). The paired t-test revealed that the trainees had a statistically significant improvement in knowledge at $p < 0.001$.

**Table 4.1** Average knowledge of the trainees before and after the training (n=94)

| Aspect | Before training | After training |
|---|---|---|
| Math online learning of the visually impaired | 2.48 (1.37) | 4.54 (0.52) |
| Properties of accessible online mathematics materials | 2.53 (1.37) | 4.55 (0.54) |
| Production of accessible online mathematics materials | 2.49 (1.36) | 4.51 (0.57) |
| Average | 2.50 (1.35) | 4.53 (0.49) |

Remark: Numbers in parentheses are standard deviations.

A comparison of the attitude of the trainees before and after the training is shown in Table 4.2. It was found that participants were aware of the need for accessible online mathematics materials for visually impaired students (average at 3.66/5.00). Nevertheless, the paired t-test revealed that the trainees had a statistically significant improvement in attitude at $p < 0.001$.

**Table 4.2** Average attitude of the trainees before and after the training (n=94)

| Aspect | Before training | After training |
|---|---|---|
| Online math materials should always be made accessible. | 3.66 (1.44) | 4.60 (0.54) |
| Accessible math materials also benefit normal students. | 3.62 (1.45) | 4.64 (0.48) |
| Published math materials should meet accessibility standards. | 3.70 (1.42) | 4.61 (0.55) |
| Average | 3.66 (1.41) | 4.61 (0.48) |

Remark: Numbers in parentheses are standard deviations.

Table 4.3 shows that the trainees were satisfied with the training program at a "very satisfied" level (4.66/5.00). The trainees were most satisfied with the lecturer's ability to explain concepts, the training format, and the content respectively. Further comments indicated that the training should be conducted in such a way that participants can follow it step-by-step.

**Table 4.3** Satisfaction results (n=94)

| Aspect | Average | SD |
|---|---|---|
| The lecturer | 4.73 | 0.47 |
| The content covered | 4.68 | 0.47 |
| Training materials (website) | 4.67 | 0.47 |
| Question and answer | 4.57 | 0.56 |
| Coordination | 4.64 | 0.53 |
| Duration (one day) | 4.64 | 0.53 |
| Format (online via Zoom) | 4.70 | 0.50 |
| Average | 4.66 | 0.42 |

The skills acquired by the trainees are shown in Table 4.4 as a result of the self-assessment after the training. The results showed that the trainees had developed skills in producing online mathematics materials that support the visually impaired. They were able to improve existing materials to support the visually impaired and were able to convey the knowledge received.

**Table 4.4** Skills acquired through the training (n=94)

| Skill acquired | Average | SD |
|---|---|---|
| Produce online materials that support the visually impaired. | 4.47 | 0.60 |
| Improve existing materials to support the visually impaired. | 4.48 | 0.62 |
| Convey the knowledge received. | 4.46 | 0.63 |
| Average | 4.47 | 0.59 |

## 5. Conclusion and discussion

This project has trained 94 mathematics teachers at a lower-secondary level in Thailand. The aim was to increase the knowledge and skills of teachers regarding the production of online mathematics materials that support visually impaired students. Also, the project aimed to foster a positive attitude toward accessible online mathematics that supports all groups of learners, regardless of their visual abilities.

The data analysis revealed that teachers were less informed about accessible online mathematics and that training programs were effective in providing such information. Training materials created using the proposed techniques are effective in providing information and showing good practices at the same time. Managing online training through the Zoom application is effective enough, considering the results. However, synchronous online training may obstruct the learning of some participants. The project organizer has a plan to create on-demand learning materials that allow participants to practice step by step and pause as needed.

Another aim of the project was to make math teachers aware that producing accessible online math materials does not increase their workload. On the other hand, accessible materials should be viewed as a proper and correct approach so that accessibility factors are taken into account throughout production. For example, teachers can choose whether to use high-contrast solid colors or gradients when creating a graph. One option will work for all learners, while the other will rule out visually impaired students. Also, the "Poor" option won't support any students if the graph is printed in black and white (e.g., for on-site examination). Another example is that many teachers type equations in Word and make screenshots while they can add equations directly to web pages if they use LaTeX commands. This is consistent with [15] that teachers already have access to useful tools. Only teachers put them to good use under the perception that the visually impaired are capable of learning, and the materials that support them are also effective for other learners.

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# Application of Machine Learning to Slow Tourism Market Segmentation: A Case Study at Nanzhuang

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**Abstract:** *Truly little is known about the characteristics of tourists to Nanzhuang township, one of four Cittaslow townships in Taiwan. This study gives implication about the distinctive styles of tourists to Nanzhuang. Today, Nanzhuang township is famous for its role as a slow city in Taiwan. It is reasonable to assume this characteristic would have influence on its tourism market. In order to understand types of tourism to Nanzhuang and to apply a beneficial advertising strategy in terms of the principles of the Cittaslow organization, we need to know the features of tourists to this township. A survey drafted according to the Cittaslow principles was performed, and a total of 222 responses were collected. Machine learning (ML) tools for data science, such as, k-means clustering, Principal Component Analysis (PCA), and one-way ANOVA were applied to do proper clustering and analysis of the data. The results have shown that the tourists can be suitably categorized into three distinct groups: Advocates of Slow Tourism (AST); Conscious of Slow Tourism (CST), and Unconscious of Slow Tourism (UST). Interestingly, the descriptive statistics between these three groups do not show any difference in their background, for instance, regarding age and education. A precise marketing strategy for slow tourism should be carefully considered accordingly.*

## 1. Introduction

Tourism is a so-called Chimneyless Industry; it can bring up the GDP for an entire nation. Prior to the Covid-19 pandemic, the tourism industry accounted for around 4~5% of the Gross Domestic Product (GDP) of Taiwan and has the possibility to increase to around 6~7 % with a surge in the number of incoming tourists. During the Covid-19 pandemic, however, this percentage has reduced, the effects of which have been felt not only in Taiwan, but the world. As we consider how to revive the tourism industry, a crucial element to observe is the current trends that are becoming apparent in this industry. Slow Tourism is one such trend. In Taiwan, the concept of slow tourism is still relatively new, however, in some cases, people are participating in this trend without awareness of their participation. According to [5], Slow Tourism can be defined as the following:

> Slow Tourism must follow two essential principles: taking time and attachment to a particular place. Taking time means modification of the daily time relationship, specifically a different perception of nature and living in harmony with a place, its inhabitants, and their culture. The environment is not merely perceived by sight, but by using all five senses. Tourists must be able to change pace, to look rather than to see, to experience the area rather than to endure it.

By such a definition, we see that the ideals of Slow Tourism can be practiced both by those who are conscious of the concept of Slow Tourism, and those who may not have awareness of this concept yet perform tourism in a way that aligns with these ideals. For example, camping is a popular activity during vacation in Taiwan, and Nanzhuang township has many ideal camping destinations. It is possible that tourists to Nanzhuang have taken advantage of these locations, regardless of their dedication to the practice of Slow Tourism.

In this study, we investigate the city of Nanzhuang, Taiwan, looking at the style and motivation for tourism of the visitors to this township. Specifically, we seek to understand what role Slow Tourism plays in these tourists' motivation for visiting, whether they are fully aware of the concepts of Slow Tourism, have some knowledge of Slow Tourism, but are unaware that they are in fact a slow tourist, or whether they are just spending their leisure time in this location, oblivious to the fact that they are performing Slow Tourism. For the local city government to properly target their advertisement regarding tourism, it is essential that they know the majority type of tourist seeking out this destination. Effective advertising will then greatly benefit the local businesses in Nanzhuang. If most tourists to this city are in fact aware of Slow Tourism, then it would be important to emphasize this concept and the specific activities and policies that reinforce Slow Tourism in Nanzhuang. If not, exposure to the concepts of Slow Tourism and the ways in which the township applies these concepts could further increase the desire of tourists to visit. Regardless of the majority type of tourist visiting Nanzhuang, we intend with this study to shed light on the type of tourism occurring at this destination, allowing for better targeted tourism advertisement for this area. This in turn, will hopefully benefit the local businesses, residents, and government by increasing the rate of tourism to the city.

In the past, such an investigation would be carried out by traditional statistical methods, however, in recent years machine learning has become popular across many domains. We would like to explore the benefits of a machine learning approach carried out in the field of tourism. Our research applies machine learning to a dataset containing responses from tourists to Nanzhuang. Specifically, we utilize two clustering algorithms to decipher whether unsupervised machine learning techniques can aid in our understanding of the types of tourists visiting this township. Furthermore, we investigate the data to understand and recognize any trends linking the visitors to Nanzhuang. The goal of this study is to understand the types of tourists that travel to this township and identify the level of importance of Slow Tourism amongst these tourists. This paper is organized as follows, we will first give a background on Slow Tourism and some related movements. We will then introduce Nanzhuang township and the dataset that was used for our experiments. This will be followed by our experimental methodology and analysis of our experimental results. Finally, we will discuss ideas for further research and give the conclusion of this study.

## 1.1 Slow Cities in Taiwan

Currently there are four slow cities in Taiwan that are members of the Cittaslow network. They are Dalin township in Chiayi County, Fonglin township in Hualien County, and two townships in Miaoli County, Sanyi and Nanzhuang. While each of these townships is unique in their culture and character, they are all representative of the ideas of the Cittaslow organization and the concept of a slow city. Each of these townships are agricultural and rely heavily on local produce, which contributes to their unique local cuisine. This research focuses on Nanzhuang township, located in the mid-north region of Taiwan. Population in Nanzhuang consists in large part of the Hakka people, and the Saisiyat people and the Atayal people, both of which are Indigenous peoples of Taiwan [2]. This township is a well-known tourist destination in Taiwan, offering a drastically distinct experience to that of the more urban destinations, such as the capital city, Taipei. Some of the most popular tourist sites include Nanzhuang Old Street, Shenxian Valley (also known as Valley of the Immortals), and Xiangtian Lake [3]. As mentioned previously, camping is also a popular activity for visitors to Nanzhuang. In fact, among the top thirty camping destinations in Miaoli County, seven of these destinations are in Nazhuang. Many tourists come to visit these popular destinations to shop, enjoy the old-style atmosphere of the township, eat local food, visit traditional temples and other activities, which is certainly a common style of tourism in Taiwan and other countries. In this way, these visitors are able to immerse themselves in the town's local culture, an important aspect of slow tourism, but does this truly constitute conscious Slow Tourism? Are these visitors aware of their participation in this style of tourism? We would like to understand the awareness amongst these tourists regarding Slow Tourism. To what extent are they seeking out such a destination because of its application of Slow Tourism?

## 2. Methodology:

## 2.1 Tourist Dataset

This research was obtained through a questionnaire that consists of two parts: the first part contains fifteen statements that are designed according to the Cittaslow Charter [1] and are used to clarify if there are distinct characteristics of tourists to Nanzhuang, based on their opinion regarding the concepts of the Cittaslow Charter. The questions are adapted from a previous study on Slow Tourism to the city, Seferihisar, Turkey [11], and a 4-point Likert scale is used as a metric for the visitors to answer (shown in Table 1). After careful inspection by specialists of the statements from Yurtseven et al., we decided to use these questions for this study with minor changes. To avoid answers without justification and to make clear the tourists' preference, we do not use the neutral (e.g., "no preference") answer, but instead we employ the 4-point scale to make a clearer distinction. For practical purposes, in order to ensure that the respondents have a clear understanding of the meaning of the statements, the statements were translated into Chinese, with some specific examples given to aid in the respondents' replies. For instance, for statement thirteen, "Promoting and Preserving local culture events," we cited a special event, Pas-ta'ay, given by the local Saisiyat tribe [2]. It is an incredibly special event that not many tourists have experienced, and it is a culture event that needs to be preserved. These statements translated to Chinese can be viewed in Table 1

The second part of the dataset contains information regarding the respondent's background, including gender, age, residence, education, duration of stay, how the visitor heard about the destination, their reason for visiting, and the amount of money spent during the visit. Each of these subjects contain two to six categorical answer options for statistical analysis.

The data was collected by my graduate student from Chung Hua University. The period of collection lasted from June 2020 to September 2020. Due to the Covid 19 pandemic, most of the activities were difficult to carry out. During the data collection period, the meaning of the statements and questions were explained if necessary, and the respondents were asked to answer based on their beliefs and experiences. In total, our dataset consists of 222 respondents.

## 2.2 Machine Learning Implemented in the Dataset

While the statistics software, SPSS, was an initial possibility for analyzing our data, we decided to use the Python programming language to apply the machine learning techniques for analysis to explore the usefulness of these techniques within the tourism field. Machine learning can be broken down into four primary categories: supervised learning, semi-supervised learning, unsupervised learning, and reinforcement learning. Due to the nature of this study, we rely on unsupervised learning algorithms. Quoted from [7]:

> Unsupervised learning subsumes all kinds of machine learning where there is no known output, no teacher to instruct the learning algorithms. In unsupervised learning, the learning algorithm is just shown the input data and asked to extract knowledge from this data.

In this study, we used the k-means clustering algorithm, a hierarchical clustering algorithm, and Principal Component Analysis (PCA). The results of experimentation with these algorithms are given in later sections. Additionally, we applied descriptive statistics to the responses from the categorical questions (part two of the dataset). These responses were used to investigate whether the resulting groups from our experiments are also defined by similar backgrounds. The outcome is given in Section 3.

The primary purpose of this study was to make proper clusters for the visitors, grouping them based on their level of understanding of the concepts of slow cities or Slow Tourism using their response from the 15-statement, thereby gaining insight on the type of tourism that is occurring at this destination. From this, a proper marketing strategy for Slow Tourism to Nanzhuang can be designed. To accomplish this clustering, we utilized the k-means clustering algorithm proceeded by a dimension reduction technique, Principal Component Analysis (PCA), and followed by the statistics application, one-way ANOVA. Quoted from [7]:

> Clustering is the task of partitioning the dataset into groups, called clusters. The goal is to split up the data in such a way that points within a single cluster are very similar and points in different clusters are different. K-means clustering is one of the simplest and most commonly used clustering algorithms. It tries to find cluster centers that are representative of certain regions of the data.

Traditionally, machine learning is applied in engineering problems, but in recent years, artificial intelligence (AI) has become more popular in the social sciences. Due to this, more use of machine learning tools in these fields in addition to engineering is expected [6].

## 3. Main Results

### 3.1 Analysis of the Experimental Results

The dataset used in this study contains 222 data points. Of these 222 data points, 53.8% were female, and 46.2% were male. While the age group varied from under 18 years of age to over 50 years of age, the largest percentage were within the 23-30 age range (37.1%), followed by the 30-40 age range, and the 18-22 age range (29% and 19% respectively). In fact, 85% of all data points fell within the 18-40 age range. Most of the respondents in the data set come from either the Northern (43%), Central (40.3%), or Southern (14.5%) regions of Taiwan (97.8% of the samples), with only 1.8% being from foreign countries, and only 0.4% being from some outlying island. Most of the respondents stayed in Nanzhuang for only a half-day (46.6%), with one day and two days, one night being the next most common duration (39.4% and 11.8% respectively). Online materials provided the widest exposure, with 37.6% of the respondents learning of Nanzhuang from online sources. This was followed closely by hearing from a friend (32.1%), with the remaining learning of the township because of proximity to their homes (16.7%), or from written materials (13.6%). Self-guided tours were the most common reason for travel (49.8%), followed by family trips and group tours (35.7% and 13.6% respectively). The largest separation occurred with the amount spent during the visit. The majority spent less than NTD 1,000 (65.2%), followed by 29.4% spending NTD 1,000-3,000. The remaining 5.4% spent NTD 3,000 or more. The graphical representations of all the descriptive statistics are given in Figure 1.

After calculating the initial descriptive statistics from the background information, the next task was our experimentation with clustering. We did not use scaling for the data since the answers are from the 4-point Likert scale, meaning there are no outliers in the data set. We used the Elbow Curve method to find the optimal number of clusters by calculating the sum of squared distances from the centroid for each number of k-clusters, see Figure 2. From this, it was determined that three clusters were best for representing our data points with k-means clustering. Another reason for not using too many clusters can be derived based on the explanation of the status of the tourists given in [10] for ecotourists, who also grouped tourists into three groups. Quoted from [10]:

> “Harder” ecotourists reflect a high level of environmental commitment and affinities with wilderness type experiences, while “softer” ecotourists are much less committed on either dimension. “Structured” ecotourists, by comparison, reveal a strong pattern of commitment but a level of desire for interpretation, escorted tours, and services/ facilities that is usually more associated with mass tourism.

Due to a large dimension (i.e., 222*15) of the dataset, it is difficult to see the outcomes in a two-dimensional plane. To overcome this, we conducted Principal Component Analysis (PCA) to reduce the dimensionality of our data for visualization and two principal vectors are eventually selected. We chose the first two principal components for modeling, whose total explanation of variance was roughly 68.1% (61.4% for the first principal component, 6.7% for the second principal component). A scatter plot representation of the results of k-means clustering after PCA shows that the data points are well separated into three clusters on the two-dimensional plane, see Figure 4. Note that, the number of points is not the same as given by the numerical outputs of the number of respondents in each of the three groups due to the fact that these points are PCA results and only show 68.1% of the original data. Three cluster groups were obtained and the number of people in those three groups are given in Figure 5. We name these three groups as: *Advocates of Slow Tourism (AST); Conscious of Slow Tourism (CST) and Unconscious of Slow Tourism (UST),* based on the average scores of the three groups. The overall average scores of the three groups are 2.61, 2.60, and 2.46, respectively; and the number of tourists in the three groups are 95, 104, and 23, respectively. Though, the average scores of the three groups do not show a strong difference, in each sub-problem, as shown in Table 1, we can see that the average scores of the three groups do show a difference. In order to be more confident about the number of groups given by the k-means algorithm, we also applied the one-way ANOVA (analysis of variance) to see if there is any difference in the means between the three groups in each sub-problem of the questionnaire. The results are summarized in Table 1. We can see that all the p-values in each question are small which confirms the rejection of the null hypothesis that the means are equal between the three groups and confirms that our assumption from the Elbow method is suitable for this dataset. This is further proof that the separation into three groups for the tourists is reasonable.

After properly separating the tourists into three groups, we further study the descriptive statistics of the tourists' background and related behavior in each of the three groups, to determine if they reveal any differences between the average scores similar to the performance of the different mean scores in the questionnaire among the three groups. For example, does the AST group have a higher educational background than the CST group or the UST group? Is the average income in the AST group higher than that of the UST or CST groups? In Table 2, we have provided both the median and the average in each of the three groups. The overall scores in either the median or average scores for the three groups do not show a strong difference, e.g., the ages of the AST group are not higher than the CST or UST groups. The median for all groups is 3.0, and the averages are 3.32, 3.28, and 3.13 for the AST, CST, and UST groups, respectively. For the level of education, the median is again 3.0 for all groups, and the averages are 2.74, 2.93, and 2.99 for the AST, CST, and UST groups, respectively. Interestingly, the mean score for educational background of the AST group is lower than the CST group. This shows that the educational background for the UST group is similar to the AST group, which is an interesting outcome. That means there are tourists who have a higher education but are still not aware of this fresh style of tourism. This implies that educational background is not a huge factor in one's appreciation of Slow Tourism and does not need extra consideration in a targeted marketing scheme. From these descriptive statistics we discovered that backgrounds of the three groups are not significantly different.

The final observation from the results begs the question, why are the numbers for both the AST and CST groups (they are 89.6% of all the tourists) relatively high? We believe it is related to the description of the questions. It is possible that the questions provide some positive images for tourists, and even though tourists may not actually know if these concepts are related to Slow Tourism, they may still favor that which is implied by the questions. That is why the UST has the fewest tourists, this outcome may not reflect the actual situation of the tourists regarding Slow Tourism, since Slow Tourism is still a new concept for ordinary tourists. If possible, in future studies, the statements in the questionnaire could be stated with positive and negative statements allowing for the respondents to think more carefully. This may provide a closer reflection of the correct percentages of each group.

## 4. Conclusion and Recommendation

Slow Tourism is a new practice in the tourism industry but is gaining popularity among tourists looking for a more relaxed and immersive experience. This practice can be related to the concept of a slow city and the principles of the Cittaslow organization.

This study used data that was collected from visitors to Nanzhuang township, Taiwan to seek more insight about the correct types of tourists visiting to this slow city nowadays, and to investigate how their motivation to visit Nanzhuang might be related to the principles of the Cittaslow. Overall, our dataset contained 222 data points, representing 222 visitors to Nanzhuang township. We were able to divide these visitors into three groups based on each respondent's responses to a 4-point Likert scale questionnaire by utilizing both the k-means clustering algorithm and PCA for better visualization purposes. One-way ANOVA was further used to make justification of the outcomes of the cluster groups by studying the p-values in each subproblem, and the results additionally confirm that the use of three clusters is the correct design. One interesting finding related to the background of the respondents was that they do not show a strong difference. The findings indicates that the backgrounds between the three groups make little to no difference.

The top two scoring groups consisted of a significantly sizable percentage of the dataset, indicating that the Cittaslow principles do influence motivation for visiting this destination, and that a substantial portion can be considered slow tourists. Though more studies should be given to ensure the real situation of Slow Tourism in Taiwan nowadays. Nevertheless, outcomes of our study reveal that the separation of the tourists into three groups is reasonable and can be further used for the design of suitable marketing strategies to attract more tourists to participate in Slow Tourism. This in turn will allow the slow cities in Taiwan to gain more economic growth through a proper and effective marketing strategy.

Further research using similar methods of the distinct types of tourists to the other three slow cities of Taiwan (Dalin township, Fonglin township, and Sanyi township) could provide interesting comparisons to our findings in this article, as well as give insight into the practice of Slow Tourism in Taiwan as a whole.

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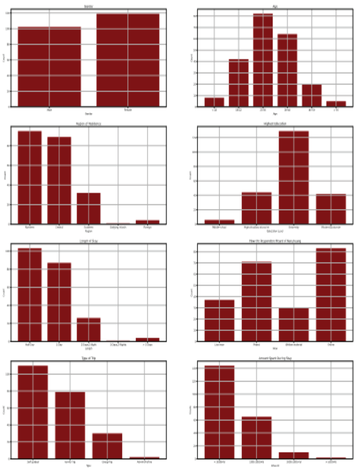

Figure 1. The Distributions of categorical background on all respondents

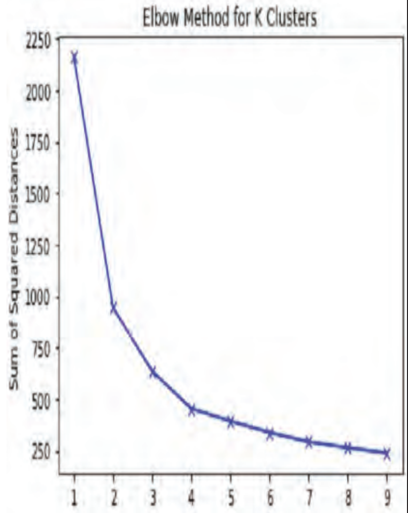


Figure 2. K-means Elbow Curve method for cluster selection

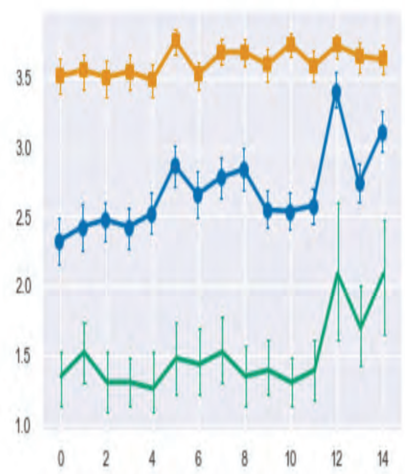


Figure 3. Means on every question (from #0 to #14) showing three lines with a large gap between these three lines

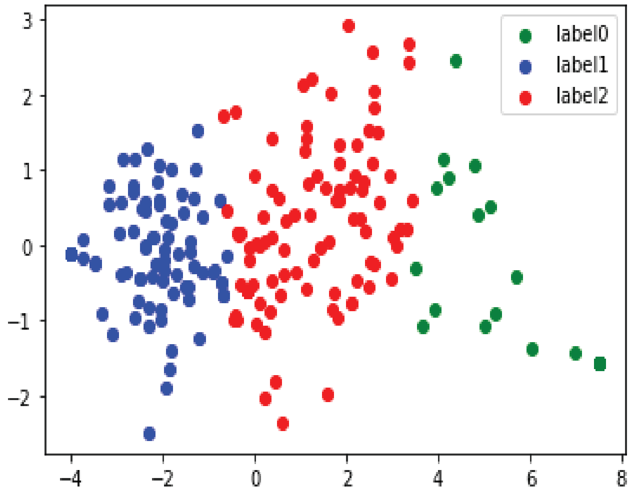


Figure 4. Three clusters produced by K-means with PCA

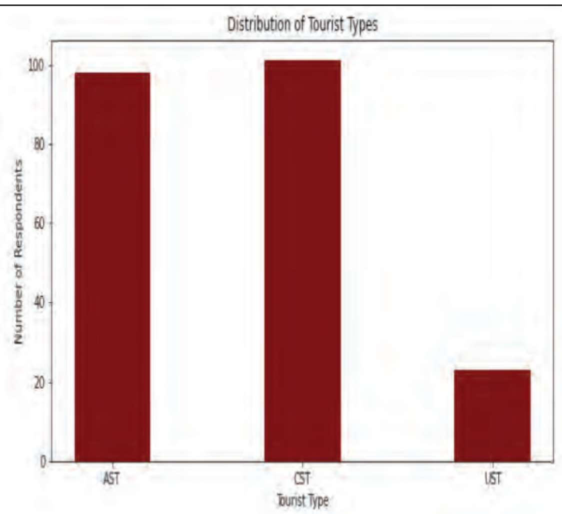


Figure 5. Three tourist groups, named AST, CST, and UST, with number of tourists equal to 95, 104, 23

Table 1: Modified Cittaslow principles (adapted from [11])

| | Mean AST | Mean CST | Mean UST | ANOVA F value | ANOVA p value |
|---|---|---|---|---|---|
| Ban on the use of O.G.M. in | 3.51 | 2.32 | 1.35 | 116.36 | 3.71e-35 |

| | | | | | |
|---|---|---|---|---|---|
| agriculture | | | | | |
| Plans for improving and for the reclamation of historical centers and/or works of cultural or historical value | 3.55 | 2.42 | 1.52 | 106.35 | 5.31e-33 |
| Quality green areas and service infrastructures (green areas, playgrounds, etc.) | 3.50 | 2.47 | 1.30 | 107.86 | 2.48e-33 |
| Plan for the distribution of merchandise and the creation of commercial centers for natural products | 3.54 | 2.42 | 1.30 | 124.25 | 8.62e-37 |
| Agreement with the shopkeepers regarding the reception and assistance to citizens in trouble: friendly shops | 3.48 | 2.52 | 1.26 | 116.52 | 3.43e-35 |
| Plans for the development of organic farming | 3.76 | 2.87 | 1.48 | 143.58 | 1.43e-40 |
| Certification of the quality of artisan produced products and objects and artistic crafts (Taiwanese Indigenous peoples' wood sculpture/knit textiles) | 3.52 | 2.66 | 1.43 | 103.60 | 2.16e-32 |
| Programs for the safeguarding of artisan and/or artistic craft products danger of extinction | 3.68 | 2.78 | 1.52 | 130.37 | 5.09e-38 |
| Safeguarding traditional methods of work and professions at a risk of extinction | 3.68 | 2.84 | 1.35 | 134.13 | 9.29e-39 |
| Use of organic products and/or those produced in the territory and the preservation of local traditions in restaurants, protected structures, and species and preparations risking extinction | 3.59 | 2.55 | 1.39 | 149.70 | 1.04e-41 |
| Favoring the activities of gastronomic Slow Food Presidia for species and preparations risking extinction | 3.74 | 2.54 | 1.30 | 239.71 | 7.03e-56 |
| Census of the typical products of the territory and support of their commercialization (updating of | 3.58 | 2.58 | 1.39 | 152.97 | 2.66e-42 |

| | | | | | |
|---|---|---|---|---|---|
| markets for local products, creation of appropriate spaces, e.g., famer's market) | | | | | |
| Promoting and preserving local cultural events (festival: Pas-ta'ay event by Saisiyat tribe) | 3.73 | 3.40 | 2.09 | 54.54 | 5.98e-20 |
| Training courses for tourist information and quality hospitality, such as training for the local government staffs or having tourist center | 3.65 | 2.74 | 1.70 | 103.66 | 2.09e-32 |
| Preparation of slow itineraries of the city (brochures, websites, home pages, etc.) | 3.63 | 3.11 | 2.09 | 45.58 | 3.45e-18 |
| Overall | 95 | 104 | 23 | | |
| N (total) | 2.61 | 2.60 | 2.46 | | |

Table 2. The median and average scores for tourists within three groups

| | Median | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Item | Sex | Age | Residence | Education | Duration | How | Style | Amount |
| AST | 2.0 | 3.0 | 4.0 | 3.0 | 2.0 | 2.0 | 4.0 | 1.0 |
| CST | 1.5 | 3.0 | 4.0 | 3.0 | 2.0 | 3.0 | 3.0 | 1.0 |
| UST | 1.0 | 3.0 | 4.0 | 3.0 | 1.0 | 2.0 | 3.0 | 1.0 |
| | Mean (average) | | | | | | | |
| AST | 1.60 | 3.32 | 4.23 | 2.93 | 1.69 | 2.31 | 3.48 | 1.36 |
| CST | 1.50 | 3.28 | 4.20 | 2.99 | 1.76 | 2.31 | 3.27 | 1.49 |
| UST | 1.48 | 3.13 | 4.30 | 2.74 | 1.57 | 2.09 | 3.13 | 1.26 |

# Learnings from the Use of Screencast Videography in Mathematics Education Research on Item-Writing

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**Abstract**: *This paper presents the viability of screencast videography (SCV) as a methodology in mathematics education research, particularly in the area of item-writing. Through the lead author's implementation of SCV in the pilot study of his dissertation project, the authors reflect on the affordances and challenges of utilizing SCV in mathematical item-writing research and its implications in mathematics education. With screencast display as its primary source of data, SCV may also utilize other sources of data, such as webcam footage and audio recording. The authors elaborate on the strengths and weaknesses of these data sources which collectively complement one another. They then share their introspections on the opportunities and limitations of SCV, and how these could be potentially addressed. In sum, SCV as a research methodology promotes corroboration and triangulation of data sources; when applied in mathematical item-writing research, it sheds light on the item-writing process and experience of mathematics teachers. This, in turn, may potentially inform the necessary support mathematics teachers need in designing assessment items that will ideally promote student learning and achievement.*

## 1. Item-Writing as a Field of Study

Research on item-writing (i.e., writing test items) for various purposes in assessment has been anchored on theories which are deeply rooted in the field of language and communication. Despite this, research on mathematical item-writing remains scant, especially with respect to the processes and experiences undergone by mathematical item-writers and their broader implications to mathematics teaching and learning. In this section, an overview of the progression of item-writing theories and models is discussed, and is then followed by a brief discussion on the knowledge gap in mathematical item-writing research.

### 1.1 Evolution of Item-writing Theories and Models

The seminal work by Flower and Hayes on the cognitive model of writing is a writer-oriented theory in which the writer and their activities related to writing are the focal point of the theory [1]. Their theory posits that writing consists of interrelated cognitive actions or thinking processes that enable the writer to navigate towards the attainment of their goal. To this, a more developed version of the model was proposed [2], which greatly differs from the former model. In this updated model, the writer and the writing processes are situated in two components: a task environment and the individual, in such a way that elements from each component interact with each other. Some of these elements are as follows: cognitive processes, working memory, long-term memory, motivation, physical environment, and social environment [3].

Further improving on the aforementioned, in [4], it is proposed that item-writing is a socio-cognitive process. In this model, the writer is positioned in a wider social context as they argue that item writing is not an isolated activity. Entities involved in the wider context are the *perceived readers* whose potential responses to the questions are considered during item-writing. The materials and resources used for item-writing, such as institutional documents (e.g., assessment framework), are considered as well. Likewise, there are efforts to further expound the field of item-writing. In [5] (as cited in [6]), item-writing was modeled as an activity consisting of three distinct phases, in which item-writing is viewed from a problem-solving perspective. These phases are as follows: initial representation phase, exploration phase, and solution phase. The initial representation phase is related to goal setting and defining the problem to be solved. Meanwhile, the exploration phase pertains to the writer's attempts to search for solutions to the problem defined previously in a purposeful approach. Lastly, in the solution phase, the writer moves towards the completion of the writing task by satisfying the constraints set for the item in the earlier phases.

In a subsequent study, item-writers were observed while going through item-writing tasks that were assigned to them by the researchers [6]. They attempted to dissect the cognitive processes of the item-writers during the task by recording and analyzing their verbal behaviors. [7] aimed to expand the model in [6] into a more comprehensive one, considering the contrast between expert and novice writers on top of their cognitive processes and knowledge structures. Such studies have been successful in pinpointing elements crucial to item-writing processes (e.g., schema activation, decomposition, and evaluation), especially duty-related item-writing activities of professionals in the academic setting (e.g., writing items for standardized assessments). Hence, such studies are necessary in advancing knowledge regarding item-writing activities, for these may provide theoretical and practical support in improving the assessment culture and practices of teachers and educators.

### 1.2 Research Gap in Mathematical Item-Writing

One notable research gap in the body of knowledge in item-writing is the absence of studies describing how the same cognitive processes apply and extend to mathematical items. The studies related to item-writing that were mentioned in the preceding sub-section discuss the production of items that mostly consist of text (e.g., a question that asks students to identify the thesis statement of a given passage). To this, minimal attention has been directed to mathematical items; an example of such is shown in Figure 1.1.

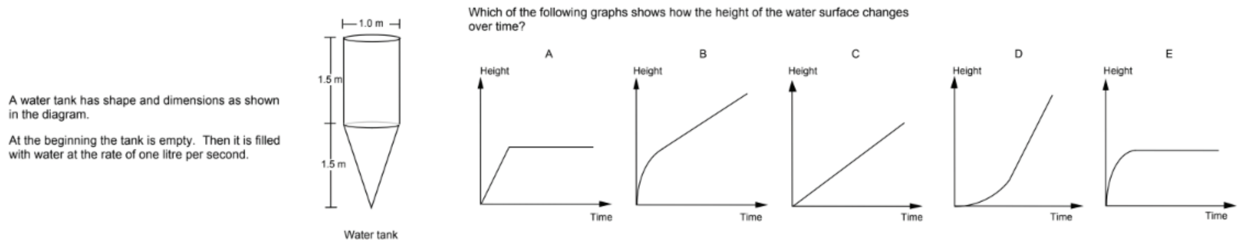


**Figure 1.1** An Example of a PISA Mathematical Item [8]

Mathematical items not only use text, but also incorporate non-textual elements, such as symbolic, notational, graphical, or tabular information. A study that focuses on how item-writers develop test items in mathematics will be helpful in uncovering the processes undertaken by academic professionals who are directly involved in this endeavor, most specifically mathematics teachers. This is essential in order to better understand mathematics teachers and their needs, so that ample support may be given and extended to them. Although teachers receive support in the development and improvement of their tests in the form of feedback from peers and mentors within the same working environment, the process that they undertake in creating the items remains widely unexamined. This is crucial to their duties, considering that teachers could spend at least a third of their time on tasks directly related to assessment – which include designing, grading, and interpreting results [9] (as cited in [10]).

More importantly, research on mathematical item-writing connects to the broader conversation on several pertinent topics of interest in mathematics education, such as task design [11], problem posing [12], and international large-scale assessments (ILSAs), among others. More so, today's educational context challenges learners to harness data, utilize technology, and integrate mathematical concepts and methods to solve non-routine problems with a critical and discerning mind. To this, examining mathematical item-writing could be seen as a valuable endeavor in further improving the quality of assessments and problems posed by teachers. This has the potential to enhance the learning experience of students. However, in the Philippines, the educational landscape is shaped by socioeconomic issues [13], in addition to haphazard curriculum modification and implementation [14]. Nevertheless, there are efforts among stakeholders to improve the quality of education across the country (e.g., scholarship programs for capacity building of STEM teachers). In light of the abysmal mathematical performance of the Philippine learners in ILSAs such as PISA 2018 [15] and TIMSS 2019 [16], the authors argue for the investigation of Filipino teachers' mathematical item-writing process, as results from such research may provide the necessary groundwork to support their professional development. This, in turn, may ideally improve the mathematics learning and achievement of Filipino learners.

## 2. Screencast Videography

Proposed by Kawaf [17] in 2019, Screencast Videography (SCV) as a research methodology aims to produce data in the form of concrete actions performed in a visual digital environment. An example of such an environment is a digital word processor which is extensively used in and ideal for document creation. In her study, Kawaf posited that the philosophical principles underlying SCV are anchored on interpreting moving images. She illustrated its application in marketing and consumer research, in the form of consumer experiences in online fashion shopping. According to her, this methodology allows for analyzing digital experience and interactions by adopting a dynamic form of inquiry that is visual-based. Data in SCV primarily consists of on-screen activities via screencast recording; this, in turn, reveals detailed records of users' digital experiences or activities. Kawaf urged researchers venturing into the use of SCV as a research method to partake in reflecting on the following aspects of their study: context, intervention level, software to be used, mode in screencasting, timeframe, and obtrusiveness.

The context pertains to the research topic and the field or industry it applies to. Meanwhile, intervention level refers to the imposition of limits or constraints onto the item-writers, such as asking them to perform specific tasks during the observation, or making certain covert actions more explicit. On the other hand, the software to be used is any package or application to be utilized in capturing experience-related behavior throughout the observation. As for the mode in screencasting, it could be in full screen mode, or streaming only certain or required parts of the screen. In terms of timeframe, it is a timeline for the observation which has been determined by the researcher beforehand, or setting indicator behaviors that determine when the observation starts and stops completely. Lastly, obtrusiveness is an aspect which might inadvertently alter the writer's natural behavior, or their tendency to show the more desirable side of their activities, as a consequence of being aware that they are being observed.

Additionally, Kawaf enumerated important considerations for analysis and interpretation, given the tendency of the data to be multimodal. One of which is the possibility for the data to be analyzed through a multimodal lens. In such cases, the visual, audio, and spatial aspects are examined altogether, including the temporal relationships among them. Hence, the method subscribes to an existential phenomenology paradigm with an overarching idea of the whole being greater than the sum of its parts; thus, the transitive parts in the continuous stream of the participants' thoughts are deemed as important as the substantive parts.

In relation to mathematics, the construction of mathematics tests using technological hardware and software could be investigated through SCV, as these require very precise formatting to reduce or avoid ambiguities in the presentation of symbols which might inadvertently cause misinterpretations among test-takers. Moreover, the use of SCV in mathematics education provides a glimpse into the teachers' thought processes and choices of action as they embark on tasks specific to instruction and assessment; these are processes that are challenging to define and concretize. Processes related to the teaching-learning cycle, in addition to the affective and behavioral dimension of mathematical item-writing, are abstract in nature. Hence, the digitalization of a teacher's actions during item-writing allows for these actions, emotions, and gestures to be observable and thus can be subject to evaluation, thereby providing valuable input to the forms of support that could be accorded to mathematics teachers.

## 3. Sample Implementation of SCV in Mathematical Item-Writing

The lead author utilized SCV in his dissertation project's pilot study in examining the item-writing process of four mathematics teachers with respect to items that are parallel to those that follow the assessment frameworks of ILSAs, specifically PISA. This is because one of his research objectives is to describe in detail how mathematics teachers construct test items in relation to their assessment literacy. He aimed to simulate the observation of processes behind writing PISA-like mathematical items through this pilot study before the actual implementation of the main study. An investigation of such processes is warranted provided that studies about ILSAs focus on student-centered variables and not so much on teacher-related factors. The teacher-participants of the pilot study were four tertiary mathematics instructors from a Philippine state university in a Northern Mindanao province. They all have academic backgrounds related to mathematics education.

In the pilot study, the teacher-participants were first oriented on the Programme for International Student Assessment (PISA) framework [18], which consists of three domains: content (i.e., quantity, change and relationships, space and shape, uncertainty and data), process (i.e., formulating, employing, interpreting, reasoning), and context (i.e., personal, occupational, societal, scientific). On the same day, they were then given at most four hours to construct exactly four PISA-like items, i.e., each item must be classified using one of the four content categories, one of the four context categories, and one of the four process categories (e.g., an item is classified as space and shape–occupational–formulating). As this was an individual item-writing activity that was simultaneously conducted for several participants, the lead author used several devices to record the screencasts of the participants over separate calls in an online conferencing platform where one device is assigned to one participant. Thus, actions performed on a digital word processor and other digital tools were recorded per participant. Moreover, the observer turned off his camera in order to subdue the obtrusiveness inherent to the research method.

Hence, three types of data were simultaneously captured through the video recordings: (1) the screencast display, (2) the webcam footage, and (3) the audio input. Utilizing an online conferencing platform allowed for the recording of the teacher-participant's screencast, webcam footage, and audio during the item-writing task. To ensure more depth of the data captured, SCV was implemented with a full screen mode of recording as this mitigates any possible loss of potentially useful data, as opposed to limiting the screencast only to applications that are of direct use to the writer, such as the word processor. While there were only three data sources, there were four teacher-participants whose item-writing session lasted between 2.35 and 3.83 hours. Data analysis began with watching the screencast recordings first, followed by the webcam footage. For each of these data sources from each participant, possibly significant moments were taken note of along with their corresponding timestamp. It was then followed by the transcription of the audio recording, in which noteworthy moments were also timestamped. It proved to be a challenge on how to consolidate and find connections among the data sources given the sheer volume of data.

With regard to the screencast recording, the authors noted that the teacher-participants were able to design their own PISA-like items, mostly via dynamic mathematical software, spreadsheet software, online calculator, web search engine, and word processor software. Additionally, their actions mostly revolved on browsing the web, editing and formatting text, images, and tables. A sample of these actions is illustrated via screenshots as shown in Figure 3.1. Scrutinizing these, it could also be observed that the teacher-participants' actions related to item-writing were deliberate, as these processes consumed most of their time spent during the item-writing workshop. Examples of such were looking for and verifying information to be utilized in the item, be it from digital or non-digital reference materials. Additionally, the participants spent significant amount of effort in the composition of the textual part of the item, as well as the integration of the non-textual elements that are useful in the item (e.g., producing a data set in tabular form, selection of accompanying graphic element).

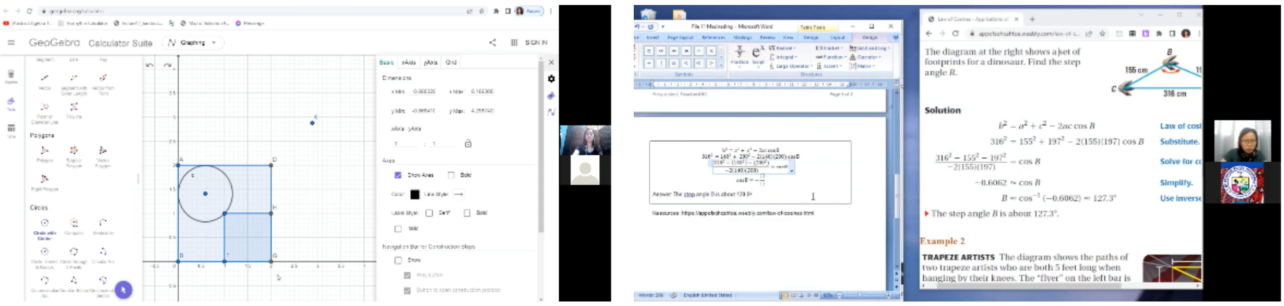

**Figure 3.1** Screenshots from the Screencast Recordings of the Teacher-participants

In terms of the webcam footage, the facial expressions of the teacher-participants showed their silent struggles (e.g., fatigue) while partaking in the item-writing activities. These mostly showed that they engaged in deep thinking manifested in physical stances or bodily postures that denote serious contemplation but with a few occasional breaks. However, it should be noted that the webcam footage was not able to capture activities that went outside its field-of-view. Lastly, the audio recordings revealed that the teacher-participants were mostly silent during the task. This was perceived by the lead author as a manifestation of focusing on the task at hand; nevertheless, thought processes that could have been verbalized were a missed opportunity. Despite this, some of them were observed to be engaging on intermittent utterances, such as self-talk or rereading portions of the partially created item that were displayed on the screencast.

In sum, the pilot study showed how SCV could be a viable methodology in observing the mathematical item-writing process of teachers. It allowed the lead author to tap into several data sources while making sense of the cognitive, affective, and behavioral dimensions of the teacher-participants' experience during the given task.

# 4. Post Facto Introspections

This section discusses the insights gained by the lead author in his use of SCV as a research methodology for examining mathematics teachers' item-writing process, along with the strengths and weaknesses of each data source as observed during the pilot study. This is followed by a discussion of the authors' reflection on the affordances and challenges of SCV and its implications in mathematics education research on item-writing.

## 4.1 Screencast Display

This source of data showed teacher-participants' actuations that led to the production of the output required by the item-writing task. This allowed the researcher to identify not only the types of resources accessed by the teacher-participants (e.g., websites, software), but also their possible patterns of action (e.g., copying text from a web page before editing it in a word processor). Their recorded acts were not limited to their online activities; their desktop and file navigation, along with their use of other digital tools, such as word processor, internet search engine, spreadsheet application, and dynamic mathematical software, also contributed to the richness of data captured via their screencast display. While this allowed the researcher to observe the teacher-participants' computer activities and, to an extent, describe the cognitive processes they enacted during mathematical item-writing, the screencast display did not provide context about their circumstances. It did not inform the researcher of the background activities that the teacher-participants undertook that may have had an impact on their decisions related to the task, such as the participant's ability to focus on the writing task given the external distractions in the immediate physical environment.

## 4.2 Webcam Footage

An advantage of capturing webcam footage is that it showed the teacher-participants' facial expressions and bodily movements that might be indicative of certain emotions that have arisen during the task. An example of bodily movement that was noted is attention span. This manifested through the intensity and duration of the participant's gaze at the screen and the frequency of eye movements. In addition, participants exhibited distress in the form of deep or heavy sighs, facial expressions that suggest tiredness (e.g., droopy eyelids or partially closed eyes), and the height at which the participant's head is rested (e.g., a low hanging head as a sign of mental fatigue).

The webcam footage also gave a glimpse of the physical environment the teacher-participants were in; this may potentially inform users of SCV whether such an environment is conducive to accomplishing item-writing tasks. In the context of the pilot study, the environments that the participants were situated were not necessarily ideal as there were some notable distractions (e.g., talking to other people in the household). They were, at the very least, able to secure a devoted physical space that is conducive enough for them to perform the item-writing task. The aforementioned observations provided the researcher with insight on the characteristics of a conducive environment for mathematical item-writing, especially with that of fatigue and productivity. Data from timestamps vis-à-vis the teacher-participants' corresponding activity and expressions indicate that breaks are necessary, and that ergonomics must be prioritized as well.

However, a possible downside of the webcam footage stems from the camera hardware. As there was only a single camera for every teacher-participant, the view of their offscreen activities was limited. For instance, writing activities or processes that were needed for the task and were done on paper may not have been captured and recorded.

### 4.3 Audio Recording

The last source of data, audio recordings, reveals the teacher-participants' verbalized thoughts and emotions during the task. It includes, but is not limited to, self-talk, which is a vital element in cognitive activities. For instance, one of the participants' self-talk behaviors was related to rereading the item and reviewing it thoroughly. Another participant was also observed to be humming and singing which is indicative of a self-soothing behavior possibly mitigating or keeping stress at bay. However, it should be noted that most of the time during the item-writing task, the teacher-participants tended to be silent as they tried to possibly focus. This is a dilemma that the lead author identified upon reflecting on his SCV implementation.

### 4.4 Opportunities and Affordances of SCV

An amalgamation of the three sources influences the collective strengths and weaknesses of SCV as a whole, including the opportunities and challenges that come with the research method, especially in the context of social sciences and the field of mathematics education. It can be seen that one source's weakness is another's strength; the three sources complement each other by filling in the limitations posed by one. To elaborate, the webcam footage and the audio recording fills in the gap of the screencast display by providing rich information about the participant's circumstance, both physically and temporally. They add depth to the item-writing processes captured primarily by the screencast display. Moreover, the screencast display informs the researcher of the actual activity being done by the teacher-participant at the exact moment pertinent to a portion of the webcam footage and audio recording. Collectively, they are able to capture a broad perspective of writing at the teacher-participant's level which is ideal for case studies where the participant's writing activities comprise the unit of analysis.

An affordance that SCV brings in mathematical item-writing research is that it allows item-writing to be viewed and analyzed from different and/or combined perspectives. From a cognitive perspective, researchers may be able to observe the possible thought processes that underlie the actions and decisions carried out by the teacher-participants with respect to their item-writing outputs. As for the affective and psychological dimensions, SCV possibly allows researchers to gain an understanding of which elements and factors associated with the item-writing process affect the emotional and behavioral states of the teacher-participants. In turn, these may inform policy and practice concerning item-writing that considers the best interests of the teachers.

Moreover, different phenomena of interest to the researcher can be observed via SCV, such as participants' overall experiences, the challenges in using technological tools, processes essential to item-writing, and how the participants carried out reference consultations throughout the item-writing process. Another affordance is that the observation can also be done remotely, which expands the pool of potential participants given that their participation is limited by their availability. Additionally, the methodology is highly replicable, which can be subject to further iterations and hence more opportunities in developing and improving it.

### 4.5 Possible Challenges and Limitations of SCV

The two authors reflected on the lead author's implementation of SCV and have identified several challenges and limitations encountered. SCV methodology is most suitable for non-participant observation studies. In such studies, observations are conducted by the researcher in the natural setting of the research participant without the possible interference of the observer [19]. Just like in any observation methodology, data from SCV can only consist of those gathered from the three sources (i.e., screencast display, webcam footage, audio recording). As such, limited data may result from the teacher-participants not being sufficiently expressive of their thoughts and emotions. Teacher-participants may also be forced to work in isolation given that they are being recorded individually. Another possible limitation is when item-writing activities are situated in social settings (e.g., consulting peers for feedback); the interaction may not be captured using this methodology. Transcribing the data from all three data sources and integrating them into one transcript is perhaps the most challenging aspect in the methodology as it requires the observer to pay close attention to three media simultaneously. This could possibly entail rewatching the recording(s) multiple times to get a good grasp of the critical moments in the item-writing process of each participant.

Furthermore, inevitable instances unavoidably mar the quality of the data captured in SCV. These were some of the unfortunate circumstances experienced by the lead author during the study: unstable internet connection, technical problems in uploading and downloading the recording, and hardware problems (e.g., malfunctioning or glitching screens). The lead author's grave source of worry during the study was the possibility of a power interruption, which is not uncommon in a developing country such as the Philippines, most especially in rural or provincial areas.

In light of the challenges and limitations, the authors offer several circumventions to address these. In terms of the three data sources possibly not being able to capture background or off-screen activities, the research participants may be requested to document such activities (e.g., through a journal or logbook), whether they deem these activities having contributed to their item-writing process or not. As for possibly protracted periods of silence during the item-writing activity, researchers may choose to implement the think-aloud protocol wherein research participants are informed ahead of time that they should verbalize their thoughts as much as possible throughout the item-writing activity [20]. However, this may require some form of training so as to familiarize the participants with the protocol.

With regard to mathematical item-writing activities carried out in groups, software applications that are geared towards collaboration may be utilized, along with other sources of data for triangulation. Such collaborative item-writing endeavors may be implemented fully online, fully onsite, or in a hybrid manner. Regardless of modality, the authors envision that it may be necessary to have at least a camera and microphone dedicated to capturing the group dynamics. Meanwhile, individual cameras and microphones will still be in place for capturing more interpersonal (or personal) interactions. More importantly, each participant's screencast display should be recorded, whether they are working on an individual or a collaborative space.

In terms of the challenge of integrating data across the various sources, the authors noted how the use of spreadsheet software is a viable means to corroborate data and facilitate later analysis. Timestamps could be placed adjacent to particular and significant observation(s) from each source. This may aid the researcher in tracking moments across the sources efficiently and effectively. Table 3.1 shows a sample snapshot of data corroboration across various sources via a spreadsheet.

**Table 3.1** Sample Data Corroboration from a Participant Recording

| Timestamp | Screencast Display | Webcam Footage | Audio Recording |
|---|---|---|---|
| 1:17:07 | Started Item Q1, filled out the item specifications for Item Q1 | | "So, question number 1. Item format, MC" |
| 1:17:21 | | | "Context" |
| 1:17:30 | | | "Process" |
| 1:17:41 | | | "Process" |

| | | | |
|---|---|---|---|
| 1:17:48 | | Motions his head to his right side | (sighs) |
| 1:23:30 | | Returned his attention to the screen briefly, then motioned his head to his right side again | |

Lastly, with respect to infrastructural issues such as power interruptions or loss of internet connectivity, the authors noted the importance of having contingency measures in place. Such plans should be designed with utmost caution and in consideration of the research participants. In the case of the lead author's SCV implementation, one contingency measure implemented was to invite a third-party observer as a meeting co-host. This is to ensure that in the event the lead author experiences any form of interruption, the continuity of recording ensues.

## 5. Conclusion

The present paper discussed the viability of screencast videography (SCV) as a methodology in mathematics education research. Tracing its roots from business research, the authors shared their reflections and insights on implementing SCV in mathematical item-writing research. The primary source of data in SCV is the recording of the screencast display, which records all the onscreen activities of the participant with respect to the task on hand. This is complemented by other sources of data, particularly webcam footage and audio recording. Triangulation amplifies the strengths accorded by each source through painting a better picture of the participant's item-writing process. Nevertheless, several limitations have been identified by the authors, to which they offered several possible workarounds. In all, the authors hope that through this paper, more research could be conducted on mathematical item-writing as it not only advances the knowledge pool in item-writing research, but may also ideally support teachers—the ones who are highly involved in the item-writing process for assessments in classroom use. Likewise, the authors envision more utilization of SCV in the field of educational research, given how the recent pandemic allowed us to embrace the opportunities accorded by technology. With remote non-participant observation becoming commonplace in conducting research via virtual means, SCV applies not only to activities that take place behind a teacher's day-to-day preparations for classes, but also to those that take place in the implementation of such planning (e.g., actual instruction in classroom settings). Further studies in mathematics education can focus due attention to the observation of such activities.

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# Morphing Tilings of the Plane into Tilings of Surfaces

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**Abstract**: *This work discusses a procedure to generate a tiling $\mathcal{T}_{S_f}$ of a 2-dimensional surface $S_f$ embedded in the Euclidean 3-space $\mathbb{R}^3$ from a tiling $\mathcal{T}$ of the Euclidean plane $\mathbb{R}^2$. We employ the computer algebra system Mathematica to generate 3D graphical images of $\mathcal{T}$ and $\mathcal{T}_{S_f}$ and render animations which create the effect of $\mathcal{T}$ morphing into $\mathcal{T}_{S_f}$.*

## 1. Introduction

Tilings of surfaces, space forms, or quotient spaces have been interesting subjects of study in discrete geometry and computer graphics [10, 13, 14, 15]. These geometric structures find applications as artworks or as mathematical models of chemical structures such as analogs of carbon nanotubes [6] or carbon nanotori [11, 12].

In recent years, technological tools have played an important role in the teaching and research of tilings. Dynamic geometry software, for example, have allowed for the investigation of properties and relationships of tilings [3], and the discovery of new families of tilings [2, 5]. Computer algebra systems have facilitated an efficient way of rendering tilings in surfaces and quotient spaces [7].

One of the important benefits of a technological tool is being able to facilitate the visualization of geometric representations. Geometric concepts can be visualized with dynamic geometric representations such as graphical images, animations, videos, applets, or with constructed or drawn representations made with a computer program [18]. Interactive applets, for example, can facilitate the deeper understanding of otherwise static geometric images, as applets allow users to explore, conjecture, make discoveries on these geometric figures, and even arrive at mathematical proofs. Computer programs can produce visual outputs, such as visualization of tilings of higher dimensions and other spaces, facilitating mathematical modeling and simulation of real-life scenarios [4].

In this paper, we present our work on tilings of surfaces, which are realized as geometric representations with the use of technology. We organize the paper as follows. In the next section, we enumerate different parametric systems of equations that define surfaces embedded in the Euclidean 3-space and discuss a procedure to generate a tiling of a given surface from a tiling of the Euclidean plane. In Section 3, we implement the said procedure in a computer algebra system to generate 3D graphical images of such tilings. We present two different ways of implementing this procedure and discuss the programming code on how each can be accomplished. In the final section, we demonstrate a technique to produce animations with a slider and animated gif files that show a tiling of the plane transforming into a tiling of a surface.

## 2. Tilings of Surfaces

A *tiling* or *tessellation* $\mathcal{T}$ of a point space $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ called *tiles* whose interiors do not intersect and whose union is the entire space $\mathbb{X}$. Although $\mathbb{X}$ can, in principle, be any set, it is usually endowed with a geometric structure. Various examples of tilings of the Euclidean plane $\mathbb{R}^2$ and their classifications in terms of symmetry and topological considerations are found in the classic text *Tilings and Patterns* [9]. One may also consult the searchable online databases *Tiling Search* [16] and *Tilings Encyclopedia* [8] to download images of tilings and retrieve useful information about them.

Tilings with rich mathematical structures and at the same time aesthetically pleasing are those consisting of tiles which either belong to a single congruence or similarity class or to a finite number of such classes. We present in Figure 2.1 examples of tilings of two point spaces. Figure 2.1(a) depicts a square patch of the famous *Ammann-Beenker tiling* $\mathcal{A}$ of $\mathbb{R}^2$ [8]. This tiling consists of congruent squares and rhombi which are arranged around the center of the plane to form a pattern with an 8-fold rotation symmetry. A star-shaped patch of an analog of this tiling on the *unit 2-sphere* $\mathbb{S}^2$ is shown in Figures 2.1(b) [front view] and 2.1(c) [back view].

(a) (b) (c)

**Figure 2.1** Amman-Beenker tilings of the point spaces (a) $\mathbb{R}^2$ and (b, c) $\mathbb{S}^2$.

In this work, we shall restrict ourselves to point spaces $\mathbb{X}$ that correspond to 2-dimensional *surfaces* embedded or lying in the Euclidean 3-space $\mathbb{R}^3$. Formally, we define a *surface* $S_f$ to be the image of a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^3$ in two variables $u$ and $v$. If we denote by $(x, y, z)$ the image in $\mathbb{R}^3$ of a point $(u, v)$ in $\mathbb{R}^2$, then $x, y, z$ may be identified with three continuous maps $f_x, f_y, f_z$, respectively, of $u$ and $v$. These identifications make $S_f$ a *parametric surface* and allow us to describe the surface conveniently using the parametric system of equations

$$\begin{aligned} x &= f_x(u, v) \\ y &= f_y(u, v) \\ z &= f_z(u, v) \end{aligned} \tag{2.1}$$

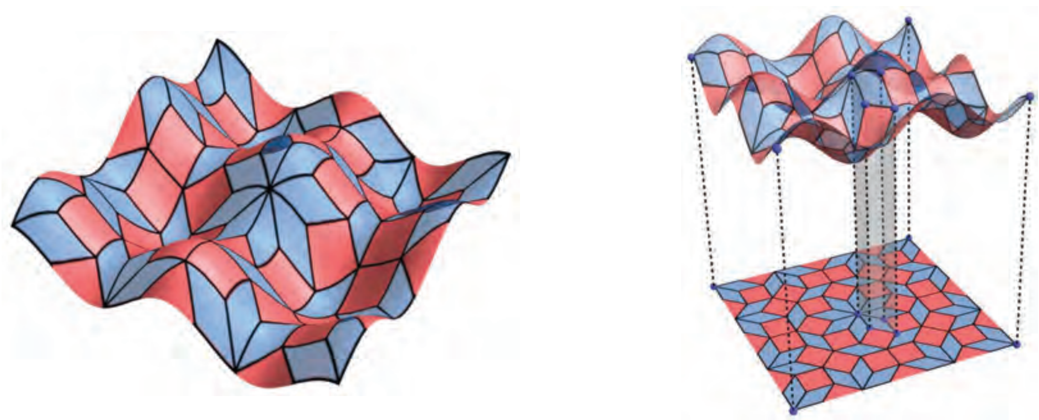
with parameters $u, v \in \mathbb{R}$. For example, if we take the maps $f_x(u, v) = u,\ f_y(u, v) = v,\ f_z(u, v) = 0$, we obtain the standard embedding of the Euclidean plane $\mathbb{R}^2$ into $\mathbb{R}^3$. This trivially implies that the Euclidean plane may also be viewed as a surface. As a matter of fact, the point space $\mathbb{S}^2$, which is the image of $\mathbb{R}^2$ under the inverse *stereographic projection*, is also a surface. Observe that under this map, which is an *inversion relative to* $\mathbb{S}^2$, the image of the small square patch of $\mathcal{A}$ in Figure 2.1(a) covers a large portion of the sphere as shown in Figure 2.1(b, c). The parametric equations defining these two and other surfaces, which we shall encounter later, are listed in Table 2.1 below.

**Table 2.1** Surfaces defined by a parametric system of equations with parameters $u, v \in \mathbb{R}$, where $a, b > 0$ are constants and $\theta_{u,v} \in [0, 2\pi]$ is the angle measured in the counterclockwise direction between the vectors $(1, 0)$ and $(u, v)$.

| **Surface** | $f_x$ | $f_y$ | $f_z$ |
|---|---|---|---|
| Plane $\mathbb{R}^2$ | $u$ | $v$ | $0$ |
| Unit 2-Sphere $\mathbb{S}^2$ | $\frac{2u}{1+u^2+v^2}$ | $\frac{2v}{1+u^2+v^2}$ | $\frac{-1+u^2+v^2}{1+u^2+v^2}$ |
| Egg Carton $\mathbb{C}_e$ | $u$ | $v$ | $\sin u + \cos v$ |
| Cone $\mathbb{C}_o$ | $\frac{a\sqrt{u^2+v^2}}{2\pi}\cos\left(\frac{\theta_{u,v}}{a}2\pi\right)$ | $\frac{a\sqrt{u^2+v^2}}{2\pi}\sin\left(\frac{\theta_{u,v}}{a}2\pi\right)$ | $\sqrt{u^2+v^2}$ |
| Cylinder $\mathbb{C}_l$ | $\frac{a}{2\pi}\cos\left(\frac{u}{a}2\pi\right)$ | $\frac{a}{2\pi}\sin\left(\frac{u}{a}2\pi\right)$ | $v$ |
| Torus $\mathbb{T}$ | $\frac{b\cos\left(\frac{v}{b}2\pi\right)}{\sqrt{a^2+b^2}-a\cos\left(\frac{u}{a}2\pi\right)}$ | $\frac{b\sin\left(\frac{v}{b}2\pi\right)}{\sqrt{a^2+b^2}-a\cos\left(\frac{u}{a}2\pi\right)}$ | $\frac{a\cos\left(\frac{u}{a}2\pi\right)}{\sqrt{a^2+b^2}-a\cos\left(\frac{u}{a}2\pi\right)}$ |

Given a tiling $\mathcal{T}$ of $\mathbb{R}^2$ and a parametric surface $S_f$, the collection $\mathcal{T}_{S_f} := \{f(t) \mid t \in \mathcal{T}\}$ of images of tiles in $\mathcal{T}$ forms a cover for $S_f$. If, in addition, the interiors of these images do not intersect, which depends ultimately on the map $f$, then $\mathcal{T}_{S_f}$ becomes a tiling of $S_f$. This happens, for instance, when $f$ is a homeomorphism between $\mathbb{R}^2$ and its image $f(\mathbb{R}^2)$. Such is the case with the maps that define the surfaces $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathbb{C}_e$ in Table 2.1.

**Illustration 2.1** We illustrate in Figure 2.2(a) the tiling of the *egg carton surface* $\mathbb{C}_e$ arising from $\mathcal{A}$.



(a) (b)

**Figure 2.2** (a) Tiling $\mathcal{A}_{\mathbb{C}_e}$ of $\mathbb{C}_e$ arising from $\mathcal{A}$. (b) The image in $\mathbb{C}_e$ of a rhombic tile in $\mathcal{A}$.

In Figure 2.2(b), eight selected vertices of $\mathcal{A}$ are marked and connected by dashed line segments with their images in $\mathbb{C}_e$. Notice that the marked rhombic tile of $\mathcal{A}$ underwent a deformation after it was mapped to $\mathbb{C}_e$. This is because the map used to define $\mathbb{C}_e$ is neither length-preserving (*isometric*) nor angle-preserving (*conformal*). Contrast this with the tiling $\mathcal{A}_{\mathbb{S}^2}$ in Figure 2.1(b) in which the angles (but not the lengths) of the tiles are preserved. This is a consequence of the fact that the inverse stereographic projection is conformal [1].

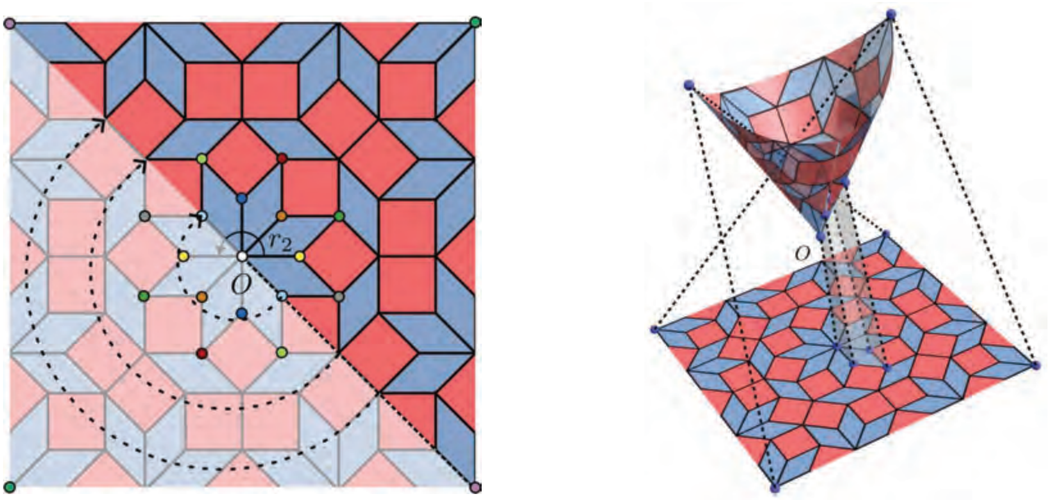
The maps in Table 2.1 may be broadly classified as either injective or non-injective. As implied earlier, the first three belong to the first class. On the other hand, the presence of the periodic sine and cosine functions in the formulas for $f_x$ and $f_y$ when $f$ defines either a *cone* $\mathbb{C}_o$, a *cylinder* $\mathbb{C}_y$, or a *torus* $\mathbb{T}$ hints at non-injectivity. Although this classification may seem purely algebraic at first glance, it has an important geometric implication when it comes to the construction of tilings of surfaces.

The cone, cylinder, and torus maps, and hence the surface they define as well, are compatible only with tilings of $\mathbb{R}^2$ that satisfy some symmetry properties. More particularly, a tiling $\mathcal{T}$ of $\mathbb{R}^2$ gives rise to a tiling of

- $\mathbb{C}_o$ if $\mathcal{T}$ possesses a $\frac{2\pi}{a}$-fold rotation symmetry $r_a$ about the origin;
- $\mathbb{C}_l$ if $\mathcal{T}$ possesses a horizontal translation symmetry $t_x$ with vector of length $a$; and
- $\mathbb{T}$ if $\mathcal{T}$ possesses both a horizontal and a vertical translation symmetry $t_x$ and $t_y$ with vectors of lengths $a$ and $b$, respectively.

The effect of any of these non-injective maps is to identify points in $\mathbb{R}^2$ that are equivalent with respect to the group of symmetries generated by $r_a$, $t_x$, or $t_x$ and $t_y$. Identified points are then "glued" together to become a single point in the resulting surface, which may now be referred to as a *space form* or a *quotient space*. Consequently, identified tiles of the plane become a single tile under the map. We discuss a specific example in the next illustration.

**Illustration 2.2** Since the Ammann-Beenker tiling $\mathcal{A}$ remains invariant under the 2-fold rotation $r_2$




(a) (b)

**Figure 2.3** (a) Points in $\mathbb{R}^2$ are identified with respect to $r_2$ to obtain (b) a tiling of $\mathbb{C}_o$.

about its center, which we assume to be the origin $O$, we can map it to the cone $\mathbb{C}_o$ by setting $a := 2$. Under this map, points in $\mathbb{R}^2$ that are sent to each other by the rotation $r_2$ are considered equivalent and get identified as a single point in the resulting cone. In Figure 2.3(a), vertices of $\mathcal{A}$ that share the same color become a single vertex in the resulting tiling of the cone. This process may be visualized as the following sequence of actions: cut the square patch in Figure 2.3(a) along the dashed segment from a corner point up to the center to form two flaps; hold one of the flaps at the corner point; and then rotate it towards the opposite corner point so that vertices of the same color are glued together to form the cone whose apex is $O$ (see Figure 2.3(b)). Observe that, as expected, points such as the opposite corner points in Figure 2.3(a) are sent to the same points of $\mathbb{C}_o$. In addition, the angles of the tiles in $\mathcal{A}$ are preserved in $\mathcal{A}_{\mathbb{C}_o}$. This is because the cone map together with the cylinder and torus maps is conformal. We note, however, that the latter two maps are incompatible with $\mathcal{A}$ precisely because this tiling is *non-periodic* [8]. That is, $\mathcal{A}$ does not admit translation symmetries in its symmetry group.

## 3. Generating 3D Images of Tilings of Surfaces

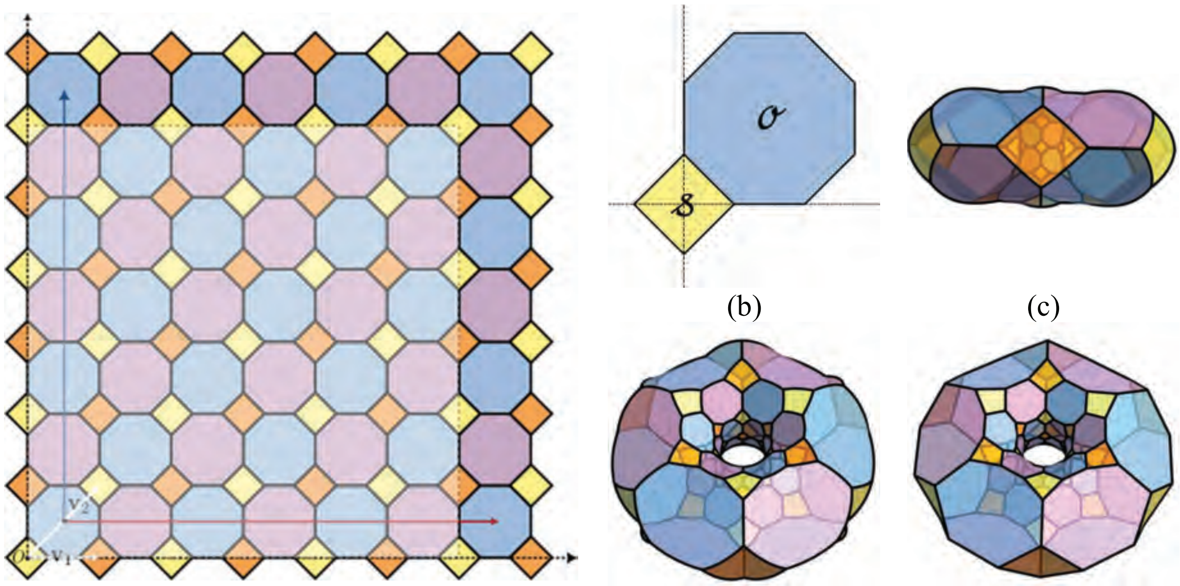
Our main objective in this and the next section is to illustrate a generic procedure to generate a 3D image of a tiling $\mathcal{T}_{S_f}$ of a surface $S_f$ from a given tiling $\mathcal{T}$ of $\mathbb{R}^2$ and render a simple animation that shows $\mathcal{T}$ morphing into $\mathcal{T}_{S_f}$. We shall implement this procedure in Wolfram Mathematica [16], a powerful computer algebra system (CAS) developed by Wolfram Research. We take advantage of this system's functional programming paradigm and its fast 2D and 3D graphics and animation rendering capabilities.

**Illustration 3.1** To illustrate the steps of the procedure, we use the face-colored *truncated quadrille tiling* $Q$ of $\mathbb{R}^2$, a patch of which is shown in Figure 3.1(a). This tiling is one of the eight semiregular *Archimedean tilings* [9] and consists of congruent regular squares and octagons with one square and two octagons arranged around each vertex. We introduce colors to the tiles of $Q$ so we can easily see and analyze the effect of a map on the tiles. Observe that the uncolored tiles of $Q$ can be generated from a single square tile $\mathcal{s}$ and a single octagon tile $\sigma$ (see Figure 3.1(b)) by successively translating them along two linearly independent directions determined by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. In this case, we begin with the tile $\mathcal{s}$ whose vertices we assign to be $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$. Basic geometry and trigonometry computations yield the vertices

$$\left(2+\sqrt{2},1+\sqrt{2}\right),\left(1+\sqrt{2},2+\sqrt{2}\right),\left(1,2+\sqrt{2}\right),\left(0,1+\sqrt{2}\right),(0,1),(1,0),\left(1+\sqrt{2},0\right),\left(2+\sqrt{2},1\right)$$

of the tile $\sigma$. With these vertices, we deduce the coordinates of the vectors $\mathbf{v}_1 = \left(2+\sqrt{2},0\right)$ and $\mathbf{v}_2 = \left(2+\sqrt{2},2+\sqrt{2}\right)$. To generate a patch of the uncolored tiling, we translate $\mathcal{s}$ and $\sigma$ along vectors of the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where the coefficients $c_1, c_2$ are integers at most some pre-assigned bounding values. The checkerboard coloring of the tiles is achieved by introducing a coloring rule based on the parity of $c_1$. More specifically, for a square or octagon tile, we assign yellow or blue, respectively, when $c_1$ is even; or orange or pink, respectively, otherwise. The Mathematica implementation of this construction appears in Code 3.1. Note that we utilized the built-in `Polygon` function to construct a tile from its list of vertices and the `Table` function to generate translates of $\mathcal{s}$ and $\sigma$. We also converted all point and vector coordinates into their numerical or decimal equivalent using the function `N`. This makes computations and graphics rendering in

Mathematica significantly faster as the system does not need to deal with exact algebraic values which often involve radical or trigonometric expressions.



(a) (b) (c) (d) (e)

**Figure 3.1** (a) A face-colored tiling $\mathcal{Q}$ of $\mathbb{R}^2$ and (b) two tiles used to generate $\mathcal{Q}$ by translations. (c – e) Images of the tiling $\mathcal{Q}_{\mathbb{T}}$ using the torus map with $a = \|6\mathbf{v}_1\|$ and $b = \|6(\mathbf{v}_2 - \mathbf{v}_1)\|$.

```
yellow = RGBColor[252/256,233/256,79/256,67/100];
blue = RGBColor[114/256,159/256,208/256,67/100];
orange = RGBColor[245/256,122/256,0/256,67/100];
pink = RGBColor[173/256,128/256,168/256,67/100];

v1 = N[{2+Sqrt[2],0}];
v2 = N[{2+Sqrt[2],2+Sqrt[2]}];

sqVerts = N[{{1,0},{0,1},{-1,0},{0,-1}}];
sqPolyYellow = Flatten[Table[Polygon[Table[sqVert+2i*v1+j*v2,{sqVert,sqVerts}]],{i,-4,4},{j,-1,7}],1];
sqPolyOrange = Flatten[Table[Polygon[Table[sqVert+(2i+1)*v1+j*v2,{sqVert,sqVerts}]],{i,-4,4},{j,-1,7}],1];

octVerts = N[{{2+Sqrt[2],1+Sqrt[2]},{1+Sqrt[2],2+Sqrt[2]},{1,2+Sqrt[2]},{0,1+Sqrt[2]},{0,1},{1,0},
    {1+ Sqrt[2],0},{2+ Sqrt[2],1}}];
octPolyBlue = Flatten[Table[Polygon[Table[octVert+2i*v1+j*v2,{octVert,octVerts}]],{i,-4,4},{j,-1,6}],1];
octPolyPink = Flatten[Table[Polygon[Table[octVert+(2i+1)*v1+j*v2,{octVert,octVerts}]],{i,-4,4},
    {j,-1,6}],1];

tiling = Graphics[{EdgeForm[{Black,Thickness[0.005]}],FaceForm[yellow],sqPolyYellow,FaceForm[orange],
   sqPolyOrange,FaceForm[blue],octPolyBlue,FaceForm[pink],octPolyPink},
  PlotRange->{{1.01*Norm[v1],7.01*Norm[v1]},{-0.021Norm[v1],4.23*Norm[v2]}}]
```

**Code 3.1** Mathematica code to generate the face-colored tiling of $\mathcal{Q}$ in Figure 3.1(a).

Generating an image of a tiling of $S_f$ from a tiling $\mathcal{T}$ then becomes a matter of replacing the coordinates of the vertices of the planar tiling with their images under the map $f$. Doing this directly, however, produces a tiling of $S_f$ whose tiles have straight edges rather than smooth curves. To remedy this, we first subdivide each edge of $\mathcal{T}$ into smaller sub-edges by introducing new subdivision points; treat the new subdivision points as the new vertices of the tiling; and then apply $f$ to these new vertices. We discuss this method for the tiling $\mathcal{Q}$ in the next illustration.

**Illustration 3.2** We first remark that $Q$ is an example of a *periodic* tiling since it possesses both horizontal and vertical translation symmetries. Consequently, it is compatible with a torus surface. For this illustration, we consider the surface $\mathbb{T}$ defined by the torus map with $a \coloneqq \|6\mathbf{v}_1\|$ and $b \coloneqq \|6(\mathbf{v}_2 - \mathbf{v}_1)\|$, the length of the red and blue vectors, respectively, in Figure 3.1(a). The additional step of subdividing each edge of the tiles in $Q$ to obtain new vertices is captured in the first five lines of Code 3.2. These lines replace the definitions of **sqVerts** and **octVerts** in Code 3.1. Here, **n** refers to the number of new subdivision points, including the original endpoints, that will be introduced per edge. The greater this number is, the smoother the edges of the resulting tiling $Q_{\mathbb{T}}$ of the torus $\mathbb{T}$ will be. The succeeding lines of code define the torus map; employ this map to obtain the vertices of $Q_{\mathbb{T}}$; and render a 3D image $Q_{\mathbb{T}}$. Two perspective views of $Q_{\mathbb{T}}$ are shown in Figure 3.1(c – d). Compare these with Figure 3.1(e) corresponding to the case **n = 1** in which no new subdivision points are introduced to smoothen the resulting bounding curves of the tiles of $Q_{\mathbb{T}}$. We remark that Code 3.2 can be easily modified to produce a tiling of another compatible surface by simply replacing **torusMap** with another map.

```
n = 10;

oldSqVerts = N[{{1,0},{0,1},{-1,0},{0,-1}}];
sqVerts = N[Flatten[Table[oldSqVerts[[Mod[k,4]+1]]+(i/n)*(oldSqVerts[[Mod[k+1,4]+1]]-
          oldSqVerts[[Mod[k,4]+1]]),{k,0,3},{i,0,n-1}],1]];
oldOctVerts =  N[{{2+Sqrt[2],1+Sqrt[2]},{1+Sqrt[2],2+Sqrt[2]},{1,2+Sqrt[2]},{0,1+Sqrt[2]},{0,1},{1,0},
    {1+Sqrt[2],0},{2+ Sqrt[2],1}}];
octVerts = N[Flatten[Table[oldOctVerts[[Mod[k,8]+1]]+(i/n)*(oldOctVerts[[Mod[k+1,8]+1]]-
           oldOctVerts[[Mod[k,8]+1]]),{k,0,7},{i,0,n-1}],1]];

torusMap[a_,b_,{u_,v_}] := (1/(Sqrt[a^2+b^2]-a*Cos[u*2π/a]))*{b*Cos[v*2π/b],b*Sin[v*2π/b],a*Sin[u*2π/a]};

sqPolyYellow = Flatten[Table[Polygon[Table[torusMap[6*Norm[v1],6*Norm[v2-v1],sqVert+2i*v1+j*v2],
       {sqVert,sqVerts}]],{i,-4,4},{j,-1,7}],1];
sqPolyOrange = Flatten[Table[Polygon[Table[torusMap[6*Norm[v1],6*Norm[v2-v1],sqVert+(2i+1)*v1+j*v2],
       {sqVert,sqVerts}]],{i,-4,4},{j,-1,7}],1];

octPolyBlue=Flatten[Table[Polygon[Table[torusMap[6*Norm[v1],6*Norm[v2-v1],octVert+2i*v1+j*v2],
       {octVert,octVerts}]],{i,-4,4},{j,-1,6}],1];
octPolyPink=Flatten[Table[Polygon[Table[torusMap[6*Norm[v1],6*Norm[v2-v1],octVert+(2i+1)*v1+j*v2],
       {octVert,octVerts}]],{i,-4,4},{j,-1,6}],1];

Graphics3D[{EdgeForm[{Black,Thickness[0.01]}],FaceForm[{yellow,Opacity[0.72],
     Specularity[yellow, 30]}],sqPolyYellow,FaceForm[{orange,Opacity[0.72],Specularity[orange,30]}],
  sqPolyOrange,FaceForm[{blue,Opacity[0.72],Specularity[blue,30]}],octPolyBlue,
  FaceForm[{pink,Opacity[0.72],Specularity[pink, 30]}],octPolyPink},Lighting->"Neutral",Boxed->False,
 Axes->False]
```

**Code 3.2** Mathematica code to generate the face-colored tiling of $Q_{\mathbb{T}}$ in Figure 3.1(d – e).

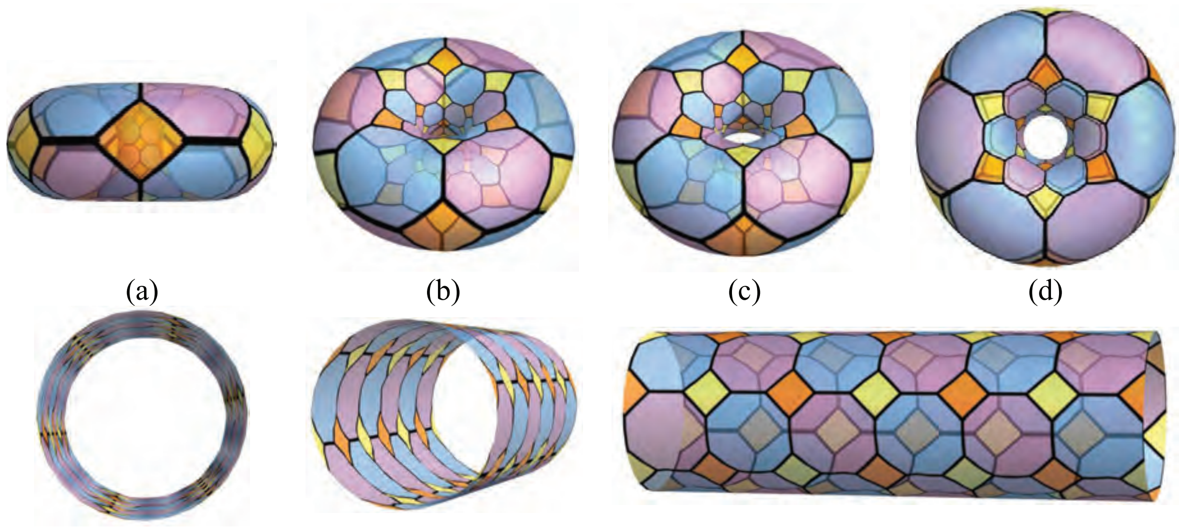
An alternative, more straightforward, and simpler way of generating a tiling of a surface in Mathematica is to use the **Texture** graphics directive within the built-in **ParametricPlot3D** function. The runtime of this second method is significantly faster and its implementation can reduce Code 3.2 to just a single line of code! We discuss how this is done in the illustration below.

**Illustration 3.3** The second method involves plotting the parametric surface by supplying its parametric system of equations and then applying an image of the tiling as texture of the surface. In the case of the tiling $Q$ and the surface $\mathbb{T}$ in Illustration 3.2, the implementation is found in Code 3.3.

```
ParametricPlot3D[torusMap[6,6,{u,v}],{u,0,6},{v,0,6},PlotStyle->Directive[Texture[tiling]],
 Lighting->"Neutral", Mesh-> None,Boxed->False,Axes->False]
```

**Code 3.3** Mathematica code to generate the face-colored tiling of $Q_{\mathbb{T}}$ in Figure 3.2(a – e).

The **torusMap[6, 6, {u, v}]** input in **ParametricPlot3D** instructs Mathematica to draw the parametric surface defined by the torus map introduced earlier in Code 3.2. The bounds **0** and **6** for both the parameters **u** and **v** indicate that the lightly shaded patch of the tiling in Figure 3.1(a) which was supplied as the sole input to **Texture** must be treated as an image consisting of pixels located within a $[0,6] \times [0,6]$ grid. Mathematica will then automatically perform any rescaling necessary for the image. Note that with this given rescaling, both the constants $a$ and $b$ get the value 6. What Mathematica will do essentially is "wrap" the patch around the surface according to the rule defined by the supplied parametric equations. That is, it assigns the color or texture of the point $(u, v)$ to be the color or texture of the image point $f(u, v)$ in the surface. This method produces the much smoother and rounder images of $Q_{\mathbb{T}}$ shown in Figure 3.2(a – d). When the torus map in Code 3.3 is replaced by the cylinder map with $a = 6$, we obtain the images for $Q_{\mathbb{C}_l}$ shown in Figure 3.2(e – g).



(a) (b) (c) (d) (e) (f) (g)

**Figure 3.2** Views of the truncated quadrille tilings of the point spaces (a – d) $\mathbb{T}$ and (e – g) $\mathbb{C}_l$.

We remark that while the second method discussed above is much faster and easier to implement than the first, the first provides more flexibility and adaptability when it comes to modifying components (vertices, edges, faces) of the tiling separately or independently from one another. This is because the second method requires a 2D image of the whole planar tiling to be used as basis for texture while the first converts each component of this planar tiling directly into a 3D graphics object. Another disadvantage of the second method is that it requires a high-resolution image for texture since the map that defines the surface stretches or deforms the image in various directions. Minimizing deformations is precisely the reason why we opt to use conformal maps, whenever possible, in lieu of ordinary maps.

## 4. Rendering Animated Tilings of Surfaces

Now that we have discussed completely how to generate an image of a tiling $\mathcal{T}_{S_f}$ of a surface in the previous section, we can now demonstrate how to render an animation that creates the effect of $\mathcal{T}$ morphing or transforming into $\mathcal{T}_{S_f}$. Mathematica offers at least two options for this to be accomplished. We can either create an animation with a slider that allows a user to manually control the animation or create an animated gif file that automatically runs the animation when opened. In either case, we create $m$ still frames for the animation where the $i$th animation frame, which records the $i$th state in the entire morphing process, shows the tiling $\mathcal{T}$ on the parametric surface

$$\left(1-\frac{i-1}{m-1}\right)\cdot(u,v,0)+\left(\frac{i-1}{m-1}\right)\cdot\left(f_x(u,v),f_y(u,v),f_z(u,v)\right), \tag{4.1}$$

for $1 \leq i \leq m$. Observe that when $i = 1$, we just obtain $\mathcal{T}$ and when $i = m$, we obtain $\mathcal{T}_{S_f}$.

**Illustration 4.1** To produce an animation showing $Q$ morphing into $Q_{\mathbb{T}}$, we run Code 4.1, which simply implements the procedure discussed earlier. In this implementation, we created the function **frame**, which generates the $i$th animation frame, and used this as an input within Mathematica's **Animate** function to produce the animation with slider in Figure 4.1(a). The **animation** list in the penultimate line creates a list of eight frames, four of which are shown in Figure 4.1(b – e). When this list is called within the **Export** function, it transforms into an animated gif file which is saved in the user's directory for later retrieval and use. The optional argument **DisplayDurations** specifies the number of seconds delay between showing successive frames.

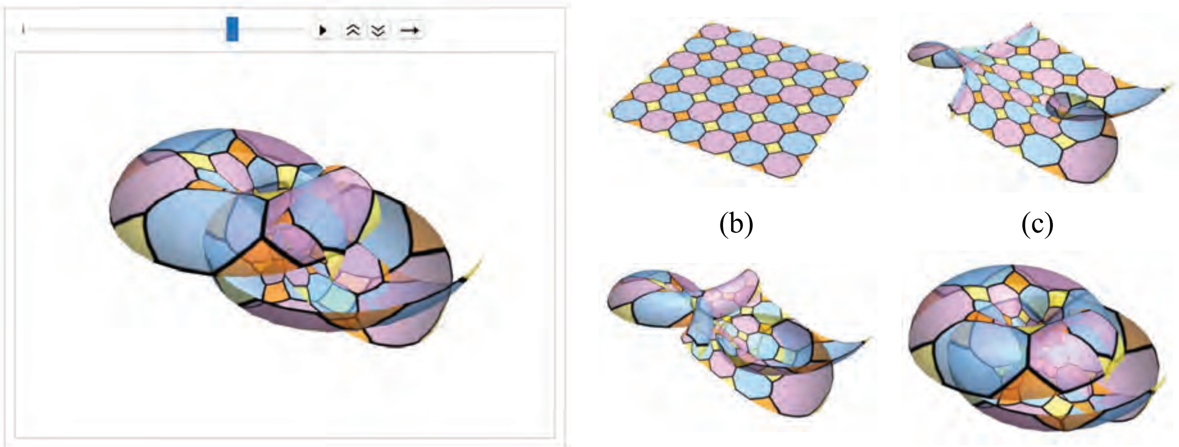
```
m = 8;

frame[i_] := ParametricPlot3D[(1-(i-1)/(m-1))*{u,v,0}+((i-1)/(m-1))*torusMap[6,6,u,v],{u,0,6},{v,0,6},
   Background->Opacity[0], PlotStyle->Directive[Texture[tiling]],Lighting->"Neutral",Mesh->None,
   Boxed->False,Axes->False];

Animate[frame[i],{i,1,m}]
Animation = Table[frame[i],{i,1,m}];

Export["animation.gif",animation,"DisplayDurations"->0.5]
```

**Code 4.1** Mathematica code to render an animation which shows $Q$ morphing into $Q_{\mathbb{T}}$.



(a) (b) (c) (d) (e)

**Figure 4.1** (a) Animation with slider showing $Q$ morphing into $Q_{\mathbb{T}}$. (b – e) Four still frames captured from the animation in (a).

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# Leveraging Technology for Effective Teaching and Learning

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**Abstract**: *During the COVID-19 pandemic, online teaching has become a major teaching method, which must involve the application of various technologies. From initial passive usage to ongoing exploration, our teaching and research team has found that teaching integrated with technology can help students better master and apply mathematical concepts to solve problems compared with traditional teaching, and technology has revolutionized mathematics education by offering innovative tools and resources that can promote active learning and enhance a motivating practice of math. In the realm of integration, technology can play a vital role in helping students visualize and comprehend fundamental concepts and applications. Firstly, the essay introduces powerful technological tools to dynamically enhance the teaching approach of complex mathematical concepts in the classroom through the utilization of TI-Nspire Software. Subsequently, it explores how technology can be integrated into curriculum design.*

## 1. Introduction

It is widely acknowledged that some calculus concepts are abstract and complex for students which has made teaching and learning challenging and even exasperating at times. Indeed, some researchers have shown that a large number of students have difficulties learning some key concepts of calculus because traditional calculus courses tend to focus more on algebraic drills and practice on calculus problems without understanding the underlying concepts [1], which would cause students to regurgitate what they have been taught and duplicate it in examinations. However, the problem is that over-emphasis on algebraic drill and practice methods will produce students who are only able to regurgitate what has been taught and duplicate it in examinations. The study conducted by Gordon reveals that algebraic solutions to problems and lengthy derivations of formulas are commonly expected among students in calculus. Gordon also pointed out that some calculus teachers felt that they were teaching algebra rather than the concepts of calculus. This phenomenon is also noted by Axtell[2] who feels that conventional teaching of calculus fails to produce students who are able to understand the fundamentals of calculus. Teachers seem to focus more on calculation techniques rather than understanding of the concepts among the students in teaching calculus. As a result, students solve differentiation or integration problems by simply applying the steps they have memorized without having a good grasp of the calculus concepts.
Understandably, Gordon and Axtell advocated that the calculus curriculum should be reformed by putting more emphasis on conceptual understanding of the fundamentals of calculus and complementing the use of graphical, numerical, algebraic, and verbal representation in the teaching and learning of calculus. Gordon (2004) further suggested that students must learn to select the right tools such as graphing calculators to help them learn calculus and use a variety of algebraic, numeric, and graphical approaches, to solve calculus problems. The purposes of the essay were therefore to examine the role of the TI-Nspire Software in teaching and learning calculus and provide some class examples in conceptual understanding and comprehensive practice based on TI-Nspire Software according to my teaching and exploration for years.

## 2. The Role of the TI-Nspire Software in Calculus

The TI-Nspire Software was used as a pedagogical tool to complement the conventional teaching approach. During observation within recent several years, the students' learning has often gone through such a process with TI-Nspire Software. TI-Nspire Software examines the setting of a given problem, formulate conjectures or hypothesis, examine the hypothesis, recheck the solutions, and finally generate a result or solution to the given problem. To facilitate exploratory learning and guided-discovery, worksheets were designed to guide the students in investigating and exploring mathematical concepts with the help of TI-Nspire Software, finally, students could understand the whole process of concepts gradually and visually at the end of the exploration. Therefore, TI-Nspire Software was also used as a confirmatory tool to verify their answers to the given problems or conjectures for a particular calculus concept.

Now, we observed that students also tried to use various methods of TI-Nspire Software to find the solutions when they were given problems in the exercise. In addition, we also found that students have a better understanding of the calculus concepts when they can use various approaches to solve problems. For instance, in the lesson on definite integral, we found that students who use graphical methods to solve stereotyped problems tended to develop a more complete understanding of the concepts of definite integral and area under the curve.

On the other hand, in solving problems involving maximum or minimum, we found that some students tended to approach the problems from a conceptual perspective using the graphical method or graphical tracing function of the TI-Nspire Software, which shows that the availability of TI-Nspire Software seemed to encourage a shift from a rigid to a more flexible technique of problem-solving among the students which is a consequence of students achieving conceptual understanding.

Through classroom observations, analysis of students' TI-Nspire Software documents, and regular reflections, we attempted to know what functions students have learned, what they would like to try, and what they would like to do with TI-Nspire Software. It was found that students used the TI-Nspire Software as a tool in several ways. For instance, students used the TI-Nspire Software as a visualization tool to better understand the behavior of graphs, new concepts taught, or problem situations. They also learned how to use the TI-Nspire Software as a confirmatory tool to verify the correctness of their answers. Table 1 summarizes the ways in which the TI-Nspire Software was actually used during the intervention program either as a pedagogical tool in teaching or as a learning tool by the students.

| Use of the TI-Nspire Software | Description |
|---|---|
| Exploratory tool | The TI-Nspire Software was used to explore and understand concepts of differentiation and integration. |
| Confirmatory tool | Students used the TI-Nspire Software to verify their answers to the questions in the exercises. |
| Problem-solving tool | using the graphing functions of the TI-Nspire Software. Students used the TI-Nspire Software to try algebraic, graphical and numerical approaches to solving a given calculus problem. |
| Calculation tool | Students used the TI-Nspire Software to calculate values or evaluate complex expressions. |
| Graphing tool | Students used the TI-Nspire Software to graph functions and to solve a given problem graphically. |

## 3. Class Design Examples

Here's a practical exploration course about the definition of definite integral and its application, and it would show how calculator works in the teaching from conceptual teaching and comprehensive practice.

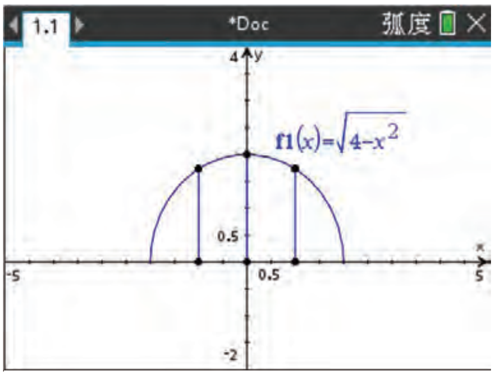
**Case 1—Definition of definite integrals**

**Activity I: Introduce the background.**

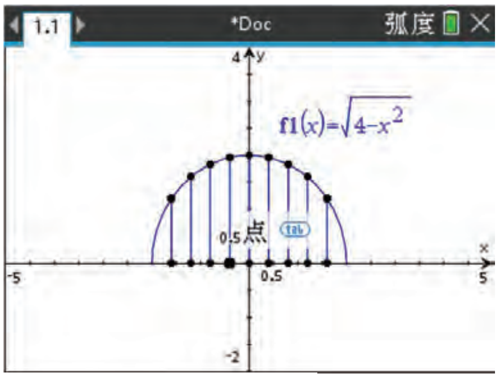
We are already aware that the area of a semicircle with a radius of 2 is equal to 2π, however, it is important to understand the mathematical reasoning behind this result.

**Activity II: Explore the problem step by step with TI-Nspire**

Step 1-Repartition the area of the curved trapezoid into smaller subdivisions.

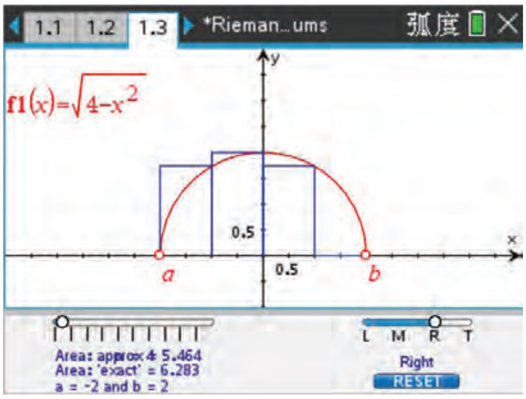



*Case 1.1-Partition 1*

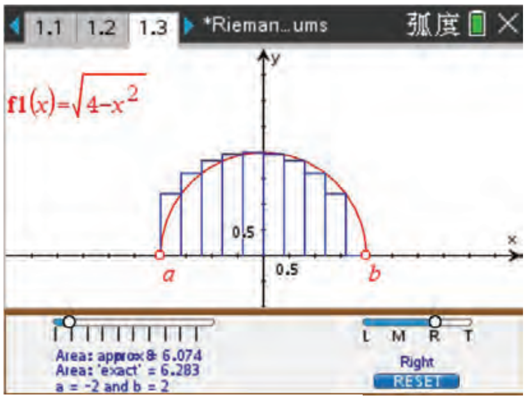



*Case 1.2-Partition 1*

Step 2-Every subdivision can be approximated by the area of a rectangle.

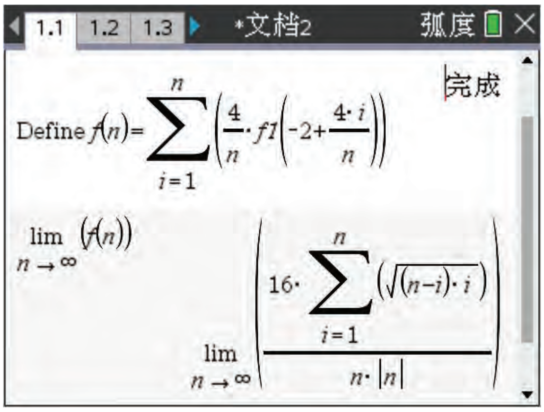



*Case 1-3 Approximation*



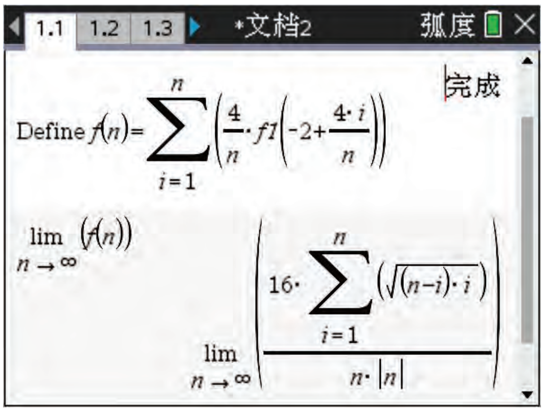

*Case 1-4 Approximation*

Step 3-Compute the total area of all rectangles.

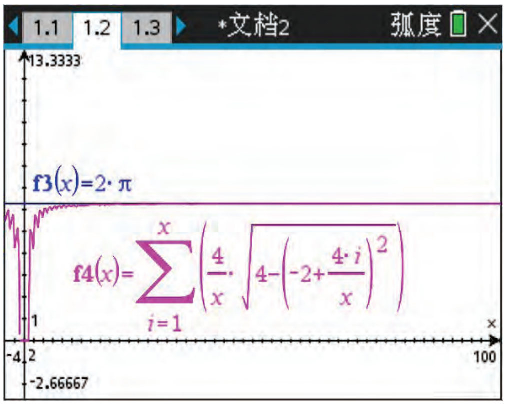



*Case 1-5 Summation*

Step 4- find the limit of sum in step 3 as the subdivision becomes infinity.




Case 1-6 Limit

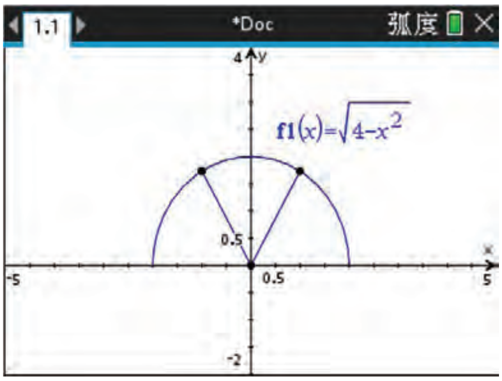



Case 1-7 Graph

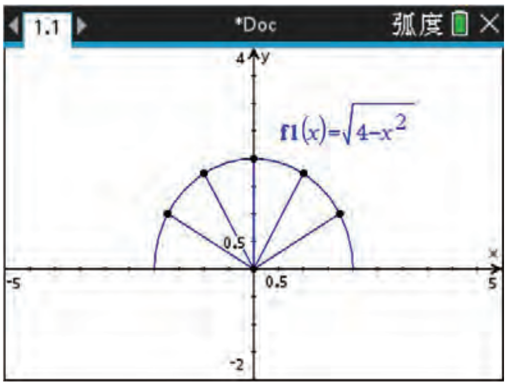
This class explores the implementation of TI-Nspire-based problem-solving techniques in the design of a comprehensive course for mathematics education. The objective is to empower students to break down definite integration into manageable steps: segmentation, approximation, summation, and limit-taking. Additionally, the study leverages the drawing and visualization capabilities of calculators to facilitate a deeper understanding of integrals from both algebraic and geometric perspectives. Notably, the limitations of calculators come into play, as direct calculation of sequence limits $(f(n))$ is not feasible, necessitating a geometric approach. However, these constraints serve as catalysts for enhancing students' critical thinking abilities. After class, students employ by TI-Nspire-based concept of definite integration to address the identified challenges and expand their mathematical proficiency.

With the help of a wireless interactive classroom in TI-Nspire Software, we found another method in partition and approximation as below:

Step 1: Segment with small sectors

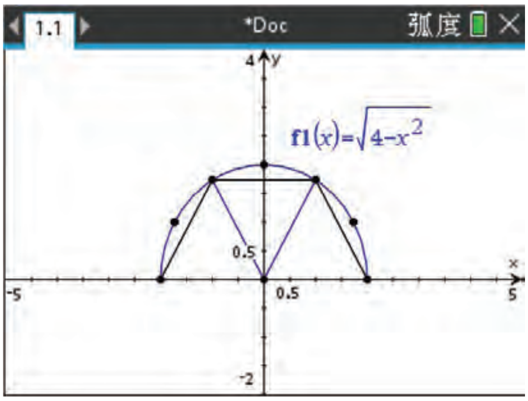



Case 1-8 Partition 2

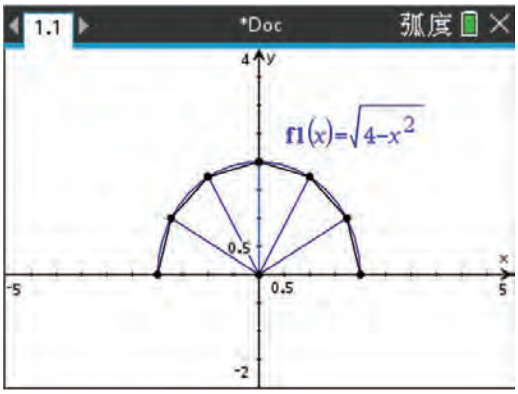



Case 1-9 Partition 2

Step 2: approximate sectors by triangles

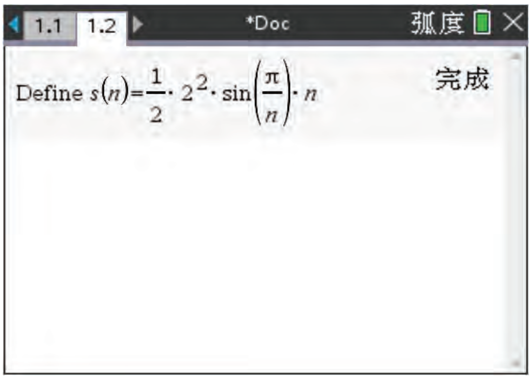



*Case 1-10 Approximation*



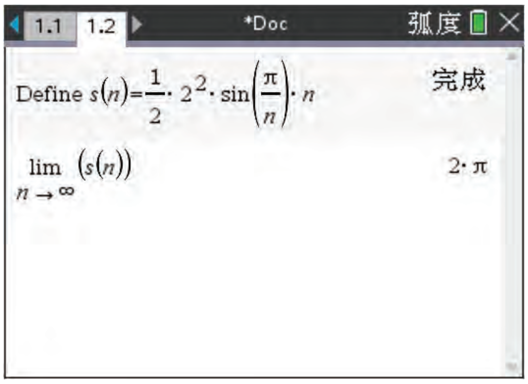

*Case 1-11 Approximation*

Step 3: sum all the areas of triangles




*Case 1-12 Summation*

Step 4: limitation




*Case 1-13 Limit*

The above process fully also explains how students master the concept of definite integral from graphically and numerically, and it also shows that an idea that partition and approximation is not unique. In addition, the second partition will also encourage students to solve the area of curve in polar coordinates later. Teachers could focus on each student's practice process and thinking methods with the function of wireless interactive classroom, which also increases the active participation and devotion of students in class.

**Case 2—Comprehensive Practice of Application of integration 1**
**Activity I: Introduce the background**

The instructor presents a series of phenomena and prompts students to identify their underlying causes, subsequently synthesizing these explanations into a common theme.

1) Why is it that when a coin is tossed, it predominantly lands on either its obverse or reverse face rather than its edge?

2) When 20 coins are thrown in a vertical arrangement, it is observed that they predominantly land on their edges rather than on their faces or backsides.

3) Why do organisms' transition from crawling on the ground to quadrupedal locomotion and ultimately to bipedalism?

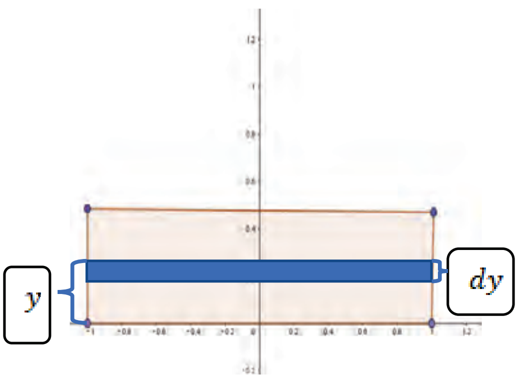
4) Look at curtain, why does it hang vertically instead of floating up?
Discussion:
These problems exhibit two primary characteristics: firstly, some are taken as given and obvious situations; secondly, they do not appear to share any common traits. However, delving beneath the surface to explore the essence of things is a crucial aspect of scientific inquiry. With this in mind, we will use coins as an example to uncover the underlying similarities behind these occurrences.

**Activity II: Mathematical modeling for the coin**

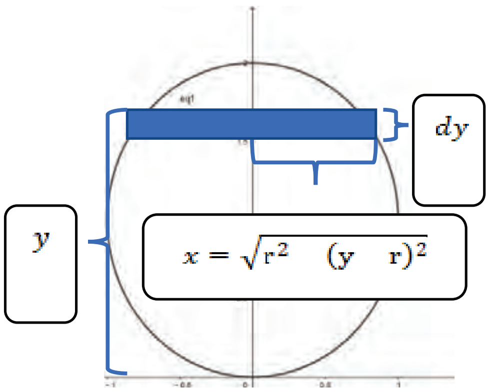
Assumption:

- A coin can be described as a right circular cylinder with a base of radius $r$
- The coin exhibits a thickness of $d$
- The density of the coin is $\rho$
- Acceleration of gravity is $g$

According to the fundamental principles of calculus and with the aid of TI-Nspire Software, students can readily compute the gravitational potential energy of the coin $W_1, W_2$




*Case 2-1 Partition in small piece*




*Case 2-2 Partition in the whole*

| Geopotential potential energy landing on the front or back: | Gravitational potential energy landing on the thin side: |
| --- | --- |
| $W_1 = \int_0^d y \cdot g \cdot \rho \cdot \pi \cdot r^2 \, dy$ <br> $= \pi\rho g \cdot \frac{d^2 \cdot r^2}{2}$ | $W_2 = \int_0^{2r} \rho \cdot g \cdot 2y\sqrt{r^2 - (y - r)^2} \cdot d$ <br> $= \pi\rho g \cdot dr^3$ |

Therefore, if $d = 2r$, then $W_2 = W_1$. If $d < 2r$, then $W_2 > W_1$. If $d > 2r$, then $W_2 < W_1$. For a coin, $d$ is significantly smaller than $2r$, thus prompting the coin to settle in a state of lower geopotential potential energy, either heads or tails. However, when considering $20$ coins arranged in a column by tape where $d$ exceeds $2r$, the water bottle also gravitates towards a state of lower energy upon landing on its side. This principle can be applied to other phenomena as well.

**Activity III: Summary and Extension**

Thus, it can be inferred that objects tend to exist in their lowest energy state. In this regard, the most natural state for humans would be lying down or walking on all fours, so that human standing upright is a form of resistance against nature. However, ultimately people cannot escape the fate predetermined by nature; much like a tossed coin will almost always land either heads or tails.

**Extensional questions from the knowledge aspect:**

1. The string assumes a singular configuration when suspended from two nails that are shorter than its own length. Is this curve representative of a quadratic function? What is the equation describing this particular shape?
2. Despite my efforts to toss them in a controlled manner, the handful of coins I hold inevitably scatter upon release rather than coalescing into a more compact formation. Why?

**Extensional thought from students' growth perspective:**

The act of human standing itself represents a form of resistance against nature, yet ultimately we cannot evade the predetermined fate bestowed upon us by nature. Similar to how a tossed coin will inevitably land on either heads or tails, humans too are subject to their own ultimate destiny. Therefore, how should we approach our existence?

Tagore's *Stray Birds*:

*Death's stamp gives value to the coin of life; making it possible to buy with life what is truly precious.*

**Case 3—Comprehensive Practice of Application of integration 2**

**Activity I: Introduce the background**

In Calculus BC, after students have learned application of integration including arc length of curve, and area bounded in polar coordinates, some practical problems make students motivated in deep learning. In the video or science museum, we have found not only circular wheels can roll smoothly, but also square wheels if provided a perfect road. Out of curiosity, they started to explore the perfect road for square wheels.

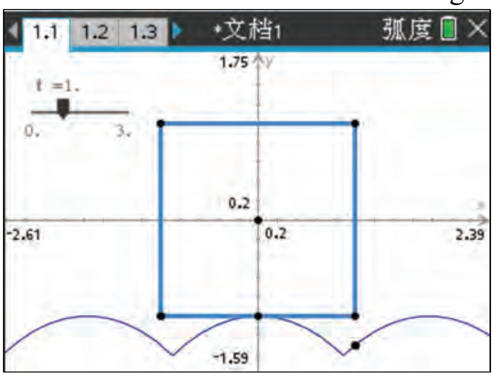
**Activity II: Transform into mathematical problems**

According to the video, teacher asked questions as followed:

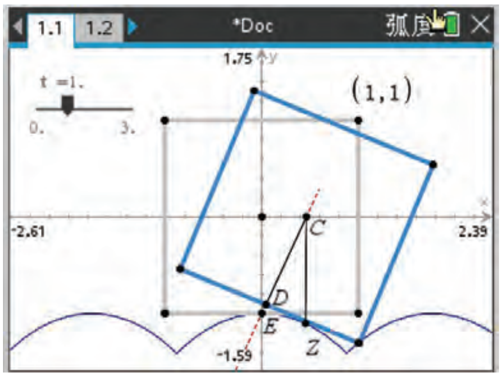
1) How to ensure smooth rolling of square wheels?

2) What is the relation of the contact point of the square wheel between the track and the axle (i.e. the center of the square)? Why?

In order to transform the practical problem into a mathematical problem, we used TI-Nspire software to simulate one wheel rolling on the road.




*Case 3-1: Start Condition*




*Case 3-2: Process Condition*

Discussion of student in groups:

Two key points to solve the problems are:

1) Specially designed track to ensure that the axle always moves in a straight horizontal line.

2) The contact point of the square wheel and the track is on a straight line perpendicular to the horizontal ground with the axle. At equilibrium, the force of gravity on the wheel is equal to and opposite to the force the track is exerting on the wheel.

**Activity III: Solve the mathematical model**

Due to the symmetry and periodicity of the wheel motion, it may be helpful to study the track equation of the wheel from the initial horizontal state to the first diagonal horizontal state. With the help of TI-Nspire Software demonstration for many times, continue the following questions:

1) Work in groups to discuss the position and geometry relationships in motion.
2) How to set up the equations?
3) How to solve?

Answer and Discussion:

1) Arc length of EZ=Line segment of DZ.
2) Let the contact point $(x, y)$, so that C $(x, 0)$.

$$\int_0^x \sqrt{1+(y')^2}dt = \sqrt{y^2-1}$$

Differentiate with $x$ on both sides,

$$\sqrt{1+(y')^2} = \frac{1}{2\sqrt{y^2-1}} \cdot 2yy'$$

Therefore, $(y')^2 = y^2 - 1$, $y' = -\sqrt{y^2-1}$ ($y' < 0$ for $x > 0$)

Solve

$$\int \frac{1}{\sqrt{y^2-1}} dy = \int -1\, dx$$

*Method 1: (Formula)*

$$\int \frac{1}{\sqrt{y^2-1}} dy = \cosh^{-1}(-y) + C, (y < 0)$$

So the solution of the differential equation: $\cosh^{-1}(-y) = -x + C$

With initial condition: $x = 0,\ y = -1$, so that $C = 0$.

$\cosh^{-1}(-y) = -x$

Therefore, $\boldsymbol{y = -\cosh(x)}$ (even function)

*Method 2: (Trigonometric Substitution)*

Let $y = \sec t$.

$$\int \frac{1}{\sqrt{y^2-1}} dy$$

As $y = \sec t < 0$.

$$= -\ln\left|y - \sqrt{y^2-1}\right| + C$$

So the solution of the differential equation:

$$\ln\left|y - \sqrt{y^2-1}\right| = x + C$$

Initial condition: $x = 0,\ y = -1$, so that $C = 0$.

As $y < 0$, so that $y - \sqrt{y^2-1} < 0$

$\sqrt{y^2-1} - y = e^x$

Therefore, $\boldsymbol{y = -\frac{e^x+e^{-x}}{2} = -\cosh(x)}$

**Activity IV: Reflection and Extension**

1) Summarize the contents AP calculus BC in this class.

2) Compare the two methods to solve the trajectory equation for square wheel in motion, what are the key points and difficulties to solve the problem?

3) Give the wide application of catenary and continue to expand learning after class.

4) Exploration:

(i) What are the perfect trajectories for the smooth motion of triangles, hexagons, ellipses, parabolas, and other shapes?

(ii) Can you find an ideal wheel for a known zigzag or sine wave track?

**Feedback from students**

Student A: Months ago, I went through a phase of what Albert Camus referred to as "absurdity." If our destiny is ultimately death, then what we do or care about now may seem meaningless. I was extremely confused and turned to reading to find some answers. I learned from "The Myth of Sisyphus," I understood why one could consider Sisyphus happy. Even though we may be tiny coins in a vast universe, we are still able to follow our personal beliefs. Our flips in the air may not cause any disturbances, but we can still be content with our own achievements.

Student B: My harvest mainly comes from the aspect of thinking. The accumulation of knowledge and concepts is sometimes novel, but the perception of essential ideas can always be memorable. Although I may not be able to understand every concept in the class, every time I understand the essence of a concept, I will have a strong sense of harvest and achievement. This may be the charm of mathematics.

Student C: Calculus is so useful in scientific research such as physics, chemistry, engineering, and economy problems, and helps us to analyze some interesting questions in our daily life. Technology has helped me to understand mathematical ideas intuitively and motivated me to focus on the formation of concepts and the core of mathematical methods. In addition, I was amazed by some practical problems to witness the process and method of mathematical modeling, which encouraged me to think of questions as a mathematician.

Student D: Technology is so powerful and helpful, which helped me to deeply understand the definition and idea of calculus, rather than calculation only from my self-study. It’s easy for me to find out the geometrical relation in some complicated kinematics simulation made by my teacher. In my class, lots of peers enjoyed high interaction with each other and teachers to complete a challenging question. And I think technology has helped us a lot, and we will make more use of it in research in the future.

## 4. Conclusion

Incorporating technology into mathematics education enhances visualization and conceptual understanding of calculus. Graphing tools, interactive applets, virtual manipulatives, and problem-solving platforms provide more opportunities to explore integration in dynamic and interactive ways for students by themselves. By leveraging these technological resources, educators can foster a deeper understanding of integration concepts, promote active learning, and connect calculus to real-world applications. In addition, technology empowers students to develop the ability to combine algebra and geometry and to improve mathematics abstract, mathematical modeling, and intuitive imagination abilities which play a significant in mathematics learning. Furthermore, educators need to pay attention to evaluation and feedback collection from students, which will help to keep track of students’ thinking processes in order to ensure the quality and effect in class.

It would be useful for teachers to find mistakes or misunderstandings so as to adapt the teaching process, and it will also encourage students to reflect on their studies.

In general, TI-Nspire Software is a powerful tool that can significantly enhance students' understanding of differentiation and integration concepts in calculus. With the help of visualization capabilities, step-by-step calculations, real-world applications, interactive activities, simulation, and error analysis, students can develop a deeper conceptual understanding of calculus, and solve practical problems by abstracting into mathematical models. Integrating technology effectively with sound pedagogical approaches will empower students to grasp these fundamental concepts with greater clarity and confidence, paving the way for success in advanced mathematics and related fields. All in all, technology not only has provided huge support for educators in teaching mathematics classrooms but also makes it more challenging for educators to explore the best way to facilitate the development of students' problem-solving abilities and mathematical core competencies in order to improve the development of students and education.

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# Students' Cognitive Developments in Learning Basic Differentiation Rules Using the Desmos Classroom Based on the Three Worlds of Mathematics

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**Abstract**

*Investigating how undergraduate students learn derivatives is crucial to supporting them in successfully continuing their studies in integral calculus. This case study investigates students' cognitive development as they learn basic differentiation rules using the Desmos Classroom (DC) based on the Three Worlds of Mathematics (TWM). This research includes 25 students enrolled in Calculus 1 at Sampoerna University during the fall semester of the 2022–2023 academic year. DC is used as a generic organizer to facilitate an embodied operation on a function's graph. DC enables students to drag the tangent line and tangency point along the graph of a function, and it allows them to magnify the screen, which helps them make sense of the tangent line and derivative concepts. Students prove the basic differentiation rules on the DC through graphical exploration, numerical computations for practices, and symbolic manipulations. Based on the TWM, the DC can contribute to the student's cognitive development by helping them learn the basic differentiation rules. All students performed well in the axiomatic formal world by proving the derivative of a trigonometric function. Most students (92%) also solved the tangent line problem, which required them to think in the proceptual world. Many students (64%) also have no limitations on graphical representation. According to this result, students' success in thinking in the axiomatic formal world does not imply their success in the proceptual world. Similarly, success in thinking in the proceptual world does not imply success in graphical representation.*

## 1. Introduction

Several strategies, including the Three Worlds of Mathematics (TWM) framework [1], have been proposed to facilitate students' learning of the derivatives. When students learn the derivative of a function, the TWM framework outlines three distinct worlds. These worlds involve the simultaneous development of conceptual embodiment, which involves the use of human perception and action; proceptual symbolism, which involves the manipulation of symbols derived from operations; and axiomatic formalism, which involves the construction of formal knowledge within an appropriate fundamental framework deduced through mathematical proof [2]. Students should commence TWM-based learning activities by experiencing the dynamic embodiment of change. Then, they should be able to perform sufficient arithmetic to obtain numerical approximations and appropriate symbolic expressions for the derivative of a function.

Tall [1] proposed using a generic organizer in calculus learning activities to actualize the TWM framework. This computer-based "generic organizer" allows students to investigate calculus concepts and generate dynamic function graphs. It enables focusing on function graphs so that every part of the graph can be seen. Tall [3] argues that computer programs are indispensable for demonstrating dynamic graphical visualization and providing precise numerical and symbolic calculations, which facilitate learning calculus derivatives. Students can visualize mathematical concepts through various graphical, numerical, and algebraic representations made possible by dynamic computer programs. Students can generate tables of values, input equations, and plot diverse functions. The dynamic nature of these mathematical objects enables students to visualize and comprehend mathematical problems or procedures in ways that traditional paper-and-pencil methods cannot [4].

Desmos Classroom (DC) is a dynamic web-based application with many advantages over other programs and applications. The program is free, user-friendly, intuitive, and extremely effective for graphic design, and encourages student participation and engagement in mathematics learning [5]. As a generic organizer, DC provides robust graphical and computational capabilities that facilitate an intuitive approach to assist students in acquiring the basic differentiation rules. Based on this justification, the researcher intended to investigate students' cognitive development in learning the basic differentiation rules using Desmos Classroom (DC) based on the TWM framework. Consequently, this case study research aims to address the following research questions:

(1) How is the students' cognitive development process in learning basic differentiation rules using DC based on the TWM framework?
(2) Can the learning activities using DC based on the TWM framework contribute to students' cognitive development?

## 2. Students' Cognitive Development According to the TWM Framework

Tall [6] developed the Three Worlds of Mathematics (TWM) framework that comprehensively analyzes the students' cognitive development as they acquire mathematical concepts. These three worlds operate and evolve differently, comprising distinct cognitive domains, as outlined below:

(1) Conceptual embodiment involves the student's perception and actions, the formation of mental images expressed in increasingly sophisticated verbal forms, and the development of an appropriate mental construct within the student's imagination.
(2) Proceptual symbolism evolves from physical actions to mathematical procedures symbolized and conceptualized as both operations to be performed and symbols that can be manipulated and calculated.
(3) Axiomatic formalism involves constructing formal knowledge within axiomatic systems using a suitable foundational framework.

The embodied approach to derivatives prioritizes forming derivative concepts through physical interactions with the function graph. It enables students to experience dynamic changes, arrive at accurate symbolic representations of a function's derivative, and make numerical estimations. According to Tall [7], the cognitive foundation of derivatives is local straightness. Local straightness is a fundamental concept of a small portion of a graph that will appear straight. Using this cognitive root, students can perceive the entire graph's slope. Figure 1 depicts how students can construct the tangent line by dragging the graph as an object and witnessing the slope changes.


*Figure 1. An Embodied Approach to the Derivative of a Function [8].*

It is essential to establish a connection between the embodied operation and the symbolic domain, which includes numerical and algebraic computations. Proceptual symbolism is the symbolic representation of functions and their derivatives [9]. According to Gray and Tall [2], a procept is a mental object consisting of a to-be-executed process and the concept that results from that process. In addition, proceptual thinking is defined as the capacity to manipulate symbols as both processes and concepts and switch between various symbols for the same mathematical object [2]. Using the same notation, the flexible and ambiguous use of symbolism to represent the dual nature of processes to be executed and concepts to be comprehended significantly improves proceptual thinking. As students advance, their internal depth, or "interiority," of the number concept grows, resulting in greater manipulation flexibility [11].

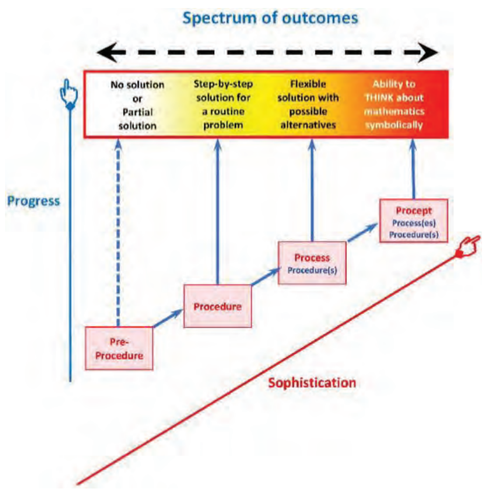



*Figure 2. The spectrum of students' cognitive development developed by Tall et al. [12].*

The spectrum of students' cognitive development outcomes at various levels of complexity are depicted in Figure 2. There are four levels on the spectrum: pre-procedure, procedure, process, and procept. This suggests that problems requiring only standard procedural responses will only serve to differentiate between students who make the transition successfully and those who do not. Multiple process paths provide alternative methods for identifying potential execution errors, including an innate awareness that something is wrong when an error occurs [13]. The procept level involves interpreting symbols as both processes to be carried out and concepts to ponder, facilitating the execution of dual processes and concepts.

## 3. Research Methods

In this study, the researcher adopted the constructivist paradigm to investigate the students' cognitive development when learning the basic differentiation rules using DC based on the TWM framework. Based on Stake's study [14], constructivism emphasizes the idea that knowledge is primarily determined by social interpretation and not by objective reality. The constructivist epistemology was used throughout the study since the researcher was able to closely interact with all research participants. The students' responses to the DC-based learning activities supported this constructivist viewpoint. This research involved 25 students enrolled in the Calculus 1 course at Sampoerna University during the fall semester of the 2022–2023 academic year.

The research design of this study was a case study, which was chosen for its suitability in answering "how" questions [15], [16]. The Desmos Classroom (DC) has been validated by experts in mathematics, mathematics education, and technology in education, and became the main research instrument. DC was used as a generic organizer based on the TWM framework to assist students in making sense of the derivative concept and applying the basic differentiation rules to solve related problems. Students' responses in the DC were analyzed to investigate their cognitive development while learning the basic differentiation rules (see Figure 2).

In this investigation, additional research instruments are utilized. Included here are the observation guidelines, and test questions. The following observation guidelines also developed by referring to the Three Worlds of Mathematics framework:

*Table 1: Observation Guidelines*

| Three Worlds of Mathematics | Sample of activity |
|---|---|
| *The conceptual-embodied world* | Students move the two points crossed by the secant line so that the two points are very close to each other and then students see the visualization of the tangent line.<br><br>Students zoom in on a graph near a particular point to see it looks like a straight line. |
| *The proceptual-symbolic world* | Determine the derivative of a function using the basic differentiation rules.<br><br>Find the slope of a tangent line.<br><br>Determine the equation of a tangent line. |

| | |
|---|---|
| *Axiomatic Formal* | Prove the basic differentiation rules. |

In terms of analysis, an interpretive theoretical approach was used to provide a framework for comprehending students' responses when learning the basic differentiation rules using DC based on the TWM framework. The methodology was inductive, and the results were predominantly descriptive [17]. Data analysis began with a comprehensive examination of the students' responses collected from DC. Each item's responses were thoroughly examined to identify emerging trends or themes. The classification of these themes according to the included descriptions allowed for a systematic organization and interpretation of the data [18].

To ascertain the outcomes of students' cognitive development, their responses to the DC activities and test-problem solutions were evaluated. Students' responses in DC revealed students' cognitive development when learning the basic differentiation rules. The purpose of the test questions (see Table 2) was to evaluate the students' cognitive development after learning the basic differentiation rules which refer to the spectrum of students' cognitive development presented in Figure 2.

*Table 2. The Test Questions*

| Problem | Expected Cognitive Outcome | Remarks |
|---|---|---|
| **Problem 1:**<br>Prove that $\frac{d}{dx}[\cos x\ ] = -\sin x$ | Axiomatic Formal | Students were expected to think in the axiomatic formal world by using formal definition of derivative to prove the derivative of cosine function. |
| **Problem 2:**<br>Find an equation of the parabola $\mathbf{y = ax^2 + bx + c}$ that passes through (0,1) and is tangent to the line $\mathbf{y = x - 1}$ at (1,0). | Procept | Students were expected to think in the proceptual world by performing the duality of mathematics-related processes and concepts to successfully solve the problem. |
| Create the graphical representation of the tangent line problem. | Graphical Representation | Students are expected to create a graphical representation of the tangent line problem. |

The data are analyzed using the spectrum of students' cognitive development outcomes devised by Tall et al. [12]. The validity of this case study research is achieved by integrating data sources that provide a comprehensive picture of the examined issue [19]. *Triangulation* involves the use of different methods to investigate the research questions and provides a means of checking validity [19]. Students' cognitive development while learning the basic differentiation rules was analyzed based on their responses or answers inputted in the Desmos Classroom. Students' verbal responses to the lecturer's instructions during the class sessions were also recorded and written on the observation sheet. To answer the second research question, the researcher also evaluated the students' solutions to two problems, requiring them to think in the axiomatic formal and proceptual worlds and produce graphical representations of the problems.

## 4. Results and Discussion

**How is the students' cognitive development process in learning basic differentiation rules using DC based on the TWM framework?**

In the previous meeting, students have experienced learning through embodied approach by exploring the graph of $f(x) = x^2$ and moving its tangent lines at several points. They plotted the slope values of the tangent line obtained from each value of $x$ (see Figures 3 and 4). The resulting plotting image is the graph of $y = 2x$, so they concluded that the derivative of $f(x) = x^2$ is $f'(x) = 2x$ (*symbolizing embodiment*). In addition, they have proven that $\frac{d}{dx}x^2 = 2x$ by applying the formal definition of the derivative ($f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ and provided the limit exists (see Figure 5).

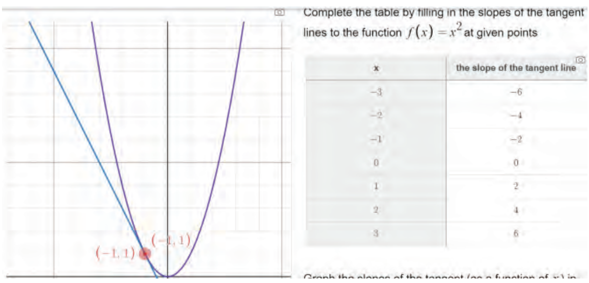



*Figure 3. Student O's responses regarding the slope of the tangent line of the function $f(x) = x^2$ for $x = -3, -2, -1, 0, 1, 2, 3$.*

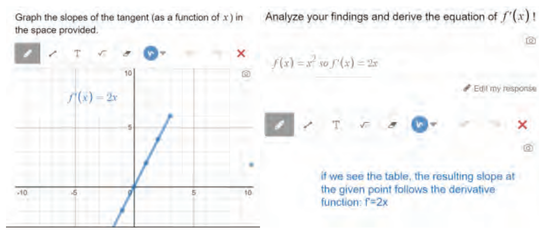



*Figure 4. Student H's responses regarding the plotting of the tangent slope values for each x value, and her conclusion.*

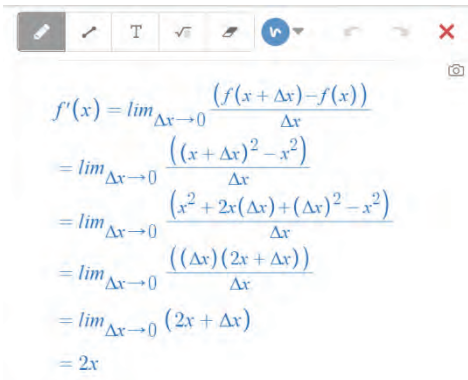



*Figure 5. Student B's response (applying the formal definition of derivatives to prove the derivative of $f(x) = x^2$)*

Since the beginning of learning activities, students were facilitated to see the visualization of the constant function and its derivative to identify that $f'(x) = 0$ (*symbolizing embodiment*). Next, students were thinking in the axiomatic formal world by proving that the derivative of a constant function $f(x) = c$ that is $f'(x) = 0$ (see Figure 6).

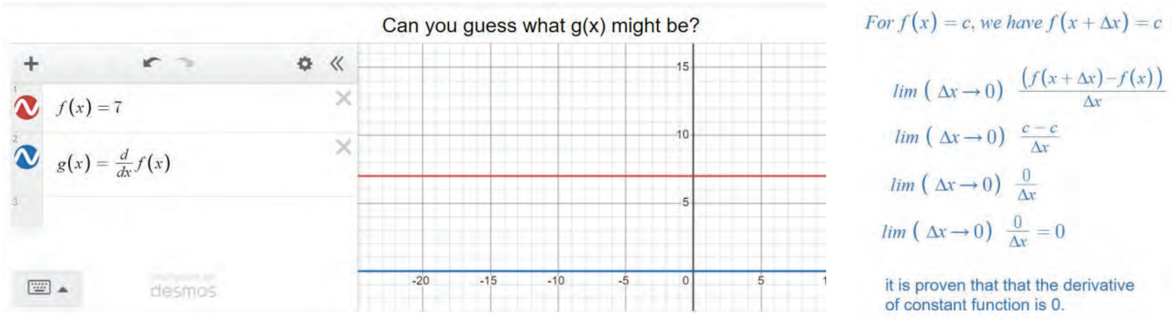
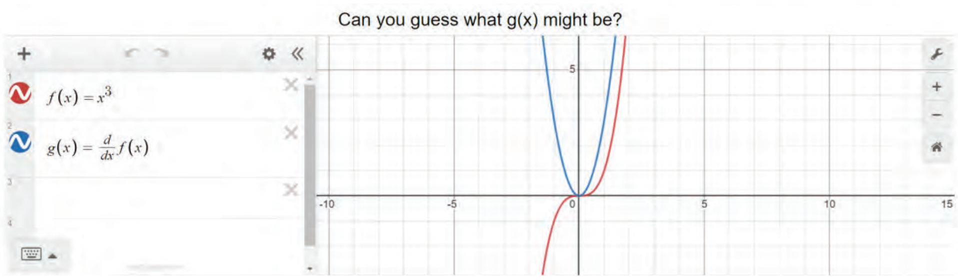
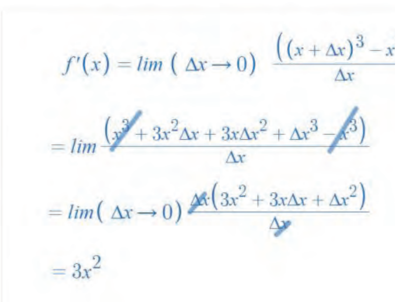
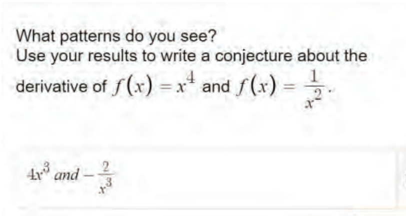



*Figure 6. Student O's response*

Similarly, students identify the derivative of the polynomial function $f(x) = x^3$ by visualizing the graph of $f'(x)$ (*symbolizing embodiment*). Next, they were rethinking in the axiomatic formal world by proving the derivative of $f(x) = x^3$, that is $f'(x) = 3x^2$ (see Figure 7). Students also prove the derivatives of polynomial functions: $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$. Students see graphical representations of the five polynomial functions and their derivatives (*embodying symbolism*). Then, they determine the derivative of the function $f(x) = x^4$ based on the pattern that emerges from the derivatives of the five polynomial functions (see Figure 8). Some students were able to generalize that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$ (see Figure 9).



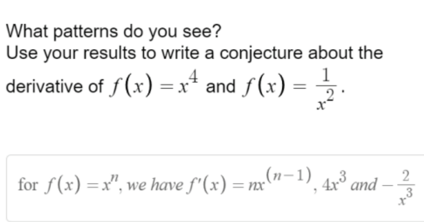




*Figure 7. Student BA's response*

*Figure 8. Student A's response*

*Figure 9. Student O's conclusion*

Students were rethinking in the proceptual world by determining the binomial expansion $(x + \Delta x)^3$ and they used the binomial expansion to the power of $n$, to verify the binomial expansion for $n$ = 2, 3, and 4. Furthermore, students were rethinking in the axiomatic formal world by proving that the derivative of the function $f(x) = x^n$ is $f'(x) = nx^{n-1}$ (*the power rule*). Then, they apply the formal definition of the derivative of a function and the binomial to the power of $n$ (see Figure 10).

$$\frac{d}{dx}\left[x^n\right] = lim\ (\ \Delta x \to 0)\ \frac{\left((x+\Delta x)^n - x^n\right)}{\Delta x}$$

$$lim\ (\ \Delta x \to 0)\ \frac{\left(x^n + nx^{(n-1)}\Delta x + .. + \Delta x^n\right) - x^n}{\Delta x}$$

$$lim\ (\ \Delta x \to 0)\ \frac{\Delta x\left(nx^{(n-1)} + \frac{n(n-1)x^{(n-2)}}{2}\Delta x\right.}{\Delta x}$$

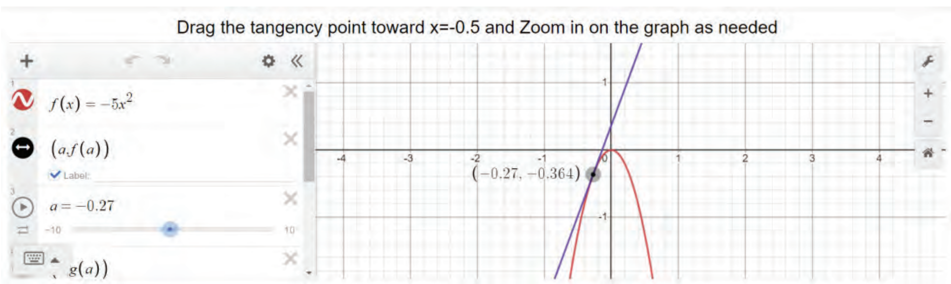
$$lim\ (\ \Delta x \to 0)\ nx^{(n-1)} + \frac{n(n-1)x^{(n-2)}}{2}\Delta x$$

$$= nx^{(n-1)} + 0 + 0 + \ldots$$

$$= nx^{(n-1)}$$

*Figure 10. Student N's response*

Students learn through the embodied approach by enlarging the graph of $f(x) = -5x^2$, moving the tangent line so that it touches $f(x)$ at $x = -0.5$. Students think in the proceptual world by obtaining $f'(x) = -10x$. Then they determine the tangent line's slope and eventually determine the equation of the tangent line (see Figure 11). Students also think in the axiomatic formal world by proving the sum rule, $\frac{d[f+g]}{dx} = f'(x) + g'(x)$ (see Figure 12).




$$f'(x) = -10x$$
$$f'(-0.5) = -10(-0.5) = 5$$
$$x_1 = -0.5 \quad y = -1.25 \qquad then\ y - y_1 = m\left(x - x_1\right)$$
$$y + 1.25 = 5(x + 0.5)$$
$$y + 1.25 = 5x + 2.5$$
$$y = 5x + 1.25$$

*Figure 11. Student H's responses*

$$lim\ (\ \Delta x \to 0)\ \frac{\left(f(x+\Delta x) + g(x+\Delta x)\right) - \left(f(x) + g(x)\right)}{\Delta x}$$
$$= lim\ (\ \Delta x \to 0)\ \frac{\left(f(x+\Delta x) - f(x)\right) + \left(g(x+\Delta x) - g(x)\right)}{\Delta x}$$
$$= lim\ (\ \Delta x \to 0)\ \frac{\left(f(x+\Delta x) - f(x)\right)}{\Delta x} +$$
$$lim\ (\ \Delta x \to 0)\ \frac{\left(g(x+\Delta x) - g(x)\right)}{\Delta x}$$
$$= f'(x) + g'(x)$$

*Figure 12. Student BA's responses*

Students learned through an embodied approach by moving the tangent line along the graph of the function $f(x) = \frac{\left(3x^7 - 5\sqrt{x} + 1\right)}{x}$, enlarging the visualization of $f(x)$ and its tangent line, and positioning the tangent line so that it touches the function at $x = 0.5$ (see Figure 13). Students were thinking in the proceptual world by using the power rule to obtain $f'(x) = 18x^5 + \frac{5}{2}x^{\frac{-3}{2}} - x^{-2}$. Students calculate the slope of the tangent line $m = f'(0.5) = 3.6335678$, and finally, they got the equation of the tangent line, which is $y = 3.6335678x - 6.8407839$.

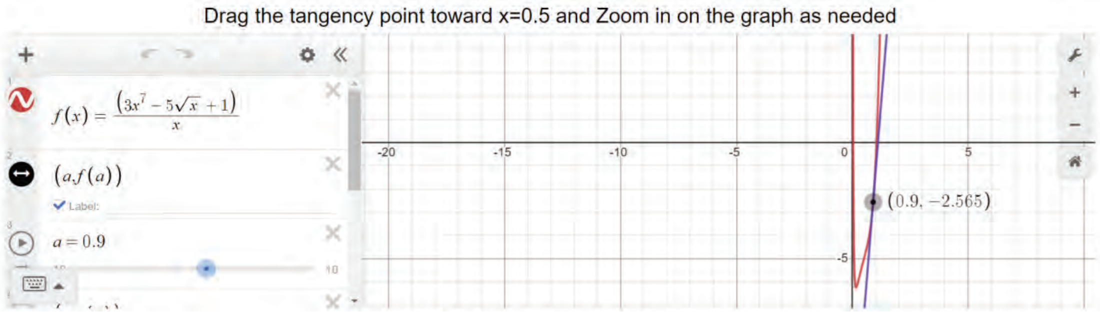



*Figure 13. Student BA's graphical exploration*

Students were facilitated to identify the derivatives of the sine and cosine functions by seeing the graphical visualization of each function and its derivative (*symbolizing embodiment*). Students were rethinking in the axiomatic formal world by proving that $\frac{d}{dx}[\sin x] = \cos x$ . The proof uses the formal definition of the derivative of the function $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, the trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$, and special limits, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (see Figure 14). Furthermore, students were rethinking in the proceptual world by applying the derivative rules of sine, cosine, and polynomial functions with degree $n$ to solve related problems.

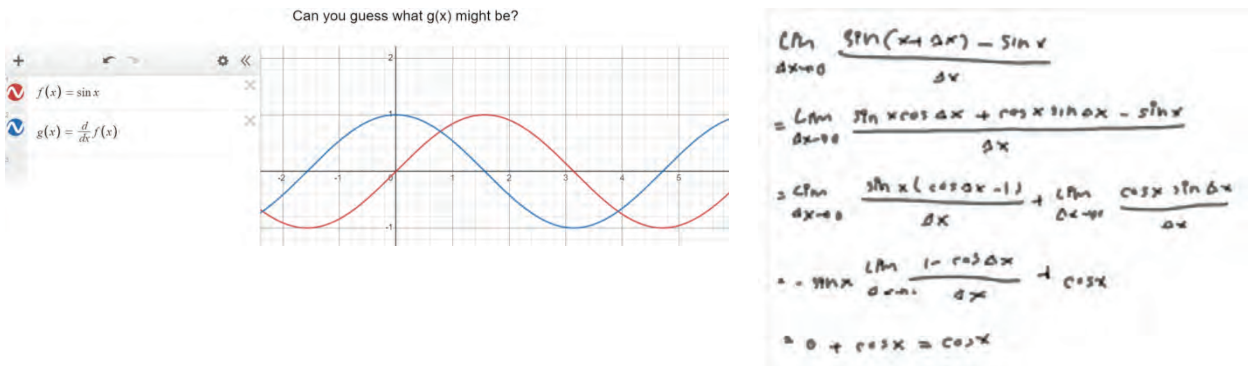



*Figure 14. Student N's responses*

Students were learning through an embodied approach by moving the tangent line to the function $f(x) = a \sin x$ to exactly touch the function at the point (0,0) and enlarging the display of the function and the tangent line (see Figure 15). Based on this activity, students concluded that the slope of the tangent line to $f(x) = a \sin x$ at (0,0) is equal to $f'(0)$ (see Figure 16).

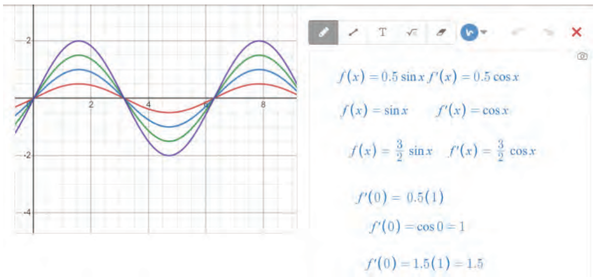



*Figure 15. Student H's Response*

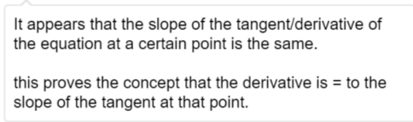



*Figure 16. Student V's Findings*

The learning process using DC based on the TWM framework focuses on developing students' perceptions of the basic differentiation rules and the derivatives as the slope of the tangent line. DC provides a graphical representation to assist students in identifying the function's derivative. After that, students think in the axiomatic formal world by proving the basic differentiation rules. They were also thinking in the proceptual world by applying the basic differentiation rules to solve related problems, facilitated through DC numerical and symbolic manipulation features. Additionally, the embodied learning activities were reflected by moving the tangent line along the function's domain until it was exactly tangent to the associated tangent point. At the same time, students also magnified the function and the tangent line (*embodying symbolism*). It reveals that DC can visualize the tangent line and compute its slope. These exploration activities assist students in finding the tangent point, the slope of the tangent line, and the tangent line's equation. According to Tall [1], learning activities that promote embodied calculus approaches, followed by numerical and symbolic manipulations, can aid students in cognitive development processes. In this study, students engaged in a series of iterative learning activities through DC that facilitated thinking in the embodied world through graphical exploration, thinking in the proceptual world by applying basic differentiation rules to solve related problems, and thinking in the axiomatic formal world by proving basic differentiation rules.

**Can the learning activities using DC based on the TWM framework contribute to students' cognitive development?**

In the first problem, it was found that all students succeeded in proving that the derivative of $f(x) = \cos x$ is $f^{'}(x) = -\sin x$. Thus, students showed their success in thinking in the formal axiomatic world. As can be seen from one of the following student solutions:

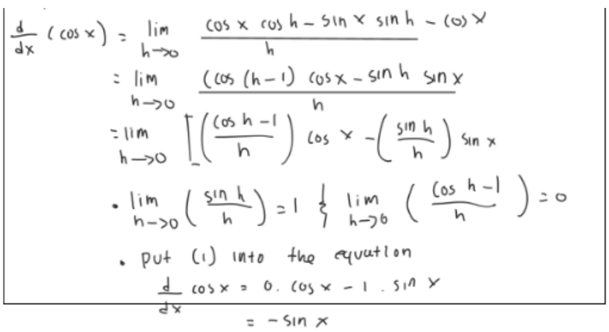


*Figure 17. Student Y's solutions for problem 1*

In the first step, Student Y used trigonometric identity to expand $\cos(x + \Delta x)$ into $\cos x \cos \Delta x - \sin x \sin \Delta x$ (see Figure 17). Next, she identified that there was a common factor, $\cos x$ which can be used to simplify the numerator to become $\cos x\,(\cos \Delta x - 1) - \sin x \sin \Delta x$cos. Then Student Y used the limit operation property to obtained $\lim\limits_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{\cos x\,(\cos\Delta x-1)}{\Delta x} - \lim\limits_{\Delta x\to 0}\frac{\sin x\sin\Delta x}{\Delta x}$. In the end, she used two special limits $\lim\limits_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x} = 0$ and $\lim\limits_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = 1$, and she proved that $\frac{d}{dx}[\cos x\,] = -\sin x$.

In the second problem, students are expected to think in the proceptual world by obtaining the equation of a parabola $y = ax^2 + bx + c$ which passes through the point (0,1) and tangent to the line $y = x - 1$ at the point (1,0). It was found that 23 out of 25 students (92%) could think in the procept spectrum when solving the tangent problem. As seen in Student J's solution (see Figure 18), he obtained the slope of the tangent line $y = x - 1$, which is $m = 1$. Next, he substituted $x = 0$ and $y = 1$ into the parabolic equation, and then he obtained $c = 1$. In the next step, he looked for the derivative of the parabolic function, which is $y' = 2ax + b$. Student J utilized the value of the parabolic's derivative at $x = 1$ , which is equal to $m = 1$, and he got $2a + b = 1$. After that, Student J substituted $c = 1, x = 1$ and $y = 0$ into the parabolic equation which resulting the second equation, $a + b = 0$. He eliminated the two equations and obtained $a = 2$ and $b = -3$. In the last step, Student J successfully obtained the parabolic equation $y = 2x^2 - 3x + 1$ , and created a graphical representation of the tangent line problem.

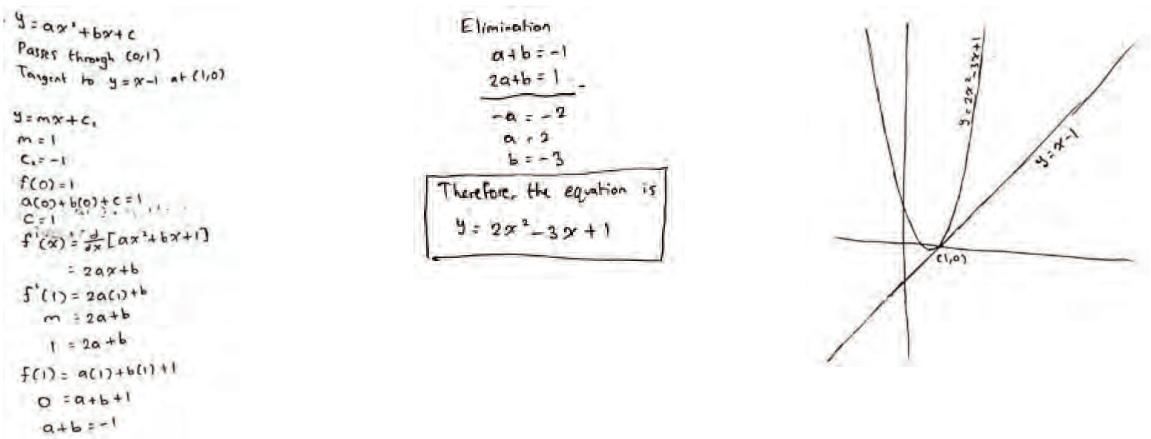



*Figure 18. Student J's solution for problem 2*

Student JA and Student J are two students who did not reach the procept spectrum when answering the second question. Student JA incorrectly interpreted the value of $b$ in the parabolic equation $y = ax^2 + bx + c$. He thought that the slope of the tangent line $y = x - 1$ is equal to $m = 1 = b$. As a result, he obtained incorrect results, such as the equation$1 = 2a(1) + b$, the values of $a$ and $b$, and even the parabola equation. Subsequently, he failed to draw the graph of the parabola. In addition, the error made by Student M was that he thought that the tangent line also passing through the point (0,1), so he substituted $x = 0$ and $y = 1$ into $y' = 2ax + b$. This led him to conclude that $y' = m = 1 = b$. This incorrect result made him fail to obtain the parabola equation and create a graphical representation of the problem (see Figure 19).

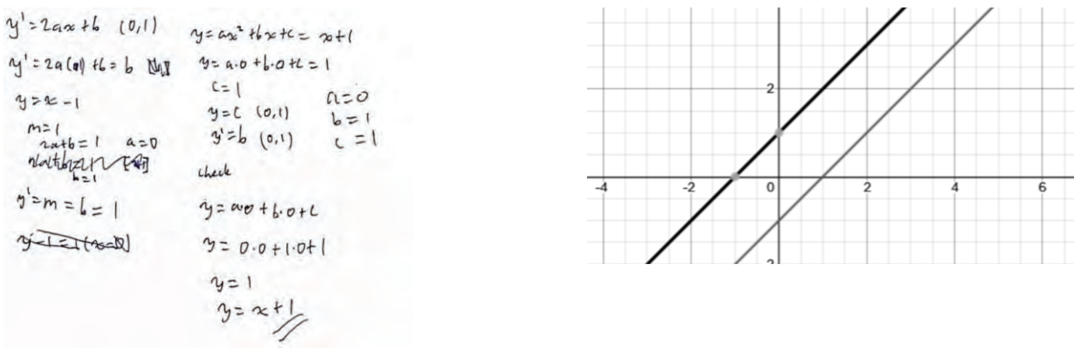


*Figure 19. Student M's answer*

Besides Student JA and Student M, 7 other students also did not produce the graphical representation of the tangent line problem (problem 2). Unlike Student JA and Student M, the seven students have obtained the parabolic equation. Thus success in thinking in the proceptual world does not imply success in the graphical representation. The students made various mistakes, and it was indicated Student S did not even try to produce the graphical representation, Student R did not succeed in drawing the graph of the parabola $y = 2x^2 - 3x + 1$, and the remaining five students (Student F, Student Y, Student A, Student MA, Student AU) could draw the graph of the parabola but did not draw the tangent line $y = x - 1$.

The general description of the students' cognitive spectrum outcomes in solving problems related to the basic differentiation rules is depicted by the following Venn diagram:

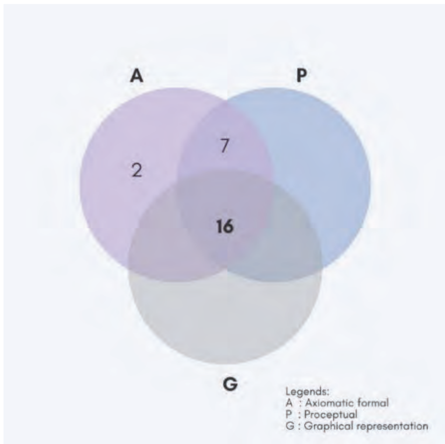



*Figure 20. Students' Cognitive Development*

- All students could think in the axiomatic formal world by proving the derivative of the function $f(x) = \cos x$.
- Two students succeeded in thinking in the axiomatic formal world but failed to think in the proceptual world. They also had difficulty creating a graphical representation of the given tangent line problem. It is because they failed to obtain the parabolic equation $y = ax^2 + bx + c$. Thus, success in thinking in the formal axiomatic world does not imply success in the proceptual world.

- Most students (23 out of 25 students, or about 92%) succeeded in thinking in the proceptual world by determining the equation of the parabola $y = ax^2 + bx + c$ through the point (0,1) and tangent to the line $y = x - 1$ at the point (1,0).
- 7 students succeeded in thinking in the proceptual world but failed to draw the graph of the parabola and the tangent line $y = x - 1$. In addition, there were 2 other students (Student JA and Student M) who were unsuccessful both in thinking in the proceptual world and in producing the graphical representation. It can be concluded that success in thinking in the proceptual world does not imply success in making the graphical representation.
- Some types of errors that were identified when students tried to create the graphical representation of the tangent line problem were: (1) Student S did not produce a graphical representation; (2) Student R did not succeed in drawing the graph of the parabola $y = 2x^2 - 3x + 1$ correctly, and (3) five others (Student F, Student Y, Student A, Student MA, Student AU) did not draw the tangent line $y = x - 1$.
- About 64% of the students (16 people) could draw the graphical representation of the parabola and the tangent line $y = x - 1$. They were also consistently successful in thinking in the formal axiomatic world and the proceptual world.

This study revealed that DC based on the TWM as a generic organizer can contribute to students' cognitive development while learning the basic differentiation rules. Although very few students struggle to think in certain cognitive worlds. This suggests that success in one cognitive world does not guarantee success in another. However, all students in this study could think in the axiomatic formal world by proving the derivative of the function. Most students (92%) succeeded in thinking in the proceptual world when solving the tangent line problem. Also, many students (64%) had no limitations in the graphical representations of the given problem. This further supports the idea that the learning activities using DC based on the TWM framework contribute to students' cognitive development. These findings were consistent with those reported in Tall's study [10],[20], which also discovered that students' capacity to sketch gradients of supplied graphs considerably increased and that their conceptualizations were extended while learning through a computer program.

## 5. Conclusion

This study has shown that the DC based on the TWM framework can be used as a generic organizer to build students' perceptual meaning of the derivatives and the basic differentiation rules. The DC based on the TWM provides recurrent activity cycles that include graphical exploration for derivative conceptual embodiment, numerical computations, and symbolic manipulations. All these activities can contribute to students' cognitive development. About 64% of the students were consistently successful in thinking in the axiomatic formal world and the proceptual world and could draw the graphical representation of the line problem. All students in this study were successfully thinking in the axiomatic formal world by proving the derivative of $\cos x$. Most (92%) also succeeded in thinking in the proceptual world when solving the tangent line problem. However, some students struggled with creating graphical representations of the tangent line problem. This suggests that success in one world does not guarantee success in another world. Therefore, lecturers should incorporate activities that target each cognitive world to support students' cognitive development.

This case study involved 25 students as research participants. The researchers suggest further research with a larger sample size to generalize the findings of this study. Utilizing DC based on the TWM framework has the potential to be implemented and studied further in other calculus topics. One is by graphically exploring the problem of higher-order derivatives of transcendental functions, which is not easy for students to guess. The extent to which DC as a generic organizer based on the TWM framework can help students make sense of the concepts of various calculus topics (embodied), gradually explore the proceptual world, and finally understand these topics symbolically (procept) can be studied holistically.

## Acknowledgements

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# Debugging on GeoGebra-based Mathematics+Computational Thinking lessons

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**Abstract**: *Computational thinking (CT) has become a buzzword recently and gained more attention from countries and researchers. Researchers realize the importance of CT and integrate it into school subjects such as mathematics, science, language, and others. Our research tries to contribute to the plugged CT activities under mathematics subjects. Collaborating with mathematics teachers, principals, and a teacher trainer, we developed a sequence of lessons in GeoGebra. Our lessons integrate CT's facets in the topic of the area of a circle. The development of the GeoGebra-based Mathematics-CT lessons incorporated educational design research methodology. We improved our lessons and implemented them for a few students. In this paper, we focused only on the debugging skill being supported by GeoGebra. Our findings show that fixing commands can be challenging as students have been through several debugging, and it can be complicated if the errors are many. This paper shows the power of GeoGebra to learn integrated CT in mathematics lessons through creating objects and debugging the program.*

## 1. Introduction

This paper is part of a larger study of integrating computational thinking in mathematics education using readily available mathematics software for mathematics teachers. Many countries have taken action to make Computational Thinking (CT) available in the school curriculum [1] and prepare their in-service and pre-service teachers with CT knowledge and skills [2], [3]. The Ministry of Education, Culture, Research, and Technology (MoERT) of Indonesia also followed the movement and developed the computer science (CS) school subject that was unavailable previously. Students will learn CT from this CS subject. Wing [4] argued that CT could be developed not only in CS courses. To contribute to this movement, we support CT through mathematics subjects. GeoGebra is a free open-source mathematics software that allows teachers and students to use it online or offline.

As CT is relatively new in education, in order to be successfully implemented, mathematics teachers need support with exemplary activities of integrating CT into mathematics lessons [5]. Our research is to develop GeoGebra-based mathematics-CT lessons for junior high school students. GeoGebra is a powerful, interactive geometry, algebra, statistics, and calculus software, intended for learning and teaching mathematics and science from primary. There are a few studies that used GeoGebra to integrate CT into mathematics lessons [6], [7] Van Borkulo et al. [6] found that GeoGebra could support CT's algorithmic thinking and generalization aspects. We want to investigate this issue more and enrich our understanding of how GeoGebra can support mathematics and CT.

Through the educational design research approach [8], the researchers collaborated with mathematics teachers, principals, and a teacher trainer to develop GeoGebra-based mathematics-CT lessons. We refer to the framework by Shute et al. [9] which improved the previous existing framework by other researchers and gave us more operationalised CT facets to be developed in our lessons.

We developed the lessons based on the Indonesian curriculum mathematics content, determining the area of a circle and its related problems. The area of a regular polygon with many sides can be used to approach the area of a circle as per Archimedes' method of exhaustion see [10]. Hence, we developed two lessons which consisted of constructing a regular polygon to inscribe in a circle. Unlike the activity proposed by King [10] that explored the circle's circumference and its formula, our study focuses on the area of a unit circle by approaching it with an inscribed polygon. Students will program to construct a manipulatable polygon inside the unit circle. We intentionally allow our students program to construct an inscribed polygon on a circle on GeoGebra and debug errors. The debugging facet will be the focus of this paper.

## 2. Theoretical Framework

Our study is based on the constructionist theory by Papert [11] who introduced computational thinking in education, especially in mathematics education. His idea of computational thinking is about interacting with a learning environment in the computer to learn or access knowledge. Papert proposed a design framework that encompasses activity engagement, ownership of ideas and learning style, and exposure. The GeoGebra-based mathematics and CT lessons were designed for students to construct objects (engagement), arrange the commands based on their ideas (ownership of ideas) and work on similar tasks or sequences (exposure).

Papert was inspired by the way Computer scientists solved difficult problems using LISP programming and he developed a similar tool for young learners to do mathematics with it [12]. This inspired Papert and his colleagues to develop LOGO, a similar tool mentioned earlier that later evolved into Turtle Geometry. Young learners used this tool to construct geometric figures. Our study incorporated GeoGebra to develop an environment where students can input their commands to construct objects and learn mathematical concepts or solve problems. GeoGebra can be used as a programming tool by its input box feature and scripts or commands. We intentionally hid the drawing or construction tools by displaying only the cursor/pointer or move tool (Figure 2. 1). We also developed the modified input box and hidden 'Algebra View' on the left side of the GeoGebra window. Students could edit, delete, or insert a command if they needed to do it.

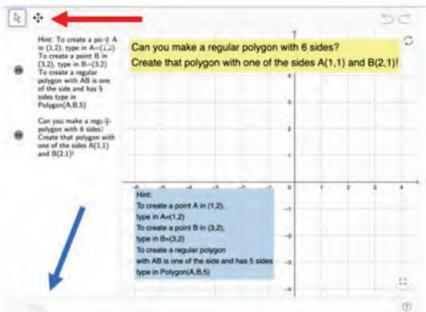


Figure 2. 1 The GeoGebra environment setting for Math-CT activity

We allowed students to interact with this GeoGebra environment and create objects, in this case, a polygon step by step, creating points and then the number of the sides of the polygon. Later, we expect them to explore the slider to observe the change in the polygon's area. In the end, students would create a unit circle with a manipulatable polygon inside it to approach the area of the circle.

## 3. Methodology

We use the educational design research (EDR) approach [8] in order to understand whether our design activity works or not and why it works or does not work. Through the iterative process of the EDR, we collaborated with mathematics teachers, principals, and a teacher educator to develop GeoGebra-based mathematics-CT lessons. We refined our activities based on our online meetings, and reflection from the pilot study. We tried our lessons with seventeen junior high school students to see how our lessons worked.

**The framework.** In this section, we refer to the CT framework by Shute et al. [9] involves six facets, which are decomposition, abstraction, algorithms, debugging, iteration, and generalisation. These facets and their operationalised definitions helped us to design our lessons. We will only describe the debugging facet in this paper due to page limitations. Debugging requires students to do systematic testing and modification. It can be seen when students can detect and identify errors, and then fix the errors. Additionally, when the solution or program does not work, they can identify, and fix the errors to make it work.

Our study promoted debugging by fixing the program to construct objects (polygon, circle, or inscribed polygon on a circle). In the debugging problem, we created a fictional character Andi who has written a program. The students should detect and identify whether Andi's command can be executed to successfully construct the intended objects. We allow students to learn how to properly use the commands or syntax to construct objects (point, polygon, slider, circle, etc.). For instance, to construct a circle with a center in A(1,1) and a radius of 3, the correct commands should be A=(1,1), Circle(A,3).

If they do not follow these commands, students will not be able to construct the mentioned circle. After they learned how to program using the correct commands or syntax, they would be exposed to incorrect commands. Later, they were challenged to recognise if the provided commands contained errors, and they had to fix the errors. Therefore, our debugging tasks followed the debugging facet.

**Debugging.** Our study wants to contribute to debugging practices as in [13] that debugging needs more attention from researchers. Debugging can form a remedial activity after writing codes or a purposively designed activity for students to explain, find, and fix bugs [14]. Our debugging tasks were more into the second form of debugging as students were required to find the errors of the existing commands (Andi's command). Visual programming tends to provide little space for debugging as it was not designed to do so [15], and block programming prevents syntax errors from happening [16]. Thus, our GeoGebra-based mathematics+CT, non-visual programming, could allow students to make and correct errors.

To detect errors, the programming tools notify the programmers directly and specifically to which part causes the errors. Robertson et al. [17] differentiate these notifications as styles of interruption (Table 3. 1). They are negotiated-style interruption and immediate-style interruption [17]. They describe that negotiated-style interruption when the programmer gets a pending message of the errors so that they will know it later. Meanwhile, the immediate-style intervention; such as a pop-up window, informs the programmer to take action immediately.

Table 3. 1 Styles of interruption in debugging

| No | Command | Types of Interruption | Description |
|---|---|---|---|
| 1 | A=(1:1) | negotiated-style interruption | Students will not see this as error as GeoGebra will run this and create a slider. |
| 2 | Circle(A,3) | immediate-style interruption | Students will get a pop-up where the command 'Circle' is not known. Thus, GeoGebra cannot make a circle as the command missed the letter 'r'. Students must immediately revise the command. |

**The context.** There are 33 tasks in the GeoGebra lessons that we developed. The lessons can be found here *https://www.geogebra.org/classroom/g7agtjdh*. The tasks that involve GeoGebra manipulations are Task 1, 5, 8, 12, 15, 17, 20, 23, 27, 31, and 33. The debugging tasks are tasks 12, 17, and 23. The goal of all the tasks is to let the students understand the approximation of pi using the area of a polygon inscribed in a circle and the area formula of a circle.

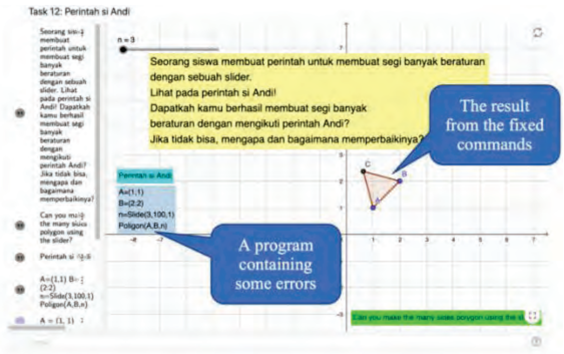


Figure 3. 1 Task 12 display

Task 12 was designed to promote debugging skills for students on the manipulatable polygon construction. Students were provided with a list of commands which included some errors. The errors started with up to three script errors. We coded the errors from what students did to solve the debugging tasks. In task 12, regarding the debugging activity, (Figure 3. 1), students must determine whether they can follow Andi's command to make a manipulatable polygon. If students cannot follow the command, then they must fix it. Three (3) errors are contained in Andi's command for this task. The errors are namely, using a dot instead of a comma for making a point, the missing letter, and the typographical error when he used the letter 'i' instead of the letter 'y'. There are four rows or lines on Andi's command, and the three are incorrect (Table 3. 2).

Table 3. 2 Errors on Task 12

| Line | Correct Command | Andi's command | Description |
|---|---|---|---|
| 1 | A=(1,1) | A=(1,1) | |
| 2 | B=(2,2) | B=(2:2) | The use of colon : instead of comma , |
| 3 | n=Slide(3,100,1) | n=Slide(3,100,1) | The missing letter: r |
| 4 | Polygon(A,B,n) | Poligon(A,B,n) | Case-sensitivity. The use of i instead of y |

The errors are in the rows 2, 3, and 4. Students had to investigate whether Andi's command was correct or not. They also had to fix the command to make it right. Students could produce a manipulatable polygon from an equilateral triangle to a 100-gon by fixing the commands' errors. The slider named n can be moved from 3 to 100 to construct the desired polygon.

Task 23 required students to do debugging from more complicated commands (Figure 3. 2). It had more errors found in rows 4,5,6,7 and errors in mathematical concepts such as the internal angle. Students had to debug n=5, a=36deg/n, Angle(A,B,a), and Polygon(B,B',a) to be n=4, a=360deg/n, Angle(B,A,a), and Polygon(B,B',n) respectively.

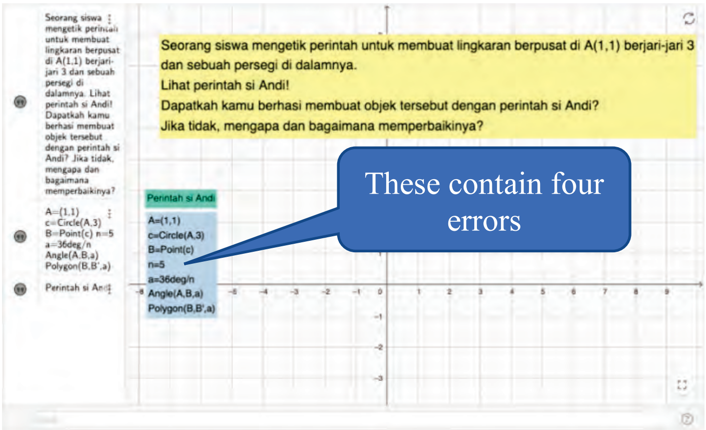


Figure 3. 2 Task 23 display

**Data Analysis.** The data were collected through video screen recording when students worked on the lesson activities as well as on students' GeoGebra files. The GeoGebra files might not show all the processes, as students might have deleted what they typed in. Thus, we could see it from the video recording. We used content analysis by Krippendor [18] to analyse the videos by making, categorising, and concluding the codes. We limit the result by using a screen video recording of one student from the pilot study. Additionally, due to page limitations, we can only show some tasks. The tasks that we presented here are related to debugging only.

## 4. Result

For task 12, the majority (12 students) could answer it successfully. It seems that at least students could fix the command for creating a point (Table 4. 1). However, in further investigation, the point command could be troublesome for some students, and they would have noticed it in the immediate-type of interruption (see Figure 4. 5).

Table 4. 1 Coding for students' responses

| Description | Code | Number of students |
|---|---|---|
| Inputted the correct commands | IC | 12 |
| Only made point A | P | 3 |
| Empty | E | 1 |
| Made some correct commands and did not continue | SC | 1 |

These are some mistakes we found in our students (Table 4. 1). They could successfully make point A and failed to make point B. If students followed Andi's command, then when they made a point B, they would not produce the correct point B. They created a slider B instead (Figure 4. 1). We coded this as making only point A."P" stands for Point, as the students produced no other objects visible on the grid.

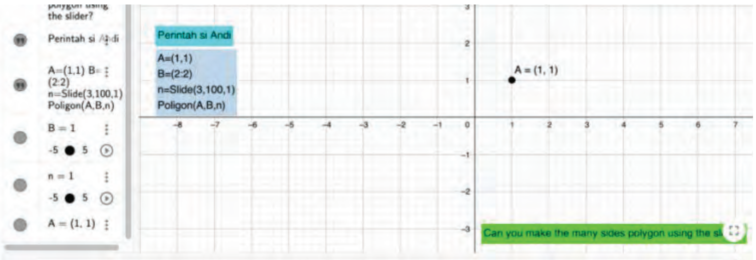


Figure 4. 1 A student made a B as a slider instead of a point

Other students also made mistakes by trying another way, such as B=(2 2) or B=(2<space>2), resulting in a number 4. In the end, students did not produce point B or other objects as they failed to debug the next errors. We coded this as making only a point A, "P", as the students had no other objects visible on the grid.

Some students stopped at a certain command after fixing or inputting the corrected commands. They successfully made some corrections, and the last command for the polygon was missing. This code belongs to making some correct commands and discontinuing to complete other commands, "SC" stands for some correct.

The successful students could make a desired object not by following Andi's command. They fixed the incorrect commands and resulted in the manipulatable polygon (Figure 4. 2). We coded this as an inputted correct command, "IC" stands for Inputting Correct. Other students did not do anything. We coded it as "E".

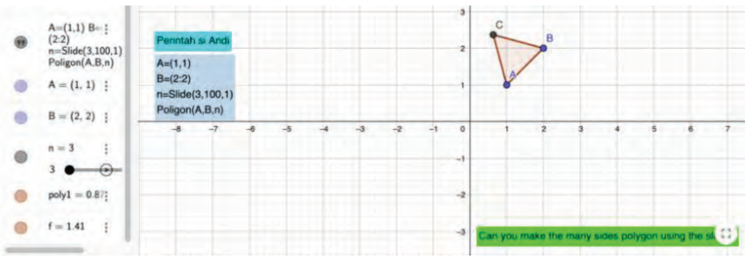


Figure 4. 2 A student successfully fixed and ran the program

The following paragraphs describe how one student in our pilot study worked on Task 12. This target student used a laptop to solve the task. When this student typed in B=(2.2), it worked but produced a slider B (Figure 4. 3). The student should have paid attention if B is a point or not, as it needs to appear on the grid, such as point A. This student then continued to type in n=Slide(3,100,1), and then it did not work; instead, the pop-up error showed up. Then, he decided to answer "No" as he could not follow Andi's command to make the object successful on Task 13.

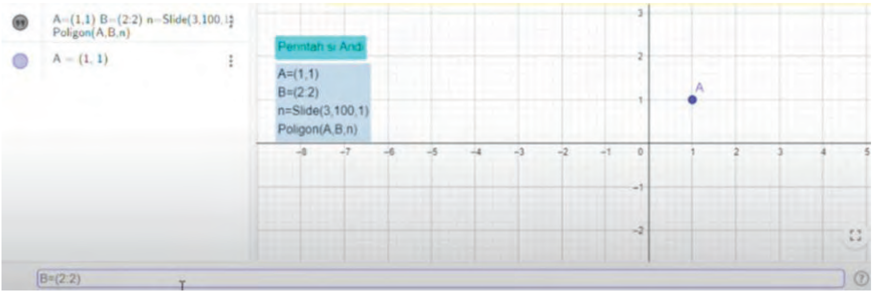


Figure 4. 3 Typing in B=(2.2) as in Andi's command as a negotiated-style interruption

He deleted point B and then typed in the correct command B=(2,2). Thus, point B appeared on the grid. He continued to type in n=Slide(3,100,1), and again, the pop-up error appeared. He deleted the n command and then typed in B=(2,1) which was not the correct command.

He then typed in n=slide(3,100,10), and again, the pop-up error showed up due to the small letter s and the missing letter “r”. He then edited the letter “s” to be “S”, but again, the pop-up error showed up due to the missing “r”. He deleted n and went to the previous task to help him. After seeing the command in the previous task, he continued with the correct n=Slider(3,100,1).

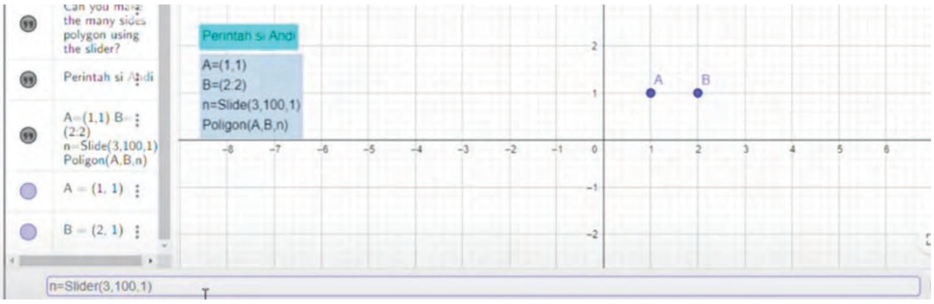


Figure 4. 4 A student made a point B incorrectly

Surprisingly, he deleted the command ‘n=Slider(3,100,1)’ to edit B as B=(2,2). But he typed in B=(2,1) which was incorrect (Figure 4. 4). He continued to make n=Slider(3,100,1) successfully. Later, he revised B into a slider as he typed in B=(2:2). Next, this student made the polygon by typing in Poligon(A,B,n), and the pop-up error showed up. He realized his error and revised the letter i into y to become Polygon(A,B,n). After typing in the correct polygon command, the pop-up error showed up to notify that B was a problem (Figure 4. 5). B is considered an illegal argument: Number B. It should be a point, not a number

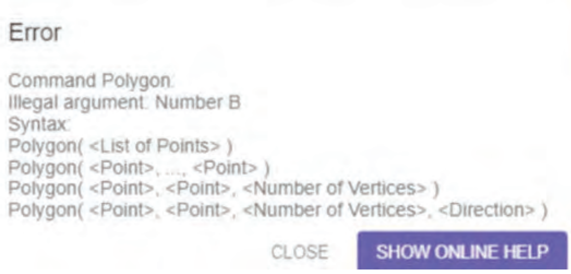


Figure 4. 5 An immediate-style interruption for the polygon to revise the B

So, he deleted the polygon command, then deleted the incorrect B and typed in B=(2,2). He typed in again Poligon(A,B,n) and a pop-up error showed up again and edited the command into the correct one ‘Polygon(A,B,n)’.

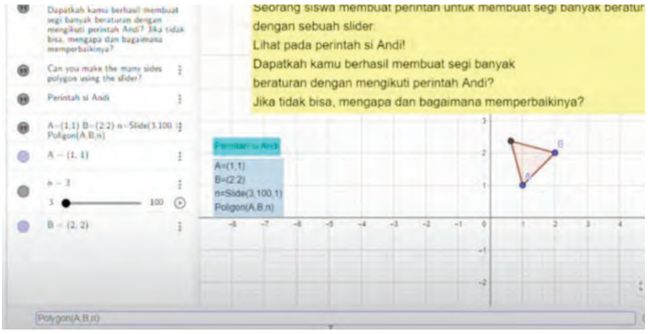


Figure 4. 6 A student successfully fixed the program

This student could make the commands run and produce the manipulatable polygon (a polygon that can be changed its number of sides by moving the slider). They could write down what they fixed. When a student engaged with task 23, he arrived at the correct commands and objects after a long process. This student spent more time accomplishing this task due to the order. The order here is the letter sequenced inside the command, for this case Angle(A,B,a). It confused this student to fix it until he checked the previous task to see the correct order. He then successfully made a square inside the circle (Figure 4. 7).

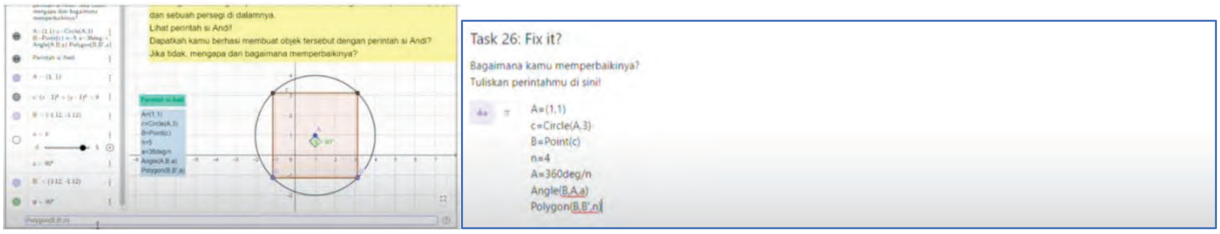


Figure 4. 7 Recognising where the errors are and the correct commands

## 5. Discussion

Papert argues that “learning consists of building up a set of materials and tools that one can handle and manipulate” [11, p. 173]. Thus, we let the students work individually on their own devices to complete all the tasks. We will discuss how our GeoGebra applet allowed students to handle and manipulate their programs and commands.

We let students have a learning sequence: learn to program, debug, and create the program. When students gained experience in debugging, they could recognise common errors (clichés) [19]. In line with this, most students could solve the debugging tasks as they get accustomed to how to fix the errors. This study witnessed students’ engagement in debugging as in [11].

Historically, programming to develop mathematical skills started in the 1970s with LOGO programming [20]. Some factors hinder LOGO's success and sustainability in education, as students and teachers have difficulty typing syntax in [21]. As we used a similar approach to LOGO, using syntax to create objects on GeoGebra, students experienced such difficulties. LOGO provided limited feedback to help learners find and reflect on their errors. Meanwhile, GeoGebra has a pop-up "debugging feature" that will appear if the syntax or command is incorrect. This has helped students in our pilot study to fix the errors.

The pop-up window in our study is related to types of interruption. Our error notifications belong to negotiated-style interruption and immediate-style interruption [17]. GeoGebra will not notify if A=(1:1) is an error immediately as it will be a slider, not a point A(1,1). Later it will become an immediate-style interruption when students cannot make a circle with the A, as it is not a point, and then a pop-up window notifies them that A needs to be fixed to be a point. Robertson et al [17] showed that the immediate-style interruption is less effective than the negotiated one. In our pilot study, students benefited from the pop-up window to fix the errors as they were new to this programming and debugging tasks.

The constructionist design could foster students' creativity and meaning-making in mathematical concepts when students create digital artefacts [22], [23]. In this case, students could utilize the mathematical concepts and observe the behaviour or characteristics. Our GeoGebra-based mathematics+CT tasks contain concepts of a Cartesian coordinate, a regular polygon, a circle, an angle, and an area of 2D shapes. It seems that the student in our pilot study could not connect the concept of Cartesian coordinate as a pair of x and y written in a standard form. It is relevant to a study by Ginat and Shmallo in [24], who found that in text-based programming, the student's lack of understanding of the underlying concepts could lead them to errors instead of syntax. It took our students some time to realize the universal form of a coordinate point as (x,y).

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# Development of an Online Training Program for Mathematics Teachers in Using GeoGebra

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**Abstract**: *For years, the Institute for the Promotion of Teaching Science and Technology (IPST), serving as the GeoGebra Center of Thailand, has been providing onsite training to teachers on using GeoGebra for mathematics instruction. However, due to the COVID-19 pandemic, the traditional onsite training has been replaced with an online alternative. The main objectives of this research were: (1) to develop an online training program for teachers focused on using GeoGebra for mathematics teaching and (2) to explore the satisfaction level of teachers who participated in the developed online training program. The study included 108 mathematics teachers as subjects, all of whom took part in the developed online training. The findings indicate that, overall, the participating teachers expressed a high level of satisfaction with the online training in using GeoGebra for mathematics instruction.*

## 1. Introduction

GeoGebra, as a dynamic software, serves as a valuable resource for supporting mathematics instruction [1]. Its capabilities make it a powerful tool in the classroom, enabling the creation of dynamic visualizations and interactive demonstrations that effectively illustrate abstract mathematical concepts [2]. Moreover, GeoGebra has proven to be highly effective in facilitating students' learning experiences. By enhancing their understanding of mathematical concepts, promoting active engagement, and fostering motivation [3], it empowers students in their pursuit of learning mathematics.

For years, the Institute for the Promotion of Teaching Science and Technology (IPST), operating under the Ministry of Education in Thailand, has served as the GeoGebra Center of the country. It offered onsite training to educators, emphasizing core tools (geometry, algebra, statistics) and practical classroom use. Unfortunately, the traditional onsite training had to be discontinued due to the COVID-19 pandemic, despite the continued need for mathematics teachers to acquire proficiency in using GeoGebra. As a result, IPST was compelled to shift its focus and develop an online training program to meet the demand for utilizing GeoGebra for mathematics instruction. This adaptation allowed teachers to continue their professional development in GeoGebra while ensuring safety and accessibility during these challenging times.

In 2022, IPST created an online training program tailored to teachers seeking to employ GeoGebra in mathematics instruction. This online program took inspiration from IPST's prior successful onsite training. The primary objectives of the new online training program remained similar to those of the onsite program, aiming to equip teachers with effective strategies for utilizing GeoGebra in mathematics instruction.

However, recognizing the unique challenges and opportunities presented by the online medium, the content, activities, and pedagogies of the training were thoughtfully enhanced to suit the virtual environment. Throughout the developmental process, the feedback and insights from participating teachers and researchers were diligently collected and analyzed. These valuable inputs played a crucial role in refining and optimizing the online training program, ensuring that it catered to the needs and preferences of the teachers and effectively fostered their proficiency in implementing GeoGebra for mathematics instruction.

In summary, this study aimed to achieve two main goals: (1) developing an online training program for adept GeoGebra usage in math teaching and (2) assessing teachers' satisfaction with this program. The next section delves into the literature review.

## 2. Relevant Literature

GeoGebra, a dynamic software with distinct benefits for math instruction, stands as user-friendly freeware accessible to all. Particularly, its impact on students is profound; it visualizes abstract math concepts, enhancing understanding [4], encouraging exploration, and idea development. Additionally, teachers find GeoGebra advantageous in classroom pedagogy. This is demonstrated by its motivating impact on students [4] [5], offering dual roles: a teaching aid and a platform for student-led math exploration [5] [6]. These benefits foster a positive learning atmosphere, generating favorable perceptions of GeoGebra among educators and learners alike [6] [7]. Consequently, IPST has chosen GeoGebra as the tool to train Thai math teachers, equipping them to proficiently wield its basic functions and integrate it into instruction.

Amidst the COVID-19 pandemic, online training has emerged as a highly appropriate alternative, offering numerous advantages. One of the key benefits is its ability to mitigate virus exposure, ensuring the safety of participants [8]. Moreover, online training significantly expands access to learning opportunities [8]. Teachers from all regions of the country can easily attend training sessions, eliminating the need for travel expenses or accommodation when the training is held far from their homes.

Additionally, online training provides participants with enhanced flexibility [9]. They have the freedom to join the training from their preferred location, whether it be their home, office, or even a coffee shop, as long as they have internet access. This added convenience allows for a more comfortable and personalized learning experience, contributing to a more effective training process.

Nonetheless, conducting online training comes with its share of challenges. Research has indicated that a common drawback of online training is the absence of face-to-face communication and interaction, leading to potential difficulties in participants fully grasping the presented content [9]. Additionally, motivating participants in the online setting can be challenging [8]. Some individuals may attend the training from unsuitable locations or during inconvenient times, such as when they are at work, resulting in distractions that hinder their focus and engagement with the training activities. These factors can collectively impact the effectiveness and overall learning experience during online training sessions.

In the pursuit of developing a new online training program to help teachers effectively utilize GeoGebra for mathematics instruction, the researchers recognized the significance of two key aspects: promoting face-to-face communication and enhancing participants' motivation towards the training. Drawing from previous studies, the researchers discovered that synchronous online training held the potential to address these concerns [10]. By implementing synchronous online training, all participants engage in real-time activities, fostering immediate interactions between trainers and participants, as well as encouraging collaboration among the participants themselves.

Moreover, the researchers also acknowledged the importance of allowing participants to share their ideas, which was identified as a vital factor in boosting motivation [11]. As a result, the training framework incorporated the concepts of synchronous online training and idea-sharing to create an environment conducive to optimal learning and engagement. Further details about the training program can be found in the subsequent section.

## 3. The Study

This study developed an online training program with the objective of preserving a structure similar to IPST's onsite training program. The program's content and activities, drawn from the onsite version, centered on vital GeoGebra components: geometry, algebra, and statistics tools. This selection recognized the time constraint teachers face in mastering all GeoGebra tools. Leveraging the software's user-friendliness, the training program aimed to equip teachers with basic tools' proficiency, empowering them to expand their skill set and generate teaching materials. Furthermore, the training included content on integrating GeoGebra into classroom instruction. To foster engagement and motivation, the program incorporated synchronous online training and idea-sharing. These strategies aimed to cultivate an immersive learning experience for all participants.

Subsequently, the researchers initiated the development of the initial draft for the online training program, tailored specifically for teachers aiming to use GeoGebra in mathematics instruction. The program was then implemented during the first phase of the study, as elaborated below.

**The first phase of the study**

During the initial phase of the study, the online training program for teachers centered around four distinct sections. The first three sections were dedicated to aiding teachers in acquiring proficiency in utilizing fundamental GeoGebra tools, including geometry, algebra, and statistics features within the software. The final section of this phase of the training program was designed to furnish concrete examples illustrating the practical implementation of GeoGebra in mathematics classroom instruction.

Within each section, the training commenced with a combination of lectures and interactive practice within the entire group. Upon completing the group activity, participants were presented with tasks or problems relevant to the content covered in that particular section. Subsequently, participants were divided into smaller groups, where they worked individually to tackle the assigned problems. After completing their individual tasks, each participant had the opportunity to present their work and share the ideas they employed to solve the problems.

Following the small group sessions, all participants reconvened in the whole group setting to collectively draw conclusions from the learning experiences and discussions held during the section. This comprehensive approach facilitated a dynamic and collaborative learning environment, enabling participants to benefit from both group interactions and individual exploration.

Examples of activities or given tasks in each sections were shown in Figure 1.

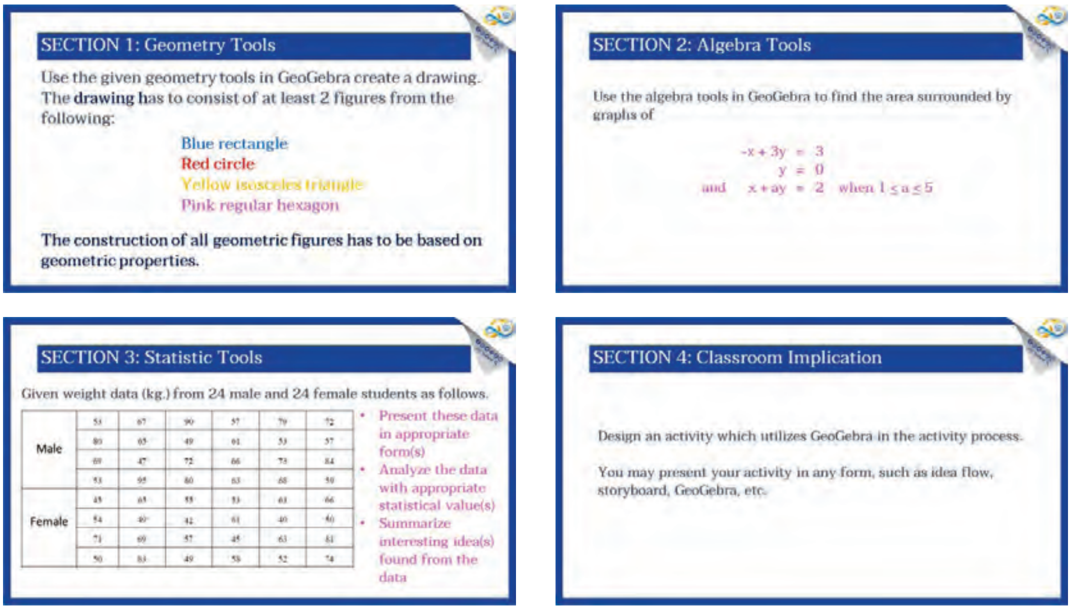


**Figure 1** Examples of activities in each section

In April 2022, the online training program for the first phase of this study was conducted via the Zoom application, spanning a duration of two days. The training group consisted of 47 secondary mathematics teachers, chosen through stratified random sampling from 900 secondary mathematics teachers. The sampling method ensured representation from various regions of Thailand, including the northern, southern, northeastern, western, central, and Bangkok regions. Throughout the online training, all participants were required to attend every section, complete all assigned tasks, and submit their work for evaluation. This approach ensured comprehensive participation and engagement from the participants, fostering a meaningful and impactful training experience.

Upon the conclusion of the training, all participants were provided with an online survey. The survey comprised three demographic questions, seeking information on gender, age, and highest level of education completed. Additionally, participants responded to 11 Likert-type questions designed to gauge their satisfaction with the online training program. For instance, one question asked about their level of satisfaction with the training activities, with responses ranging from highly dissatisfied (1) to highly satisfied (5). Furthermore, the survey featured two open-ended questions, encouraging participants to provide their recommendations and feedback on the online training program.

The data collected from the demographic questions and the Likert-type questions were subsequently analyzed in terms of frequencies and percentages. This analysis aimed to offer valuable insights into the participants' basic characteristics and assess their overall satisfaction levels with the online training program.

Moreover, the recommendations and comments received from the open-ended questions were carefully summarized and considered for enhancing the training program during the second phase of the study. This feedback played a crucial role in refining the program to better meet the participants' needs and preferences, ensuring an improved and more effective training experience in the subsequent phase.

**The second phase of the study**

After analyzing the results from the survey, the researchers observed that incorporating small group activities significantly boosted participants' engagement during the online training. Nevertheless, it became apparent that the participants required additional time for the training, and they expressed the need for more examples of GeoGebra's implementation in mathematics classroom instruction.

In light of these findings, adjustments were made to the structure of the training program for this phase. The number of sections was reduced from four to three, with a primary focus on the fundamental tools within GeoGebra software, which were geometry, algebra, and statistics tools. Moreover, to address the participants' request for more practical guidance, examples of classroom implementation were integrated into all sections in form of discussion. This modification aimed to provide a more streamlined and comprehensive learning experience, allowing participants to gain deeper insights into GeoGebra's applications in mathematics instruction. Taking an algebra section for an example, once participants completed the assigned task resulting in the instructional material depicted in Figure 2, subsequent discussions would encompass topics such as:

- Where would you apply this material?
- How would you structure the activity if you were to implement this material?
- What questions would you pose if you conducted the activity using this material?
- How would you improve this material to be more beneficial for your activity?

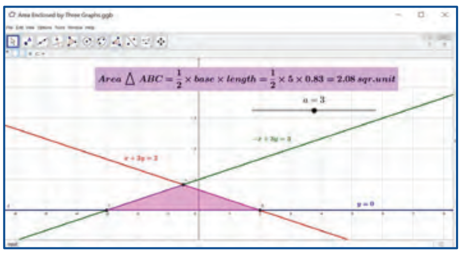


**Figure 2** Example of completed task

The training in the second phase followed a similar approach to that of the first phase, encompassing various content and activities. These included lectures and interactive activities within the whole group, task assignments, division into small groups, individual work within these groups, presentations and discussions, and sharing of ideas followed by group conclusions within the whole group. Furthermore, the integration of GeoGebra into mathematics classroom instruction was a central aspect of the discussion, both in small group sessions and in the whole group setting across all sections.

As in the first phase, all participants in the second phase were required to engage in activities across all sections, participating in both whole group and small group settings. They were expected to complete all assigned tasks and submit their completed work for evaluation. Moreover, each participant was requested to complete an online survey similar to the one employed in the first phase. Subsequently, all collected data were analyzed for frequencies and percentages. Recommendations, feedback, and comments were summarized.

The findings derived from the implementation of the online training program designed for teachers to employ GeoGebra in mathematics instruction, encompassing both the first and second phases, are expounded upon in the subsequent section.

## 4. Results

The study's findings were from the first and the second phases of the study. More details of these results are provided in the following subsections.

**Results from the first phase of the study**

Out of the 47 secondary mathematics teachers who took part in the first phase of the study, 39 teachers (82.98%) completed and submitted the online survey. This group comprised five males (12.82%) and 34 females (87.28%). Approximately 40% of these participants were aged above 40 (38.46%), while another 25.64% fell within the age range of 26 to 30. In terms of educational qualifications, roughly half of the participating teachers had completed their bachelor's degree (51.28%), and the other half had attained their master's degree (48.72%).

Regarding participants' satisfaction with the online training program during the first phase, the rating for the content presented in whole group activities exceeded the 'satisfied level.' Detailed information is provided in Table 1.

Table 1

*Frequencies and percentages of participants in each satisfied level for content presented in whole group activities in the first phase of the study*

| Indicator | Satisfied level | | | | |
|---|---|---|---|---|---|
| | Highly dissatisfied | Dissatisfied | Neutral | Satisfied | Highly satisfied |
| Content in geometry section | 0 (0.00%) | 0 (0.00%) | 2 (5.13%) | 11 (28.21%) | 26 (66.67%) |
| Content in algebra section | 0 (0.00%) | 0 (0.00%) | 2 (5.13%) | 12 (30.77%) | 25 (64.10%) |
| Content in statistics section | 0 (0.00%) | 0 (0.00%) | 1 (2.56%) | 12 (30.77%) | 26 (66.67%) |
| Content in classroom implementation section | 0 (0.00%) | 1 (2.56%) | 2 (5.13%) | 10 (25.64%) | 26 (66.67%) |

Table 1 illustrates that about 60 percent of the participating teachers expressed high satisfaction with the content presented in the geometry, algebra, statistics, and classroom implementation sections. This indicates that the content within these sections was well-suited for online training. However, it is noteworthy that one teacher felt dissatisfied with the content provided in the classroom implementation section.

Furthermore, the participating teachers' satisfaction with the small group activities during the first phase of the online training program surpassed the 'satisfied level,' as indicated in Table 2.

Table 2
*Frequencies and percentages of participants in each satisfied level for small group activities in the first phase of the study*

| Indicator | Satisfied level | | | | |
|---|---|---|---|---|---|
| | Highly dissatisfied | Dissatisfied | Neutral | Satisfied | Highly satisfied |
| Small group trainer's ability | | | | | |
| Explanation | 0 (0.00%) | 0 (0.00%) | 3 (7.69%) | 4 (10.26%) | 32 (82.05%) |
| Language used | 0 (0.00%) | 0 (0.00%) | 2 (5.13%) | 6 (15.38%) | 31 (79.49%) |
| Response to question | 0 (0.00%) | 0 (0.00%) | 1 (2.56%) | 9 (23.08%) | 29 (74.36%) |
| Task given in | | | | | |
| Geometry section | 0 (0.00%) | 0 (0.00%) | 1 (2.56%) | 12 (30.77%) | 26 (66.67%) |
| Algebra section | 0 (0.00%) | 1 (2.56%) | 1 (2.56%) | 11 (28.21%) | 26 (66.67%) |
| Statistics section | 0 (0.00%) | 0 (0.00%) | 1 (2.56%) | 11 (28.21%) | 27 (69.23%) |
| Classroom implementation section | 0 (0.00%) | 0 (0.00%) | 1 (2.56%) | 13 (33.33%) | 25 (64.10%) |

As shown in Table 2, over 70 percent of the teachers expressed high satisfaction with the trainer's abilities to explain concepts, use language effectively, and respond to participants' questions in the small group setting. Regarding the tasks given in each section, around 65 percent of the teachers reported high satisfaction, while approximately 30 percent expressed satisfaction. However, it is worth noting that around five percent of the teachers rated their satisfaction at unsatisfied to neutral levels. This suggests that some participants may have encountered difficulties in solving the tasks provided in each section.

Additionally, two significant recommendations emerged from both the participating teachers and the researchers who closely observed each small group. Firstly, the participants expressed a need for more time in each section. They required additional time during the whole group activities, particularly when learning to use geometry, algebra, and statistics tools. Moreover, they also requested more time to collaborate in small groups, allowing them to effectively complete the assigned tasks.

Secondly, participants pointed out that the examples provided in the classroom implementation section predominantly focused on algebra tools. They expressed a desire to see more examples that were related to geometry and statistics tools, enabling them to gain a comprehensive understanding of GeoGebra's application across various mathematical concepts.

Taking these valuable recommendations and comments into account, the researchers proceeded to revise the online training program, incorporating the necessary adjustments and improvements. This revised program was then implemented during the second phase of the study, aiming to create an enhanced and tailored learning experience for the participants.

The findings from the second phase of the study are outlined in the next subsection.

**Results from the second phase of the study**

Out of the 61 secondary mathematics teachers who participated in the second phase of the study, 42 teachers (68.85%) submitted the online survey. This group comprised two male teachers (4.76%) and 40 female teachers (95.24%). The majority of teachers fell within the age categories of 20 to 30, more than 40, and 36 to 40 (30.95%, 26.19%, and 23.81% respectively). Nearly all participants had completed either a bachelor's or master's degree (27.62% and 47.62% respectively). Additionally, 4.76% of the participants had attained a degree higher than a master's degree.

Regarding participants' satisfaction with the online training program in the second phase, the rating for the content presented in whole group activities exceeded the 'satisfied level.' Further details are provided in Table 3.

Table 3

*Frequencies and percentages of participants in each satisfied level for content presented in whole group activities in the second phase of the study*

| Indicator | Satisfied level | | | | |
|---|---|---|---|---|---|
| | Highly dissatisfied | Dissatisfied | Neutral | Satisfied | Highly satisfied |
| Content in geometry and the implication section | 0 (0.00%) | 0 (0.00%) | 2 (4.76%) | 16 (38.10%) | 24 (57.14%) |
| Content in algebra and the implication section | 0 (0.00%) | 0 (0.00%) | 2 (4.76%) | 14 (33.33%) | 26 (61.90%) |
| Content in statistics and the implication section | 0 (0.00%) | 0 (0.00%) | 2 (4.76%) | 18 (42.86%) | 22 (52.38%) |

Table 3 reveals that approximately 95 percent of the participating teachers expressed satisfaction and high satisfaction with the content presented in the geometry and implementation, algebra and implementation, and statistics and implementation sections. This highlights the highly appropriate content within these sections for online training.

Moreover, the participating teachers' satisfaction with the small group activities during the second phase of the online training program exceeded the 'satisfied level,' as indicated in Table 4.

Table 4
*Frequencies and percentages of participants in each satisfied level for small group activities in the second phase of the study*

| Indicator | Satisfied level | | | | |
|---|---|---|---|---|---|
| | Highly dissatisfied | Dissatisfied | Neutral | Satisfied | Highly satisfied |
| Small group trainer's ability | | | | | |
| Explanation | 0 (0.00%) | 0 (0.00%) | 3 (7.14%) | 10 (23.81%) | 29 (69.05%) |
| Language used | 0 (0.00%) | 0 (0.00%) | 3 (7.14%) | 10 (23.81%) | 29 (69.05%) |
| Response to question | 0 (0.00%) | 0 (0.00%) | 2 (4.76%) | 10 (23.81%) | 30 (71.43%) |
| Task given in | | | | | |
| Geometry and the implication section | 0 (0.00%) | 0 (0.00%) | 2 (4.76%) | 15 (35.71%) | 25 (59.52%) |
| Algebra and the implication section | 0 (0.00%) | 0 (0.00%) | 3 (7.14%) | 12 (28.57%) | 27 (64.29%) |
| Statistics and the implication section | 0 (0.00%) | 0 (0.00%) | 4 (9.52%) | 14 (33.33%) | 23 (54.76%) |

Table 4 displays that around 70 percent of the teachers expressed high satisfaction with the trainer's abilities to explain concepts, use language effectively, and respond to participants' questions in the small group setting. Regarding the tasks given in each section, about 60 percent of the teachers reported high satisfaction, and approximately 30 percent indicated satisfaction. Notably, no teachers reported feeling highly dissatisfied or dissatisfied with the tasks provided in the small group activities. This suggests that the given tasks across all sections were deemed appropriate by the participants in terms of content and time allocation.

In summary, participants' comments from the online survey indicated that the training duration in the second phase (3 days) was considered appropriate. Furthermore, several recommendations surfaced, which could prove beneficial for future training endeavors. Firstly, teachers preferred future training sessions to be scheduled on weekends or holidays to avoid conflicts with their weekday school responsibilities. Secondly, teachers expressed a desire to delve deeper into the functionalities of algebra, geometry, and statistics tools in GeoGebra, including learning about writing command language and scripting language.

Thirdly, participants expressed a need for future training that specifically focused on using GeoGebra to create mathematical media for certain subjects, such as 3-D geometry, geometrical analysis, and calculus. Lastly, there was interest in training sessions tailored to the implementation of GeoGebra in classroom instruction, specifically catering to participants who were already familiar with the basic tools in GeoGebra. These valuable recommendations are valuable insights that can help shape and enrich future training programs to better meet the needs and preferences of the participants.

## 5. Conclusion and Discussion

The primary focus of this study was on developing an online training program specifically designed for teachers to effectively use GeoGebra in mathematics instruction. Drawing inspiration from IPST's successful onsite training in this area, the developed online program aimed to achieve two main objectives: facilitating teachers' understanding of the basic tools available in GeoGebra software, including geometry, algebra, and statistics tools, and demonstrating how GeoGebra can be effectively implemented in classroom instruction.

During the developmental process, the researchers recognized and addressed two major challenges commonly encountered in online training: the potential lack of participant interaction and motivation. To deal with these issues, the study incorporated two key pedagogical strategies into the training program. Firstly, the concept of synchronous online training was adopted, ensuring real-time interactions and fostering engagement among participants. Secondly, the training program encouraged active idea sharing, allowing participants to exchange thoughts and insights. These pedagogical concepts were implemented in both the online training programs used during the first and second phases of the study, aiming to create a dynamic and enriching learning experience for all participants.

During the first phase of the study, the online training program, consisting of four sections (geometry, algebra, statistics, and classroom implementation), effectively met the participants' needs. Additionally, upon reviewing the tasks submitted by the participants, the researchers observed that the teachers engaging in the training demonstrated proficiency in using basic tools within GeoGebra and displayed creative ideas for implementing GeoGebra in classroom instruction.

However, a significant constraint encountered by the participants was time limitation. The allocated time posed challenges in both learning the basic tools of GeoGebra and completing the given tasks. Despite this constraint, the time pressure was slightly decreased during the small group activities. In these sessions, participants benefited from close observation and peer assistance, provided by both group members and the group trainer. Such support contributed to a more conducive and interactive learning environment, allowing participants to navigate the constraints more effectively.

In the second phase of the study, the focus shifted towards the application of GeoGebra in classroom instruction, in addition to learning the basic tools. Consequently, the online training program for the second phase comprised three sections: geometry and its implication, algebra and its implication, and statistics and its implication.

Compared to the first phase, the training duration for the second phase was extended to three days, providing more time for comprehensive learning and exploration. Overall, the participating teachers in the second phase expressed satisfaction with the online training program. As in the first phase, they exhibited proficiency in using basic tools within GeoGebra and demonstrated creative ideas for implementing GeoGebra in classroom instruction. Moreover, the participants expressed their interest in attending future training sessions (either onsite or online) focused on the usage of GeoGebra, particularly in creating instructional media for specific mathematical content.

However, the results on the participants' satisfaction with the online training, primarily centered around synchronous learning and idea sharing, was as anticipated. Previous research has consistently shown that in online learning, participants tend to express greater satisfaction with synchronous learning environments as they offer enhanced opportunities for discussions and idea exchange, ultimately boosting learning outcomes [12] [13].

Nevertheless, several variables could have influenced the findings. For instance, the participants' abilities in using GeoGebra tools might have been influenced by the increased digital literacy among teachers during the COVID-19 pandemic, as many had to adapt to online teaching. Additionally, the teachers' satisfaction with the online training program might have been influenced by the lack of training experiences for a long time during the COVID-19 pandemic. To ensure the integrity of future implementations of the online training program, these variables should be carefully controlled and taken into consideration.

In conclusion, the developed online training program for teachers in using GeoGebra for mathematics instruction proved to be effective in helping teachers acquire proficiency in using basic tools within the GeoGebra software. Moreover, the training provided valuable insights into the implementation of GeoGebra in mathematics instruction. Significantly, the participating teachers expressed high satisfaction with the online training program.

However, to fully understand the advantages of this online training program under normal circumstances, it would be beneficial to repeat the study after the end of the COVID-19 pandemic. This future study could provide a comprehensive comparison of the training program's effectiveness and outcomes in a non-pandemic setting, offering further insights into its impact on teacher learning and classroom instruction.

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# Incorporating Digital Interactive Figures: Facilitating Student Exploration into Properties of Eigenvalues and Eigenvectors

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**Abstract:** *Linear algebra is a key topic in mathematics and many other disciplines. In this paper, we consider a set of digital interactive figures (I-figs) using Mathematica created for linear algebra students in introductory and advanced courses, which prioritize pattern-seeking and examples over arithmetic processes. The figures discussed here are a part of digital worksheets designed to facilitate students' ability to visualize and work with eigenvalues and eigenvectors. Minimizing student computation for the benefit of conceptual focus while looking at an unlimited number of examples is an overarching theme. Different design intentions are explored for each of the four example figures. The worksheets provide a foundation for motivating students to participate in a system of observation, conjecture, proof, and theorem. Students were further supported through classroom lessons and additional homework activities.*

## 1. Introduction

Linear Algebra is a key topic in mathematics, is used in many other disciplines, provides the foundation for many AI and data analytic techniques, and plays a significant role in industry.

In teaching and learning linear algebra, the use of Computer Algebra Systems (CAS) such as MATLAB, Maple, Python, or Mathematica has a valuable place (Lay, Lay, & McDonald [7]). For example, in working with matrices and vectors, solving systems, matrix multiplications, finding inverses, and reducing a matrix to a reduced echelon form, to name a few, technology can help with accurate and fast calculations as well as allow one to concentrate on the deeper conceptual aspects of the discipline that are often difficult for students to grasp. Technology can help to build a visual image of the details, as well as the big picture, and allows for predictions, investigations, and producing conjectures.

Linear algebra students' ways of thinking about abstract notions of linear algebra, and in particular eigenvalues and eigenvectors, have also been a topic of study in some studies (e.g., Thomas & Stewart [12]; Salgada & Trigueros [9]; Wawro, Watson, & Zandieh [13]). In the early 90s, the Linear Algebra Curriculum Study Group (LACSG) recommended that "faculty should be encouraged to utilize technology in the first linear algebra course" [3]. The linear algebra recommendations by LACSG 2.0 also emphasized the importance of technology in teaching linear algebra [11].

Despite national initiatives in the US on teaching linear algebra with MATLAB, for example, the ATLAST project initiated by Steven Leon and his colleagues (Leon et al. [8]) in the early 90s, systematic studies of linear algebra instructors' thought processes and their students' feedback on the effect of technology on their understanding are lacking. In a survey paper by Stewart, Andrews-Larson, and Zandieh [10], the authors looked at linear algebra education papers from 2008-2017 in mathematics education journals and found almost no systematic classroom studies on the effectiveness of the use of Mathematica, Maple, Python, or MATLAB in the classroom, with the exception of a study of using MATLAB in mathematical modeling (Dominiques-Garcia et al. [4]). However, the survey paper found studies on eigentheory using Geometer's Sketchpad (e.g., Gol Tabaghi [5]; Caglayan [2]), and GeoGebra (Beltran-Meneu et al. [1]) focused on the geometric aspects of the topic.

In this paper, we present and discuss the use of interactive figures (I-figs) embedded into a student worksheet and used by the authors in both introductory and advanced linear algebra courses. Additional I-figs can be found in the interactive electronic version of the textbook *Linear Algebra and its Applications* [7]. Our emphasis is on conjecturing and formal understanding and less on geometry, in line with Harel's [6] notion of the shortcomings of using geometry as a foundation to build linear algebraic ideas. Instead, we focus most of our visuals on rectangular arrays of numbers, representing matrices and vectors, and setting up the I-figs to illustrate various patterns in the relationships of the objects presented.

The research team (the authors) regularly teach introductory and advanced courses in linear algebra. The first and second authors have started using the I-figs in teaching their classes, with guidance from the third author on how to collect data in order to research their effectiveness and improve their use.

The introductory course typically starts with representing a system of equations as a matrix and solving the system using row operations, followed by a discussion of linear dependence and independence, span, basis, linear transformations, vector spaces and subspaces, determinants, eigenvalues and eigenvectors, and orthogonality. The predominant visual for the course is rectangular arrays, which are used to illustrate vectors, matrices, and linear transformations. In the introductory course, the homework is built on computation and understanding concepts, with the expectation that students can be asked to do simple proofs on their own. The more advanced course, however, emphasizes the vector spaces and linear transformations at a more rigorous level, leading to a more detailed development of eigenspaces and orthogonality. The homework in the advanced course is predominantly proof based.

## 2. The Objectives and the Design of Interactive Figures

### 2.1 Student Need for Familiarity with Linear Algebra Objects

For many mathematics students, linear algebra is often their first exposure to mathematics and mathematical objects outside of their experience in traditional algebra and calculus courses. Linear algebra is then potentially their first exposure to variables denoting more than just scalar values or scalar valued functions. A succinct encapsulation of this limitation is in the fundamental eigenvalue/eigenvector equation $A\mathbf{x} = \lambda\mathbf{x}$, where $A$ represents an $n \times n$ matrix, $\mathbf{x}$ is a vector in $\mathbb{R}^n$, and $\lambda$ is a scalar. This equation requires students to view each of these variables as separate entities equatable because both operations, the transformation of $\mathbf{x}$, represented by $A\mathbf{x}$, and the scaling of $\mathbf{x}$, represented by $\lambda\mathbf{x}$, form vectors in $\mathbb{R}^n$.

Furthermore, when introduced to the notion of a vector in linear algebra, students may tend to rely upon geometric representations of vectors, common in physics or vector calculus. This can become an impediment to student understanding as they work with higher dimensional systems that cannot be geometrically represented. While there is certainly a place for geometric thought and interpretation in linear algebra, we build many of our I-figs to encourage students to see matrices and vectors in the form of rectangular arrays as the primary visual. This approach lifts to any (finite) dimension and is an asset to students learning computational techniques such as row operations and calculating determinants.

### 2.2 Using Technology to Experience Mathematics Through Observation, Conjecture, Proof, and Theorem

Much of a mathematician's work can be described in the general process of observation, conjecture, proof, and theorem (not ascribing a perfectly sequential process, instead inviting repeated iteration upon these steps). It is worth mentioning that we order a proof before the theorem despite students typically seeing textbooks that display theorems with their proofs following. In this mathematician's system of observation, conjecture, proof, and theorem, we view the theorem as necessarily existing only after the proof validates it, and the final theorem is the product of condensing the result of the proof into a succinct and useable statement.

However, many linear algebra students have little to no experience reading, analyzing, or constructing such formal mathematics. One intention of these worksheets is to help students establish a practice of experimenting mathematically and to form conjectures about the objects they study. In particular, we have developed and will continue to iterate a series of digital worksheets meant to give students a structured environment to experiment with vectors and matrices so that they may begin making observations and, ideally, formulating conjectures.

To facilitate this experimentation and observation, a part of what we wish to accomplish with these worksheets is to eliminate any unnecessary arithmetic that would normally restrict or distract students from pattern-seeking. These I-figs do not seek nor advise to eliminate the arithmetic practice of processes like matrix multiplication or eigenvalue calculation. Instead, these actions are performed by the digital worksheet to remove them as a cognitive load for the student so that they might focus on more general structural patterns in the algebra. Other homework and coursework should provide sufficient opportunity for these rote practices.

### 2.3 Role of Digital Worksheets

For content delivery, the purpose of this series of I-figs is to provide students a venue to encounter concepts and numerical or algebraic patterns in linear algebra that are common and will be expanded upon in future lessons in a more rigorous fashion. As inherently introductory (or even pre-introductory) activities, these digital worksheets were aimed to be fairly succinct, requiring only 15 to 20 minutes to allow students to meaningfully engage with the presented ideas. The design of these worksheets was guided by two user-oriented principles: minimal investment and ease of use, and two content delivery principles: example generation and encouragement to conjecture.

A central philosophy for these worksheets is that they may be approached by students with minimal investment on the students' part. In computational I-figs, the worksheet performs operations such as scaling or multiplying matrices, or as in this worksheet, more complex actions such as computing eigenvalues, reducing student cognitive load. As such, students do not necessarily need to know how to calculate eigenvalues in hopes that they may focus only on the relationship between the scalar matrix exponent and its effect on eigenvalues. This also serves to allow students to view any desired number of examples in a reasonable and accessible amount of time.

Each activity is designed so that example vectors or matrices (as appropriate) are generated by the worksheet rather than by manual student input, though students are typically allowed to control the size of the matrix generated (usually $2\times2$ up to $4\times4$) and to generate new examples. By intentionally avoiding incorporating manual input of matrix values, the worksheet aims to minimize the amount of effort and time required of students to begin pattern-seeking.

In piloting this worksheet, we see students making valuable observations but without the tools or confidence to pose their observations as conjectures or to attempt to prove a result. To address this, our current lesson system involves providing a digital worksheet either before or directly after the first lesson in a supported content area. Students are invited to experiment with the worksheet, make any observations they can, and record their thoughts and questions as a reflection assignment. In subsequent class time, the worksheet can be revisited together in order to collect and organize the various conjectures that can be made from student observations, paying attention to carefully modeling how an observation can be framed as a conjecture. For those conjectures with accessible proofs, the next step would be to collectively build such a proof or proof outline.

Naturally, some students will be inclined to investigate beyond the scope of the worksheet and would like to experiment with their own input objects. The I-figs are intentionally restricted in the amount of student input required and hence allowed, and so do not permit students to design their own experiments. This is in line with the goal for these worksheets to foster and encourage student conjecturing. In addition, students can be encouraged to perform calculations by hand themselves or via other resources such as Wolfram matrix labs or MATLAB.

### 2.4 Mathematics Software Requirements

The digital interactive figures (I-figs) being used were constructed using Mathematica, however, another CAS could be used. Important goals for the instructors in creating the I-figs and worksheets include less investment of time and coding ability by students, the ability to create worksheets containing the I-figs with commentary and questions, and no additional cost for students to use the figures and worksheets.

The robust Manipulate command enables these I-figs to be fully self-encapsulated. Interacting with the figure only requires students to manipulate slide bars, click buttons, and select from drop-downs. This is important for the ease of use for students of various educational backgrounds who may or may not have any coding experience.

Mathematica is structured to allow various cell types to both coexist and nest compatibly. This allows the I-figs to be narratively structured with instructional or descriptive text blocks and images to be neatly arranged, creating a conversational worksheet for students.

The I-figs and worksheets run on Wolfram Digital Player, which is an application developed and supported by Wolfram and is available to download for free. While programming in Mathematica does require a license (which a given university or college may or may not possess), the accessibility of Wolfram Digital Player removes cost as a barrier for classroom and student use of the I-figs and worksheets once they are created.

## 3. The Eigenvalue/Eigenvector Worksheet and Instructors' Reflections

In this section, we will show examples of I-figs ranging from illustrating one idea per figure to more sophisticated I-figs that display multiple patterns and invite several possible conjectures or counterexamples. In all examples to follow, different algebraic objects are displayed as appropriate, allowing students to visualize and contrast the different structural forms of scalars, vectors, and matrices that coexist in the linear algebraic equation $A\mathbf{x} = \lambda\mathbf{x}$. Furthermore, we expect this experience to facilitate students' handling of these objects in subsequent lessons, assignments, and settings beyond the course.

### 3.1 Focused I-Figs: Displaying the Effect of a Single Algebraic Operation on Eigenvalues

Consider the below example of two worksheet modules (Figure 1) used in the introductory course. The participants are able to dictate the desired size of the matrix within the range of

2×2 to 4×4, and to choose key scalars as appropriate. A "new matrix" button allows the student to generate new examples. In each of these two figures, there is a single pattern intended to be emphasized. Broadly, this compartmentalization is meant to eliminate distraction and allow students to reasonably quickly make observations, keeping time investment minimal and pattern recognition as their primary focus.

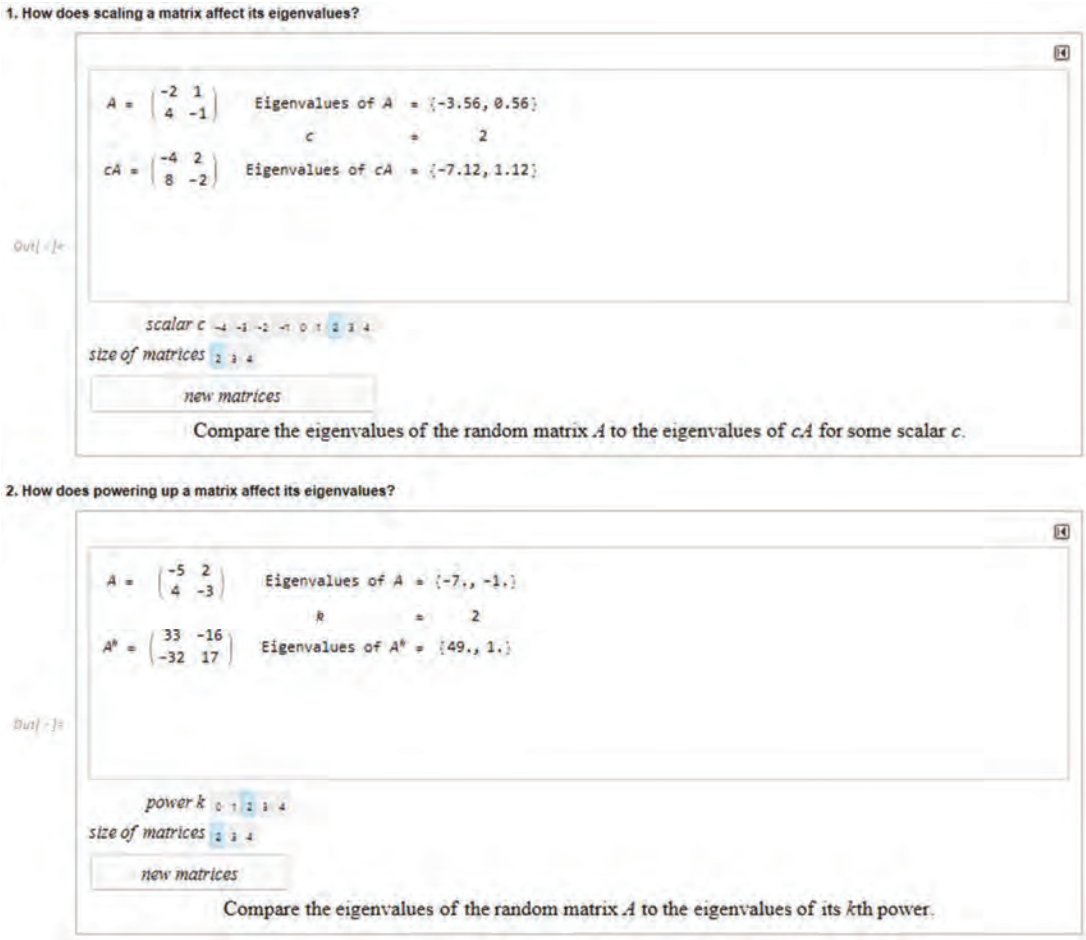


**Figure 1** First Modules in Eigenvalue Worksheet

### 3.2 Building some Geometric-Algebraic Connections

Other activities have been made with the intention of emphasizing a connection between geometric and algebraic ideas. Consider the following example (Figure 2.1). The vector **v** is displayed, both geometrically and as a rectangular array, along with the transformed $A\mathbf{v}$. Students are given control of the second component of **v** via a slider (which naturally changes $A\mathbf{v}$ as well). While the transformation matrix $A$ is not specified, students at this point in their linear algebra education could potentially calculate $A$ if curious. However, the salient interest of this figure is to motivate students to connect the linear algebraic equation $A\mathbf{v} = \lambda\mathbf{v}$ with both geometric and algebraic representations of that scalar relationship. These two eigenvalue-eigenvector relations can be found by experimenting with the slider for **v** or using the two buttons provided to move directly to those values (Figure 2.2).

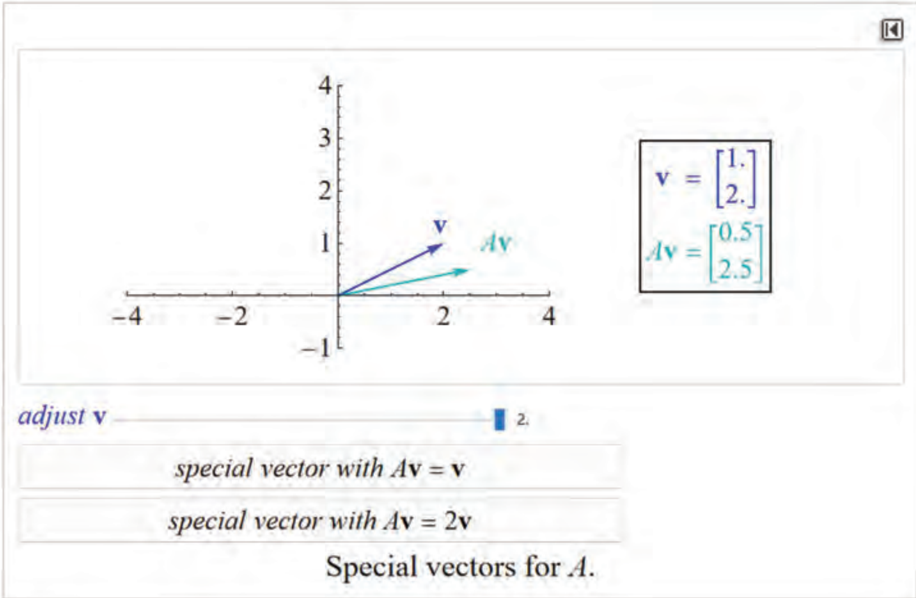


**Figure 2.1** Geometric and Algebraic Connection of Eigenvalues and Eigenvectors

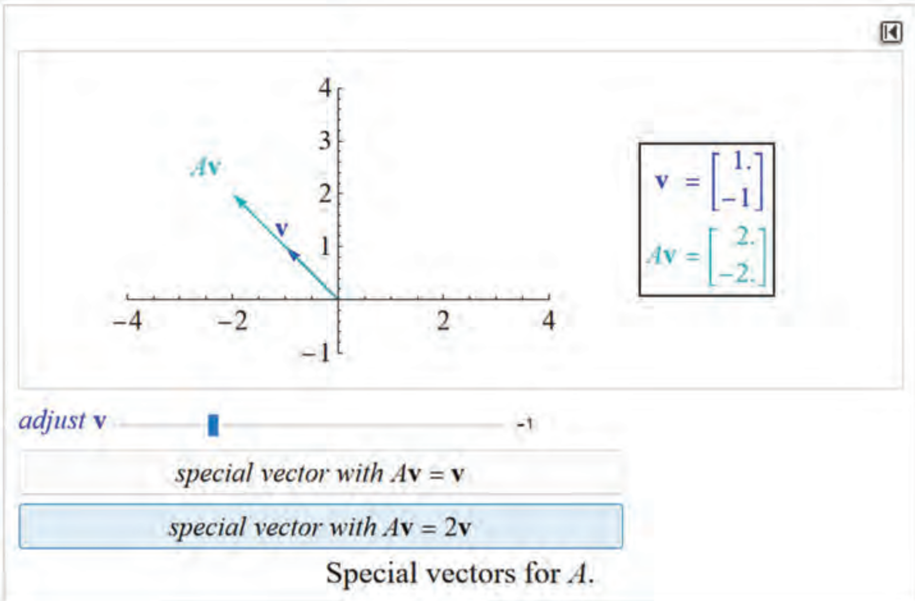


**Figure 2.2** Displaying Eigenvalue 2 Geometrically

### 3.3 Increasing Sophistication: Multiple Potential Conjectures or Counterexamples

Next, we will look at a more complex I-fig (Figure 3) which possesses more than one distinct pattern for students to observe and conjecture. Students may observe that the eigenvalues of $AB$ and $BA$ match, or $A$ and $B^{-1}AB$ share eigenvalues.

This figure also contains several common patterns that students might expect to observe that do not hold, such as the eigenvalues of *AB* or *A*+*B* being directly related to the eigenvalues of *A* and *B* separately. An equally important skill alongside conjecturing and proving conjectures is the ability to appraise and counter false conjectures. Our hope for this particular I-fig is to allow students to question a reasonable-looking hypothesis (for example, a direct relationship between *A*, *B*, and *A*+*B* eigenvalues) and to generate examples via the worksheet for the purpose of finding convincing counterexamples.

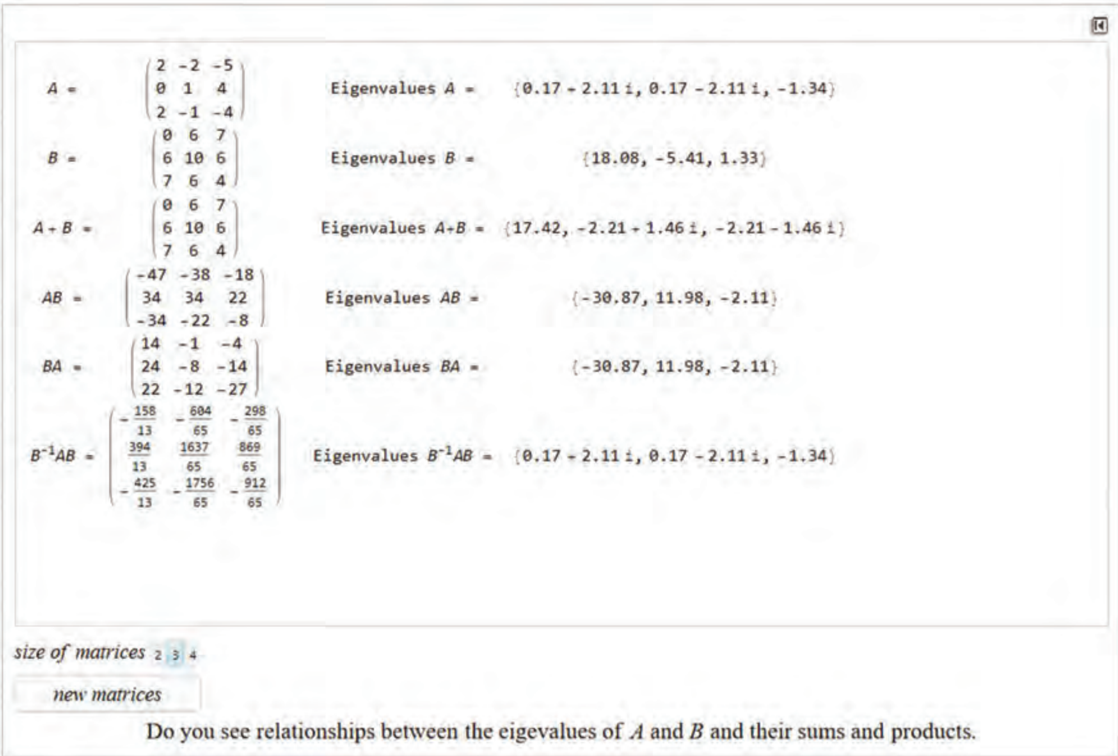


**Figure 3** Digital I-fig with Multiple Points of Interest

### 3.4 Progressive Example: Advanced Students are Expected to Formally Justify Their Observations.

The last figure (Figure 4) is an example of how I-figs can be constructed to curate. In Figures 1 through 3, used in the introductory course, operations on the matrix are the primary focus. The eigenvalues produced from the generated integer matrices could be complex and were rounded to two decimal places. In Figure 4, used in the more advanced course, the focus of the I-fig is on the eigenvalues and eigenvectors. A random set of integer eigenvalues and integer linearly independent eigenvectors are created first and used to construct each matrix. Such a construction creates messy rational matrices, as seen, however, allows the students easier access to assessing and pattern-seeking among the eigenvalues and eigenvectors. In particular, for this I-fig, if the matrix were generated first and yielded irrational and complex eigenvalues, it may be exceedingly difficult for students to recognize the connection between the eigenvalues of *A* and $A^{-1}$.

Since this figure was used with a more advanced group of students, in the homework that followed, they were asked to prove that for any invertible matrix $A$, the eigenvalues of $A^{-1}$ are the reciprocals of the eigenvalues of $A$. Students also observed that the two matrices have the same eigenvectors, helping them see a possible route to proving that the observed relationship between the eigenvalues of the two matrices must always exist.

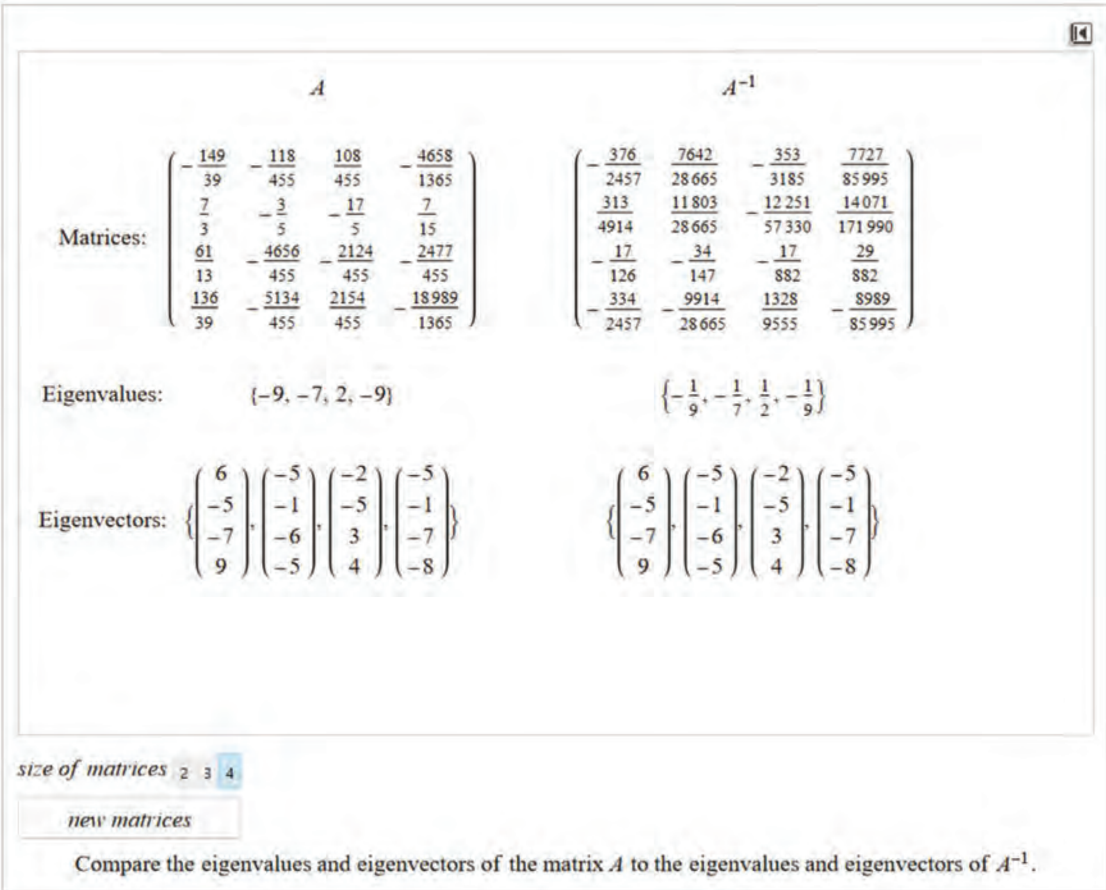


**Figure 4** Digital I-fig with Curated Eigenvalues

## 4. Concluding Remarks and Future Work

This paper discusses a way to operationalize interactive figures, such as those developed in the textbook *Linear Algebra and its Applications* [7], into student worksheets, and study the effectiveness of the proposed use. Our I-figs and worksheets were designed to provide students with the opportunity to work through the mathematical process of observation, conjecture, proof, and theorem.

In using the eigenvalue and eigenvector worksheet in teaching, the authors provided students with an opportunity to look at an unlimited number of examples quickly and effortlessly. This leverages the computational benefits of CAS to allow students to focus and think about linear algebraic properties without the arithmetic burden common to linear algebraic calculations, hence enhancing their knowledge base.

As was found in the linear algebra education survey paper [10], studies about the effectiveness of employing common linear algebra CAS, such as MATLAB, that alleviate the cognitive load of performing arithmetic operations and interacting with linear algebraic objects, were rare. To fill this gap in the literature, careful research studies about effective ways to integrate technology into the classroom in ways that improve students' understanding of linear algebra are needed.

To evaluate the effectiveness of this worksheet, we conducted a research study in the first author's introductory linear algebra course. As part of this study, the worksheet on eigenvalues and eigenvectors was administered to students around the first lecture covering these concepts. Student-centric data was collected, consisting of written reflections following their participation in the worksheet assignment, as well as follow-up surveys in subsequent meetings, including a post-chapter review. Currently, we are in the process of analyzing this data. For a more effective evaluation of students' thought processes, based on our experiences and analysis of the data, we will modify the I-figs and worksheets, as well as our research designs, as needed. In any review scenario, we aim to maintain a mentality of persistent improvement rather than any attempt to confirm a final form for these digital worksheets.

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# Learning guidance based on elimination singularity phenomenon

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**Abstract**

*In the study of mathematics, when two items have concepts in common, understanding the relationship between them can be subject to "overgeneralization." In this study, a technique introduced in previous research [1] to analyze overlap singularity through a simulation using a neural network on the loss surface is extended to elimination singularity and the effect of overgeneralization.*

*In the elimination singularity examined here, students who answered a series of questions correctly immediately after studying the topic of combinations showed a superficial understanding of the topic, while students who answered the questions partially correctly following the passage of time after learning combinations were affected by overgeneralization. Findings from this analysis are used to formulate learning guidance for teachers of mathematics.*

## 1 Introduction

In learning theory, plateauing is a phenomenon in which learning is affected by singular regions and stagnates. When two concepts, A and B, are learned in the order of A and then B, it is possible that the later-learned B may affect the concept understanding of the earlier-learned A. This is called the "overgeneralization of A as B". In a previous study of overlap singularity, students who gave partially correct answers immediately after learning combinations exhibited a natural way of understanding as part of the developmental learning process, while students who answered partially correctly with the passage of time after learning combinations displayed a deeper understanding [1]. To address overgeneralization stagnation, we consider a singular region due to elimination singularity, defined later as the state where only the group of the two student groups.

## 2 Preparation for analysis

In this study, we used the concept of overgeneralization to investigate changes in student group understanding of permutations and combinations [1]. To consider the loss surface and singularity in learning, we use Kullback information as the error function. Fundamental problems in learning theory have been described in the literature [1], [2], [3]. To test semantic comprehension in the field of permutations and combinations in the unit "number of cases" in the high school mathematics curriculum, we created a set of test questions as follows: (Questions (a) and (c) are permutation problems, question (b) is a combinatorial problem, and question (d) is an interrelationship problem.) (a) "How many ways can three people be taken from a group of nine people and arranged in a row?" (b) "How many ways are there to choose two cards out of seven?" (c) "How many ways can a group of four students arrange themselves in a row?" (d) "How many ways are there to choose one chairman, one vice-chairman, and one secretary from the ten members of a committee? Concurrent posts are not permitted." An examination of scoring and problems of semantic comprehension and interrelationships is reported elsewhere [1].

We define two learning stages : (1) The stage in which partially correct answers are influenced by the study of combinations between the first and second examinations. (2) The stage in which correct answers are obtained by considering the interrelationship with combinations through the re-learning of permutations between the second and third examinations. Three student groups are defined based on questions (a) through (c): (i) Students who achieved full points in the permutation problems and partial points in the combination problem. (ii) Students who achieved partial points in the permutation problems and full points in the combination problem. (iii) Students who achieved full points in both the permutation and combination problems. We compared student groups (i) and (iii) in learning stage (1), and student groups (i) and (ii) in learning stage (2).

**Definition 1** *For inputs $x$, $\theta_0$, the output $Y$ of a two-layer neural network is defined as follows:* $Y := f(x, \theta_0) = w_3 \tanh(w_1 x) + w_4 \tanh(w_2 x)$.

**Definition 2 (Elimination singularity)** *Elimination singularity is defined as passing through the parameter region* $R_1 := \{\theta \in \mathbb{R}^4 | w_3 = 0\} \cup \{\theta \in \mathbb{R}^4 | w_4 = 0\}$.

In this case, the learning model can be expressed as follows: $f(x, \theta) = w_4 \tanh(x, w_2)$, $w_3 \tanh(x, w_1)$. The coordinate transformation from the parameters $\theta = (w_1, w_2, w_3, w_4)$ to the new parameters $\xi = (a, b, v, w)$ can be defined as follows: $a = w_2 - w_1$, $b = \frac{w_3 - w_4}{w_3 + w_4}$, $v = \frac{w_3 w_1 + w_4 w_2}{w_3 + w_4}$, $w = w_3 + w_4$. According to Amari [2], we set $(v, w)$ to the optimal solution $(v^*, w^*)$, and then consider the trajectory of the parameters $(a, b)$ to the optimal solution.

For the two student groups, let c be the average score for the combinatorial problem, d be the average score for the permutation problems, and e and f be the number of students assigned these problems, respectively. The weights of the neural network can then be determined as follows: $w_1' = c$, $w_2' = d$, $w_3 = \frac{e}{e+f}$, $w_4 = \frac{f}{e+f}$. Next, we consider students in each of the three examinations as a subset. In order to keep the parameters $v$, $w$ for the overall set and for the three subsets constant, we applied corrections to the subset weights $w_1'$, $w_2'$. For the two groups of students, the score obtained by subtracting 0.5 from the average of the sum of the permutation and combination scores from the first to the third examinations is the $g$ point. In each examination, the scores obtained by subtracting 0.5 from the average of the sum of the permutations and combinations are $h_1$, $h_2$, $h_3$, respectively. At this time, the correction is determined by $w_i = w_i' \times \frac{g}{h_i}$.

Table 1 shows the average score, number of students, and the correction coefficients for students who gave correct answers (left two columns) and partially correct answers (right two columns) to the interrelationship question (d) in the three examinations.

Table 1: Examination results

| 1st to 3rd examinations | (i) (full, partial) | (iii) (full, full) | (i) (full, partial) | (ii) (partial, full) | 2nd examination | (i) (full, partial) | (iii) (full, full) | (i) (full, partial) | (ii) (partial, full) |
|---|---|---|---|---|---|---|---|---|---|
| Average $c, d$ | 0.228571429 | 0.5 | 0.22826087 | 0.279411765 | Average $c, d$ | 0.25 | 0.5 | 0.25 | 0.276666667 |
| Number of people $e, f$ | 70 | 57 | 23 | 85 | Number of people $e, f$ | 3 | 23 | 8 | 60 |
| Overall average $g$ | 0.350393701 | | 0.268518519 | | Overall average $h_2$ | 0.471153846 | | 0.273529412 | |
| Total number of people | 127 | | 108 | | Total number of people | 26 | | 68 | |
| | | | | | Correction factor $\frac{g}{h_2}$ | 0.743692753 | | 0.981680605 | |
| | | | | | Correction $w_1, w_2$ | 0.185923188 | 0.371846376 | 0.245420151 | 0.271598301 |
| 1st examination | (i) (full, partial) | (iii) (full, full) | (i) (full, partial) | (ii) (partial, full) | 3rd examination | (i) (full, partial) | (iii) (full, full) | (i) (full, partial) | (ii) (partial, full) |
| Average $c, d$ | 0.234126984 | 0.5 | 0.214285714 | 0.245 | Average $c, d$ | 0.125 | 0.5 | 0.21875 | 0.313333333 |
| Number of people $e, f$ | 63 | 5 | 7 | 10 | Number of people $e, f$ | 4 | 29 | 8 | 15 |
| Overall average $h_1$ | 0.253676471 | | 0.232352941 | | Overall average $h_3$ | 0.454545455 | | 0.280434783 | |
| Total number of people | 68 | | 17 | | Total number of people | 33 | | 23 | |
| Correction factor $\frac{g}{h_1}$ | 1.381262125 | | 1.15564932 | | Correction factor $\frac{g}{h_3}$ | 0.770866142 | | 0.957507895 | |
| Correction $w_1, w_2$ | 0.323390735 | 0.690631062 | 0.24763914 | 0.283134083 | Correction $w_1, w_2$ | 0.096358268 | 0.385433071 | 0.209454852 | 0.30001914 |

Taking parameter $a$ as the difference between the average score for the permutation problems and the average score of the combinatorial problem, we considered the balance between the overgeneralization of permutations as combinations and the overgeneralization of combinations as permutations. Parameter $b$ is defined as the ratio between the number of students in the two groups to be considered.

In terms of the two concepts (permutations and combinations), we can visualize the understanding (learning trajectory) of the student group on the learning loss surface. More details on the construction of neural networks and a learning loss surface are given in the literature [1], [4]. Here, the state of understanding is displayed as points for the 1st examination (blue), 2nd examination (red), 3rd examination (green), and 1st to 3rd examinations (yellow). Student groups (i) and (iii) are shown on the left side of Figure 1; student groups (i) and (ii) are shown on the right.

# 3 Singularity phenomena in mathematics education

We considered elimination singularity, near-elimination singularity, and first convergence in information science. Cross-overlap singularity is described elsewhere [4].

We will first examine elimination singularity using the ratio of the number of students in the two student groups to be considered. Here we consider only the group of students who scored full points for the permutation problems ($b = 1$) or the combination problems ($b = -1$).

Next, we consider near-elimination singularity using the ratio of the number of students in the two student groups. This approaches the situation described by $b = 1$ or $b = -1$ above. Finally, we consider fast convergence for permutation and combination questions, whereby the difference between the mean score for students with partially correct permutation answers and that for students with partially correct combination answers is given a partial score. If the difference is large ($a > 0$), it becomes even larger without passing through $a = 0$.

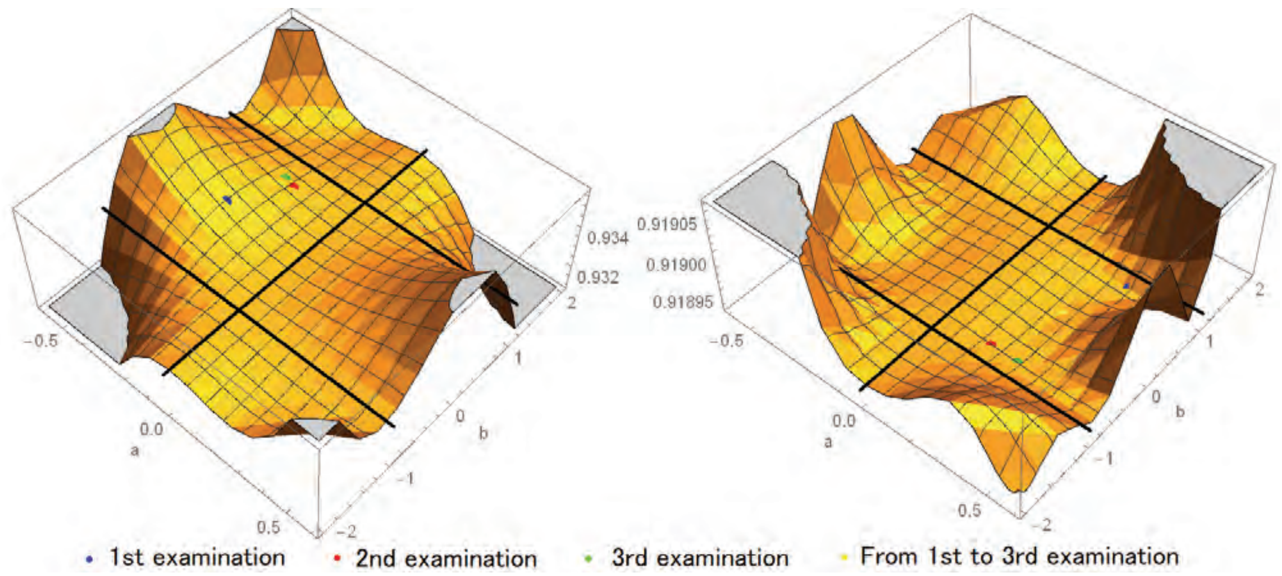


Figure 1: Learning loss surface for semantic understanding

# 4 Simulations

In the following sections, the results obtained from real data are used to perform simulations in which the initial values are changed in 0.05 increments from −0.6 to 0.6 for $a$ and from −1.1 to 1.1 for $b$ [1], [4]. In the figures, the initial values for the first (blue), second (red), and third (green) examinations are indicated by ⊚, and the true distribution is indicated by ×. For the dynamics from the first to the second examination, the initial value (simulation) is shown in blue, and the value after the simulation is shown in red. For the dynamics from the second to the third examination, the initial value (simulation) is shown in red, and the value after the move is shown in green.

## 4.1 Analysis of students who answered correctly with only superficial understanding

Immediately after learning combinations, students answered the interrelationship question (d) using permutations. We analyzed the change from the first to the second examination to target the students who gave correct answers. We now consider the process of understanding the two concepts for students who gave completely correct answers to the permutation problems, partially correct answers to the combination problem, and completely correct answers to both the permutation and the combination problems.

In the following, we consider the case where learning about combinations begins after learning about permutations. At the beginning of the study of combinations, we assume a situation that involves only the group of students who scored full points for the permutation problems, and overgeneralization using combinations as permutations occurs. Since there are few students who fully understand combinations, overgeneralization of permutations as combinations does not occur. Therefore, we consider this unbalanced state to be an elimination singularity.

### 4.1.1 Simulation for changing $a$

First, we consider the dynamics of the simulation in which $a$ is varied. The evolution of parameters $a$, $b$ on the $a\,b$ plane and the dynamics on the loss surface are shown in Figure 2.

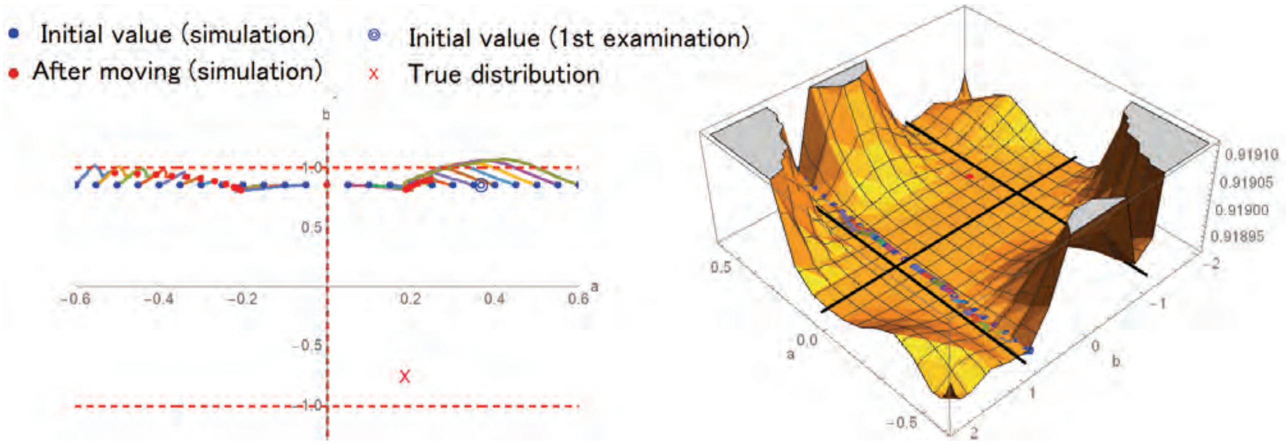


Figure 2: $a$ change: correct answer

The simulation results and transformation for the schema are shown in Table ?? .

Table 2: Simulation results and schema transformation

| Simulation | Result | Schema transformation |
|---|---|---|
| Overgeneralization of combinations as permutations increases | Only the group of students who scored full points for the permutation problems | Permutations |
| Overgeneralization of permutations as combinations increases | Proportion of students does not change | None |

Learning begins from the state $b = 0.85$, where the proportion of students who answer the permutation questions correctly and the combination questions partially correctly is larger and more unbalanced than the proportion of the group of students who answer both the permutation and combination questions correctly.

When the overgeneralization of combinations as permutations increases ($a$ decreases), the critical line $b = 1$ is affected, and elimination singularity occurs. Only students who answer the permutation questions correctly and the combination questions partially correctly are included; there are no students who answer both the permutation and combination questions correctly. If the overgeneralization with combinations as permutations is smaller than $a < -0.3$, the learning of combinations does not progress and a transformation of the permutation schema takes place.

As overgeneralization of permutations as combinations increases ($a$ increases), near-elimination singularity occurs due to an approach to the critical line $b = 1$ and a return to the original state, stagnating at $a = 0.3$. Since the ratio of students who answer the permutation questions correctly and the combination questions partially correctly to students who answer both the permutation and the combination problems correctly does not change from the initial state, the learning of combinations does not progress, and a transformation of the permutation schema does not occur.

### 4.1.2 Simulation for changing $b$

Next, we consider the dynamics of the simulation when $b$ is varied. The evolution of the parameters $a$, $b$ on the $a\,b$ plane and the dynamics on the loss surface are shown in Figure 3.

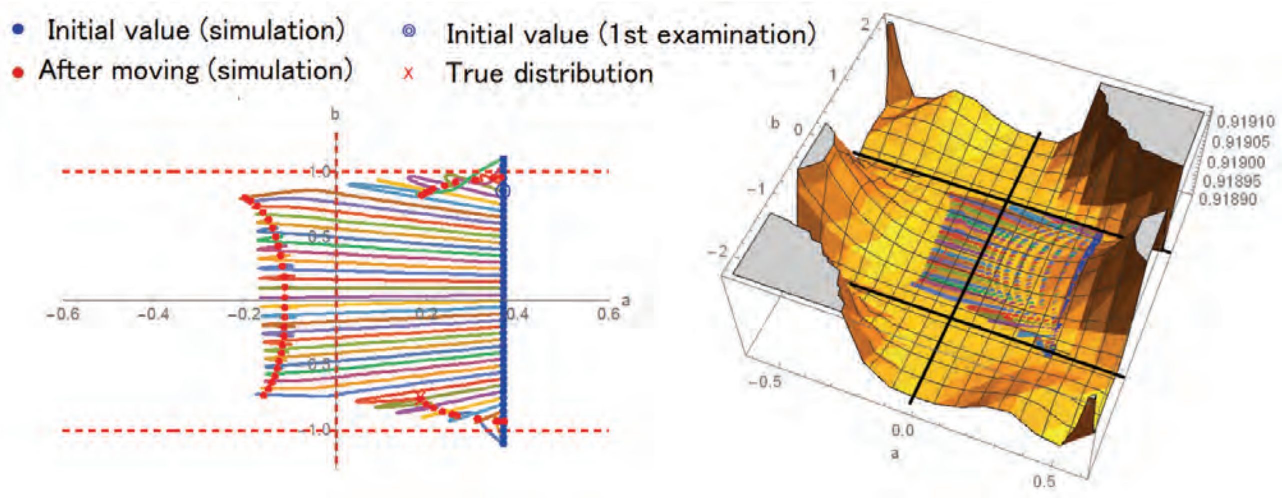


Figure 3: $b$ change: correct answer

The simulation results and transfer are shown in Table 3.

Table 3: Simulation results and transfer

| Simulation | Result | Transfer |
|---|---|---|
| Increase the percentage of students who answer the permutation question correctly and suppress overgeneralization of permutations as combinations | Positive transfer | |
| Increase the percentage of students who answer both the permutation and combination questions correctly | Overgeneralization of combinations as permutations becomes larger | Negative transfer |

Learning starts from the state $a = 0.36$, where the overgeneralization of permutations as combinations is large.

At first, as the proportion of students who answer the permutation questions correctly and the combination questions partially correctly becomes larger ($b$ increases), a near-elimination singularity occurs due to an approach to the critical line $b = 1$ and a return to the original state. A positive transition occurs because the large overgeneralization of permutations as combinations is suppressed.

As the proportion of students who answer both the permutation and combination questions correctly becomes larger ($b$ decreases), a cross-overlap singularity occurs beyond the critical line $a = 0$. The state of overgeneralization of permutations as combinations is large, $w_1$, $w_2$ becomes 0.35 from $v = 0.35$, and there is no difference from the overgeneralization of combinations as permutations. Furthermore, the overgeneralization of combinations as permutations becomes large, which leads to a negative transition.

#### 4.1.3 Examination of teacher's teaching strategy

By simulating the change in $a$, $b$, the transformation of the schema and transfer can be assumed, allowing the teacher to devise an effective teaching strategy.

By increasing the overgeneralization of combinations as permutations from the change in $a$, the schema of the permutation changes, but the learning of combinations does not progress. From the $b$ change, increasing the proportion of students who answer both the permutation and combination questions correctly increases the overgeneralization of combinations as permutations, so that a negative transition occurs even as the learning of permutations progresses.

Although the problem is answered correctly (absent the influence of "choosing", students think in terms of permutations), it is not affected by the overgeneralization of permutations as combinations due to an insufficient understanding of combinations. This can be considered a superficial understanding of the developmental process.

Guidance in combinations is necessary. Furthermore, it is necessary to provide guidance for deepening understanding by considering the interrelationships between permutations and combinations. Students need to be taught to make correct judgments in problems that require understanding the interrelationships between permutations and combinations.

### 4.2 Analysis of students who give partially correct answers due to overgeneralization

In the interrelationship problem (d), after learning permutations, the student who was correct by using permutations was partially correct by mistakenly using combinations after some time had passed since learning permutations. Students who scored partial points for the permutation problems and full points for the combination problem were compared with students who scored full points for the permutation problems and partial points for the combination problem. Consider the process of understanding the concept. After learning combinations, time passes. The case in which it is time to start re-learning permutations is considered below:

Assume a situation in which, after learning combinations and time passes, there is only the group of students who scored full points for the combination problems, and overgeneralization using permutation as combinations occurs. Since there are no students who fully understand permutations, overgeneralization of combinations as permutations does not occur. Therefore, we consider this unbalanced state to be an elimination singularity.

### 4.2.1 Simulation for changing $a$

First, we consider the dynamics of the simulation in which $a$ is varied. The evolution of parameters $a$, $b$ on the $a\,b$ plane and the dynamics on the loss surface are shown in Figure 4.

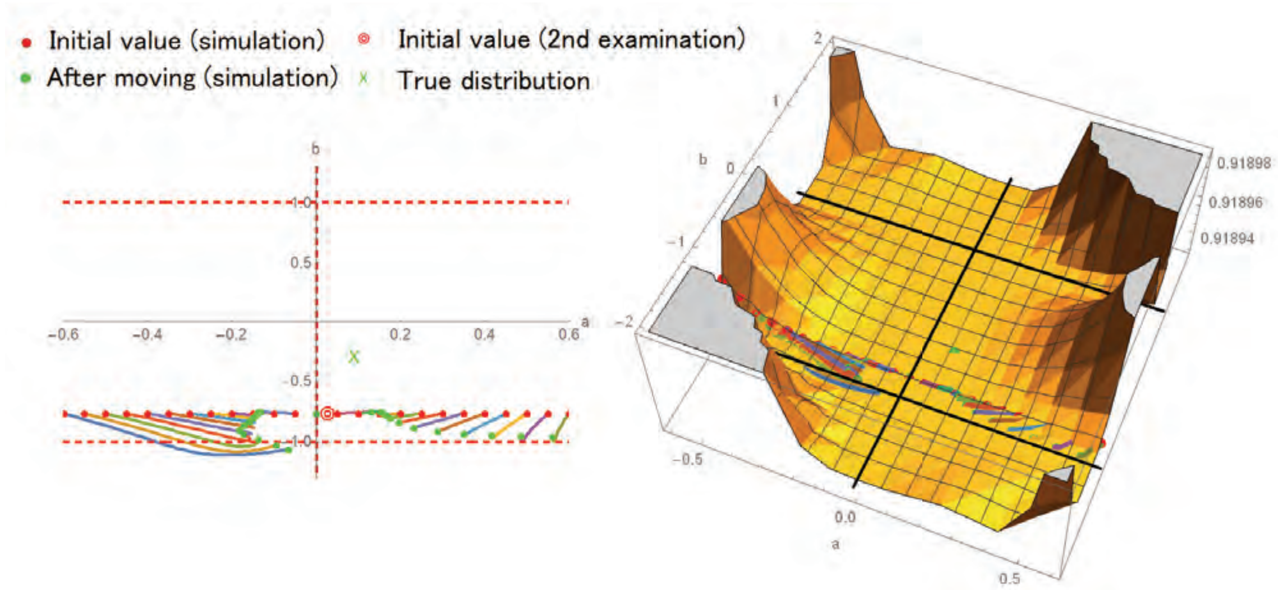


Figure 4: $a$ change: partially correct answers

The simulation results and transformation of the schema are shown in Table 4.

Table 4: Simulation results and schema transformation

| Simulation | Result | Schema transformation |
|---|---|---|
| Overgeneralization of permutations as combinations increases | Only the group of students who scored full points for the combination problems | Combination |
| Overgeneralization of combinations as permutations increases | Proportion of students does not change | None |

Learning begins from the state $b = -0.76$, where the proportion of students who answer the combination questions correctly and the permutation questions partially correctly is larger and more unbalanced than the proportion of students who answer the permutation questions correctly and the combination questions partially correctly.

When the overgeneralization of permutations as combinations increases ($a$ increases), the critical line $b = -1$ is affected, and an elimination singularity occurs. There are only students who answer the combination questions correctly and the permutation questions partially correctly. There are no students who answer the permutation questions correctly and the combination questions partially correctly. If the overgeneralization with permutations as combinations is larger than $a > 0.3$, the learning of permutations does not progress and a transformation of the combinations schema takes place.

As the overgeneralization of combinations as permutations increases ($a$ decreases), a near-elimination singularity occurs due to an approach to the critical line $b = -1$ and a return to the original state, stagnating at $a = -0.15$. Since the ratio of students who answer the combination questions correctly and the permutation questions partially correctly to the students who answer the permutation questions correctly and the combination questions partially correctly does not change from the initial state, the learning of combinations does not progress, and a transformation of the permutation schema does not occur.

### 4.2.2 Simulations for $b$ change

Next, we consider the dynamics of the simulation when $b$ is varied. The evolution of parameters $a$, $b$ on the $a\,b$ plane and the dynamics on the loss surface are shown in Figure 5.

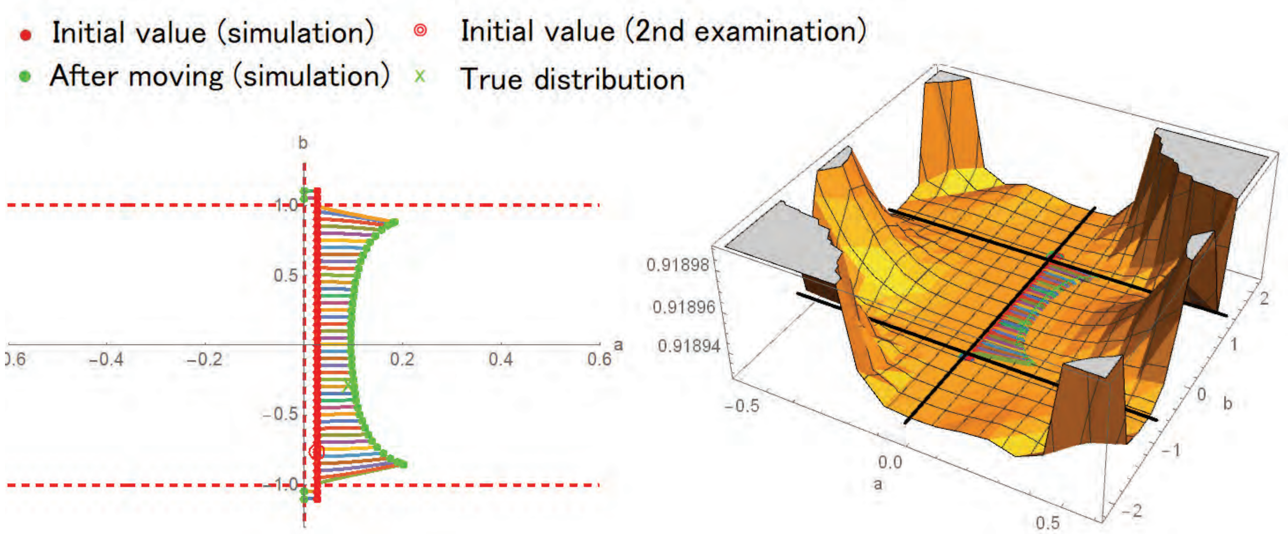


Figure 5: $b$ change: partially correct answers

The simulation results and transfer are shown in Table 5.

Table 5: Simulation results and transfer

| Simulation | Results | Transfer |
|---|---|---|
| Increase the proportion of students who score full points for the permutation problems and suppress large overgeneralization of permutations as combinations slightly | Negative transfer | |
| Increase the proportion of students who score full points for the combination problems | A large overgeneralization of combinations as permutations becomes even larger | Negative transfer |

Here, we start learning from $a = 0.026$, a state in which the overgeneralization of permutations as combinations is highly biased.

As the percentage of students who score partial points for the permutation problems and full points for the combination problem increases ($b$ decreases), fast convergence occurs without the influence of the critical line $a = 0$. The difference from the overgeneralization of permutations as combinations is even greater, so that a negative transition occurs.

When the proportion of students who score full points for the permutation problems and partial points for the combination problem increases ($b$ increases), fast convergence occurs without the influence of the critical line $a = 0$. The large overgeneralization of permutations as combinations is suppressed, so that a negative transition occurs.

By relearning permutations, the overgeneralization of permutations as combinations can be suppressed from $a = 0.18$ to $a = 0.094$.

### 4.2.3 Direction of teacher guidance

By simulating the change in $a$, $b$, the transformation of the schema and transfer can be assumed, allowing the teacher to devise a teaching strategy.

By increasing the overgeneralization of permutations as a combination from the change in $a$, the schema of the combinations changes, but the learning of permutations does not progress. From the $b$ change, increasing the proportion of students who score full points for the permutation problems increases the overgeneralization of permutations as combinations, so that a negative transition occurs even as the re-learning of permutations progresses, but the overgeneralization of permutations as combinations can be slightly suppressed.

Although the problem is answered partially correctly (under the influence of “choosing,” students think in terms of combinations), it is greatly affected by the overgeneralization of permutations as combinations. This can be considered a partially correct answer affected by overgeneralization. Therefore, it is considered necessary to re-learn permutations, paying attention to understanding permutations even after learning combinations.

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# Problem Research and Use of ICT in Mathematics Education

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**Abstract**: *In Japan, standards have been established for curriculum organization. In this paper, we describe the standards and the transition of the use of ICT. We also discuss the SSH project (Super Science High School), a national project to promote science and mathematics education, and present a case study of the use of ICT in one of the educational activities in the project, based on the author's experience, and describes its educational effects. Furthermore, based on the discussion of the case study, the direction of the use of ICT in statistics and mathematics education in the future will be proposed.*

## 1. Introduction

In Japan, there are curriculum standards called "Courses of Study," which have been established by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) to ensure that a particular standard is maintained in all schools throughout Japan. These standards stipulate general curricular considerations, the number of class hours, and each subject area's general goals, contents, and content. Each school determines the textbooks and timetables for its students based on these standards. The Courses of Study are reviewed every ten years to indicate the qualities and abilities necessary for children to live, and to consider social changes such as globalization, rapid informatization, and technological innovation. In high schools, education under the new Courses of Study will begin in 2022, and "inquiry" has been identified as a keyword for learning in the overall curriculum.

In addition, in this revision, science and mathematics education is required to enhance learning activities for scientific inquiry through observation, experimentation, and statistical education for analyzing data and solving problems.

Therefore, this paper focuses on "inquiry" learning activities in science and mathematics education and efforts to enhance statistics education. The inquiry process involves problem setting, information gathering, organization and analysis, and summarization and expression. We believe that one of the educational activities that participate in the inquiry process is the Problem Research[*1] conducted at Super Science High Schools (hereafter, SSH[*2]). Based on our experience working at SSH-designated schools from 2008 to 2020, we will summarize the changes in the Courses of Study and the status of the use of ICT and introduce examples of the use of ICT in Problem Research. We want to discuss the case studies from the viewpoint of statistical education and make suggestions for future statistical education and the use of ICT.

## 2. Changes in the Objectives of Mathematics and the Use of ICT

We summarized the objectives of the mathematics department in the Courses of Study from the time when the SSH project started to the present. We focus on how the objectives are expressed in terms of what kind of skills students should acquire through the study of mathematics. At the same time, we summarize the efforts and the status of the use of ICT at A High School.

### 2.1 Objectives of Mathematics*[3] and Classroom

| Notification period Implementation period | Objectives of Mathematics *[3] | Classroom Practice |
|---|---|---|
| Notification in March 1999 Implemented on an annual basis since 2003 | To help students deepen their understanding of the basic concepts, principles, and rules of mathematics, to develop their ability to think and express phenomena mathematically, cultivate a basis for creativity through mathematical activities, recognize the values of mathematical ways of seeing and thinking and develop an attitude to use these values actively. | [Class] Do not use ICT.<br>[Equipment] No screen or projector in the classroom. Teachers bring their own it. A wired LAN is gradually being installed. |
| Notification in March 2009 Implemented on an annual basis since 2013. | Through mathematical activities, to help students deepen their understanding of the basic concepts, principles, and rules of mathematics, develop their ability to think and express phenomena mathematically, cultivate a basis for creativity, recognize mathematical values, and develop an attitude of making judgments based on mathematical arguments by actively using these values. | [Class] Use graphing software for presentations and conduct classes in the computer room. In Problem Research, students use tablet PCs in groups to search the Internet and use Excel and PowerPoint. Use graphing and geometry software and a graphing calculator. |

Goals of the Mathematics Department in the current Courses of Study and the current ICT environment in schools

| | |
|---|---|
| Notification in March 2018 Implemented on an annual basis since 2022 | The goal is to develop the following qualities and abilities to think mathematically through mathematical activities, using mathematical ways of seeing and thinking.<br>(1) To understand the basic concepts, principles, and rules of quantity and shape, and to acquire the skills to mathematize, mathematically interpret, mathematically express, and mathematically process phenomena.<br>(2) Cultivate the ability to examine events logically using mathematics, to find properties of quantities and figures and to examine them in an integrated and developed manner, and to express events concisely, clearly, and accurately using mathematical expressions.<br>(3) Cultivate an attitude to think persistently and apply mathematics to daily life and learning by realizing the enjoyment of mathematical activities and the value of mathematics and evaluating and improving the problem-solving process in retrospect. |
| The school environment | Each student is given one tablet PC and a Google account. Each classroom is equipped with a projector, screen, and Wi-Fi network. Both students and teachers can use it every hour. |

### 2.2 Promotion of teaching through mathematical activities

From the changes in the Courses of Study to date, it can be seen that "mathematical activities" are of great significance in mathematics education in Japan. Furthermore, in those activities, ICT is practical and currently indispensable.

*"Mathematical activities"* are activities in which the students conduct the learning process of arithmetic and mathematics. The image of this is shown below (see [1]).

It is described as moving between the real world and the world of mathematics. In the real world, the student can perceive everyday life and social events mathematically, process them mathematically, and solve problems. One can think about mathematical events in an integrated and developed manner and solve problems. These processes are shown to move between the real world and the mathematical world by focusing on the problem expressed mathematically and obtaining results.

The Problem Research is an activity that can realize this learning process and we have been practicing it. The following is an example of practice with ICT.

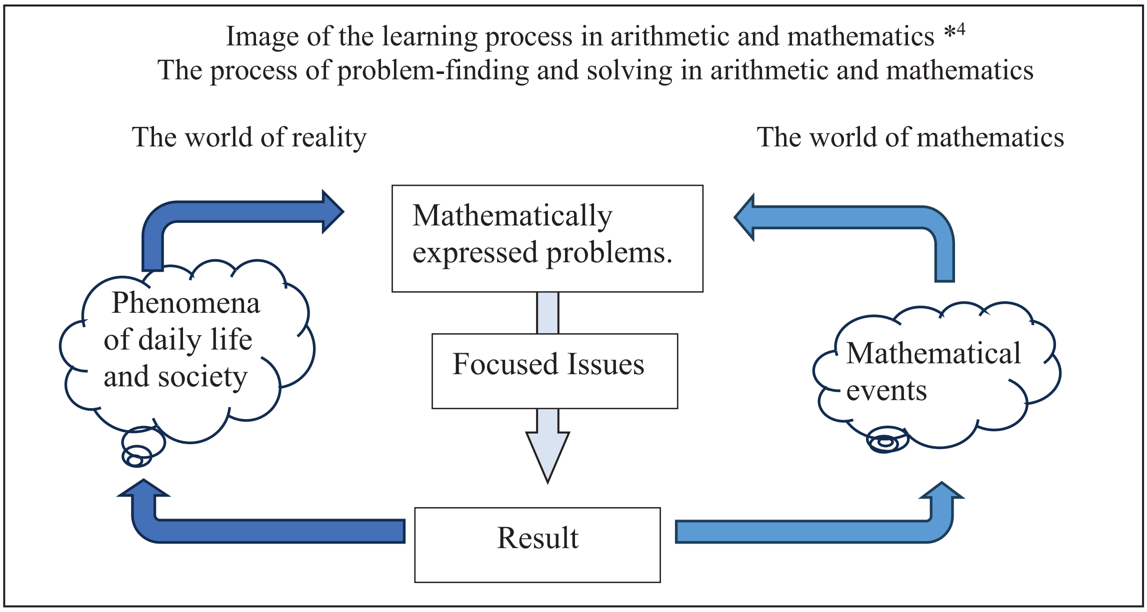


**Figure 2.1** Image of the learning process in arithmetic and mathematics*4

## 3. The practice in Problem Research at SSH school

The author worked at two SSH schools, the first being School A (April 1, 2008-March 31, 2015), currently in its Phase IV designation as an SSH-designated school. The author worked at the time of the I and II designations.

The second school is School B (April 1, 2016 - March 31, 2020), currently in its Phase Vth designation as an SSH-designated school. The author worked at School B during the IV designation. In both cases, Problem Research on an issue is positioned, and students set an issue and research it individually or in groups. In addition to delving deeply into specialized content in mathematics, there is a strong tendency to use mathematics to research issues. The author describes the classroom practice at School A.

## 3.1 Class Outline

### 3.1.1 Problem Research: "Data analysis of photovoltaic power generation and its future use"

Solar panels were installed on the roof of the school building, and temperature, solar radiation, and the amount of electricity generated were measured, these data had been stored on a computer in the school since March 2004. Students learned of the existence of this data and analyzed it. The computer spreadsheet software Excel was used.

Students were interested in energy issues. Since solar power generation has been attracting attention from all over the world as an environmentally friendly method of power generation, and research has been conducted to improve its efficiency, they investigated, based on data accumulated at the school, whether solar radiation is proportional to power generated, whether the temperature is related to power generated, and what can be done to improve the efficiency of power generation, and what should be done to improve the efficiency of power generation. As a method, data on electricity generated by solar panels was imported into a computer and graphed for annual and monthly comparisons. The following data dealt with the data at that time.

| date | solar radiation (kwh/㎡) | temperature (℃) | the amount of electricity generated (kwh) |
|---|---|---|---|
| 1 | 1.476 | 23.8 | 14.41 |
| 2 | 0.893 | 20.9 | 7.96 |
| 3 | 4.233 | 23.8 | 40.35 |
| 4 | 4.845 | 23.9 | 44.77 |
| 5 | 5.471 | 26.2 | 50.68 |
| 6 | 4.79 | 26.8 | 44.29 |
| 7 | 3.991 | 28.9 | 37.06 |
| 8 | 2.191 | 27 | 20.76 |
| 9 | 2.408 | 27.7 | 22.97 |
| 10 | 1.827 | 25.4 | 17.52 |
| 11 | 3.377 | 25.6 | 33.16 |
| 12 | 4.457 | 27.1 | 41.39 |
| 13 | 2.278 | 27.6 | 21.24 |
| 14 | 4.458 | 30 | 40.92 |
| 15 | 3.468 | 28.7 | 31.61 |
| 16 | 1.888 | 24.3 | 17.55 |
| 17 | 2.098 | 25.6 | 19.42 |
| 18 | 2.133 | 26.9 | 20.36 |
| 19 | 1.975 | 27.3 | 18.45 |
| 20 | 4.291 | 26 | 39.99 |
| 21 | 1.69 | 26.3 | 16.67 |
| 22 | 3.524 | 26.1 | 33.68 |
| 23 | 2.18 | 23.8 | 20.36 |
| 24 | 3.697 | 26.8 | 35.06 |
| 25 | 1.872 | 27.3 | 18.03 |
| 26 | 2.594 | 25.4 | 24.37 |
| 27 | 2.503 | 25.3 | 23.43 |
| 28 | 1.897 | 25.2 | 18.58 |
| 29 | 3.774 | 28.8 | 35.45 |
| 30 | 2.275 | 24.9 | 21.14 |
| 31 | 1.974 | 24.4 | 18.91 |

**Figure 3.1** Data of photovoltaic power generation

The data on solar radiation, temperature, and electricity generated were graphed. The graphs were graphed over a day to identify trends. The weather was also used as a reference. There were differences between sunny and rainy days. (See [1] and [2], [3]) On sunny days, the amount of solar radiation and electricity generated were higher than on rainy days.

In addition, a similar graph was made to grasp the trend to know the annual change. There were differences depending on the season.

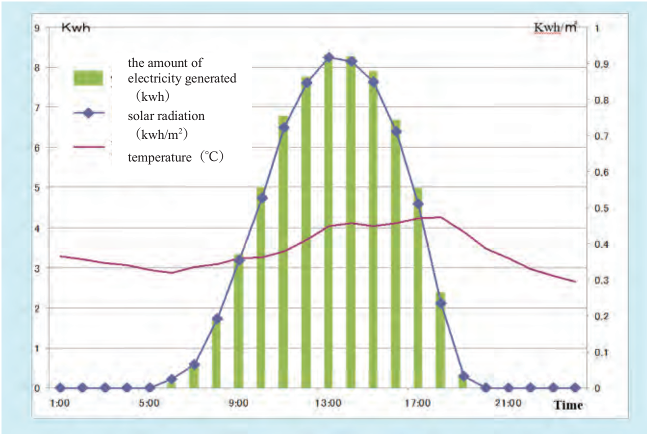


**Figure 3.2** Graphs of the amount of electricity generated, solar radiation and temperature (sunny)

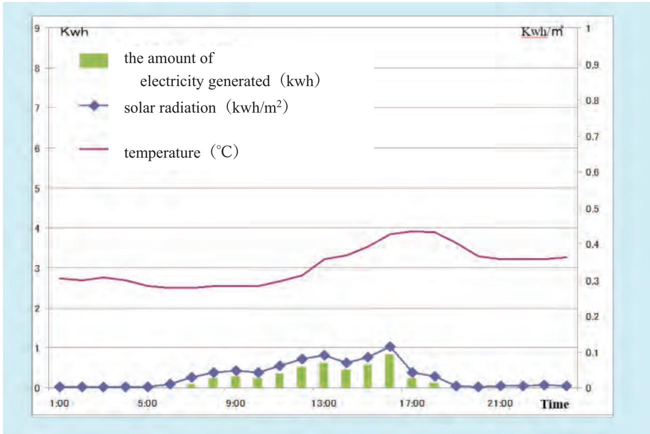


**Figure 3.3** Graphs of the amount of electricity generated ,solar radiation and temperature (rainy)

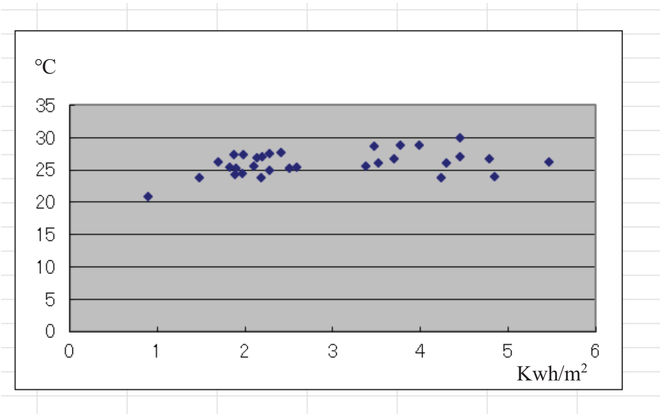


**Figure 3.4** Scatter diagram of temperature and solar radiation

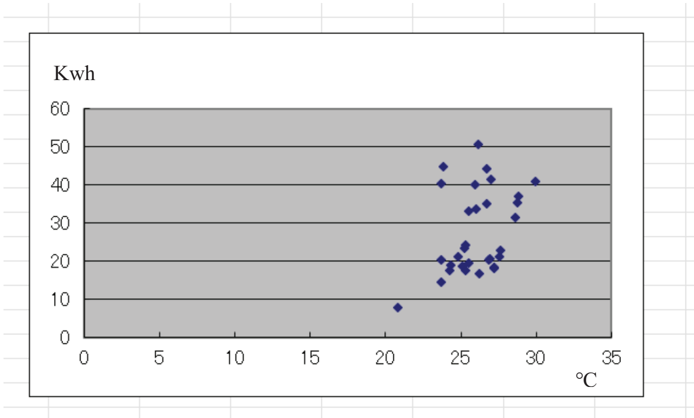


**Figure 3.5** Scatter diagram of the amount of electricity generated and temperature

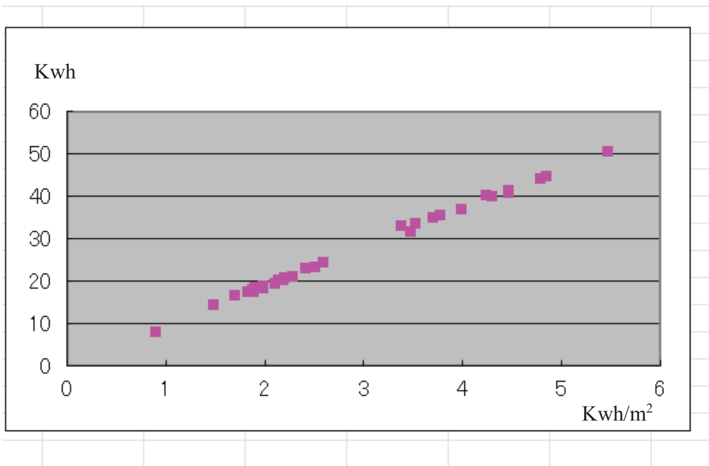


**Figure 3.6** Scatter diagram of amount of electricity generated and solar radiation

Next, Students also examined the correlation between average monthly temperatures the amount of electricity generated, and solar radiation to determine whether temperature affects them. Scatter plots were created using data from the past five years. It was found that there is no direct relationship between temperature and the amount of electricity generated and between temperature and solar radiation. (See [4] and [5])

Regarding solar radiation and electricity production, data analysis from the past five years shows they are proportional. These were compiled throughout the year. (See [6]) This indicates that the amount of solar radiation affects the electricity generated.

As a discussion, the graph shows that the shape of the graph is mountainous in shape in all years, indicating that the summer season tends to have higher temperatures, solar radiation, and electricity generation. The reason for the lower amount of electricity generated from November to February, the winter season, was that geographical factors resulted in fewer sunny days due to higher winter precipitation. A new finding was that electricity generation was higher in the spring months of March through May, rather than in the summer when there are more sunny days. They were able to confirm that in some years this was beyond the summer months. Typhoons and fall rains influenced the lower amount of electricity generated in September in the fall, while the higher amount in October could be attributed to the influence of clear autumn weather. It was also found that there is a correlation between the amount of solar radiation and the amount of electricity generated, which can be said to be proportional to the amount of electricity generated.
(See [7] and [8])

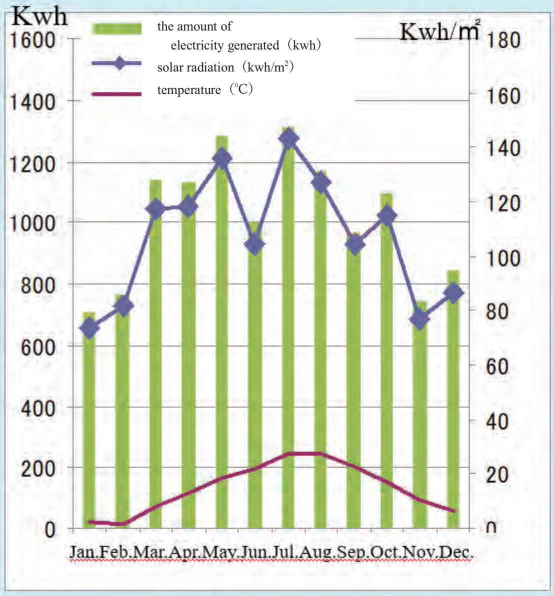


**Figure 3.7** Graphs of the amount of electricity generated, solar radiation and temperature (year)

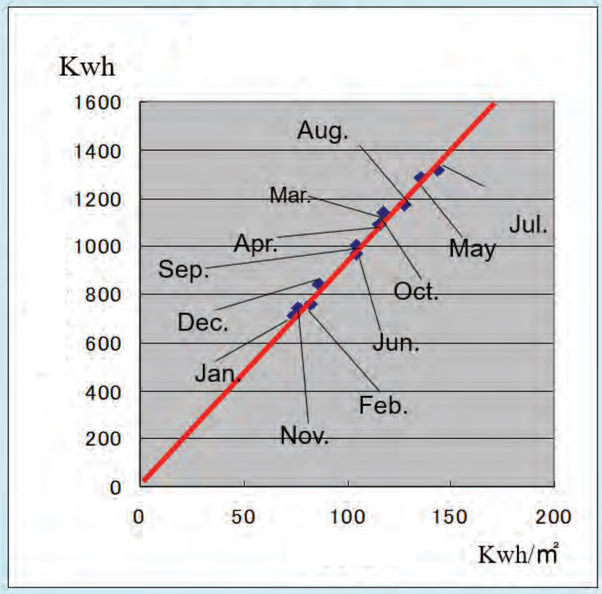


**Figure 3.8** Scatter diagram of the amount of electricity generated and solar radiation (year)

For future development issues, Students said that would like to research the relationship between solar panel temperature and power generation efficiency, the relationship between weather and power generation, and confirmation of whether there is a relationship between light wavelength and power generation capacity based on measurements of changes in power generation at different light wavelengths and whether it is practical to install solar panels in homes, considering installation costs and electricity costs. The difference in the amount of electricity generated by solar panels and irradiation angles. Since the panels become hot while the sun is shining, they would like to research more effective solar panels, including the use of solar heat.

**3.1.2 Course "Fundamentals of Mathematics"**

The goal of the course is two points:

1. Acquire knowledge and skills in data processing using a personal computer.
2. Acquire methods to analyze data accurately.
3. Acquired the ability to process and read information accurately and develop an attitude to apply it to real life.

The course aimed to give students with the basic knowledge and skills necessary to research an issue. To acquire the basic knowledge and skills necessary for research, the students used presentation software using a PC and spreadsheet software in data processing to write a report using a PC. The students experienced the hypothesis, verification, and summary process in the computer room.

**3.2 Consideration of Classroom Practice**

Problem Research 3.1.1 is an issue study where the mathematical activities described in 2.2 can be practiced. This example of setting an issue is from familiar data, formulating a hypothesis, analyzing, and verifying it, summarizing the results, and discovering a new issue – the case study of using ICT in this process. By using ICT to process a large amount of data, they could find relationships that confirm the results of previous studies and discover the characteristics of the area in which they live and future issues based on these characteristics. Data analysis brought the mathematical world closer together in familiar places in the real world.

ICT is effective in advancing research using data, and course 3.1.2 is a case study aimed at developing a foundation for actively using ICT to use data.

ICT was actively utilized in both cases to promote the Problem Research further. Next, since "data analysis" is a common theme in these cases, we will discuss them from the viewpoints of statistical education and the use of ICT.

**3.3 Consideration from the Perspective of Statistical Education and the Use of ICT**

In the current Courses of Study, emphasis is placed on the enhancement of statistical education, and the theme of "analysis of data" as a study content in the statistical field requires students to understand the basic ideas of statistics and to be able to organize and analyze data and identify trends using these ideas. The content of the study is data scattering and data correlation.

In the practical example, a graph was created for the scatter of the data. They also created a scatter plot of the correlation of the data to grasp the trend. In junior high school, students learned how to capture trends in data through histograms and representative values under the theme of organizing data, and in high school, they handled more data. They had the experience of making predictions about the scatter of data and relationships among data, analyzing and verifying them, and finding further issues from them. In addition, they handled data from solar power installed at the school to deal with concrete data from their immediate surroundings. By using something familiar, students were more motivated to learn and could experience the benefits of using ICT. These were practical examples of enhancing statistical education emphasized in the current Courses of Study.

In addition, Excel, a computer spreadsheet software, was used as the tool for analysis. Students realized that data can be quickly processed using spreadsheet software. ICT is effective in data analysis, allowing them to instantly and visually capture the relationships among the data. This motivated students to further research to find new relationships using the data, and to formulate and verify hypotheses that setting new conditions might make a difference. Thus, using ICT was very effective in promoting the students' research. In addition, the students' future issues, such as the temperature of solar panels and power generation efficiency, could be further researched using new measurement equipment. The relationship between the temperature of solar panels and changes in the irradiation angle of the panels could be mathematically modeled for use in ICT. Furthermore, it may be possible to develop this into cross-curricular content, such as physics and economics. Considering the above, this was a practical example that allowed students to realize the effectiveness of using ICT and will enhance the statistical education required in the future.

## 4. Future tasks

Using statistical analysis software other than Excel and the handling of various data also effectively enhances Problem Research and statistical education. We want to continue to practice using ICT and verify its educational effects.

Regarding enhancing statistical education, activities are required to process a large amount of data using ICT to grasp trends and solve or discover problems. It is also stated that dealing with familiar things for such data is necessary. In addition to Excel, there is R for statistical analysis software, which is a simple language to code and does not require variable declarations. Many libraries have been created on the web, so we can use complex methods simply by calling the libraries. Despite the algorithm's complexity, it can be written in just a few lines. In addition, information such as teaching materials can be obtained from the Statistics Bureau of the Ministry of Internal Affairs and Communications website. In this regard, along with the deployment of tablet terminals, many educational materials and tools related to statistics. The fact that data science is attracting attention is also significant. We want to make actual use of this information.

In addition, one textbook company has the following theme for problem-based learning on data analysis. The theme deals with a bird's body weight and wing area. The bird must obtain force from its wings to lift its body weight to fly. Let us investigate the relationship between birds and wing area using data from various birds. In this assignment, 29 types of birds are shown in a scatter plot with body weight on the horizontal axis and wing area on the vertical axis, and a box-and-whisker plot of body weight is also shown along the axes. The assignment in the textbook is to find outliers in body weight from the box-and-whisker plots of body weight. After removing the weight "outliers," the remaining data is used to create another scatterplot, from which the weight "outliers" are again removed, and another scatterplot is created. The correlation between body weight and wing area gradually becomes weaker. The question is, "How can this phenomenon be explained? We want to work on this kind of data as well.

Furthermore, the Japanese Ministry of Education, Culture, Sports, Science, and Technology stated that the ideal form of children's learning that we aim for is "individualized, optimal and collaborative learning that draws out the potential of all children. The basic idea regarding the use of ICT for building school education is that ICT is indispensable as a fundamental tool for school education and that it is necessary to change school education, solve various problems, and improve the quality of education through the optimal combination of existing practices and ICT The paper is based on the following principles.

Currently, students are using tablets as "stationery" daily. Students input and express their opinions and visualize them using the whiteboard function and exchange opinions in groups or as a whole class. Students input and calculate data using spreadsheet software and summarize and discuss the results. Teachers and students communicate with each other via tablet terminals, and students collaborate.

The use of ICT in mathematics education has two aspects. One uses ICT to support the development of mathematical content, such as graph and graphic drawing software, spreadsheet software, and programming languages, which are used to deepen mathematical thinking. The other is to use the software like a paper notebook, emphasizing a variety of output and expression.

It is also a future task to verify in what situations those two aspects can be used well and what educational effects can be achieved.

**Acknowledgments**

This paper is an introduction to the mathematical activities undertaken by the students under the author's guidance. The author wants to express my gratitude.

**Appendix**

*1 Problem Research is an activity that presents an answer to an issue with a rationale. Students set their assignments. Activities conducted by individuals or groups.

*2 Super Science High School (SSH) is a national project that has been implemented since 2002, with the Japan Science and Technology Agency (JST) as the primary implementing agency, with the objective of "conducting research and development focusing on science and mathematics education to foster future global human resources for science and technology. SSH is an epoch-making initiative in that it allows the promotion of science and technology human resource development in specific schools to begin at the secondary education level and was established as a symbolic project following the birth of the Ministry of Education, Culture, Sports, Science and Technology (MEXT) following the reorganization of central government ministries and agencies. The program was established as a symbolic project symbolizing the birth of the Ministry of Education, Culture, Sports, Science and Technology (MEXT) after central ministries and agencies.

*3 For the Courses of Study, we referred to the "English Translation of the Courses of Study for Junior High Schools (Provisional Translation)" posted on the website of the Ministry of Education, Culture, Sports, Science, and Technology
(Retrieved July 30, 2023,
https://www.mext.go.jp/a_menu/shotou/new-cs/youryou/eiyaku/1298356.htm).

*4 We translated some image diagrams into English based on p. 26 of Reference [3].

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# The Study of Geodesic Computations on Sphere and Spheroid with Optimized Numerical Methods

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**Abstract**: *This article explores the problem of determining the shortest distance on spheres and spheroids, which is fundamental for calculating geodesic paths and establishing the computational relationship for all possible routes on a surface. Geodesic paths hold significant importance across a wide range of geometric applications. The study classifies geodesic paths on surfaces into two categories: analytical and numerical and compares the Runge-Kutta and Euler methods as computational techniques. These methods are employed to solve nonlinear ordinary differential equations governing geodesic paths. Based on the comparative analysis, the Runge-Kutta method is identified as the preferred approach for accurate calculations in this study. To ensure efficient computation and swift determination of values, the proposed calculations are implemented using the Google Colab platform, leveraging its capabilities for efficient numerical computations. Convergence rates analysis highlights the Runge-Kutta method's superior accuracy, making it the preferred choice when exact solutions are elusive. In summary, this research underscores the Runge-Kutta method's significance in approximating solutions to differential equations, offering valuable guidance for method selection in various applications involving geodesic paths.*

## 1. Introduction

Geodesic paths, which represent the shortest distance between two points on a surface [1, 2, 16], hold immense significance across diverse fields like computer graphics, geometric modeling, and surface segmentation [11]. While extensive research has been dedicated to geodesics on spheres, investigating geodesics on more complex surfaces poses a formidable challenge due to their fluctuating curvatures and non-differentiable nature [5, 15]. The curvature of a curve in a plane stands as a fundamental measure of its departure from a straight line [3]. Remarkably, a straight line manifests zero curvature [4], whereas a circle with radius r possesses a curvature of 1/r, clearly illustrating the inverse relationship between curvature and radius. Understanding the interplay of geodesics and curvature yields profound insights into the geometric attributes of surfaces.

The problem of determining the shortest path or distance between two points on a surface S in three-dimensional space, termed the geodesic path and geodesic distance, has undergone extensive exploration in the realms of differential geometry and computational geometry [5]. Traditional methods for computing geodesics can be categorized into three classes: solving the nonlinear differential equations that govern geodesics via numerical integration techniques, such as the Runge-Kutta method [6]; computing discrete geodesics on surfaces through specific rules; or constructing geodesics on smooth surfaces using geometric methods [7, 8, 16]. In the realm of differential geometry, the Euler-Lagrange equations have played a pivotal role, initially introduced by Euler and later refined by Lagrange to establish a mathematical foundation for the Principle of Least Action.

These equations serve as a versatile framework for analyzing a wide array of functions and have found application in the study of geodesic paths, including those subject to nonstationary constraints. While the first approach, employing numerical integration methods, yields precision, the inherent complexity of the differential equations governing geodesics often renders them arduous to solve. Discrete geodesics have garnered attention with the advancement of computational capabilities, yet they cannot be directly computed on the original smooth surface [7, 16]. Moreover, there exists limited research on directly constructing geodesics on smooth surfaces using geometric methods [9, 10].

This paper endeavors to explore geodesics on diverse surfaces and meticulously compare the accuracy of two numerical methods: Euler's method and the fourth-order Runge-Kutta method [6], as applied to the approximation of these geodesic paths. The computation and analysis of geodesics play an indispensable role in unraveling the properties and behaviors of surfaces. This study encompasses the investigation of geodesics on multiple surfaces, accompanied by theoretical elucidations of geodesic theory. The employed numerical methods, Euler's method and the fourth-order Runge-Kutta method, serve to compute geodesic paths [6], and their accuracy is subjected to rigorous testing. Additionally, we delve into the examination of geodesic distances through numerical calculations on several intriguing surfaces.

The successful execution of this experimental endeavor is made possible thanks to the generous financial support extended by Prince of Songkla University, Surat Thani Campus. The authors wish to express their profound gratitude for the invaluable backing and reinforcement that facilitated the realization of this research.

## 2. Preliminaries

**Definition 2.1 [Parameterized Surface]**: Consider a continuously differentiable $\mathbb{R}^3$ valued function of two variables. The two straight lines passing through the origin, given by the following equations:

$$G(u,v) = (x(u,v), y(u,v), z(u,v)) \tag{2.1}$$

**Definition 2.2 [Geodesic Curve]:** A geodesic curve $\gamma \in P(x,y)$ is such that $L(\gamma) = d_M(x,y)$.

**Definition 2.3 [Geodesic Distance]:** Given some Riemannian space $(M,H)$ with $M \subset \mathbb{R}^s$, the geodesic distance is defined as

$$d_M(x,y) :\overset{\text{def}}{=} \min_{\gamma \epsilon P(x,y)} L(\gamma) \tag{2.2}$$

where $P(x,y)$ denotes the set of piecewise smooth curves joining $x$ and $y$ for any $x, y \in M^2$

$$P(x,y) \overset{\text{def}}{=} \{\gamma \backslash \gamma(0) \text{ and } \gamma(1) = y\}. \tag{2.3}$$

The shortest path between two points with the Riemannian metric is called a geodesic. If the metric $H$ is well chosen, then geodesic curves can be followed salient features on images and surfaces.

A geodesic curve between two points might not be unique for instance about two antipodal points on a sphere. To perform the numerical computation of geodesic distances, we fix a set of starting points $S = (x_k)_k \subset M$ and consider only distance and geodesic curves from this set of points.

**Definition 2.4 [Geodesic Euler Method]:** Let $\tilde{S}$ be a polyhedral surface with a polyhedral tangential vector field $v$ on $S$, let $y_0 \in S$ be an initial point, and let $h > 0$ a (possibly varying) stepsize. For each point $p \in S$ let $f(t_i, y_i, y(t_0))$ denote the unique straightest geodesic through $p$ with initial direction $v(p)$ and evaluated at the parameter value $t$. A single iteration step of the geodesic Euler method is given.

Euler method can be classified as an easy-to-understand method for solving this ordinary differential equation. Principles for solving this ordinary differential equation. We find the approximate solution of $y_{i+1}$ at $t_{i+1}$ from the result. $y_i$ is known at $t_i$, where $h$ is the width of the calculation. Substitute the values in the equation

$$y_{i+1} = y_i + f(t_i + y_i)h \quad \rightarrow y_{i+1} = y_i\emptyset(t_i, y_i, h)h\,, y(t_0) = y_0$$
$$y_{i+1} = y_i + \emptyset(t_i, y_i, h)h\,, y(t_0) = y_0$$
$$\emptyset = a_1k_1 + a_2k_2 + \cdots + a_nk_n$$

where $a_1, a_2, \dots, a_n$ are constants and
$$k_1 = f(t_i, y_i)$$
$$k_2 = f(t_i + p_1h, y_i + q_{11}k_1h)$$
$$k_3 = f(t_i + p_2h, y_i + q_{21}k_1h + q_{22}k_1h)$$
$$\vdots$$

where $p_1, p_2, \dots, q_{21}, q_{22}, \dots$ are constants.

**Theorem 2.5 [Geodesic Runge-Kutta Method]:** Let $\tilde{S}$ be a polyhedral surface with a polyhedral tangential vector field $v$ on $\tilde{S}$ , let $y_0 \in \tilde{S}$ be an initial point, and let $h > 0$ a (possibly varying) stepsize. For each point $p \in \tilde{S}$ let $f(x, y, y(x_0))$ denote the unique straightest geodesic through $p$ with initial direction $y_0$ and evaluated at the parameter value $x$. A single iteration step of the geodesic Runge-Kutta method is given. This estimates that is closer to the real value than the Euler method. With the following methods for $y' = f(x, y), y(x_0) = y_0$ where $f$ There is only one possible answer on a range with $x_0$ in $y_{n+1} = y_{(x_{n+1})}$ at $x_{n+1} = x_0 + (n+1)h$ where $n = 0, \dots, N-1$.

We will get the formula $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where
$$k_1 = hf(x_n, y_n),$$
$$k_2 = hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1\right),$$
$$k_3 = hf\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2\right),$$
$$k_4 = hf(x_n + h, y_n + k_3),$$
$$x_{n+1} = x_n + h.$$

**Property 2.6 [Sphere]:** The properties below are with respect to the parameterization $r$ given above. Parameter-curve tangent vectors the parameter-curve tangent vectors are

$$r_\theta(\theta, \varphi) = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix}, \quad r_\varphi(\theta, \varphi) = \begin{pmatrix} -r\sin\theta\sin\varphi \\ r\sin\theta\cos\varphi \\ 0 \end{pmatrix}. \tag{2.4}$$

The standard unit normal vector field is

$$\widehat{N}(\theta, \varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{2.5}$$

The prolate and oblate spheroidal infinite elements are based on a mathematically rigorous formulation, viz. a multiple expansion (a power series in 1 /r, where r is a spheroidal radius, multiplied by an oscillatory factor)

**Property 2.7 [Prolate spheroid]:** We now turn to the case when $S$ a prolate spheroid is surrounding the structure. Introduce prolate spheroid coordinates $(r, \theta, \emptyset)$ defined by

$$\begin{cases} x = \sqrt{r^2 - f^2}\sin\theta\cos\emptyset \\ y = \sqrt{r^2 - f^2}\sin\theta\sin\emptyset \\ z = r\cos\theta \end{cases} \tag{2.6}$$

and let $S$ be given by $r = r_1$(as before), where $r_1 > f$. That is, $S$ is the surface $\frac{z^2}{r_1^2} + \frac{x^2}{r_1^2} + \frac{y^2}{-f^2} = 1$ in Cartesian coordinates, which is the surface obtained by rotating an ellipse with semi-major axis $r_1$, and foci at $(0,0,\pm f)$ about its major (z) axis.

**Property 2.8 [Oblate spheroid]:** Finally, we turn to the case when $S$ is an oblate spheroid, where the appropriate coordinates $(r, \theta, \emptyset)$ are defined by

$$\begin{cases} x = r \sin\theta \cos\emptyset \\ y = r \sin\theta \sin\emptyset \\ z = \sqrt{r^2 - f^2} \cos\theta \, . \end{cases} \tag{2.7}$$

We again let $S$ be given by $r = r_1 (r_1 > f)$, yielding the surface $\frac{x^2+y^2}{r_1^2} + \frac{z^2}{r_1^2 - f^2} = 1$ in Cartesian coordinates, which corresponds to rotating the same ellipse as before about its minor axis.

Numerical analysis is a branch of mathematics that deals with the study of methods and procedures used to obtain approximate solutions to mathematical problems. EndreSull and David Mayer defined the numerical analysis as a branch of mathematics that provides the theoretical foundation for the numerical algorithm, we rely on to solve a multitude of computational problem in mathematical models or the study of algorithms that use numerical approximation [18]. Numerical analysis naturally finds applications in all fields of engineering and the physical science, but in this 21st century, the life science and even the arts have adopted elements of scientific computations [19]. The overall goals of the field of numerical analysis in the design and analysis of techniques to give approximate but accurate solution are hard to get. It is, therefore, important to be able to estimate the error involved in such approximation. Thus, the aims of this work were to compare between Euler and Runge-Kutta methods to a rigorous analysis in order to demonstrate the efficiency of the methods to other similar techniques. It was also examining the effect of the steps on the accuracy of the techniques. Secularity band differences in the results of some numerical methods with the standard Euler's method of order three and four was examined.

To study experimental numerical methods such as implementing and running computational codes. It is necessary to understand in detail the properties/behavior of the numerical algorithms considered. Which can be represented in the form of an algorithm for computation and ease of programming. By setting the constant values as $x_0 = 0$, $y_0 = 1$, $h = 0.025$, and $x = 1$. Importing the calculation process through the program side in the equation (2.5).

From our previous discussion, it is clear that errors play an important role in numerical analysis. The error is of various types and arise from different sources. We have already seen two types of errors namely, truncation errors and roundoff errors. Among the other types of errors are input errors, which enter the computation through input data. This could happen, for example, if some inputs are known only approximately or determined, say, experimentally. Such errors are not the subject of study in numerical analysis. Neither are mistakes or blunders, because they cause the whole computation to become worthless and numerical analysis can do nothing to save the situation.

Data conversion also introduces errors, which are input errors. Data conversion error belongs to the category of round-off errors, to be discussed presently. We are interested here in numerical errors which, as we have seen, are of two kinds, namely, round-off errors and truncation errors.

**Definition 2.9 [Error]** : We formally define

$$\text{Error} = \text{True value} - \text{Approximate value} \tag{2.8}$$

and

$$\text{Relative Error} = \frac{\text{Error}}{\text{True value}} \tag{2.9}$$

In this study we use an algorithm to infer the relationship between Euler's method and Rung-Kutta method by the value used to calculate each method of the formula, respectively.

## 3. Methods and Results

To compute the geodesic path using the Euler's and Runge-Kutta methods, we first need to define the geodesic equations for the specific surface on which we want to find the geodesic path. The geodesic equations are differential equations that describe the curves representing the shortest distance between points on the surface.

Let us consider a surface $S$ with parameterization given by two parameters u and v, and the geodesic path is represented by a curve with coordinates $(u(t), v(t))$, where t is the parameter along the geodesic path. The geodesic equations can be expressed as a set of first-order differential equations:

$$du/dt = f(u, v) \tag{3.1}$$
$$dv/dt = g(u, v) \tag{3.2}$$

where $f(u, v)$ and $g(u, v)$ are functions that depend on the specific curvature properties of the surface S. To compute the geodesic path using the Euler method, we discretize the geodesic equations and update the coordinates $(u, v)$ iteratively for small time steps. Here are the general processes:

**Algorithm 3.1** Euler's Method
**1. Start**
**2. Define function**
**3. Get the values of x0, y0, h and xn : Here x0 and y0 are the initial conditions**
**h is the interval**
**xn is the required value**
**4. n = (xn – x0)/h + 1**
**5. Start loop from i=1 to n**
**6. y = y0 + h*f(x0,y0)**
**x = x + h**
**7. Print values of y0 and x0**
**8. Check if x < xn**
**If yes, assign x0 = x and y0 = y, Otherwise, go to 9.**
**9. End loop i**
**10. Stop**

It is essential to choose a small enough time step to ensure accuracy in the approximation. Larger time steps may lead to inaccuracies, especially if the geodesic path has sharp curves or encounters regions of high curvature. It is important to note that the Euler method is a first-order numerical method and may not be as accurate as higher-order methods like the Runge-Kutta method. However, for simple surfaces and short geodesic paths, the Euler method can still provide reasonable approximations. Keep in mind that the specific form of f(u,v) and g(u,v) will vary depending on the surface $S$ and its curvature properties, so the geodesic equations need to be adapted accordingly for each surface under consideration.

**Algorithm 3.2** Runge-Kutta Method

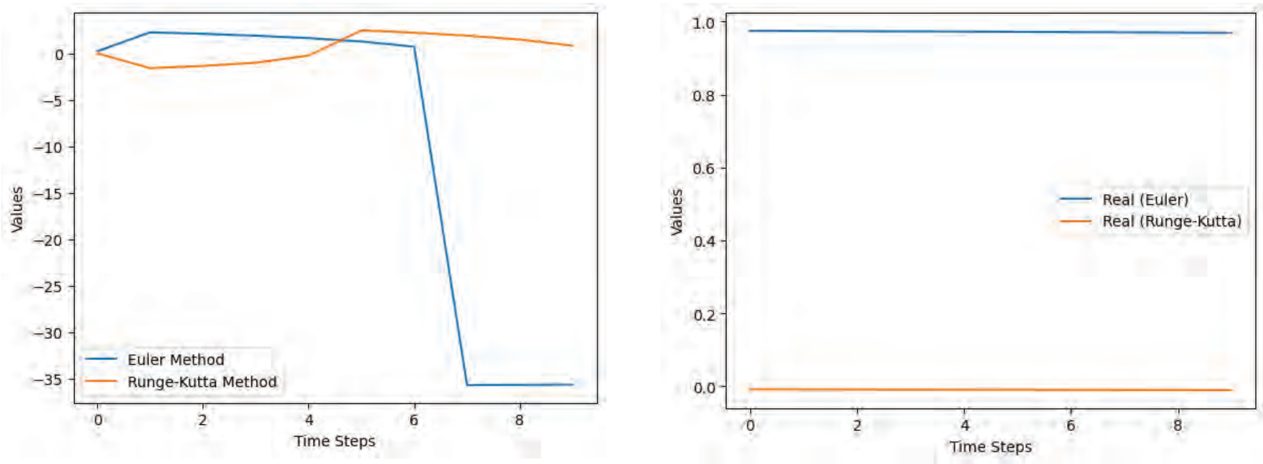
```
1. Start
2. Define function f(x,y)
3. Read values of initial condition (x0 and y0)
        number of steps (n)
        calculation point (xn)
4. Calculate step size (h) = (xn - x0)/n
5. Set i=0
6. Loop      k1 = h * f(x0, y0)
             k2 = h * f(x0+h/2, y0+k1/2)
             k3 = h * f(x0+h/2, y0+k2/2)
             k4 = h * f(x0+h, y0+k3)
             yn = y0 + k
             i = i + 1
             x0 = x0 + h
             y0 = yn
   While i < n
7. Display yn as result
8. Stop
```

The Runge-Kutta method uses four intermediate slopes (k1, k2, k3, k4) to compute the update, resulting in a more accurate approximation compared to the Euler method. The method effectively considers multiple points within each time step, capturing more information about the behavior of the geodesic path on the surface. The choice of a small-time step $\Delta t$ is still important for accuracy in the approximation. By using the fourth order Runge-Kutta method, we can achieve better accuracy even for complex surfaces and longer geodesic paths. Again, it's important to adapt the specific functions $f(u, v)$ and $g(u, v)$ according to the surface S under consideration, as the curvature properties and geodesic equations will vary for different surfaces.

**Table 3.3** Exact numerical values from our study

| Round | Euler | Runge | Real (E) | Real(R) | error(E) | error(R) |
|---|---|---|---|---|---|---|
| 0 | 0.22958 | 0.02464 | 0.97500 | -0.00833 | 0.74542 | 0.03297 |
| 1 | 2.27594 | -1.57748 | 0.97438 | -0.00853 | 1.30157 | 1.56894 |
| 2 | 2.11864 | -1.34199 | 0.97375 | -0.00874 | 1.14489 | 1.33325 |
| 3 | 1.91677 | -0.99420 | 0.97313 | -0.00894 | 0.94364 | 0.98526 |
| 4 | 1.65444 | -0.22553 | 0.97250 | -0.00915 | 0.68194 | 0.21638 |
| 5 | 1.29667 | 2.48888 | 0.97188 | -0.00936 | 0.32479 | 2.49823 |
| 6 | 0.72231 | 2.23764 | 0.97125 | -0.00956 | 0.24894 | 2.24720 |
| 7 | -35.68496 | 1.92211 | 0.97063 | -0.00977 | 36.65559 | 1.93188 |
| 8 | -35.66473 | 1.50234 | 0.97000 | -0.00997 | 36.63473 | 1.51231 |
| 9 | -35.64273 | 0.83337 | 0.96938 | -0.01018 | 36.61210 | 0.84355 |
| AVE. %error | | | | | 11.52936 | 1.31700 |

In the result of our study, from Table 3.3, it shows that the results were significantly improved with the Runge-Kutta method. The table below summarizes all the results for the method. In comparing the two methods. This is trivial that the Runge-Kutta method is better than Euler especially when multiple xn values are required. If only a few xn values are required, the Runge-Kutta technique is superior to the Euler method as it produces slightly better results: See the following figure:




(a) (b)

**Figure 3.1** Graph of approximate solution of standard Euler and Runge-Kutta compare to (a) exact solution and (b) real value

The concept of convergence rate serves as a fundamental criterion for assessing the performance of numerical methods. It quantifies how quickly a numerical method converges to the true solution as the step size decreases, with higher values indicating faster convergence. In our analysis, we will examine the convergence rates of two numerical methods: Euler and Runge-Kutta. These rates will provide valuable insights into the methods' respective abilities to approach accurate solutions with decreasing step sizes. Subsequently, we will discuss the implications of these convergence rates, considering the trade-offs between accuracy and computational efficiency, to make informed decisions regarding the choice between the Euler and Runge-Kutta methods in specific applications. See the following figure:

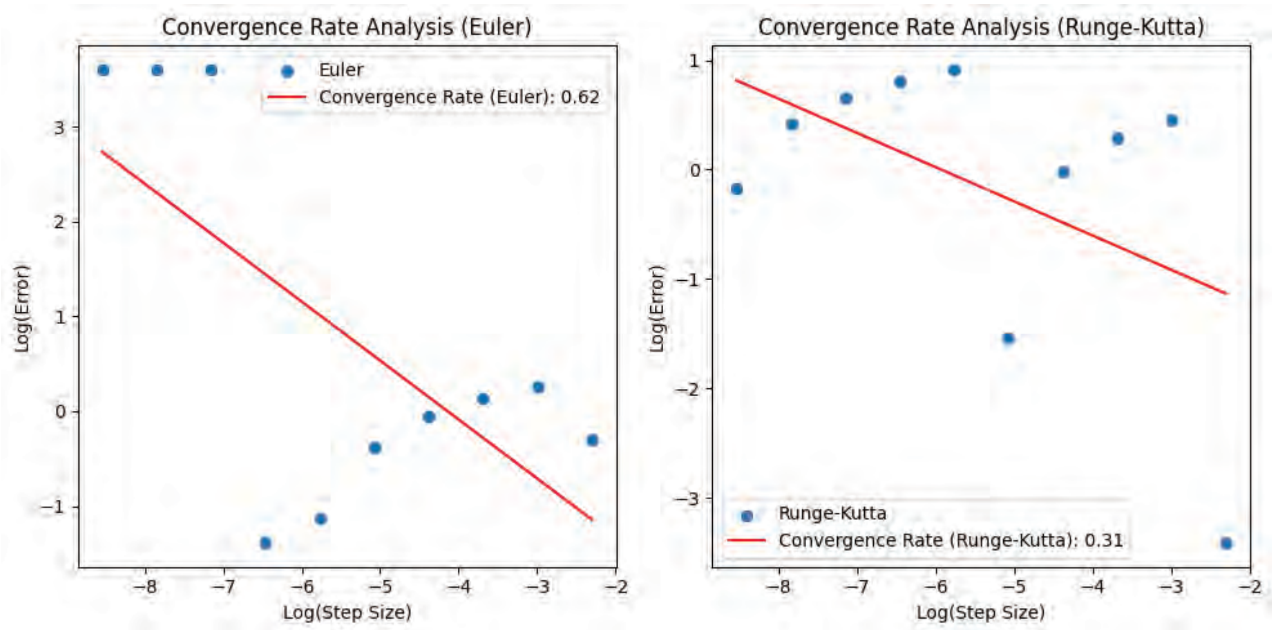



**Figure 3.2** Graph of standard Euler and Runge-Kutta compare to Percentage-error

The convergence rates, Convergence Rate (Euler): 0.62 and Convergence Rate (Runge-Kutta): 0.31, provide valuable insights into the performance of these numerical methods. Convergence rate measures how quickly a method approaches the true solution as the step size decreases, with higher values indicating faster convergence. In this context, the Euler method outperforms the Runge-Kutta method with a higher convergence rate (0.62), signifying faster convergence as step size decreases. Conversely, the Runge-Kutta method has a lower convergence rate (0.31), indicating slower convergence and requiring finer discretization for the same accuracy level. These convergence rates prompt considerations of practicality; the Euler method converges faster but may demand smaller step sizes and increased computational effort, while the Runge-Kutta method, with slower convergence, may offer efficiency advantages with larger step sizes. Therefore, selecting the most suitable method depends on balancing accuracy and computational efficiency in your specific application, with these convergence rates guiding your decision-making process.

## 4. Conclusion

In this study, we have conducted a thorough comparison between two numerical methods, Euler's method and the Runge-Kutta method, for approximating solutions to differential equations. These methods share the common iterative approach of updating variables over time; however, they diverge in crucial aspects that profoundly affect their accuracy. Euler's method initiates with an initial variable value and subsequently updates it based on the derivative at each step, signifying the rate of change at a specific point. The new value is calculated by multiplying the derivative by a small time interval and adding it to the previous value, iterating until the desired solution is reached. In contrast, the Runge-Kutta method introduces a new default value and systematically refines it. By incorporating multiple derivative approximations at various points within each time interval and calculating weighted averages of these approximations, it achieves heightened accuracy. Our experiments, conducted in a web service environment such as Google Colab, consistently demonstrate the Runge-Kutta method's superior performance over Euler's method. This superiority arises from its capacity to leverage a wealth of information regarding variable behavior within each time interval, resulting in significantly more precise approximations. Consequently, the Runge-Kutta method consistently exhibits smaller error values compared to Euler's method, indicating its superior accuracy.

In practical applications where exact solutions are often elusive, numerical methods like Euler's and the Runge-Kutta method prove invaluable. While exact solutions can serve as benchmarks for accuracy assessment, they are frequently unavailable, especially for intricate or nonlinear differential equations. In such scenarios, the accuracy of these methods becomes paramount, and it can be gauged effectively through rigorous testing and error analysis.

Our comprehensive analysis demonstrates that the Runge-Kutta method consistently excels, providing heightened accuracy across a spectrum of scenarios, including both linear and nonlinear solutions, even extending to fourth-order differential equations. This exceptional accuracy positions the Runge-Kutta method as the preferred choice in situations where precision plays a pivotal role.

In summary, our findings strongly support the Runge-Kutta method's superiority in terms of accuracy when approximating solutions to differential equations. Its advanced approximation techniques and consistent excellence in precision underscore its significance in practical applications demanding superior accuracy.

**Acknowledgements** The author would like to thank her colleague and reviewers for their valuable comments and input.

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# Virtual reality to study geometrical aspects of architectural heritage

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**Abstract**

*In this paper we address the construction of virtual reality scenarios for the NeoTrie VR software, to make some 3D models of architectural objects manageable anywhere. Students will be able to see the pieces (muqarnas, vaults and arches and towers) in a 3D model, manipulate and assemble them, to understand their spatial arrangement. They will also be able to use Neotrie tools to build and thus better assimilate the underlying geometric structures.*

## 1 Introduction

Islamic architecture in Andalusia has provided us with some of the most beautiful architectural objects and intricate geometry. All of these objects have very useful characteristics in the teaching of mathematics, but some are also especially suitable for understanding geometric processes in 3D [6], [7]. This is the case of the muqarnas domes, where hundreds, and sometimes thousands, of pieces are added to form a complex dome given by the union of the pieces. We also deal with crossed ribbed vaults, made for the first time in Córdoba, and which allow the construction of domes with interlocking arches. Finally, some architectures, such as that of the Torre del Oro in Seville, a separate tower that gave welcome along the Guadalquivir river in Seville, are especially suitable for construction in 3D. We will detail these constructions in Section 2.

However, 3D tools are necessary to transfer and understand these concepts with students. Our main goal in this article is to explain the process of designing and creating various virtual reality scenarios with interactive activities.

The chosen tool for this task is NeoTrie VR (briefly, Neotrie) [1], a multiplayer virtual reality software, developed by Virtual Dor (spin-off of the University of Almería), which allow users to create, manipulate and interact with geometric objects and 3D models in general, of different types. Before thinking about its implementation in virtual reality, the 3D models (Section 3) were designed for 3D printing to perform manipulative workshops [8].

[1] `https://www2.ual.es/neotrie/`

Recent implementations by the developers team of Neotrie (see [9, 4]) have allowed to generate new scenarios, with friendlier interfaces, menus and new functions, such as arcs creation.

Hence, students will be able to see the pieces in virtual reality, manipulate and assemble them, to understand their spatial arrangement. The VR scenes will also include interactive activities to understand the architectural structures by building them themselves using the dynamic geometry tools provided by the software (see Section 4).

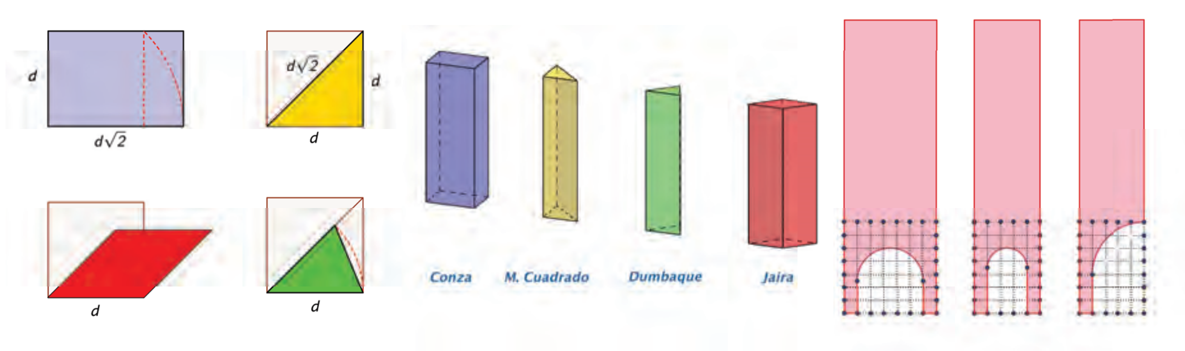
These interactive activities can be better carried out simultaneously with the manipulative workshop with the printed pieces, usually in the same room that hosts the exhibition Paseo *Matemático Al-Ándalus*, as it was done the University of Almería in October 2022, and the Instituto Cervantes of Fez in May 2023.

## 2 Mathematical description

The muqarnas vaults are among the most complex 3D objects in decoration in islamic architecture. They are formed from the aggregation of simpler pieces [3, 5, 10]. Muqarnas are marvelous geometric art creations that challenge our spatial understanding by their capacity of interrelation.

To create them we start from 4 basic prisms, with sections based on $\sqrt{2}$ that fit together (Figure 1a). These prisms are called *conza*, *medio cuadrado* (half square), *dumbaque* and *jaira* (Figure 1b). Other singular pieces such as the *estrella* (star) or the *almendrilla* (little almond) are also added to these. The star usually forms centers with radial symmetry (total or local) around it.

The final pieces or *adarajas* are made by producing some concave cuts of these basic prisms, at one end, in one or both directions. The cuts are based on integer divisions of 7 and 5 (1/7 is very close to $\sqrt{2}$). At the ends of a piece there remain elongated parts of 1/7 of the length or width of the piece, or square, 1/7 x 1/7 size of section. These pins, called *patillas*, are where the small arcs of the cut ends. In Figure 1c we show the *conza* piece with the corresponding cuts.



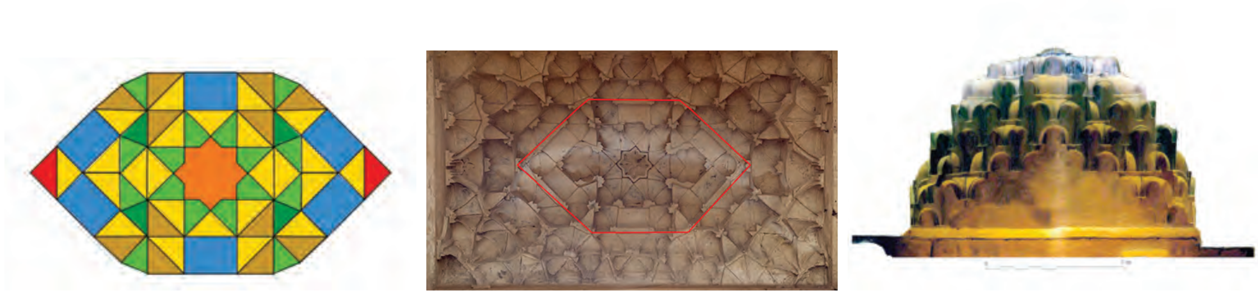

(a) Muqarnas prisms plan. (b) Muqarnas prisms. (c) Convex cutting of a conza.

Figure 1: Geometry of muqarnas.

When the prisms are assembled they form, in plan, a tessellation of a rectangular or polygonal area (see Figure 2a). There are innumerable ways to cover such a space with these prisms, which together with the creativity of the craftsmen, gave rise to a great multitude of possible designs. But all of them maintain properties of radial symmetry around the stars and axial symmetries in the basic axes of the polygon we are dealing with.

After a layer is added around a central piece, new layers are added at lower levels, thus forming level contour lines. A muqarnas dome can have up to dozens of layers at different levels. The patillas or pins are joined together making clusters. This arrangement, together with the concave cuts, makes the whole look very pleasing to the eye. Is dome structure reminiscent of stalactites in a cave. It is said that the muqarnas domes are inspired by these natural formations (Figure 2c).

In this work we have modeled the central part of the *Puerta del Lagarto* ceiling (XII A.D.), a door in the main mosque of Seville (Spain), over which the present cathedral was built. The original ceiling is the first and simplest built in Andalusia. It has 255 pieces arranged in 7 different levels. (Figure 2b)



(a) Muqarnas vault plan. (b) Picture from bottom. (c) Photogrammetric model showing dome-like structure.

Figure 2: Muqarna vault of *Puerta del Lagarto*.

Another of the pieces that we use in VR scenarios are ribbed domes. These are vaults formed by interlaced arches that cross each other. In the Mosque of Córdoba (VIII A.D.) these domes were tested in a pioneering way. There we can find three different ribbed dome types, although all of them are formed by 8 arches [1].

The first type has 4 crossing points, and in each of them 3 arches of different lengths and curvature cross simultaneously. Although geometrically it is the simplest to construct, it nevertheless has practical problems in its execution, due to the resulting complex crossings. The second type has 8 crossing points, with two overlapping arcs at each point, and where all arcs are equal. It forms in its projection, on a plane parallel to the ground, a stellate polygon *[8/2]* (Schläfli symbol for stellate polygons). It is the most colorful dome and the one that leaves more space in the center, and yet it was not the most replicated in history [2].

This honor goes to the third type of dome, which although geometrically more complicated, it is easier to execute due to the greater number of crossings, which makes the crossing pieces easier to design. It also has 8 equal arcs, but each arc crosses with the 4 consecutive arcs, if these are arranged circularly. It forms, in its projection on a plane, a stellated polygon of type *[8/3]*. In Figure 3 we show the geometrical structure of the three domes. In Figure 4 we show the photogrammetric view of the three types of domes in the Mosque of Córdoba, and in Figure 5 the 3D models for each one of these types.

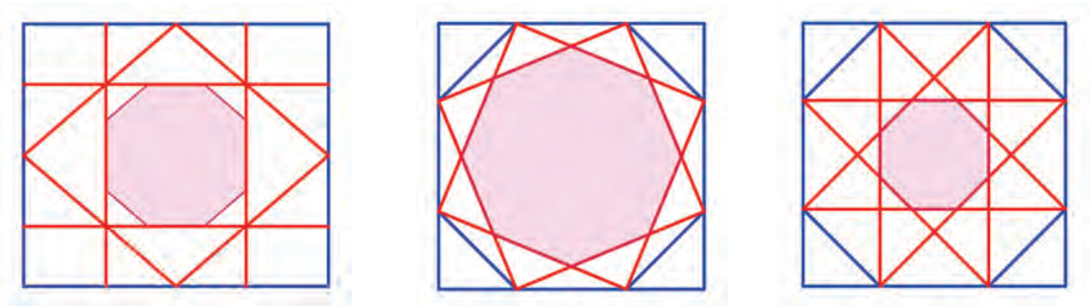


Figure 3: The geometry of the 3 ribbed domes. All with 8 arches, but with 4, 8 and 12 crossing number, respectively.

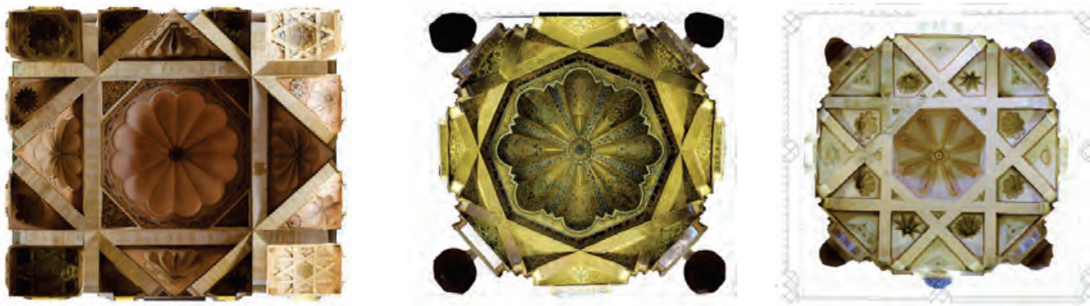


Figure 4: Photogrammetric view of the 3 types of ribbed domes. From left to right: Villaviciosa chapel, Maqsura and lateral Maqsura domes.

Figure 5: Rendering of the 3 types of ribbed domes: lattice + diamond, and stars [8/2] and [8/3].

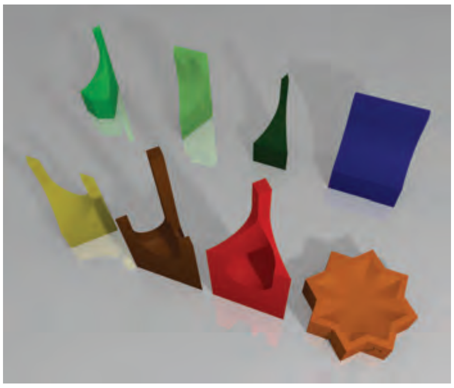
Finally, in the third VR scenario we address the construction of the *Torre del Oro* in Seville. It is an interesting prismatic tower for the manipulation of prisms and simple 3D constructions. It is the only dodecagonal tower in the medieval islamic world (dodecagonal regular prism).

In it, an external dodecagon (whose diametrical section is $\sqrt{2}$, surrounds a hexagonal prism that contains a spiral staircase inside. The construction of this tower in its spatial form allows us to understand the processes of duplication from a hexagon to a dodecagon and to manipulate them spatially.

# 3 3D models

Prior to the development of the virtual reality scenarios we modeled all the pieces in order to 3D print them. Those 3D prints have been used to not only ensure that the proportions of the pieces are correct but also to guarantee that they will fit correctly together. Moreover, the pieces have also been used in different workshops with students and the general public where we noticed the interest of the participants in the manipulation of these objects to fully understand their geometrical properties and relations [8]. The modeling have been done using FreeCAD (`https://www.freecad.org`), a free parametric modeler which is quite suitable for this kind of pieces that are well described by its geometrical properties.

For the muqarnas scenario we selected the central part of the muqarnas ceiling of the *Puerta del Lagarto* in the Cathedral of Seville. That section contains 8 different pieces (Figure 6a). All of them part from a simple prism extruded from a rectangle, a star-shaped polygon or a triangle. Then some parts of the prism are cutted out by means of the application of boolean operators with some other basic volumetric parts (cylinders, spheres, cubes...). For example, in Figure 7 one of the muqarnas is constructed from a prism of rectangular base to which a cylinder and another rectangular prism have been substracted. In addition to the muqarnas themselves, a layered box was also designed to help participants to mount the muqarnas ceiling replica that are attached to it by means of some neodymium magnets. In Figure 6b we show the 3D printed muqarnas being used in a practical workshop.



(a) The modeled muqarnas of the central part of the ceiling in the *Puerta del Lagarto*

(b) The 3D printed muqarnas used in a workshop with high-school students.

Figure 6: Geometrical modeling of muqarnas.

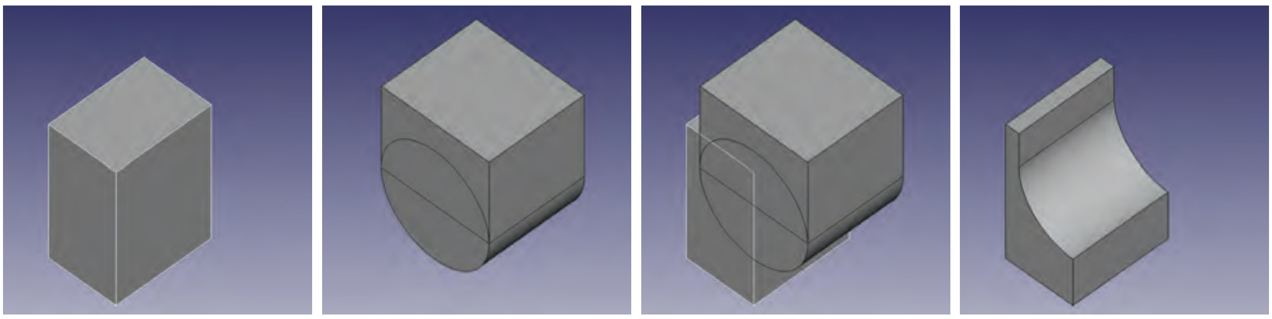
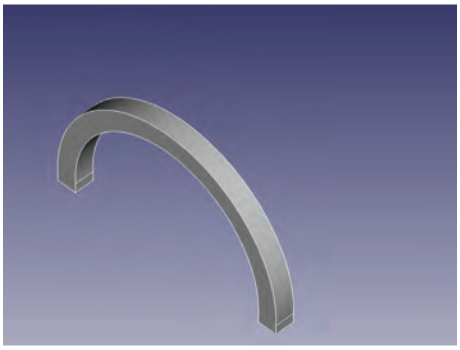


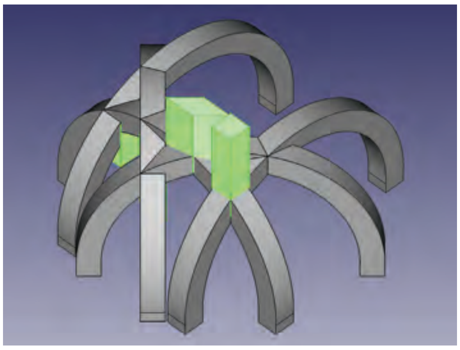
Figure 7: Geometrical modeling of one of the muqarnas.

For the crossed ribbed vaults scenario we wanted to model the arches in such a way that 8 pieces of a single model of an arch allow the construction of the ribbed vault. To do so we started with a simple extruded piece from a 2D model of the shape of the arch and then applied different boolean operators to make some cuts and additions in the intersection points of the arches that allow the interlocking of the adjacent arches (Figure 8).

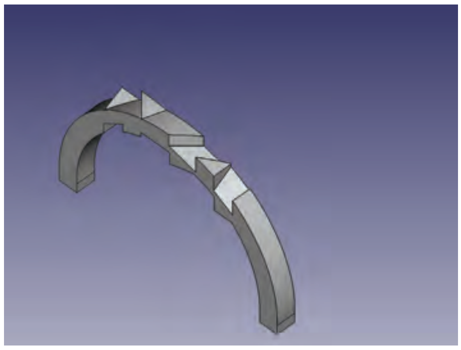


(a) The initial extruded arch from a 2D shape.

(b) The adjacent arches on the vault.



(c) Some prisms are added or subtracted to produce some cuts



(d) Final model of the arch.

Figure 8: Modeling an arch for the crossed ribbed vaults scenario.

In addition to the arches themselves we have also modelled the domes that rest over the arches. This last piece is used in the workshop once the participants have been able to engage all the arches (Figure 9).

Figure 9: Complete model with interlocked arches and upper dome. Picture of the 3D printed model.

# 4 The VR scenarios

Once in the game, there is a box *Paseo Matemático Al-Ándalus* to access the activities of this project. Each scenario contain pictures in this article, short explanations and instructions, the described 3D models (STL format) to assemble, and some interactive activity to work on the geometry associated with the corresponding architectural structures.

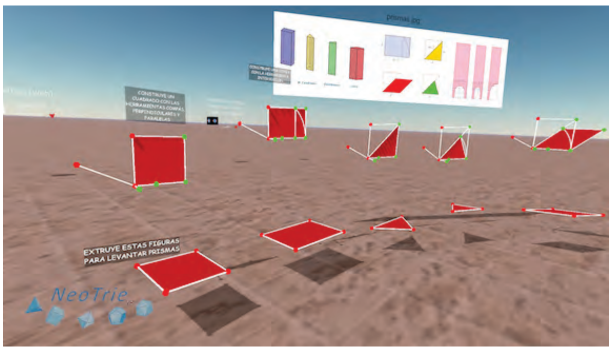
The first attempt is usually started with simpler exercises for the user, in which the patterns and organization of the architectural elements are recognized in broad strokes. Later, in a math workshop led by the teacher, students deepen and assimilate the necessary concepts to better understand the geometric structures. To this end, the teacher presents more complex exercises using dynamic geometry tools, such as the compass or perpendiculars and parallels, to build simple models of these pieces and STL structures.

By carrying out these activities the students between the ages of 12 and 16 years work on perpendicularity and parallelism in space, the notions of straight prisms with regular bases, practice and learn about different proportions, build circles and arcs in space, spheres, cylinders and cones.

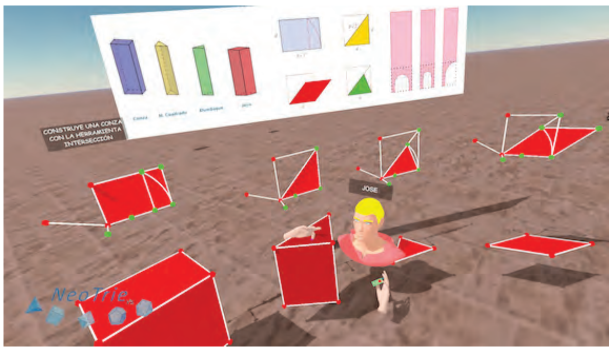
The first scenario (Figure 10) is devoted to muqarnas. Here the player must make copies of the pieces and place them on the virtual ceiling following the pattern indicated on a template (Figures 10b). One can also construct the prisms that generate the muqarnas, by using the compass, perpendicular and parallel tools, and then extruding. However, the software does not yet allow to make spherical or cylindrical cuts on the prisms, which would allow to obtain each of the muqarnas.


(a) General view.


(b) Cloning a piece with the copy seal tool.




(c) Build the basis of prisms for muqarnas.

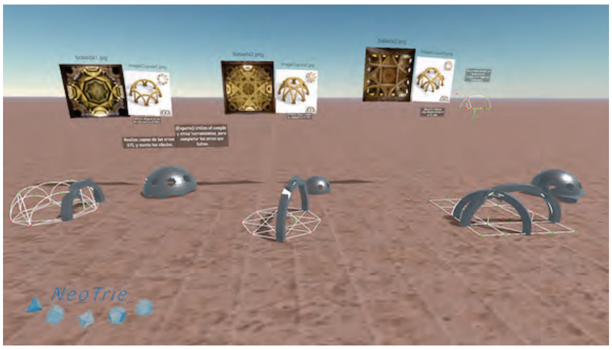



(d) Extrude the polygons to get the prisms.

Figure 10: VR scenario for muqarnas dome.

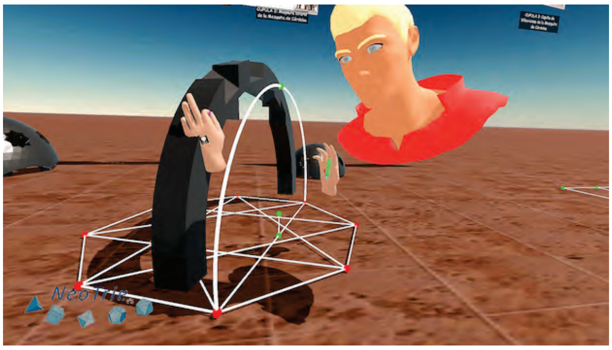
The second scenario (Figure 11) contains the 3 vaults described in Section 2. The player observes the geometrical bases of each vault and makes copies of the arches and put them in place (Figure 11b). Then, one can further propose to students to build the base figure of the dome or to build the missing circled arcs with the compass tool or to build a Carpanel arc.

The real domes consist of carpanel arches. In the scene the user can see one to try to reproduce it. The difficulty in constructing these arcs in space is that we need auxiliary perpendicular directions that indicate the plane where the arc is located (see Figure 11d).

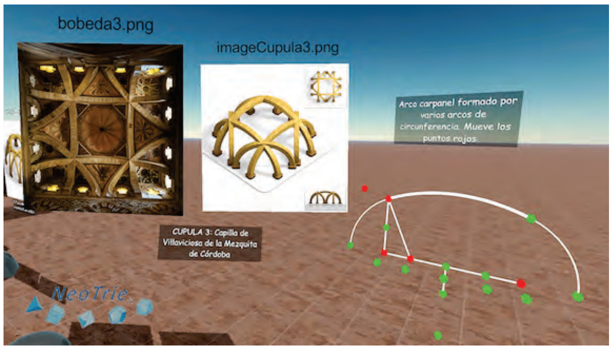



(a) General view.


(b) Assembling a dome.



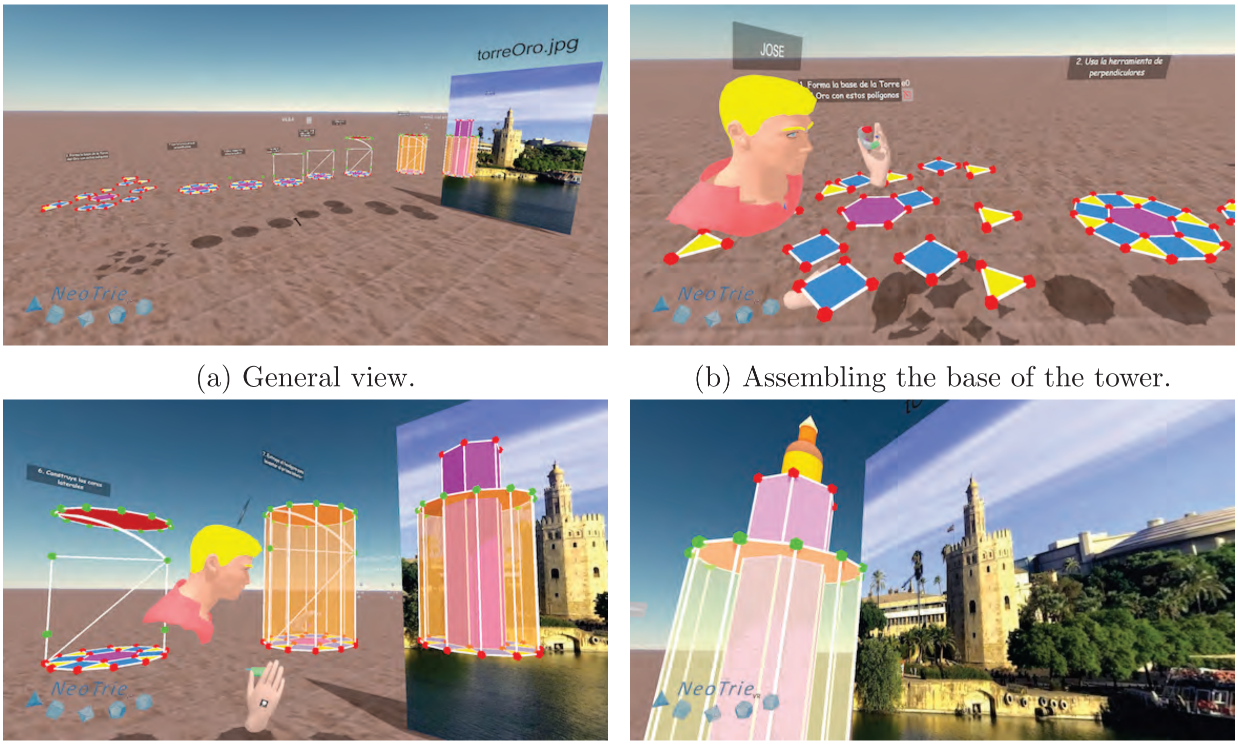

(c) Make the base figure, the circled arcs and place the STL objects correctly.




(d) Build a dynamic Carpanel arc with the compass, perpendicular and parallel tools.

Figure 11: VR scenario for vaults and arches.

In the third scenario (Figure 12) the player can build a simple model of the *Torre del Oro* step by step. Starting with regular polygons (1 hexagon, 6 squares and 6 equilateral triangles), he must use the magnet tool to form the polygonal base of the tower (Figure 12b). Then, lift the prism with the perpendicular, parallel and compass tools, to finish by extruding the inner hexagon. This process can be seen step by step in the video `https://youtu.be/2BkjtrMByNo`. Further exercises could be (see Figures 12c and 12d) to add more geometric elements at the top of the tower: sphere, cylinder, cone, to use the scale copy seal tool to get a model of the tower with real measures and to build a spiral staircase in the interior of the hexagonal prism.




(a) General view. (b) Assembling the base of the tower.

(c) Last steps of the construction. (d) Adding more elements to the top.

Figure 12: VR scenario for the Torre del Oro.

We would like to point out that the implementation of arcs in space has required the use of shadders in Unity. This has been implemented by Hernandez in the compass tool (see[4, Section 5.2]), which already used the case of a circle passing through 3 points and the circle, giving 4 points, the 4th fixing the normal direction to the circle (Figure 13)

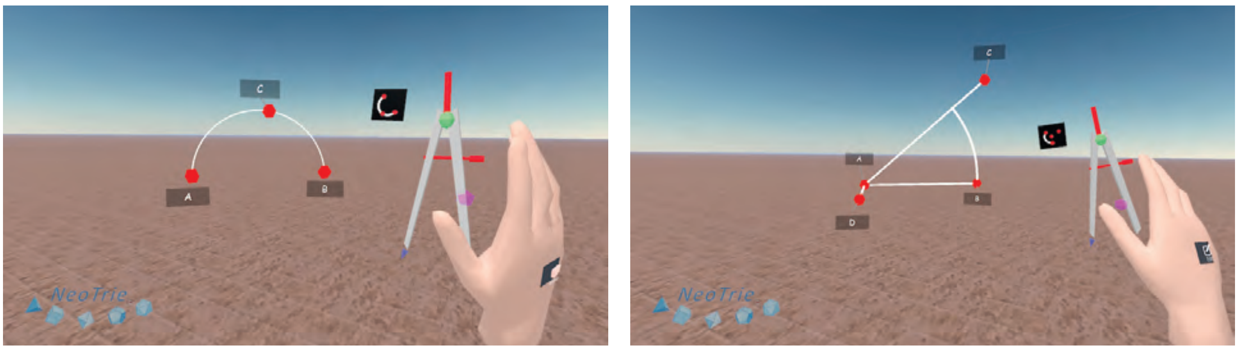



Figure 13: Arc by touching $A$, $B$, and $C$; Oriented arc by touching $A$, $B$, $C$, $D$.

# 5 Conclusion

In this paper we have described a motivating learning situation that combines history, art and geometry, with the help of new technologies (3D printing and virtual reality). Students can learn by building by themselves various structures involving important geometry concepts:

proportion, parallelism, perpendicularity, symmetries, projections, tessellations, spatial curves and surfaces, etc.

We conclude that the power of VR, together with the challenge of its assembly and the admiration aroused by its composition make NeoTrie VR, along with these scenarios, an ideal tool for teaching mathematics together with art and technology, giving rise to a STEAM workshop.

# 6 Acknowledgements

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# Mathematical Problem-Solving with GeoGebra: An Analysis of Students' Processes

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**Abstract**: *This paper reports the students' processes when tasked to solve mathematical problems in Geometry using GeoGebra. Twelve Grade 10 students divided into 6 dyads participated in the study. Each dyad solved 3 problems using a laptop and their verbal conversations, social interactions, and screen activity were recorded and videotaped. The conversations were transcribed word for word and the transcripts were supplemented by the description of their GeoGebra activities. The qualitative analysis focused on the mathematical problem-solving with technology (MPST) processes. The results revealed that grasping, analyzing, exploration, planning, creating, verification, and dissemination were the MPST processes which were usually observed. Challenging problems were characterized by a series of explorations and verifications which were made possible by the features of GeoGebra.*

## 1. Introduction

The problem-solving process in mathematics is usually characterized by four stages. Understanding the problem comes first, followed by devising and implementing a strategy, and then looking back to see whether the correct answer is obtained [1]. [2] expanded Polya's stages into six problem-solving processes which he called episodes namely: reading, analysis, exploration, planning and implementation and verification. Since then, a plethora of research has been written to investigate and improve students' problem-solving skills. Problem-solving has evolved from just merely a pen and paper activity into a more sophisticated activity that uses technology.

The emergence of computers, laptops and handheld devices led to the creation of dynamic geometry software (DGS) which allows someone to perform geometric constructions, sketch graphs, explore, and create mathematical models in a dynamic environment. An example of a DGS commonly used nowadays is GeoGebra. In GeoGebra, dynamic aspects from DGS are combined with the capabilities of a computer algebra system (CAS). The software allows users to input equations, functions, and coordinates; to keep record of the details in the algebra part of the computer screen; and to visualize everything in the geometry part of the computer screen. GeoGebra enables a user to work dynamically with points, lines, segments, or rays by clicking and or dragging them or by directly changing an equation [3]. Because of these features, GeoGebra became a widely used software for teachers' instruction, and students' modelling and problem-solving activities. It has been utilized to aid teaching, to understand difficult concepts, to visualize mathematical relationships, to provide proofs and to solve problems.

With the increasing demand on the use of computer software such as GeoGebra and its incipient application in the teaching and learning process, the need to investigate the procedure involved is of great value. This paper discusses the micro-level analysis of the problem-solving process followed by students with the use of a GeoGebra.

## 2. Conceptual Framework

[4] proposed a framework for describing techno-mathematical fluency to investigate the mathematical problem-solving with digital tools in a beyond-school mathematical competition. The techno-mathematical fluency emerged from 'problem-solving with technologies' exercises which were characterized by the interplay between mathematical knowledge and technological fluency. Techno-mathematical fluency is based on [2] episodes of mathematical problem-solving and processes of digital literacy framework [5]. The processes of digital literacy were analyzed whilst problem-solving episodes were happening. Each digital literacy process was carefully placed and analyzed in each of the five problem-solving episodes. The analytical tool that describes the techno-mathematical fluency of the students is collectively termed as MPST or mathematical problem-solving with technology. The processes underlying MPST are grasping, noticing, interpreting, integrating, exploring, planning, creating, verifying, and disseminating. The table below shows how [4] categorized the problem-solving processes of the student by merging the problem-solving episodes and the digital literacy processes.

**Table 1**

Processes underlying mathematical problem-solving with technology (MPST)

| Processes | Descriptions |
|---|---|
| Grasp | Appropriation of the situation and the conditions in the problem, and early ideas on what it involves (Read[a], Statement[b]) |
| Notice | Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary (Analysis[a], Identification[b], Accession[b]) |
| Interpret | Placing affordances in the technological resources on pondering mathematical ways of approaching the solution. (Analysis[a] ; Evaluation[b] , Interpretation[b] ) |
| Integrate | Combining technological and mathematical resources within an exploratory approach. (Exploration[a] ; Organisation[b], Integration[b]) |
| Explore | Using technological and mathematical resources to explore conceptual models that may enable the solution. (Exploration[a] ;Analysis[b] ) |
| Plan | Outlining an approach to achieve the solution based on the analysis of the conjectures explored. (Planning and Implementation[a] ; Synthesis[b] ) |

| | |
|---|---|
| Create | Carrying out the outlined approach, recombining resources in new ways which will enable the solution and create new knowledge objects, units of information or other outputs which will contribute to solve-and-express the problem. (Planning and Implementation[a] ; Creation[b] ) |
| Verify | Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources. (Verification[a] ) |
| Disseminate | Present the solutions or outputs to relevant others and consider the success of the problem-solving process. (Verification[a] ; Reflection[b] , Dissemination[b] ) |

[a] Phase of mathematical problem-solving as proposed by [2]
[b] Process of digital technology problem-solving as proposed by [5]

The MPST framework was used in shedding light on the problem-solving processes of the subjects in a collaborative setting.

## 3. Methodology

The population of this study is the set of Grade 10 students currently enrolled in a Science, Technology and Engineering (STE) high school institution. One section was randomly chosen and students from this group were ranked according to their third quarter academic performance. Three sets of students were identified: high performing (top 4 students), average performing (the middle 4 students) and low performing (bottom 4 students). Hence, a total of 6 dyads participated in the study. The pairings of students in each group were done randomly. The pseudonyms assigned to each student in presented in Table 2.

**Table 2**
Pseudonyms assigned to each student

| High Performing Group | Average Performing Group | Low Performing Group |
|---|---|---|
| Kobe and Liz | Anna and Marie | Franz and Topher |
| Ian and Grace | Jed and Kat | Rox and Sab |

Each group answered three mathematical problems using GeoGebra. The three problems of varying degrees of difficulty (easy, average, difficult) were created by the researcher.

The following were the problems given to each group.

1. A line passes through the center of the circle and intersects it at points $(-1, 2)$ and $(7, 4)$. Find the equation of the circle.
2. Find the equation of the circle inscribed in the triangle enclosed by the line $y = -2x + 5$ and the coordinate axes.
3. Construct a rhombus whose one side lies on the line $y = -2x + 5$. Check all properties that will prove that the figure you constructed is really a rhombus.

To ensure that the participants' skills or lack of skills in using GeoGebra will not affect that problem-solving attempts, a 3-session training on GeoGebra was conducted outside the students' regular classroom schedule. The sessions were conducted in the school's computer laboratory. All necessary procedures and permissions (from the students' guardians and the school head) were secured before conducting the orientation. The training materials used were based on the manual published by GeoGebra on their website.

One computer unit (laptop) was used by each dyad. The latest version of GeoGebra was installed in the laptop. The sequence of the problem-solving sessions was as follows: first 2 dyads of the high performing group, 2 dyads of the average performing group, and lastly, the 2 dyads of the low performing group. Each dyad was given a maximum of 45 minutes to answer each problem. As the groups answered the problem, they were asked to talk with each other and verbalize whatever they were thinking and the processes they employed. Each of the students' problem-solving activity was videotaped and was supplemented by the researcher's field notes.

As the students worked in pairs, their actions, dialogue, and gestures toward each other and toward the computer were videotaped. The screen activity was recorded using the Free Screen Recorder application. The final outputs of the students were the GeoGebra files which contained their answers to the problems and the construction protocol.

## 4. Results

### 4.1 Problem-solving outcomes

The outcomes of the problem-solving activities of the students highlight their score, number of steps in the construction protocol and the time spent. The score given to the final answer are as follows: '0' when there's no answer; '1' if the answer was incorrect; '2' when there was an estimated correct answer but the construction failed the drag test; '3' if the answer was correct but the construction failed the drag test; '4' when the answer was correct, the construction passed the drag test but was not able to present all possible solutions; and '5' when the answer and the construction in GeoGebra were both correct.

Table 3 shows the results of the problem-solving episodes. The first number on the table indicates the score received by the dyads, the second number pertains to the number of steps in the construction protocol and the third is the problem-solving time. For instance, Kobe and Liz tailed '5, 5, 2 min' on problem 1 which means that they were able to obtain the correct solution with a valid construction using 5 GeoGebra steps within 2 minutes.

**Table 3**

Problem-solving outcomes of all dyads

| Dyad | Names | Problem 1 | Problem 2 | Problem 3 |
|---|---|---|---|---|
| 1 | Kobe and Liz | 5, 5, 2.0 min | 2,15, 23.9 min | 4, 15, 5.9 min |
| 2 | Ian and Grace | 5, 5, 2.9 min | 2,14, 28.8 min | 2, 27, 9.5 min |
| 3 | Maria and Anna | 5, 5, 3.8 min | 1,12, 15.7 min | 1, 26, 31.6 min |
| 4 | Jed and Kat | 5, 5, 5.7 min | 1,21, 22.2 min | 2, 39, 14.05 min |
| 5 | Franz and Topher | 1, 4, 1.8 min | 3,11, 10.4 min | 2, 14, 22.03 min |
| 6 | Sab and Rox | 2, 7, 10.3 min | 0, - , 13.9 min | 4, 26, 22.2 min |

Legend (for final answer):
0 – no answer
1 – incorrect answer
2 – with an estimated correct answer but fails the drag test
3 – with a correct answer but fails the drag test
4 – with correct answer, passed the drag test but failed to present all possible solutions
5 - Correct answer and correct construction

Table 3 shows that, for nearly all three problems, the high performing group outperformed the other two groups in terms of scores, number of construction protocol steps, and solution time. Problem 1 was the easiest of the three problems. Only the low performing dyads were not able to get the correct answer. All dyads had difficulty understanding Problem 2, particularly on finding the meaning of the words enclosed and coordinate axes. Only Dyad 5 were able to arrive at the correct answer, but the construction failed the drag test, that is, changing a movable component of a construction and still the mathematical properties were maintained. Two dyads (1 and 6) were able to solve Problem 3 successfully. However, the number of steps in the construction protocol and the amount of time needed to finish the problem varied greatly.

Figure 1 shows a final output of a dyad. The left pane is called the algebra view which contains the coordinates of the points and the equations of the lines and the circle. The right most area is the construction protocol which highlights the steps and tools used by the student. The middle area is the graphics view. This is where students do the actual construction.

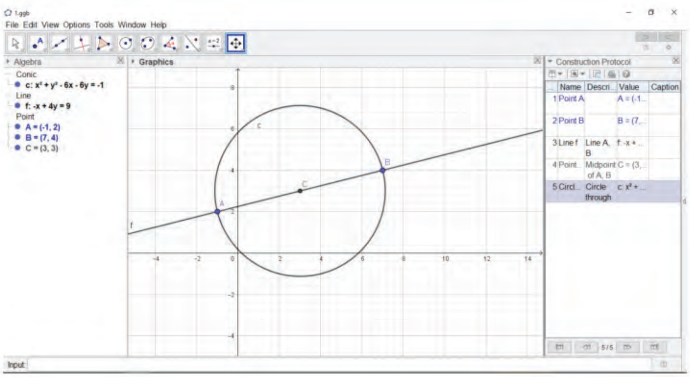


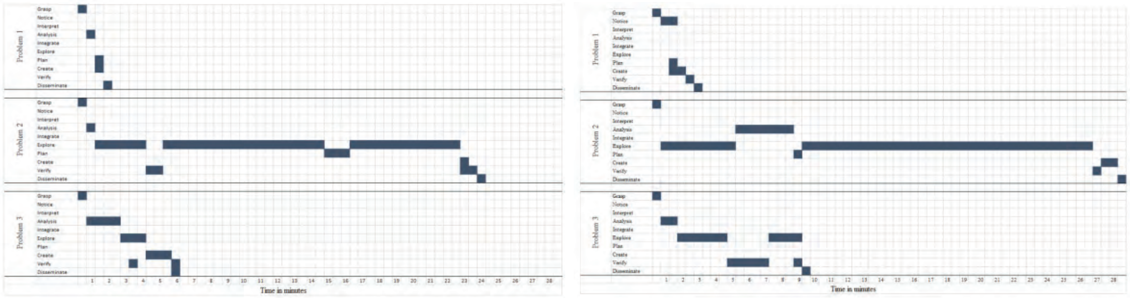

**Figure 1**. A final GeoGebra output for Problem 1.

### 4.2 Mathematical problem-solving processes

The time-line presentation of each problem-solving session by each dyad is presented below. The vertical axis lists the processes, and the horizontal axis has the time in minutes. Each grid shows a 30-second time interval.

The summary of the MPST processes of Dyad 1, one of the two dyads under the high performing group, is shown in Figure 4.1(a). It can be gleaned from the figure that, for Problem 1, the dyad had an easy time answering the question. After the *grasp* phase, they *analyzed* the question quickly and moved to the *plan* and *create* phase and finally the *disseminate* stage. The second problem, however, took them almost 24 minutes to finish. The processes involved were *grasp, analysis, explore, verify, plan, create* and *disseminate*. It can be noticed that they spent most of their time exploring for the right solution. The dyad had a hard time looking for the center of the inscribed circle. In the last problem, the pair spent nearly twice the time it took to finish Problem 1. The processes involved are *grasp, analysis, explore, create, verify* and *disseminate*. The analysis phase was filled with discussion of what a rhombus is and the explore part was spent on trying to create a rhombus by constructing first an equilateral triangle.

The other dyad under the high performing group, also did well in Problem 1 solving the item in less than 3 minutes. The session includes *grasp, notice, plan, create, verify* and *disseminate* processes. Similarly, the pair spent a lot of time answering Problem 2. This session was filled with a lot of *exploration*, and *analysis* activities. The other processes for session 2 were *grasp, plan, create,* *verify* and *disseminate*. They had difficulty in finding the center of the enclosed circle. Also, it took them a while arguing on what enclosed meant and the role of the coordinate axes. The third problem session included *grasp, analysis, explore, verify* and *disseminate* actions. The whole problem-solving session of the dyad is presented in Figure 4.1(b).

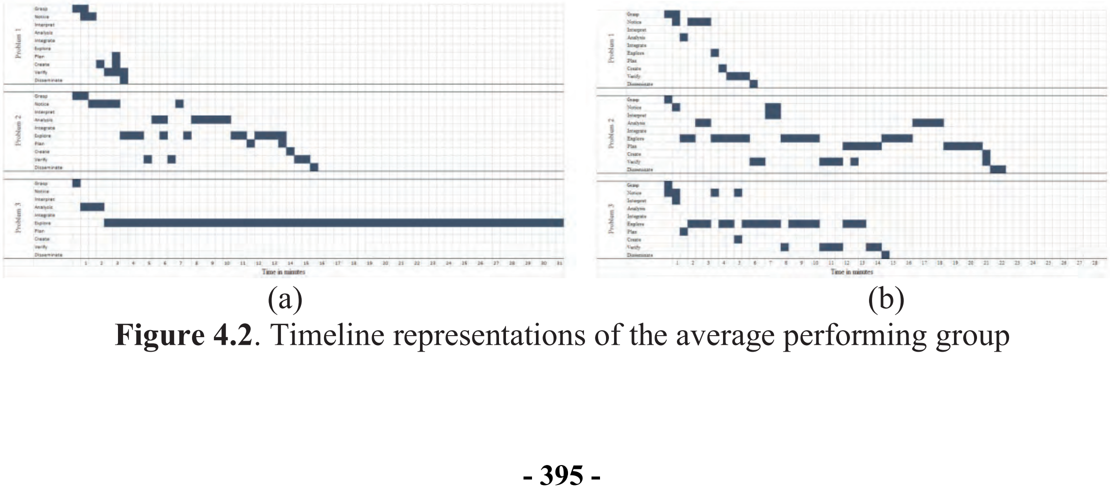


(a) (b)

**Figure 4.1**. Timeline representations of the high performing group

Figure 4.2(a) summarizes the processes Dyad 3 underwent in solving the three problems. For Problem 1, after reading (*grasp*) and analyzing the tools to be used (*notice*), they immediately constructed (*create*) the circle. But after the verification (*verify*), they found some errors and jumped into planning(*plan*) and then quickly constructed (*create*) the circle, checked its appropriateness and finally reported (*disseminate*) the answer. They were able to finish this in less than 4 minutes. The second problem-solving session included almost all the processes. Even if they were not able to solve the problem, the session was filled with processes such as *grasp, notice, explore, verify, analysis, create*, and *disseminate*. They were not able to understand what it meant to be enclosed. Up to the latter part of the session, their output showed a triangle inscribed in a circle instead of the other way. Problem 3 only included *grasp, analysis* and *explore* processes. The dyad was not able to come up with the correct construction of a rhombus, although there were traces of verification along the activity, the researcher still put the whole process into exploration primarily because of the utterances of the students.

The dyad of Jed and Kat also obtained the correct answer to the first problem. They progressed through the processes in almost a linear manner from *grasp* to *disseminate*. As for problem 2, the tandem struggled to understand on meaning of "enclosed" . They did not take into account the coordinate axes. They had a hard time incorporating the given line and finally ended with a wrong answer. The session was filled with varied processes with almost no recognizable pattern. The processes noted for session 3 were *grasp, notice, interpret*, a lot of *exploration, plan, create, verify*, and *disseminate*. The dyad spent a lot of time working with parallel lines and adjusting the position of the lines so that the parallelogram will have equal sides. The problem-solving processes is summarized in Figure 4.2(b).

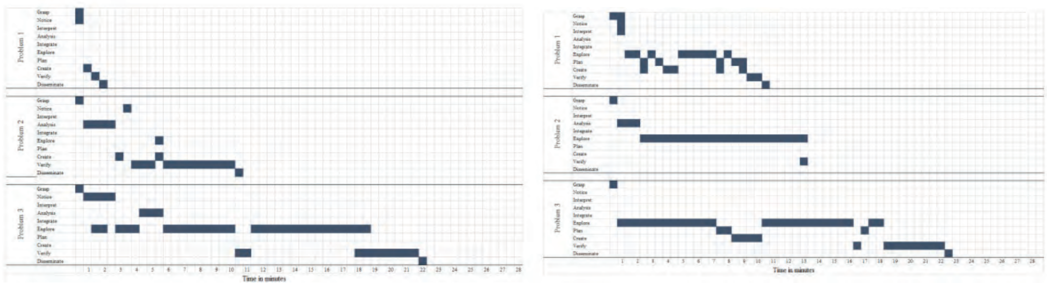


(a) (b)

**Figure 4.2**. Timeline representations of the average performing group

The summary of the problem-solving processes transpired in the sessions with Franz and Topher is presented in Figure 4.3(a). It can be gleaned from the figure that, for Problem 1, the session included *grasp, notice, create, verify*, and *disseminate*. The processes were very straight forward but the dyad was not able to answer the problem correctly. The processes observed for Problem 2 were *grasp, analysis, create*, a lot of *verification, explore* and *disseminate*. Franz was quick to recognize that the intersection of the angle bisectors is the location of the center of the inscribed circle. Also, the analysis part included argumentation on the meaning of "enclosed". For Problem 3, the session was filled with a lot of exploration. The dyad spent a lot of time working with parallel lines. They were able to construct a parallelogram with equal sides but failed to have the opposite angles congruent.

Finally, the problem-solving undertakings of Dyad 6 is summarized in Figure 4.3(b). The analysis of data revealed that the first session was characterized by grasp, notice, interpret, and a couple of exchanges between explore, plan, create, verify and disseminate. The dyad's first answer used the point (-1, 2) as the center. But they realized that they should find the midpoint and they did but on the final construction of the circle, The tool "circle with a center through a point" was used and correctly clicked on the midpoint but did not use any of the given points to set the radius. Session 2 included only grasp, analysis, explore and verify processes. They gave up after several explorations. The processes observed during the last problem-solving session included *grasp*, a long chain of *exploration, plan, create, verify* and *disseminate*.



(a) (b)

**Figure 4.3**. Timeline representations of the low performing group

## 5. Discussion

The timeline for each group showsthat not all processes were recognized during the students' problem-solving exercise. One factor contributing to this is the difficulty in categorizing some student acts, particularly during the phases of notice and interpretation. Most of the time, students proceeded right away in using GeoGebra rather than discussing first what tools to be used and their function before utilizing them. Additionally, it is very challenging to separate the integrate process from explore since, once again, students will do the activity without mentioning the tools that should be coupled with and for what purposes. The process of categorizing the problem-solving activity is deemed challenging because there was typically no explicit indication that a process has changed.

In this study the timelines of the problem-solving activity suggest that the processes may not occur linearly. This observation is different from the result in [4] and [6]. This variation could be explained by the collaborative nature of the task.

Finally, students were able to spend a lot of time in exploration because the features of GeoGebra allowed them to do so. This is specifically true for the challenging problems. However, if students do not have the mathematical skills that complement the technological skills, obtaining the correct solution may prove to be difficult still.

## 6. Summary

The MPST framework provides a potent analytical tool for comprehending the microlevel processes that underlie the usage of technology in a mathematical problem-solving activity. In this study, results showed that grasping, analyzing, exploration, planning, creating, verification, and dissemination were the frequently observed processes. In most cases, these processes were not seen in a linear way. The affordances of GeoGebra helped the students to explore and verify when solving difficult problems. However, these features can be maximized if a student has the mathematical skills to interpret the visual representations during these processes.

## 7. Recommendations

Analyzing student work completed in a classroom context using the MPST framework is possible. If student work is graded, results may be different because students tend to work harder and pay closer attention if they are aware that their performance will have an impact on their marks. The activities can be done individually or in a group. A comparison of the problem-solving processes between the two set-ups can also be made

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# Perception of Learners on Virtual Learning Environment in Mathematics in Nepali Context

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**Abstract**

*This paper aims to identify the perception of learners of virtual learning environments in Mathematics at the university level in the context of Nepal. Mid-West University was the study site. This is a qualitative case study design. All the students who were studying Mathematics in different semesters of graduate and undergraduate level in the year 2021 of faculty of education were the population of this study. Ten students were selected as participants using purposive sampling to represent the different ten districts of Karnali Province. In-depth interview was used as the tool for data collection. The interview was conducted using guidelines through phone and online calls using mobile. The interview was recorded in my mobile device, and the points were noted in my notebook as well. All the collected information was transcribed, translated, and categorized to developed themes and analyzed in a descriptive manner. The result indicates that learners perceive virtual learning in higher-level Mathematics as a necessity and an opportunity. Moreover, the participants said that virtual learning is a useful learning process, and it is better than face-to-face mode because it develops knowledge as well as skill, it also develops the habit of searching for resources, increases self-confidence and independence, it provides permanent learning, saves expenditure and provides an opportunity to earn. Similarly, virtual classes can be continued in strikes and other difficulties like lockdowns, and experienced and busy professors can take classes in their leisure time, even for those with disabilities.*

## 1. Introduction

Virtual learning is a technology-based instructional method that enables learners to participate in learning from a distance in synchronous or asynchronous modes. In another sense, it is a web-based teaching-learning platform to ensure all sorts of teaching-learning components. Virtual learning platforms allow content management, curriculum mapping and planning, learners' engagement and administration, communication and collaboration, and real-time communication.

During the change of time, we normally get changes in the pedagogy of teaching and learning mathematics in higher education. Without the use of recent development, we get a lack of completing tasks. Technology innovation is directly related to societal transformation as we find the change or modification of social framework. It is the change in attitude, behaviors, and activities connected with cultural and economic values. According to [9], computers and other technological innovations influence our lives largely all the time. It is believed that the development of computer technology is a paradigm in the field of teaching and learning and is regarded as technological innovation. We use computers or other digital devices to gain educational objectives determined by the specific learning environment.

Virtual learning is a procedure of personal renovation of content, which is carried out in the purpose of and grounded on the reasoning structure of learning. The elements that make up this structure are basic cognitive skills, specific knowledge of an area, learning strategies, metacognitive abilities and self-regulation, affective and motivational factors, goals and expectations. The situation deals with classroom behaviors and another essence of learning variables as the established values of classroom learning. This demands constant updating of subjects related to virtual learning to incorporate them during the design of VLEs [4].

Looking back into the history of Nepali Education system, the College of Education started an adult education program through the radio in 1958. It is considered to have been the first initiative towards distance education in Nepal. Another important initiative was the launching of a radio education teacher training project in 1978 by the Ministry of Education (MoE) with technical and financial support from USAID. The project started radio broadcasting since 1980 and was focused on enhancing the professional capabilities of in-service primary teachers having qualifications under the School Leaving Certificate (SLC). Likewise, as provisioned in the National Education Commission 1993, the Distance Education Centre (DEC) was established under MoE in 1994. The center conducted teacher training and education awareness programs through radio broadcasting. After the unification of DEC with the National Centre for Education Development (NCED) in 2005, professional development training courses for primary to secondary level teachers, SLC support, and radio programs on education information are being conducted. Policy and directives to regulate open education and distance learning in Nepal are Open Education and Distance Learning Policy 2007 and Directives on Distance Education/Open Learning Program 2007 with its third amendment 2014. Formulating policies and other documentation on distance education and open learning programs is supportive of establishing open schools and formalizing distance learning programs [2].

Some popular approaches to offer flexible learning are online or distance study. Other examples of flexible learning options are opportunities for learning from work and employer engagement, part-time study, web-based or blended learning, time-driven programs at students' pace, contact sessions, workshops, and seminars. In the Nepali context, some of these options are available to a limited number of programs of higher studies at two leading universities: Tribhuvan University (TU) and Kathmandu University (KU). TU offers ODL for bachelor and master programs and KU offers ODL for master programs. These ODL programs from both universities are for teacher education. Students at the secondary level also have distance learning options for SLC. Besides that, some schools, colleges, and universities also offer flexibility in schedule: evening, day or morning classes. Students working full-time during the day can join morning or evening classes. For the students in the Himalaya regions, there is a provision of Mobile Schools. Such schools shift their location seasonally (winter and summer) as people living in the Himalaya region move their places of residence [6].

Nowadays, all Nepali students connected to the Internet have equal opportunity to choose a variety of formal and non-formal or flexible learning. They can pick any course of their choice from the University of their Dreams that exists anywhere in the world. In addition, Nepali students have limited options to get an education with flexible options at domestic institutions. Starting from the secondary level to the tertiary level which includes initiatives of open schools and ODL at HEIs. The limited flexibility of higher studies is being expanded because every year new courses and programs are offered by TU and KU through ODL. Since 2012, the Open University Infrastructure Development Board has been working to establish Open University in Nepal. As a result, Nepal Open University (NOU) was established in 2017 to provide higher education on Bachelors', Masters' and M. Phil levels through a virtual learning environment.

Mid-Western University (MWU) was established in 2010 as the regional University in Nepal. It has launched all its programs through a face-to-face mode of learning. In the pandemic situation of Covid-19, more teachers have been taking classes through virtual mode. Department of Mathematics Education, Faculty of Education has been conducting virtual classes on graduate and undergraduate levels. As the current issue of the new learning paradigm, the researcher became interested in identifying learners' perceptions of virtual learning modes. More studies have been conducted on the perception and challenges of virtual learning that are expressed in the literature review session. It is a new and relevant problem to discover learners' perceptions of virtual learning in higher education mathematics in Karnali Province. Guided by the question—how do the students perceive virtual learning environment in higher level Mathematics learning? Thus, this study aims to identify learners' perception in a virtual learning environment in higher level Mathematics learning.

The study 'Perception of Learners on Virtual Learning in Higher Mathematics' is a case study at Mid-Western University. Mid-Western University, established in 2010 in Nepal, has launched all of its programs through face-to-face mode and it has aimed to launch a year B.Ed. and one-year M.Ed. program through virtual mode. During the pandemic period of COVID-19, mathematics teachers at MWU conducted virtual classes on Mathematics. As the teacher of Mathematics Education at MWU, the researcher is concerned about this issue. This study will be helpful to the concerned students, teachers, program coordinators, curriculum designers, and all the people who are interested in e-learning. This study aims to identify the perceptions of learners on virtual learning in higher education Mathematics. So, from the view of the researcher, this study will be beneficial to MWU and all the other university authorities and concerned people in the virtual learning environment.

## 2. Literature Review

In virtual learning environments must be spaces for coaching and producing mastering which can be pedagogically modeled and incorporated with numerous components including technological platforms, activities, and fabric, which all aim to generate knowledge [8]. In flip, the interplay of the community with the aid of manner of technological gear enriches the exceptional of gaining knowledge of the concerned subject matter. [7] states that virtual learning is a system of private reconstruction of a content material that is executed in the characteristic of and based at the cognitive shape of gaining knowledge among the elements that make up this shape are simple cognitive capabilities, specific information of a place, learning strategies, meta-cognitive talents and self-regulation, affective and motivational elements desires, and expectancies. All those factors and how a student utilizes them can cause satisfactory mastering". The changed paradigm is represented through virtual receiving. This is why teachers, directors, technical and help bodies of workers, as well as the institution itself, discover themselves faced with a new and one-of-a-kind shape of learning management system.

As mentioned in [4], Coaching-gaining knowledge is not a closed domain including a lecture room, to paint for various kinds of college students, the improvement of VLE calls for an attempt to migrate from a closed system to a new reality. This demands constant up-courting of topics that get related to virtual gaining knowledge off or you to contain them at some point in the design of VLE. [1] developed a framework for virtual internalization and as indicated in the study digitalization, collaboration, and virtual learning situations were explored in the international dimension. The virtual mode was found as an alternative to the other modes as it is not affected by any boundaries and the interested learners could join from a distance and they can use the Learning management system as prescribed. Seeing the motive of transnational studies, I like to develop this problem in the context of Nepal, which is not covered in the studies that have already done [3] researched 'Virtual Learning Environment (VLE) in Mathematics Education'. The study found that the major opportunities that VLE contributed in F2F classes were to improve study habits, make study more active, and provide the opportunity to learn, re-learn and clarify the concepts. This implies that the F2F courses need to be designed to integrate VLE in an appropriate way.

## 3. Methodology

Methodology refers to the overall procedure of research. It includes the research design, study site and population, method of sampling, data collection tools and procedure, and data analysis and interpretation method. Following [2], the methodology provides a guideline to develop strategy, plan of action, process or design lying behind the choices. The methodology of this research paper is described as follows:

### 3.1. Study Design

The design of the study is a case study. This is the case of the virtual learning context in the mathematics group of Mid-West University. The Case study, through predominately a qualitative study design, is also prevalent in quantitative research. [5] says a case could be an individual, a group, a community, an instance, an episode, an event, a subgroup of a population, a town, or a city. Through predominately a qualitative study design, the case study is also prevalent in quantitative research. To be called a case study, treating the total study population as one entity is important. A flexible and open-ended technique of data collection and analysis characterizes a case study. The advantage of the case study design is that the research can be much more detailed, but the disadvantage is that it is difficult to generalize the findings.

### 3.2. Study Site

Mid-West University is the study site. As a teacher at MU, the researcher thought it easy to collect primary data from students at MU. As a personal study in a short period, the researcher could not include other universities of Nepal that have launched virtual learning in higher education.

### 3.3. Population, Sample Size, and Strategy

The population of the study is the students studying Mathematics as a major at undergraduate and graduate level in Mid- West University.  The researcher knows about all the units of the population. The total number of students is about fifty-two. Ten of them have been selected using purposive sampling to represent the different ten districts of Karnali province. I have used the symbol/pseudonyms S, D, J, S, R, K, H, J, M, and $D_1$ to indicate the participants from Surkhet, Dailekh, Jajarkot, Salyan, Rukum, Kalikot, Humla, Jumla, Mugu and Dolpa respectively.

### 3.4. Study Tools

As per the purpose of the study, I have used interview guidelines. In-depth interview was conducted through phone and online call with the participants from each district of Karnali province of Nepal. As a teacher at MU, I was in contact with all the participants through phone, email, and Facebook groups.

### 3. 5. Data Collection Procedure

I have used the following procedure to collect the data from the respondents—at first, I built the rapport and clarified the research problem to the respondents. Then I formed an interview guideline for in-depth interviews and conducted interviews with ten participants from different districts of Karnali province through phone and online calls using mobile. At last, I thanked the participants and said goodbye. The responses were collected and noted in my diary. Also, the audio was recorded with the permission of the participants.

### 3. 6. Data Analysis and Interpretation

Data was collected through in-depth interviews. I recorded the call by getting permission from the participants for analysis purposes. I also noted the important points in my diary in the process of conversation. I often listened to the call recording and *transcribed* the audio recordings to note the required information. The interview was conducted in Nepali language. After listening to the call recording many times, I *translated* the required information into the English language without changing the meaning. I translated most of the interviews and further conversations into English. After listening to the audio several times carefully, I explored the emerging themes. The non-formal discussion and interview helped me to identify the perceptions of learners in virtual learning in higher mathematics. I categorized the perceptions of the participants to develop a theme and analyzed them critically and thematically.

### 3.7 Ethical Consideration

Ethics are the moral principles a person must follow, regardless of place or time. Behaving ethically involves doing the right thing at the right time. Research ethics focuses on the moral principles that researchers must follow in their respective fields of research. I have spent sufficient time to build a good rapport with the participants. I have made the participants confident about privacy and have not compelled them to participate in my study.

## 4. Analysis and Discussion

The learners who participated in the interview talked about the usefulness of virtual learning in Mathematics in higher education. They explained that the learners can read from their own home /room and physical attendance in class is unnecessary. On the basis of the participants' view, I developed the following themes regarding the perception of learners on the virtual learning environment in Mathematics in the university level:

### 4.1. Useful Learning Procedure and an Opportunity

The participants of this study perceived the virtual learning environment as an opportunity for higher education in the context of Nepal because most of the job holders and housewives cannot attend college regularly in physical mode. The participant K said:

> *Continuing higher education in face-to-face mode is not an easy job in the context of Nepal. Most of the students from remote areas and poor economic backgrounds should start work to fulfill the needs of family before higher education. It has been a compulsion and a trend of Nepalese youth. In this situation, virtual mode in higher education has been useful and a great opportunity for learners like me.* He went on to explain that interested learners who are unable to attend the class physically due to his/her own physical problems can continue higher education easily in virtual mode.

All the participants in an in-depth interview said that virtual learning is a useful learning process. It is useful in the sense that the learners can read in their own home /room. Physically attendance in class is not needed. All the interested learners in remote areas can join class in their own home/workplace. Interested learners who are unable to attend the class physically due to his/her own problems can continue higher education easily.

On the other hand, there is a trend of early marriage in Nepalese girls. One female student (D) said, *the childbearing female also can continue higher education in virtual mode. So online mode has become a golden chance for them to continue higher education in their own home with babies*. The virtual mode in higher education is useful not only for job holders and childbearing females, it is equally useful to physically disabled students. After talking about the usefulness of virtual learning, I asked the following question to the participants: *"In what ways virtual learning situation is regarding better than face-to-face learning?"* All the participants expressed that virtual mode is better than face-to-face mode in higher Mathematics. They shared their different feelings in the Nepali language. I translated the views, categorized them, and developed the following different themes.

### 4.2. Develop Knowledge and Skill

The experience of learners in virtual mode is that it develops skills as well as knowledge. Learners should be updated in online resources and procedures. Online application, online admission procedure, online registration form, online exam form, and online assessment, are the main aspects of virtual learning. These aspects provide computer skills that are needed in every field of life in this modern era. Without proper knowledge of computers and technology, no one can become successful in a profession like as academic field, banking, business, bureaucracy, or any other. Participant H shared his feelings as *Virtual learning develops the habit of searching resources and ICT skills as well as the English language.*

The habit of searching for resources makes the learners active and engaged in learning. Without the activeness of learners and searching habits, virtual learning is impossible. Students do not expect knowledge from teachers/ facilitators as in face-to-face classes. So, the virtual mode develops the habit of searching for resources according to the common view of the participants.

### 4.3. Increases Self-confidence and Independency

The habit of searching for resources, checking authentication, and choosing the best materials develops self-confidence and independence. Course facilitators provide instructions as needed and the learners are concerned with e-library, websites, and other links to find the required materials. Participants shared that the virtual mode of learning mathematics developed their independence and confidence more than before. Almost all the participants advocated increasing self-confidence and independence from online learning rather than traditional face-to-face learning.

### 4.4. Provides Permanent Learning

The knowledge obtained from doing is more permanent than listening. Virtual learning motivates students to search and do themselves. All the participants expressed that virtual learning is the best option for permanent learning. Participant J focused on "*knowledge obtained from doing is more permanent than listening.*"

According to the view of the participants, teachers provide sufficient learning materials to the learners and the learners do the work as instruction. Learners get feedback and correct their mistakes themselves. Identifying mistakes and correcting them as needed makes permanent learning and decreases errors.

### 4.5. Virtual Learning Saves Expenditures and Provides Opportunity to Earn

Virtual learning saves the amount of rent, food expenditure, transportation cost, and other costs of students for face-to-face learning. Learners need not leave home and need not leave jobs/work. So, it saves money and provides an opportunity to earn. Learners can read the materials in their leisure time. Participant R said, *we need not leave home and job/work to learn. We can do the work to earn in the daytime and read in our leisure time. Teachers upload recorded videos of class in Google Classroom so that the students can learn in offline mode and in leisure time.*

Learners can do work of earning in the day and they can read in leisure time. As the view of the participants, teachers upload recorded videos of class in Google Classroom or face Facebook group, or messenger groups so that the students can learn in offline mode and in their leisure time.

### 4. 6. Virtual Classes can be Continued in Strikes and Other Difficulties like as Lockdown

From the view of participants, virtual class is the best option in the context of Nepal. Due to different issues strikes, lockdowns and other obstacles arise in face-to-face learning, but these obstacles do not affect the virtual classes. No unfavorable situation affects virtual classes and can be continued even on holidays and festivals.

### 4.7. Experienced and busy Professors can take Classes in their Leisure time and even in those Disabilities

According to the participants, virtual classes are possible in leisure time. There is a lack of efficient and experienced professors of Mathematics in higher education in Nepal. Some professors of Mathematics are busy and cannot take more classes in regular face-to-face mode. So virtual classes are possible at the time they want. Retired professors who are skilled in technology can take virtual classes from their own homes. They can continue virtual classes in the situation of physical disabilities and other difficulties.

The above opinions indicate that the learners perceived the virtual process as essential in higher education mathematics. The participants expressed the need for and importance of virtual learning in higher mathematics in the Nepalese context. I found the common theme about the perception of learners on virtual learning that virtual learning in mathematics in higher education is the necessity of jobholders and childbearing female students in Nepal. In the same way, most of the mathematics teachers at the school level cannot continue higher education due to their jobs. The government of Nepal has implied SSRP but due to the lack of qualified Mathematics teachers, applied mathematics could not be compulsory in classes 11 and 12. It is also the result of the success percentage in Mathematics at SEE level. According to the view of the participants, virtual learning in Mathematics in higher education in needed.

### 4. 8. Virtual Mode is Difficult in Solving Long Numerical Problems in Mathematics

Virtual classes in Mathematics are more challenging than other theoretical subjects. Participants said that they felt more difficulty in solving numerical problems. Teachers also cannot present all the steps of long numerical problems in slides. They give only the instruction to solve. Learners could not show the teachers how to find the mistake and correct it immediately. Participant S emphasized numerical problems and said, *I have sent the image of my handwriting three times in numerical contents. Images were not clear at first, in the second time I could not manage the serial. So, I sent it again after managing all*. It is not easy to type all numerical steps quickly and show the teachers.

Sometimes the learners solve by copying and sending images to the teachers. The image of the solution may or may not be clear and the teachers find it difficult to check out images and identify mistakes. A long numerical problem may have more images and if the serial of images could not be managed, the solution becomes zigzag. This process is difficult in a virtual class of Mathematics class. Participant $D_1$ shared his experiences by showing the following example on the difficulty of solving long numerical problems step by step in a virtual mode of learning as follows:

**Qu**. Find the curvature and torsion of the curve x= a*cos(t), y= a*sin(t), z= a*t*cot(α).

Solution: Position vector r = (x, y, z) = (a*cos(t), a*sin(t), a*t*cot(α)

First derivative $\dot{r}$ = (-a*sin(t), a*cos(t), a*cotα)

Second derivative $\ddot{r}$ = (-a*cos(t), -a*(sin(t), 0)

Third derivative $\dddot{r}$ = (a*sin(t), a*cos(t), 0)

Now, the curvature (κ) = $\frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3}$ and torsion ($\tau$) = $\frac{[\dot{r}\ddot{r}\dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$

$D_1$ said, "I feel that it is so difficult to solve such problems in an online class because I cannot type easily, and I do not have sufficient knowledge and skills in using a mathematical software package." Other participants also emphasized this issue and said that scalar triple product and cross product of vectors takes more symbolic steps, and it is not easy to solve and show the e-copy immediately to our teachers.

### 4. 9. Difficulty in Solving Problems with Multiple Tables, Pictures, and Diagrams

According to the participants, long steps of numerical problems with complex operators and symbols are difficult to type and make slides. The participant H stated:

> *Some of the problems of mathematics are pictorial, and graphical, and need a table. Creating graphs, pictures, and tables takes more time and it is difficult to show in PowerPoint. Sometimes, PowerPoint breaks the long tables, pictures, and figures, creating difficulty in understanding. He gave the following example:*

**Qu.** Maximize $z = 8x_1 + 10x_2 + 7x_3$ subject to $x_1 + 3x_2 + 2x_3 \leq 10$ , $x_1 + 5x_2 + x_3 \leq 8$ and $x_1, x_2, x_3 \geq 0$

Solution: Let $s_1$ and $s_2$ be slack variables. The standard form of the above system is:

Maximize $z = 8x_1 + 10x_2 + 7x_3$ subject to $x_1 + 3x_2 + 2x_3 + s_1 = 10$ , $x_1 + 5x_2 + x_3 + s_2 = 8$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

**Step 1: Setting up the tableau.**

| $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $z$ | c |
|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 1 | 0 | 0 | 10 |
| 1 | 5 | 1 | 0 | 1 | 0 | 8 |
| -8 | -10 | -7 | 0 | 0 | 0 | 1 |

Based on the participants' view, the above types of problems with long steps and more tables are difficult to teach and learn in online classes. The learners should do the numerical solution in copy, and it is difficult type to but the teachers like e-copy of solutions than images of handwriting because the image of handwriting may not be clear and may not be in serial. It is the problem of virtual classes in Mathematics.

## 5. Conclusion

Virtual learning is a technology-based pedagogy applicable to children and adults in the modern era [11]. It is a widely used approach in developed countries. Developing countries like Nepal have also practiced it in higher education. It has come into practice more in the pandemic period at the school level. There are more advantages of virtual learning with some challenges.

This study was done to identify the perception of learners on virtual learning in mathematics at a higher level. Ten participants were selected using purposive sampling from different districts of Karnali Province. An in-depth interview was the tool to collect data. Collected rough data were transcribed, translated, coded, categorized, and developed themes. From the study, it can be concluded that the learners perceive virtual learning in higher mathematics as a necessity and an opportunity. Moreover, the participants said that virtual learning is a useful learning process, and it is better than face-to-face mode because, it develops knowledge as well as skill, develops the habit of searching for resources, increases self-confidence and independence, provides permanent learning, saves expenditure, and provide opportunity to earn. Similarly, virtual classes can be continued during strikes and other difficulties like lockdowns and experienced and busy professors can take classes in their leisure time even with disabilities.

Hence, virtual learning in higher mathematics is a useful learning process and an opportunity. However, teaching long numerical problems through virtual mode is not as easy as theoretical content. Mathematics teachers should be updated and well-known in mathematical software and alternative tools to teach long numerical problems easily through virtual mode.

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# Enhancing Students' Achievement and Investigating Students' Satisfaction in Learning Mathematics By using a Flipped Classroom

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*Abstract: The objectives of this classroom action research were to enhance students' mathematical achievement and to survey students' satisfaction with learning by using a flipped classroom. The participants were 32 grade 11 students who enrolled in the second semester of the academic year 2019 at a secondary school in Bangkok, Thailand. The topic used in this study was Vectors in Three Dimensions. The instruments were 7 lesson plans using flipped classrooms and a satisfaction survey. Before class, students studied online learning through video clips, handouts, homework, and quizzes. During class, students discussed the contents that they had studied from home, solved harder problems, and got individual help from the teacher. Learning lasted 14 periods with 50 minutes in each period. There were three cycles of action plans. Data were collected from pretest, posttest, and satisfaction surveys. Data were analyzed by using the Effectiveness Index. The results showed that: 1) the Effectiveness Index of the flipped classroom was 0.80 which revealed that students' achievement increased by 80 percent from the beginning; and 2) students' satisfaction in three categories: students' understanding category, learning activities category, and learning atmosphere category by using flipped classroom were at least satisfied, satisfied, and very satisfied, respectively.*

## 1. Introduction

Mathematics is a school subject that stimulates student's thinking. It enhances reasoning, creative thinking, planning, deciding, and solving problems. Mathematics is a tool for studying science and technology. Therefore, mathematics is useful for living and developing the quality of life (Academic and Educational Standards Affairs, 2005).

The Basic Education Core Curriculum implemented in B.E. 2551 (A.D. 2008) aimed to enable all children and youths to continuously learn mathematics with their potential. This aim was applied from Grade 1 to Grade 12 while the contents extended wider as students moved up to higher levels. The contents comprised six strands: Strand 1 Number and Operation, Strand 2 Measurement, Strand 3 Geometry, Strand 4 Algebra, Strand 5 Data Analysis and Probability, and Strand 6 Mathematical skills and processes. Today, The Basic Education Core Curriculum implemented in B.E. 2551 (Revision B.E. 2560) was developed by the Institute for the Promotion of Teaching Science and Technology (IPST) delegated authority from the Ministry of Education which prescribed the contents in three strands: Strand 1 Number and Algebra, Strand 2 Measurement and Geometry, and Strand 3 Statistics and Probability [1]. Schools can make their own decision from the National Curriculum to produce a school-based curriculum in which schools can choose contents and sequence to teach on their own.

For Thailand, the educational reform has introduced decentralization and delegated authority to schools as a school-based management system. Schools are supposed to develop a school-based curriculum based on learning outcomes and indicators illustrated in the Basic Education Core Curriculum in the Mathematics Area. IPST has promoted a new method of teaching science, technology, engineering, and mathematics to foster students for standards of learning outcomes, indicators, and 21st-century skills by emphasizing knowledge and skills that are suitable to real life and careers. Moreover, the Institute for the Promotion of Teaching Science and Technology (IPST) developed an additional mathematics curriculum for high school mathematics to provide mathematical knowledge and skills to students studying science programs such as complex numbers, matrices, vectors in three dimensions, analytic geometry, trigonometric functions, and basic calculus.

By the reform and by the government setting of specialized IPST, it should be expected that student's achievement in science and mathematics was at a satisfactory level. However, the average mathematics O-NET (Ordinary National Education Test) scores in Grade 12 were 24.88%, 24.53%, and 37.50% in 2016, 2017, and 2018 respectively [2]. The low average scores clearly indicated that high school students in Thailand need improvement and higher quality of mathematics instruction. Students should have an opportunity to increase their mathematics scores at both national and international levels. In order to achieve this, mathematics instruction needs reform by changing from the traditional approach to a new approach.

The same situation of unsatisfactory student achievement happened in the school that the researcher had worked with. From discussion with teachers in the mathematics department, it could be concluded that the causes were: 1) limited class time, 2) school activities and events reduce class time and 3) insufficient time allocated to the discussion about important mathematical concepts and engaging in problem-solving.

## 2. Flipped Classroom

Nowadays, technology has made great advances. Online activities such as online conferences, online marketing, Google Classroom, and online learning are rapidly growing. Two American secondary school teachers, Bergmann J. and Sams A. [3] used the advantages of online learning to help their students who had to leave the classes during sports competition events. This practice gradually developed into a new teaching and learning method called flipped classroom. The method was adopted by many practitioners and institutions in various fields such as mathematics, science, engineering, and languages [4], [5]. The results from applying flipped classrooms were positive to students' achievement [6].

The flipped classroom can extend class time. In online learning, students can replay video clips as many times as they want and work on some parts of the lessons in advance. So, it allows more time for class discussion which can lead to a deeper understanding of the contents and encourage students to engage in learning. Therefore, many teachers, in recent years, have adopted flipped classrooms and expanded class time to develop students' understanding. Students have more time to think critically about mathematical concepts and ideas through collaboration, justifying, and explaining their processes while the teacher facilitates and guides them [7].

In teaching mathematics, the researcher has been assigned to teach vectors in three dimensions for Grade 11. The idea of vectors is important in applied mathematics because many quantities used in physics such as force, motion, the flow of fluid electric current, light, and sound are concerned with vectors. The researcher studied various textbooks to be used as a basis for the instruction of vectors in three dimensions. Those textbooks were Vectors and Related Topics in Schaum's Outline Series in Vector Analysis Second Edition [8], [9], Cambridge Additional Mathematics [10], New Additional Mathematics [11], and Calculus textbook [12]. Moreover, the researcher has studied textbooks about the teaching of vectors such as Teaching of Vectors [13] and Handbook for Grade 11 Mathematics Teachers developed by IPST [1].

In addition, the flipped classroom was a new approach to learning and teaching methods for students. By this method, the participants had to adjust themselves to the new method. Before class time, they must control themselves by watching online video clips, reading math handouts, and doing worksheets and quizzes. During class time, key concepts would be revealed, and students studied and discussed more contents, examples, problem-solving, and knowledge extension. These activities needed time and concentration. It might cause tension to them. So, the researcher wanted to know whether they were satisfied with this method.

From the above problems and the benefits of flipped classrooms as mentioned, this study aimed to use flipped classrooms to enhance mathematics achievement and to survey students' satisfaction in learning mathematics.

## 3. Conceptual Framework

Action research is conducted by the researcher to solve a specific problem(s) in a specified area or by practitioners to improve their practices. Classroom action research is also conducted by teachers as researchers to solve the problem(s) in the classroom or to improve their own teaching. It sometimes creates ideas for grounded theory which are bases for the main theory. Actually, in action research, the researcher neither identifies the sample nor the population because the researcher aims to solve problems in a specific area and also does not aim to generalize the research results. In teaching mathematics, the problems fixed with students' low achievement are related to understanding, reasoning, proving, and problem-solving. Now, teachers can extend class time for more discussion and problem-solving by using the internet which gradually developed to be a new method of teaching and learning called a flipped classroom. The flipped classroom is composed of three steps: 1) before class, 2) during class, and 3) extending skills or knowledge. Before class teacher assigns students to study basic concepts online through video clips, study handouts, and quizzes in advance. During class time, the teacher reviews important concepts, discusses more examples, and problem-solving. From various studies and ideas from flipped classrooms, this classroom action research aimed to enhance students' achievement by using flipped classrooms and to investigate students' satisfaction in learning mathematics by using flipped classrooms. The researcher adopted the action research model presented by Kemmis and McTaggart [14] which involves spiral and recursive steps of: 1) planning a change, 2) acting and observing the processes, 3) reflecting on these processes, and 4) revising the plan.

## 4. Research Methodology

This research took the model of classroom action research with three cycles to enhance students' mathematical achievement and to investigate students' satisfaction in learning mathematics by using flipped classrooms. Participants in this research were 32 Grade 11 students studying in the second semester, of the academic year 2019 at a secondary school in Bangkok, Thailand.

## 5. Research Instruments

The instruments used in this study consisted of lesson plans, achievement tests, satisfaction surveys, and teacher's reflections.

(1) Lesson plans: There were 7 lesson plans on vectors in two dimensions and three dimensions integrated with the flipped classroom. The contents were 1) Introduction of geometric

vectors, 2) Coordinate systems in two dimensions and three dimensions, 3) Vectors in two dimensions and three dimensions, 4) Addition, subtraction, scalar multiplication of vectors, and their properties, 5) Unit vectors and direction cosines, 6) Dot products, cross products, and their properties, and 7) Application and problem-solving. At the end of periods 3, 5, and 7, the researcher reflected on teaching results for improvement in the following periods. All lesson plans were commented on by an expert. Then, they were revised according to the comment.

(2) Achievement test: The researcher developed the mathematical achievement test. It was composed of 10 multiple-choice items for 10 points and 3 written tests for 10 points. This test covered all the topics used in this study. The Index of Item Objective Congruence (IOC) was used to measure the congruence between learning objectives and the test items. The IOC of this test was measured by three experts. For this achievement test, the value of IOC in multiple-choice items was 0.97, but IOC in written tests was 0.89.

From 6 cognitive domains presented by Anderson and Krathwohl [15] who presented the revised Bloom's taxonomy, the researcher considered testing students only 4 cognitive domains (remember, understand, application, and analysis) presented by Anderson and Krathwohl.

(3) Satisfaction survey: The rating scale was a 5-point Likert scale to survey student satisfaction in learning using a flipped classroom. There were 20 items that can be categorized into three (3) groups: students' understanding, learning activities, and learning atmosphere. This survey had Cronbach's internal confidence of 0.89. The questions used in this research were shown as follows:

Category 1: Students' understanding

Item 1: I understand the content more because I do it alone, not only listen to teachers.
Item 2: I can remember the content longer.
Item 3: I can understand the content by myself.
Item 4: I can apply this learning process to other subjects.
Item 5: I'm proud of myself when I can understand the hard content.
Item 6: I can decide by using reasoning.
Item 7: I can understand my friends more.
Item 8: I can work with others.
Item 9: I think this learning process makes me learn more than only listening to teachers in the classroom.

Category 2: Learning activities

Item10: Learning management suits the content.
Item11: Learning management helps me to exchange knowledge with others in the classroom.
Item12: Learning management helps my thinking and decision.
Item13: Learning management makes me brave when questioning or answering.
Item14: Learning management helps me to comment in the classroom.
Item15: Learning management makes me understand the content more.
Item16: Learning management helps me and my friends learn together.

Category 3: Learning Atmosphere

Item17: The learning atmosphere makes me participate in learning management.
Item18: The learning atmosphere makes me responsible for myself and others.
Item19: The learning atmosphere makes me studious.
Item20: The learning atmosphere makes it easier to talk and to ask questions with the teacher.

The data were collected from the following sources: pretest, posttest, and satisfaction survey and analyzed by using the effectiveness index: mean, mode, and standard deviations.

## 6. Research Results

### Results on the enhancement of achievement

Table 1: Students' scores from pretest, posttest, and Effectiveness index.

| Student's score from the pretest | | Student's score from the posttest | | Effectiveness Index |
|---|---|---|---|---|
| Mean | S.D. | Mean | S.D. | |
| 8.88 | 2.47 | 21 | 1.48 | 0.8 |

From Table 1, there was a large gap between the mean of the pretest and that of the posttest. It showed the enhancement of students' achievement. On the other hand, the standard deviation of posttest scores was smaller than that of the pretest. It showed the narrowed distribution of the posttest scores. The effectiveness index was 0.8 which is higher than the accepted effectiveness index (0.5). The effectiveness index of 0.8 meant that the student's achievement was improved by 0.8 compared to that at the beginning. It could be concluded that using flipped classrooms could be able to enhance students' achievement.

### Results from the satisfaction survey

Table 2 Results from students' satisfaction survey

| **Question No.** | **Number of Students** | | | | | **Satisfaction level** |
|---|---|---|---|---|---|---|
| | **Very Satisfied** | **Satisfied** | **Neutral** | **Dissatisfied** | **Very Dissatisfied** | |
| Category 1: Students' understanding | | | | | | |
| 1 | 7 | 14 | 11 | 0 | 0 | Satisfied |
| 2 | 5 | 19 | 8 | 0 | 0 | Satisfied |
| 3 | 13 | 19 | 0 | 0 | 0 | Satisfied |
| 4 | 18 | 14 | 0 | 0 | 0 | Satisfied |
| 5 | 28 | 4 | 0 | 0 | 0 | Satisfied |
| 6 | 9 | 18 | 5 | 0 | 0 | Satisfied |
| 7 | 2 | 18 | 12 | 0 | 0 | Satisfied |
| 8 | 3 | 18 | 11 | 0 | 0 | Satisfied |
| 9 | 21 | 11 | 0 | 0 | 0 | Very Satisfied |
| Summary of this category | | | | | | Satisfied |

| Question No. | Number of Students | | | | | Satisfaction level |
|---|---|---|---|---|---|---|
| | Very Satisfied | Satisfied | Neutral | Dissatisfied | Very Dissatisfied | |
| Category 2: Learning Activities | | | | | | |
| 10 | 2 | 22 | 8 | 0 | 0 | Satisfied |
| 11 | 5 | 18 | 9 | 0 | 0 | Satisfied |
| 12 | 13 | 19 | 0 | 0 | 0 | Satisfied |
| 13 | 11 | 21 | 0 | 0 | 0 | Satisfied |
| 14 | 17 | 14 | 1 | 0 | 0 | Very Satisfied |
| 15 | 6 | 16 | 10 | 0 | 0 | Satisfied |
| 16 | 6 | 21 | 5 | 0 | 0 | Satisfied |
| Summary of this category | | | | | | Satisfied |
| Category 3: Learning Atmosphere | | | | | | |
| 17 | 24 | 8 | 0 | 0 | 0 | Very Satisfied |
| 18 | 9 | 16 | 7 | 0 | 0 | Satisfied |
| 19 | 17 | 10 | 5 | 0 | 0 | Very Satisfied |
| 20 | 12 | 14 | 6 | 0 | 0 | Satisfied |
| Summary of this category | | | | | | At least Satisfied |
| Summary of all categories | | | | | | Satisfied |

From Table 2, the results showed that students were satisfied with the understanding category and learning activity category while they were at least satisfied with the learning atmosphere category. In summary, students were satisfied with all categories of learning by using a flipped classroom**.**

**Additional results from students' Internet access**

The number of online devices and the time used by students for internet access. Before the experiment, the researcher investigated online devices used by students and the time used for internet access. The results are shown in Table 3 and Figure 1 below.

Table 3: The number of students who had online devices

| Online devices | Number of students | |
|---|---|---|
| | Have | Don't have |
| Cell phone | 31 | 1 |
| Computer | 29 | 3 |
| Tablet | 15 | 17 |

Table 3 revealed that the number of students who had a cell phone, a computer, and a tablet were 31, 29, and 15 respectively. One student didn't have a cell phone but he had a computer at home, so every student could access the online learning.

The time used by students for internet access per day is shown in Figure 1 below.

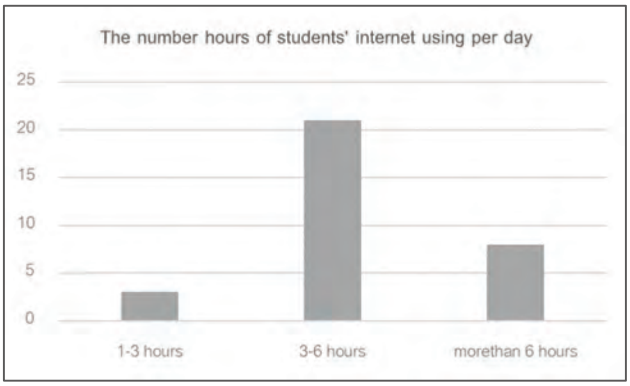


Figure 1: Number of Hours Students used the Internet per Day.

Figure 1: shows that students spent at least 1-3 hours on internet access. Most students (21 students) accessed 3-6 hours a day. So, each student was accustomed to internet access.

## 7. Conclusion

For students' achievement, the effectiveness index from this study was 0.80 which was higher than that of the accepted index 0.50. This indicated that students' achievement was enhanced after using flipped classrooms. The index of 0.80 meant that students' achievement was enhanced by 0.80 compared to that from the beginning of the pretest. For students' satisfaction, the results revealed that by using a flipped classroom, students were satisfied with math learning in all three categories: students' understanding, learning activities, and learning atmosphere.

## 8. Discussions

Students' achievement was enhanced because of using flipped classrooms. This enhancement was the results of the following: 1) class time was extended from a before-class study such as watching video clips online, studying handouts, doing homework, and quizzes; 2) more class time for discussion and problem solving from worksheets; and 3) extending new skill/knowledge such as practice harder problems and more application. Reflection in each cycle helped the researcher to improve the teaching process and to talk to students who didn't do quizzes or didn't understand the contents. In addition, line groups supported students' learning. Participants could discuss problems, and solution processes and communicate through it. All of those activities encourage students to learn and to spend more time learning.

The enhancement was consistent with many studies. Karadag and Keskin [16] studied the effects of the flipped learning approach which revealed that it positively affected students' academic achievement and attitudes toward mathematics. Furthermore, for some components of flipped classrooms in mathematics, many studies showed that they could enhance students' mathematical achievement. Khairiree. K [17] found that students who studied in using flipped classroom were more engaged than those in a traditional classroom. The flipped classroom helped students manage time in doing classroom activities more efficiently and improved their achievement. Unakorn and Klongkratoke [18] stated that flipped classrooms engaged students in self-learning and increased opportunities for students to achieve learning objectives of learning statistics topics in Grade 11. Poomorn, A. [19] also stated that for students' satisfaction with flipped classroom instruction, most of the students strongly agreed that they communicated with the teacher more often. They also agreed that they had greater opportunities to communicate with other students. They could apply out-of-class experiences with the lesson and could learn more from practical applications.

One crucial point to consider was about preparing teaching and learning materials such as handouts and quizzes. The researcher applied the practices and recommendations from experts and professional organizations for the benefit of students' understanding, computational skills, reasoning, and problem solving. The recommendations were about: 1) content selection and sequencing, and 2) teaching vector concepts, important points to be careful of, and some pitfalls to be avoided [13]. The following were some points integrated with teaching: 1) visualization of vectors, vector operations, negative and unit vectors by drawing directed line segments to represent them; 2) reasoning such as why $\frac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$ or why $(\vec{u}\cdot\vec{v})\cdot\vec{w}$ is meaningless; 3) computation in vector operations; 4) prove some theorems left for students; 5) prove harder problems; 6) applications in physics for dot product (the work done) and cross product (the torque); and 7) extended knowledge such as equations of spheres and lines in space. For students' satisfaction with flipped classrooms, the results showed that most students were satisfied with learning. These results were consistent with the studies of [20], [21], and [22].

From limitations in classroom action research, normally there will be no generalization of the results. Applying the practices must be done with great care. This study was concluded in a highly economical area. Students and parents were ready for high educational competition. Achievement enhanced and students' satisfaction resulting from applying flipped classrooms in different areas may or may not occurred.

## 9. Recommendations

**Recommendations for teachers**

Until now, teachers and students experienced online learning, so it was a good time to use flipped classrooms by extending from online learning. There were two parts to be considered, the teachers' part and the students' part. For the teacher's part, using flipped classrooms needed much time and effort in preparing lesson plans, video clips, handouts, homework, quizzes, learning activities, and problems. Part of students, needed class time, concentration, and self-study ability by watching video clips, studying handouts, and doing homework and quizzes. Considering both parts, teachers at the beginning of using flipped classrooms might try just one-fourth or one-half of a course and look back to evaluate both teachers' and students' parts and improve them. Then, teachers could develop and expand the whole course.

**Recommendations for future research**

Further repetitive research about using flipped classrooms was still needed to fulfill various points that were not answered in this study. Some of them were the following:

1) Effectiveness of flipped classrooms to high school students at normal or average ability.

Samples in this study were in leading high schools and were in high economic areas. They showed above-average learning abilities. What was the effectiveness of the average student?

2) Effectiveness of flipped classroom to lower-secondary school students (Grade 7 – Grade 9 students).

Many research and practices were performed with high-school or undergraduate students but rarely with lower-secondary students.

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# Secondary Level Mathematics Teachers' Critical Reflections on the Use of GeoGebra for Teaching Trigonometry

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## Abstract

*Technology-integrated pedagogy creates an engaged learning environment that supports conceptual, relational and procedural understanding. This study explores the roles of the GeoGebra Application (GA) in teaching trigonometry. The data were collected after and before the seven-day online training programs on using GA in teaching trigonometry through reflective experiences. We used critical reflective practice, cognitive theory and social constructivism to interpret and make meaning. This study revealed that teachers had disempowering and negative images towards trigonometry and its teaching before training. They believed trigonometry is an interpretation-free discipline and teaching as preparation for the final examination. Likewise, they lacked conceptual and relation understanding of the trigonometric knowledge and concepts that severely affected their teaching-learning activities. After the training program, the participant teachers reported that GA is a handy technological tool that helps visualize abstract trigonometrical concepts, supports to develop positive images toward trigonometry, and fosters an engaged learning environment. Finally, this study signified that integrating GA in trigonometric teaching can positively affect respective teachers' thinking, knowing and doing.*

## 1. Introduction

Student disengagement in mathematics learning has become one of the significant problems for the ensuing quality of mathematics education in Nepal [1]. Researchers have explored that mathematics teaching-learning activities focus on imparting abstract mathematical formulae, concepts and theorems without considering conceptual and relational understanding [2-4]. The pouring and banking approaches of pedagogy [5] suffer mathematics education practices in schools and universities, resulting in negative images [6] and attitudes and beliefs [3]. It instigates mathematical anxiety among the learners that adversely affects mathematics education and creates the vicious circle of disliking, disengagement, underachieving, failure and dropout [7-8] and supports to spread of negative images of mathematics in society as abstract, difficult, mysterious, cold and dry, masculine and elite [33]. The pedagogical approach is one of the responsible factors for producing these images [2]. To bring a paradigm shift in mathematics pedagogical practices, we must break the boundary of conventional informing and banking pedagogy by integrating modern technology into mathematics teaching-learning activities [9].

Technology has become integral to various aspects of human life, including education. Many research studies have explored that teachers need technological knowledge to enrich the quality of mathematics education [10][13]. Technology covers a wide range of software and hardware designed for specific mathematical purposes, such as GeoGebra, Geometer's Sketchpad, Cabri Geometry, Graphing Calculators, Mathematica, MATLAB, etc., that are applicable in different devices and platforms (tablet PC, spreadsheets, virtual blackboards, PowerPoint, Java Applets, etc.). One of the technologies that is widely used in education and has a significant potential to enhance conceptual and relational understanding is the interactive GeoGebra application/software (GA) [11-12].

GA is a technological tool that supports mathematical thinking and learning. It enables practitioners to create, visualize, and demonstrate mathematical objects, formulae, concepts, and theorems and establish relationships creatively and intuitively. Interactive and dynamic features of GA significantly support and encourage learners to engage in mathematical activities independently and/or collaboratively [13]. Likewise, GA helps bring variability in teaching-learning activities by demonstrating a single concept dynamically. It is one of the best ways to boost the pupils' cognitive capabilities by incorporating alternative thinking. It signifies that interactive GA is a learning environment that scaffolds the learners with the dynamic interplay of seemingly static and rigid mathematical ideas.

Despite having such potential of GA for enhancing mathematical learning, it has not been reached in school mathematics classrooms in the context of Nepal. Until now, most of the mathematics teaching-learning activities in Nepali schools have incorporated chalk and talk and learning by doing without reflection, resulting in the rote memorization, recitation and reproduction of mathematics mathematical facts, formulae, concepts and theorem [16]. It un/intentionally creates a barrier to engaging in conceptual and relational understanding through critical reflective practices. Against this backdrop, this study explores the participants' reflections on integrating GA in trigonometry teaching at the secondary level.

## 2. Purpose and Research Questions of the Study

The purpose of the study is to explore the secondary-level teacher's critical reflections on the use of GA in trigonometry teaching before and after participating in seven-day training on "GeoGebra for Teaching Trigonometry at the Secondary Level." To address the purpose and navigate our research, we formulated the following research questions:

i) What practices and presumptions did secondary-level teachers have about teaching trigonometry before training?
ii) How did secondary-level teachers reflect on their experiences integrating GA after the training?

## 3. Theoretical Referents

To explore the experiences of secondary-level mathematics teachers from multiple perspectives and make meaning accordingly, we used critical reflective practices, cognitive learning theory, and the theory of social constructivism as theoretical referents. We used critical reflective practice as a way of writing to help us make holistic meaning by interconnecting three levels of experiences: reflection-on-action, reflection-in-action, and reflection-for-action [15]. The first level of reflection offers ways to analyze the participants' stories and experiences, what they did, the meaning perspectives of these activities, who benefited, and whose voices were heard [16]. Secondly, the reflection-in-action provides the lens for exploring contemporary practices. Finally, the third stage of critical reflection focuses on imagining and envisioning future tasks by improvising their past and present actions for the betterment of the overall programs.

Likewise, Brookfield also provided four procedures of critical reflection [17]. First, researchers and practitioners deeply engage in contemporary theory and research to understand the present discourses. In the second stage, they critically reflect these theoretical and philosophical underpinnings. In the third stage, practitioners offer the peer-group reflection on their ways of knowing and doing to improve their actions, which is used mainly in our training session to improve methods of constructing, presenting and discussing the trigonometrical concepts. Finally, critical reflective practitioners offer students reflections on their actions to improve classroom practices.

Cognitive learning theory focuses on developing the cognitive capabilities of learners through diverse perspectives. It uses the immediate environment as the source of information that underpins rational reasoning and intuitive thinking. Internal and external factors influence learning, but the rational and intuitive power of the learners has played a significant role in cognitive development [18]. The new environment and interaction create disequilibrium with their prior knowledge, and then learners comprehend the situation by rational and intuitive reasoning to make sense. After comprehending it, learners store new knowledge in their cognitive structures for future application [19]. From this perspective, the GA learning environment provides ample opportunity for the learners in which they interactively engage in learning mathematical concepts utilizing dynamic visual presentation, variability of representations, and their relations with other concepts, indeed evoking the learners to revive their prior knowledge by integrating the new ideas and concepts, that contribute to cognitive development [20].

Social constructivism denies an approach to learning as accumulating mathematical facts, concepts, skills and muted symbols, but it is a process of social and cultural interactions. Cognitive capability is not only the individual endeavour but also the product of their culture and social interactions. Social constructivist learning theory rejects that learners can learn mathematical ideas after cognitive development but believes that cognitive development occurs during learning [21]. From this perspective, learning is shifting the lower boundary of the zone of proximal development (ZPD) through the scaffolding process [21-22]. The GA learning environment provides alternative ways of performing, demonstrating and interpreting mathematical concepts and ideas, which certainly contribute to shifting the ability to be independently involved in the inquiry process to enrich their professional development.

## 4. Methods of Data Generation and Interpretation

The required data were generated during the training sessions on using GeoGebra in teaching trigonometry at the secondary level, organized by the Council for Mathematics Education (CME) Chitwan, Nepal, during COVID-19. The government of Nepal declared a lockdown in the country to prevent the rapid spreading of COVID-19. CME Chitwan Branch decided to organize training for the respective teachers and appointed the first author as a program coordinator. The second and third authors helped develop the detailed training session plan. Through the Google form, we offered the proposal of interest from teachers teaching trigonometry at the secondary level and taking online classes during COVID-19. We received approximately 60 applicants' forms. We selected only 21 teachers on a first come, first served basis because selecting large numbers may create difficulty in handling and possibly disrupt making the class more interactive.

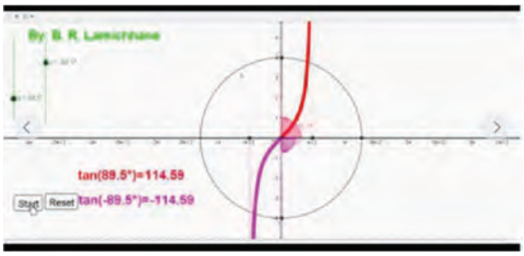

Fig. 1. Screenshot of tracing of tan curve in positive and negative direction

To generate the required data, we asked the participants to critically reflect on their experiences of teaching-learning activities throughout their academic and professional journey. They seemed hesitant to express their experiential experiences. Then, I (the first author) told my story. I learnt the trigonometrical relation by rote memorization through the techniques given by my mathematics teacher, such as "*Pandit Badri Prasad Hara Hara Bole*", which helped me to remember trigonometrical relations without making any sense and meaning. The techniques of remembering were that we have to take the first alphabet of the above mantra as P, B, P, H, H, B, and the first three alphabets constitute the numerators and the remaining three work as denominators of the trigonometrical relations. Then they look like $sin\theta = \frac{p}{h}$, $cos\theta = \frac{b}{h}$ and $tan\theta = \frac{p}{b}$. And for the other relations, we reverse these relations. It is one example of how the first author learned trigonometry relations from the first trigonometric class. After that, participant teachers freely shared their experiences, which helped us to capture the essence of their presumptions and practices.

We have used the online classroom discussions in a training session, participant teachers' stories and their lived experiences [14] in trigonometry teaching as data sources. The participants were divided into three groups to systematize the discussion and sharing session. The team leaders are assigned to collect the queries of the other participants through the chat box. The three-hour online session was conducted on a one-day alternative basis to provide enough time for the participants to engage deeply in GA environments. At the beginning of every session, approximately 20-30 minutes were provided for the participants to share their experiences. The team leaders shared the experiences of the teaching-learning activities, and each participant was also allowed to share their experiences if any, distinct experiences from the shared ones. We recorded all the activities of the training sessions facilitated by the first authors by obtaining consent from the participants, assuring them that the data would be used only for research purposes, and strictly maintaining confidentiality. Grounded in the theoretical referents, we have generated the themes and interpreted the data with the help of second and third authors.

## 5. Discussion and Interpretations

This section briefly discussed the participants' views on trigonometry and its teaching before and after the online training sessions. We have discussed the overall findings in two parts—before and after training.

### 5.1. Reflection of Teachers before Training

To explore the participant teachers' experiences, the first author opened the discussion by asking questions.

Me: *How do you demonstrate the value of* $tan(90^o) = \infty$ *in your classroom?*

Participants*: Sir, we can easily show it. All participants argued in the same fashion. The value of* $tan(90^o) = \frac{Sin(90^0)}{cos(90^0} = \frac{1}{0} = \infty$.

Me: *Do the students understand the value of* $\frac{1}{0} = \infty$*? How do they represent* $tan(90^o)$ *Geometrically? Can you give some examples that visualize the value of* $tan(90^o)$*?*

Participants: *Sir, we do not think from this perspective. Our curriculum does not demand such types of skills and concepts. We believe students do not have difficulty remembering the value of trigonometric functions at the given range.*

Me: *I respect your argument. Let's forget students for a while. Can you visually present the value of* $tan(90^o)$*? We break the session into three rooms for discussion and ask them to come to a common understanding of the group.*

Participants: *Three groups share their understanding. However, there is no variability in interpretation and presentations. They repeated the above arguments. One of the groups presents it with a graph automatically generated by the GA.*

Me: *How do you respond to the students, if they ask, how* $tan(90^o) = \infty$**.**

Participants: *As we have already said, it is a mathematical formula and needs no further discussion. We have never raised such questions during our academy journey and have never faced such types of questions. Please don't waste our time discussing such a topic.*

Me: *Ok, it is fine. However, I have one more query for you, and then we start our main session. Do you realize pedagogical practices in mathematics education are responsible for generating students' popular negative images or attitudes towards mathematics/trigonometry?*

Participants: *We are somehow responsible. Curriculum experts/designers, the Ministry of Education, local governments, educational authorities and other line agencies are more responsible than teachers. Teachers are innocent, powerless and obedient; they have fulfilled their responsibilities.*

From the above discussion, we have captured the teacher's conceptions and teaching experiences before training, which are presented in the following subsections.

#### 5.1.1. Trigonometry as an Interpretation-free Discipline

All participants have the same presumptions that trigonometry is a branch of mathematics with already established formulae, concepts and theorem. Trigonometric knowledge is like other mathematical knowledge that has existed and remained somewhere in the universe [23]. Teachers' tasks are to bring such mysterious knowledge, formulae, and concepts into a classroom as it is and transmit them to the students. Likewise, they view teaching as reproducing the given trigonometrical knowledge [35] that does not need further interpretation. The already established knowledge, concepts and ideas do not demand further interpretation. To do so is a waste of time because trigonometrical knowledge is universal.

#### 5.1.2. Teaching as/for Preparation of Examination

Most teachers agreed that teaching is the preparation for the upcoming final examination. They think teaching-learning activities need not go beyond the narrowly framed textbooks. Teaching means solving the textbooks' questions by following the rigid algorithm. Its mechanistic way of finding the solution to bookish questions restrains creative, critical and imaginative thinking in mathematics, and learners become muted followers. They never raised questions regarding its genesis, values and relation to the other disciplines. Students become passive receivers of knowledge and teachers as knowledge depositors [5]. Through the long-run practices of informing pedagogy, it appeared to be prevalent in school trigonometric teaching. Teachers frequently reject the necessity of alternative ways of presentation and discussion to prepare the students for the final examination.

#### 5.1.3. Lack of Conceptual and Relational Understanding

One of the research participants reflects on his/her experience. *As a mathematics student, I read trigonometry for the first time in grade IX. I did not make sense of the trigonometric functions and their values. Teacher entered the room and drew a right-angled triangle on a backboard. He denoted the sides of the triangle by p, h, and b, wrote the formulae of sin, cos, tan, cosec, sec, and cot, and urged us to remember them. Likewise, a few days later, the teacher made a matrix of values of standard angles of the trigonometric functions and gave some tricks to memorize them. Then, we unquestionably adopt the same learning and teaching procedures.*

It indicates that teachers were rarely orients to create alternative discourses in teaching trigonometry. They hardly realized the importance and roles of variability principles in teaching trigonometry for cognitive development [20]. Due to the lack of teachers' conceptual and relation understanding of trigonometrical concepts, most teachers could not relate the trigonometric concepts to real-world problems. From the other perspective, our teachers have been deskilled by the education system of the country in general and university in particular so that teachers can only perform the files duties [24]. From the beginning of the modern education system in Nepal, teaching-learning activities tied within external recognition, such as success and failure in content domains, were determined through externally executed paper-pencil tests. Consequently, our future teachers become the disseminators of bookish knowledge.

### 5.2 Reflection of Teachers After Training

After listening to the teachers' presumptions and experiences, we started our session from the beginning of the trigonometric functions and their values. It is impossible to present all the materials developed during the training sessions. So, we select some exemplary cases that could evoke the participant teachers to think alternatively and become energized to be creative, critical, imaginative and collaborative learners.

*Based on the participant's interest, we chose the case to demonstrate trigonometric values. In doing so, I provided them with a construction protocol and requested them to follow the procedures. First, we construct the dynamic graph of the trigonometric functions of sine, cosine, and tangent that demonstrate the values of the trigonometric function in different ways (see Fig. 2 (i), 2 (ii) and 2 (iii)) and ask them to prepare the dynamic graph collaboratively in their groups that can differently visualize the values of the trigonometric function. After completing the construction, we discussed its pedagogical implications. To illustrate the value of the* $\tan(90^0) = \infty$*, we create dynamic graphs (see Fig. 3(i), 3 (ii), 3 (iii)) and discuss them.*

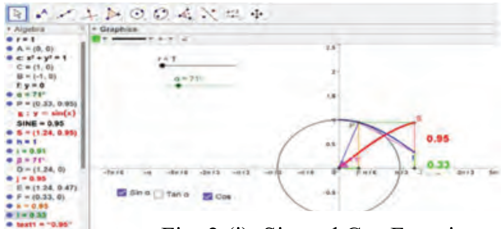

Fig. 2 (i). Sin and Cos Function

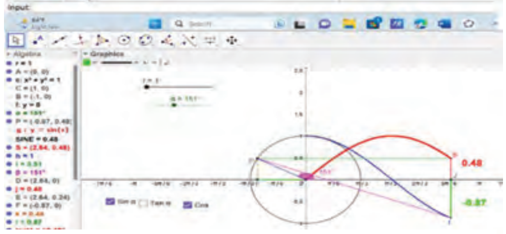

Fig. 2 (ii). Sin and Cos Function in 2nd Quadrant

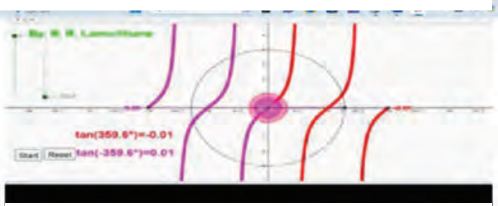

Fig. 2 (iii). Tangent Function

Me: *We have just prepared the graph to demonstrate the value of* $\tan(90^0) = \infty$ *on the standard unit circle. Look at figure 3(i); the tangent line at point P on the circle extended to the X and Y axes, which meet axes at points E and F, respectively. In this figure, we can see the values of* $\tan(45^0)$ *and* $\cot(45^0)$*. What are the values?*

Participants: *Both values are equal.*

Me: *Which parts of the graph represent the values of* $tan(45^0)$ *and* $cot(45^0)$*?*

Participants: *The values of* $tan(45^0)$ *and* $cot(45^0)$ *are represented by the line segments PE and PF, respectively.*

Me: *What happens when we increase the value of the angle from* $45^o$ *to* $85^o$ *(see Fig. 3(ii))?*

Participants: *The values of tan increased from 1 to 11.43, and the value of cot decreased from 1 to 0. 087.*

Me: *Please increase the value of the angle from* $85^o$ *to* $89.99^o$ *and note the values of tan and cot (see Fig. 3(iii)).*

Participants: *Sir, the value of tan rapidly increases, but the value of the cot tends to zero. When the angle value increases gradually, the tangent line tends to parallel to the X-axis. It does not meet the X-axis. Oh! It's astonishing! Similarly, when the value of the angle is* $90^o$*, the value of the cot becomes zero. That is, the distance between PF (cotangent) becomes zero.*

*Me: Yes, you can't see point E in the figure, and the line becomes asymptote to a parallel line to the X-axis at point P (it seems parallel in figure 3 (iii)). Do these line PE and X-axis meet at the X-axis as in figures 3(i) and 3(ii)? In this case, what would be the values of PE? Can you determine it?*

*Participants: No sir, we cannot determine the point E on the X axis, i.e. we cannot find the value of PE.*

*Me: That is why it is infinity. You can see the visualization of the value of* $\tan(90^o) = \infty$. *Again, drag a slider and construct a matrix of the values of the sin, cos and tan, and observe the trends of values of these functions.*

*From this, you can also observe that sine and cosine values tend to be one and zero. When one is divided by the small number between one and zero, it becomes infinitely large. It is also other ways of understanding the values of* $\tan(90^o) = \infty$ *(see Fig. 4). Other tools, such as Desmos, Mathematica, MATLAB, etc., help you demonstrate the value of trigonometric functions. In these ways, we can visualize the value of trigonometric functions differently (See Fig. 1, 2(i-iii), 3(i-iii) and 4). Can you draw a graph to illustrate the values of these trigonometric functions differently? They designed the graph differently in a dynamic version and presented it in a group (see Fig. 5). They described the nature of the sine function, its periodical values, and negative and positive angles with curve tracing.*

After the end of the sessions, we asked them to reflect critically on the impact of training that will help us design training in the coming days. From their reflections, we have captured the following themes.

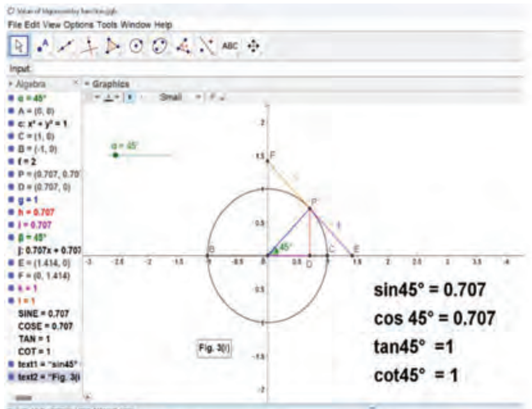


Fig. 3 (i) Value of Sin, Cos and Tan, at $45^0$

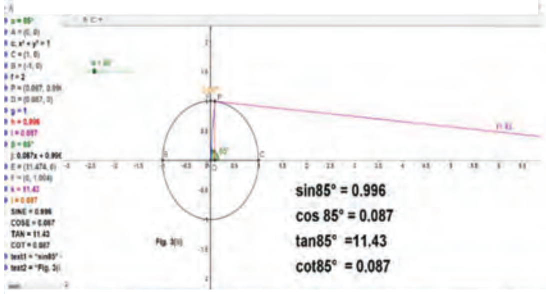


Fig. 3 (ii). Value of Sin, Cos and Tan, at $85^0$

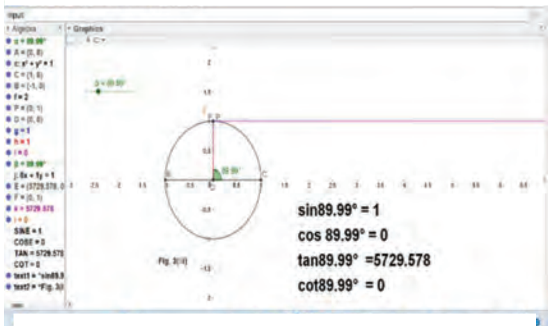


Fig. 3 (iii) Value of Sin, Cos and Tan, at $89.99^0$

### 5.2.1 GA helps Visualize Abstract Trigonometric Concepts

Most teachers view trigonometry as an abstract, vapid, decompensate and mysterious subject before engaging in the training session. Describing the roles of the GA in teaching trigonometry, one of the participants from group A narrated his/her experience. *We are really in an illusionary state. Our university and government institutions have not taught and trained us through technology during pre-service and in-service teacher training programs. We also do not try to enhance our content knowledge and technological skills independently. We forcefully compelled our students to memorize the trigonometric formulae, concepts and algorithms for the recitation in the upcoming final examination. I realize, we have only learned to imitate and taught the students to do so. This training is really fruitful for us. We have learnt to visualize the abstract trigonometric formulae like the value* *of tan 90$^0$, relations between and among the trigonometric concepts, double angle formula (sin2A) and others.*

This reflective experience explores that the most challenging tasks for teachers and students are visualizing seemingly abstract trigonometrical concepts. The GA helps demonstrate abstract concepts in a dynamic visual mode effectively [34]. Likewise, incorporating the GA in teaching mathematics supports enhancing conceptual, relational and procedural understanding [25] and the development of positive images of practitioners toward the subject.

| Angle Measure | sin | cos | tan |
|---|---|---|---|
| 30° | 0.5 | 0.87 | 0.58 |
| 60° | 0.87 | 0.5 | 1.17 |
| 86° | 0.998 | 0.07 | 14.301 |
| 89.51° | 1 | 0.009 | 116.927 |
| 89.99° | 1 | 0.00017 | 5729.57789 |
| 90° | 1 | 0 | ∞ |

Fig. 4: Value of sin, cos and tan at different degrees generated by GeoGebra

### 5.2.2 GA Develops Positive Images toward Trigonometric

Participant teachers' reflective narratives before training indicated that most teachers have negative images towards mathematics in general and trigonometry in particular. Many researchers [26-27] have also explored that images of mathematics significantly affect their classroom teaching-learning activities. There is a high possibility of transmitting negative images, anxiety, and beliefs from teachers to their students that adversely affect the teaching-learning activities, resulting in low achievement in mathematics [7].

In this regard, the reflective experience narrated by one of the participants from group B is worth including. *I have experienced during my academic journey of mathematics education that mathematics is a mental game played by some elite people. In my opinion, trigonometry is nothing more than the accumulation of formulae, symbols, axioms, and theorems that remain far from our lives. After participating in this training, I regret my previously held beliefs and images of mathematics and trigonometry that hinder my teaching-learning activities and possibly discourage my students from adapting mathematics as their future career subject. I am astonished that GA makes abstract trigonometrical knowledge and concepts dynamically visualize and establish the relations between and among the concepts. It amazingly explores the beauty of mathematics. This training supports eliminating the stereotypical views of mathematics and teaches us that many alternatives exist to solve or represent the same trigonometrical concept.*

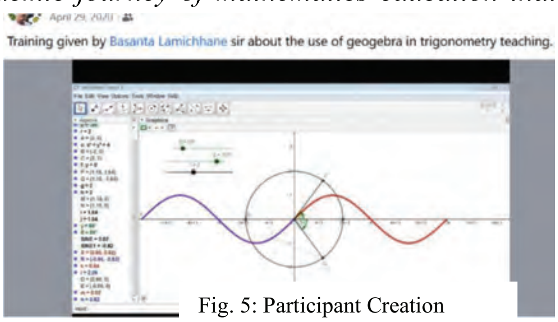


Fig. 5: Participant Creation

This representative narrative of teachers has unearthed many untold realities of trigonometric teaching-learning. Due to the lack of technological knowledge and its intertwined pedagogical skills, teaching-learning activities are incarcerated within the transmissionist, machinist and disempowerment realms [28]. In this regard, GA helps bring the variability in teaching-learning activities, fosters the collaborative learning environment, promotes the learner-centred pedagogical approaches and disrupts the notion of learners as passive receivers of knowledge [29]. Arrive at this point of inquiry, we have realized that GA has the potential to create an engaged learning environment.

### 5.2.3 GA Promotes Engaged Learning

The reflection of one of the participants from group C highlights the key issue of engaged learning enhanced by the training. *Now, I have realized probably all of my colleagues also agree with my views (no one disagreed with him) that trigonometry is one of the most useable branches of mathematics, but we teachers make it difficult, abstract, obscure, and mysterious. We victimize students because we cannot enable them to be creative, critical, imaginative, independent, and collaborative learners, but we compel them to imitate the disempowering notion of learning. I did not feel bored and tedious in such a long three-and-a-half-hour session, and the 7-days passed like a blink. I thoroughly enjoyed it. It helps us widen our cognitive thinking, develop the skills to work in a group, enhance our conceptual and relational understanding and create a learning environment in which we are emotionally engaged without external motivation.*

Learners' engagement in learning is a significant attribute for enhancing and promoting the learners' academic achievement [30] by addressing the issues of disliking, anxiety, dropping out, and boredom in teaching-learning. To promote an engaging learning environment, we must focus on the learners' academic, cognitive, social and affective dimensions [30]. The GA-integrated learning environment enhances cognitive development because it scaffolds the learners to deploy contrast and variation approaches and dynamic visualization. Likewise, the GA environment encourages the learners to work collaboratively by cooperating, feedbacking, commenting and encouraging each other to bring the desired outcomes. Finally, GA supports addressing the affective dimensions by enriching the emotional attachment of learners to the learning process and self-reporting learning values [31].

## 6 Conclusion and Implication

The disengagement in learning mathematics, in general, and trigonometry, in particular, is one of the major problems in the context of Nepal [32]. This study has indicated that factors behind the disengagement in learning trigonometry are teachers' negative images and beliefs towards trigonometry, presumptions of teaching as the transmission of an already existing body of knowledge to the passive minds of students, and lack of technological expertise and its intertwined pedagogical skills. This study further explored how integrating GA in teaching trigonometric functions helped address the disempowering notions above. The GA-integrated approach supports visualizing abstract trigonometrical concepts, resulting in conceptual, relational and procedural understanding. It also played a significant role in developing positive images toward trigonometry, possibly preventing the vicious circle of disliking, underachievement, anxiety and dropping out. Finally, the GA environment underpins engaged learning by incorporating the most significant aspects of learning: academic, cognitive, social and affective. We concluded that integrating technological tools in trigonometric teaching is necessary in the 21st-century complex era. It enhances the learning environment that explicitly contributes to developing positive attitudes, beliefs and images towards the subject, consequently contributing to the meaningful learning of trigonometry rather than imparting the discrete knowledge to prepare for the final examination.

## 7. Funding

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# AI Chatbots as Math Algorithm Problem Solvers: A Critical Evaluation of Its Capabilities and Limitations

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## Abstract

*AI-based chatbots are appearing as a powerful tool for solving mathematical algorithm problems. These chatbots, trained on extensive datasets and natural language models of text and/or code, can understand and generate mathematical expressions. They can show step-by-step solutions to math problems and explain the associated concepts. This paper evaluates AI chatbots like Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha as problem solvers for math algorithms for learners, teachers, and teacher educators from school to university. We discuss how AI recognizes mathematical expressions and equations from simple arithmetic, algebra, trigonometry, and statistics examples. We explore how AI-based chatbots solve basic to advanced math problems, their ability to offer personalized solutions, and their potential to improve students' math learning. We highlight their capabilities and limitations. Challenges faced by AI chatbots include a limited understanding of natural language models and prompt engineering, an inability to solve complex math problems, and the potential for bias. Future research should focus on improving AI chatbots' accuracy, reliability, and problem-solving capabilities. Despite the advancements in AI chatbots, students should continue interacting with human teachers to develop their cognitive math skills and conceptual understanding.*

## 1. Introduction

AI chatbots, such as ChatGPT, Google Bard, and Bing Chat, are advanced natural language models capable of producing responses that closely resemble those generated by humans [1-2] whereas Wolfram Alpha is an AI-based tool equipped with computational intelligence. On the other side, those trained on massive datasets of text and code can answer various questions, including math algorithm problems. Likewise, ChatGPT, Google Bard, and Bing Chat are also based on statistical approaches to training large language models (LLM). These chat tools have undergone extensive training using large-scale text data and possess the capability to address a wide range of inquiries, encompassing mathematical algorithmic problems. Indeed, the proficiency exhibited by AI models—natural language model and computational intelligence in solving complex equations and calculations has prompted inquiries regarding the potential uses of chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha, among others, in assisting with mathematical assignments. In this regard, Rawat and Mishra discussed the important role that mathematics plays in the development of AI [3]. They argued that mathematics is essential for AI in several ways for providing tools for data analysis and modeling [3]. AI algorithms need to be able to process large amounts of data to learn and make predictions. Mathematics offers data analysis and modeling tools like linear algebra, calculus, and statistics. Likewise, mathematics is constantly being used to develop new AI algorithms that are more powerful and efficient. Mathematics can also be used to understand the limitations of AI. For example, it is possible to use mathematics to prove that there are some problems that AI will never be able to solve.

Moreover, chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha are known for their improved math skills and ability to help students do better in mathematics of any level. Students explore this AI tool by offering basic to advanced knowledge and information on various math topics, as well as thorough instruction and help with geometry [4]. For instance, Lee and Yeo used design-based research to develop and evaluate the chatbot [5]. In the first iteration of the chatbot, researchers worldwide qualitatively analyzed the training data to identify the most common types of questions that preservice teachers (PSTs) would ask. They then created corresponding responses for each type of question. In the second iteration, the researchers changed the chatbot's responses based on feedback from PSTs. Likewise, Wang and Lester further argued that the rapid development of AI technologies calls for designing and developing K-12 AI literacy curricula to support students entering a profoundly changed labor market [6]. Wang and Lester further defined AI literacy as the ability to understand the basic concepts and principles of AI, to evaluate AI systems critically, and to use AI tools and techniques to solve problems [6]. With the above background and gap, in this paper, we explore the potential uses of AI chatbots, including Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha, as problem solvers for mathematical algorithms across school to university level for mathematics learners. Also, we delve into how these AI chatbots recognize and solve mathematical problems ranging from basic arithmetic to complex statistics. Lastly, we highlight AI chatbots' capabilities of solving mathematical problems and critically evaluate the capabilities and limitations of using algorithms for solving such problems in mathematics.

## 2. Recognize Math Expressions and Equations

An expression in mathematics consists of numbers, variables, and operations. A mathematical statement that establishes the equality of two expressions is known as an equation. 5 + 2; 3x - 4; and/or (2x + 1) (3x - 4) are examples of expressions. $2x + 3 = 7$, $x^2 - 4x + 4 = 0$, and $y = mx + b$ are examples of equations. The above numbers, variables, and operations can be solved by using AI chatbots. Likewise, AI also can be used to solve mathematical equations and expressions. For instance, see below to prompt in ChatGPT of the question of arithmetic—and embed code.

**Prompt to ChatGPT**: Ms. Bhandari wants to buy 2000 Bahrain Dinar paying Nepalese currency. If buying and selling rates of 1 Bahrain Dinar is Rs. 271.73 and Rs. 273.33. How much Nepali currency is required for the transaction?
**Response in embed code**: When you're buying a foreign currency, you will generally need to use the "buying" rate, which in this case is Rs. 271.73 per 1 Bahrain Dinar.
Ms. Bhandari wants to buy 2000 Bahrain Dinar. To find out how much Nepali currency is required for the transaction, you would multiply the amount of Bahrain Dinar she wants to buy by the buying rate:

```
\[
\text{Total Cost in Nepali Rupees (Rs.)} = 2000 \, \text{Bahrain Dinar} \times 271.73 \, \text{Rs./Bahrain Dinar}
\]
\[
\text{Total Cost in Nepali Rupees (Rs.)} = 543,460 \, \text{Rs.}
\]
```

So, Ms. Bhandari would need 543,460 Nepali Rupees for the transaction to buy 2000 Bahrain Dinar.

Further, Rawat and Mishra suggested some examples of how mathematics is used in AI, such as in linear algebra. At the same time, linear algebra is used to represent data in a way that AI algorithms can process [3]. For example, images can be represented as matrices, and text can be represented as vectors. Calculus is used to develop AI algorithms that can learn and make predictions. For example, gradient descent is a calculus-based algorithm used to train neural networks. Statistics are used to evaluate the performance of AI algorithms. Thus, chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha can process a fundamental part of mathematics such as equations and expressions.

Opposing the above, a chatbot like this can carry out simple mathematical operations like addition, subtraction, multiplication, and division. Next, Supriyadi and Kuncoro discussed the potential of ChatGPT, a large language model chatbot developed by OpenAI, to revolutionize mathematics teaching and argued that ChatGPT can be a powerful tool for mathematics education [7]. ChatGPT, Google Bard, and Bing Chat can generate personalized feedback, provide explanations of mathematical concepts, and help students solve problems. For instance, see Figure 1 to see the factorization processes in detail.

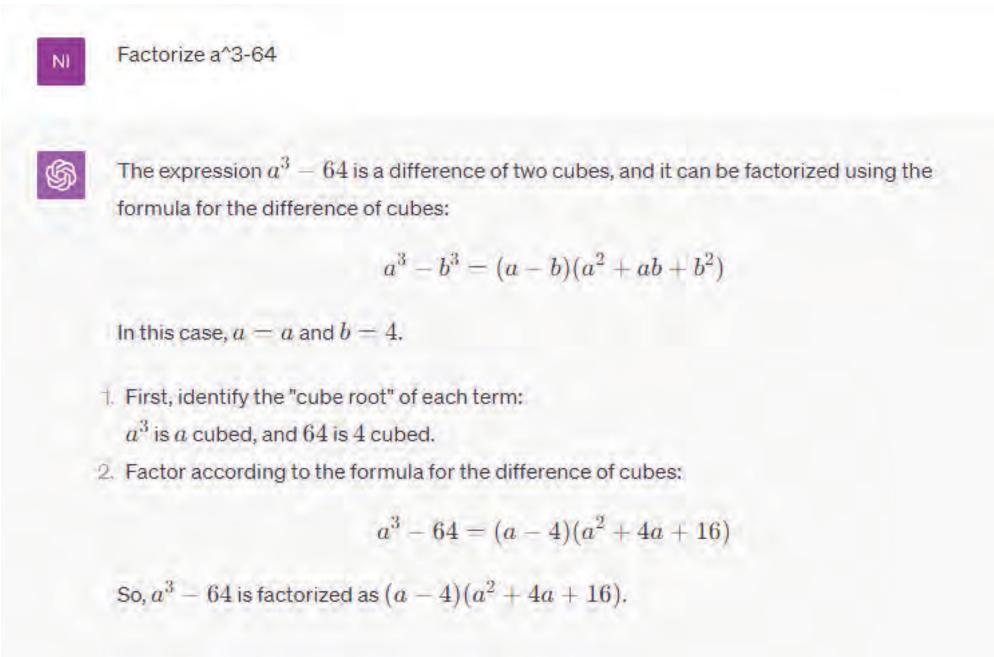



Figure 1: Factorized by ChatGPT

It can also be used to create engaging and interactive learning experiences. Instead, Qiu et al. explored the effectiveness of AI in improving students' deep learning in primary school mathematics teaching in Fujian Province and acknowledged that studying mathematics can be challenging for primary school students and suggested that AI can be used to evaluate individual student's learning challenges and provide personalized support to optimize their success in mathematics classes [8]. So, these chatbots can resolve complex equations, including systems of linear equations and quadratic equations. Overall, Monte-Serrat and Cattani argued that the ethical design of AI systems requires a deep understanding of the relationship between mathematics and human behavior [9]. They call for mathematicians and computer scientists to work together with ethicists and social scientists to develop ethical guidelines for developing and using AI systems.

## 3. Solving Basic and Advanced Math Problems

Simple math problems like addition, subtraction, multiplication, and division can be solved by chatbots such as ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha. For instance, if you ask any of the chatbots mentioned above what 25+25 is, they will all display the right answer, 50. In the same way, if you ask it what 200-95 is, it will give you the correct answer of 105. See Figure 2 below as an example responded to by Google Bard.

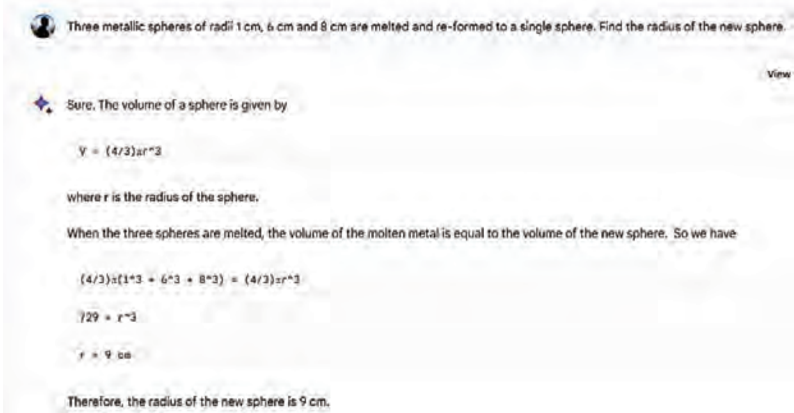



Figure 2: Response by Google Bard

Furthermore, chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha can resolve more challenging mathematical problems like calculus, differential equations, geometry, and trigonometry. Martínez-Sevilla and Alonso concluded by arguing that AI has the potential to revolutionize mathematics education [10]. They call for more research on the use of AI in mathematics education and for developing new AI-enabled learning tools that can help students learn mathematics more effectively. Next, Davies et al. added that "the practice of mathematics involves discovering patterns and using these to formulate and prove conjectures, resulting in theorems." [11, p. 70]. Moreover, these chatbots can comprehend the issue and produce a solution using deep learning algorithms. But, because these chatbots are natural language models equipped with computational intelligence, they might not always be able to find the right solution, particularly if the problems are difficult or call for a particular approach or formula. In contrast, chatbots like ChatGPT, Google Bard, and Bing Chat will be able to perform math to a certain degree of accuracy.

## 4. How to Learn Math with Chatbots

Qawaqneh et al. investigated the impact of virtual laboratories (VLabs) based on AI on students' motivation toward learning mathematics and concluded that AI-based VLabs can effectively increase students' motivation toward learning mathematics [12]. Likewise, Martínez-Sevilla and Alonso added that the potential of AI to support mathematics education on monumental heritage and suggested that AI can be used to create new learning experiences that are more engaging and interactive and that can help students to develop a deeper understanding of mathematics [10]. Further, Chang et al. investigated the effects of online STEAM hands-on learning on AI learning, creativity, and creative emotions with 25 preservice teachers and found that STEAM hands-on learning can effectively improve AI learning, creativity, and creative emotions [13]. Hence, chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha can explain the underlying ideas and formulas and offer step-by-step solutions to math problems. This can help students understand the reasoning behind the solution and develop their mathematical abilities. Martínez-Sevilla and Alonso remarked that AI has the potential to revolutionize mathematics education by creating new learning experiences that are more engaging, interactive, and personalized [10].

Chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha can also generate plenty of math practice problems for students to finish, which will help them get better and/or unique solutions. With the help of chatbots like ChatGPT, Google Bard, Bing Chat, and Wolfram Alpha, students can take on more of the role of the teacher. Making the most of the potential that these platforms have offered students can be accomplished by using customized instructions. See Figure 3 for the solution generated by Bing Chat.

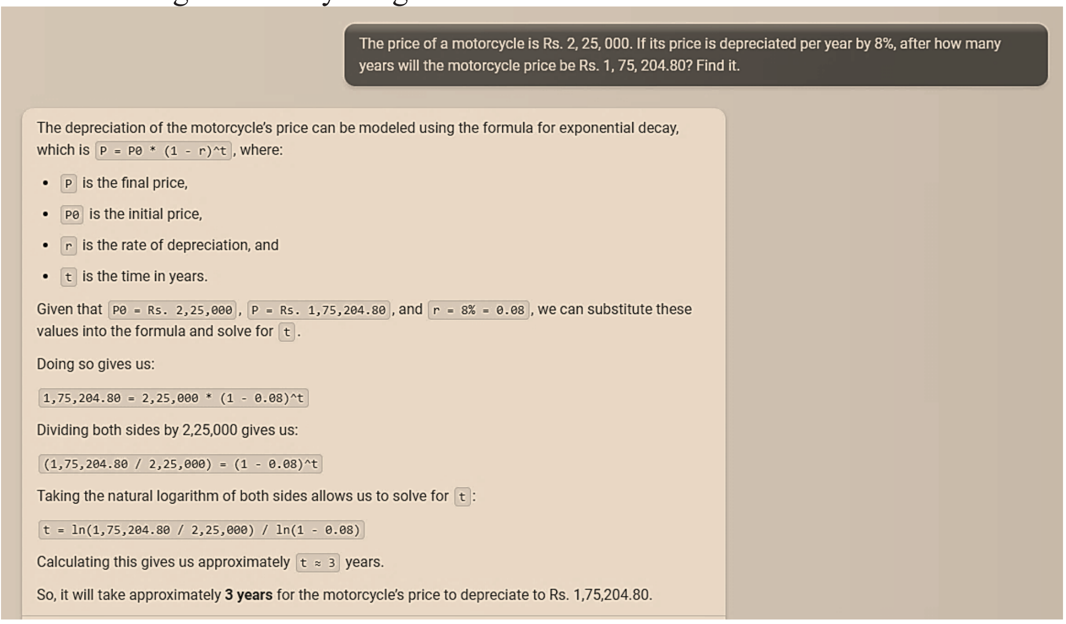



Figure 3: Solution Generated by Bing Chat

## 5. Capabilities and Limitations of Chatbots for Math Problem Solving

Monte-Serrat and Cattani argued that mathematics plays an important role in the development of AI, which has implications for the ethical design of AI systems [9]. So, AI chatbots have been designed to solve a wide range of math problems, from basic arithmetic to complex calculus from basic to university level among the different areas of mathematics. They can also generate proof for mathematical theorems and help students understand complex concepts. In this regard, Wang and Lester argued that AI literacy is essential for preparing students for future jobs and ensuring that they can participate in a democratic society in which AI plays an increasingly important role [6]. So, the wide range of capabilities of AI chatbots in solving math problems is the ability to understand natural language and computational intelligence, the ability to generate step-by-step solutions, the ability to explain mathematical concepts, the ability to explain mathematical concepts, and the ability to generate proofs. See below for one more computation of the mean generated by ChatGPT.

**Prompt to ChatGPT**: Find the mean of [200-300, 40], [300-400, 55], [400-500, 68], [500-600, 77], [600-700, 80].

**Response in embed code**: To find the mean of the given grouped data, you can use the following steps:
### Finding the Mean:
1. First, find the mid-point of each class interval:

- \( (200+300)/2 = 250 \)
- \( (300+400)/2 = 350 \)
- \( (400+500)/2 = 450 \)
- \( (500+600)/2 = 550 \)
- \( (600+700)/2 = 650 \)

2. Next, multiply each mid-point by its corresponding frequency:
   - \( 250 \times 40 = 10000 \)
   - \( 350 \times 55 = 19250 \)
   - \( 450 \times 68 = 30600 \)
   - \( 550 \times 77 = 42350 \)
   - \( 650 \times 80 = 52000 \)
3. Sum up these products:
   \( 10000 + 19250 + 30600 + 42350 + 52000 = 154200 \)
4. Sum up the frequencies:
   \( 40 + 55 + 68 + 77 + 80 = 320 \)
5. Calculate the mean:
   \( \text{Mean} = \frac{154200}{320} = 481.875 \)

So, the mean of the given data set is 481.875.

AI chatbots can understand natural language questions with computational intelligence about math, even if they are not phrased in a formal mathematical way. This makes them accessible to a wider range of students, including those who are not fluent in mathematical notation. Likewise, AI chatbots can generate step-by-step solutions to math problems, which can be helpful for students who are learning to solve problems independently. Next, AI chatbots can explain mathematical concepts concisely, which can be helpful for students who are struggling to understand a particular concept. Also, AI chatbots can generate proofs for mathematical theorems, which can be helpful for students who are learning how to prove theorems on their own.

On the contrary, powerful tools for math problem-solving, chatbots like Google Bard, ChatGPT, Bing Chat and Wolfram Alpha, have some limitations. One drawback is that these chatbots might only occasionally be able to resolve challenging mathematical equations and big statistical analysis that call for a particular approach or formula. See below to distinguish the simplification of $\frac{1}{y+b}+\frac{1}{y+c}+\frac{1}{y+d}+\frac{by}{y^3+by^2}+\frac{cy}{y^3+cy^2}+\frac{dy}{y^3+dy^2}$ by researchers and Wolfram Alpha—AI-based tool equipped with computational intelligence in Figure 4.

$$=\frac{1}{y+b}+\frac{1}{y+c}+\frac{1}{y+d}+\frac{by}{y^2(y+b)}+\frac{cy}{y^2(y+c)}+\frac{dy}{y^2(y+d)}$$

$$=\frac{1}{y+b}+\frac{1}{y+c}+\frac{1}{y+d}+\frac{b}{y(y+b)}+\frac{c}{y(y+c)}+\frac{d}{y(y+d)}$$

$$=\frac{1}{y+b}+\frac{b}{y(y+b)}+\frac{1}{y+c}+\frac{c}{y(y+c)}+\frac{1}{y+d}+\frac{d}{y(y+d)}$$

$$=\frac{y+b}{y(y+b)}+\frac{y+c}{y(y+c)}+\frac{y+d}{y(y+d)}$$

$$=\frac{1}{y}+\frac{1}{y}+\frac{1}{y}$$

$$=\frac{3}{y}$$

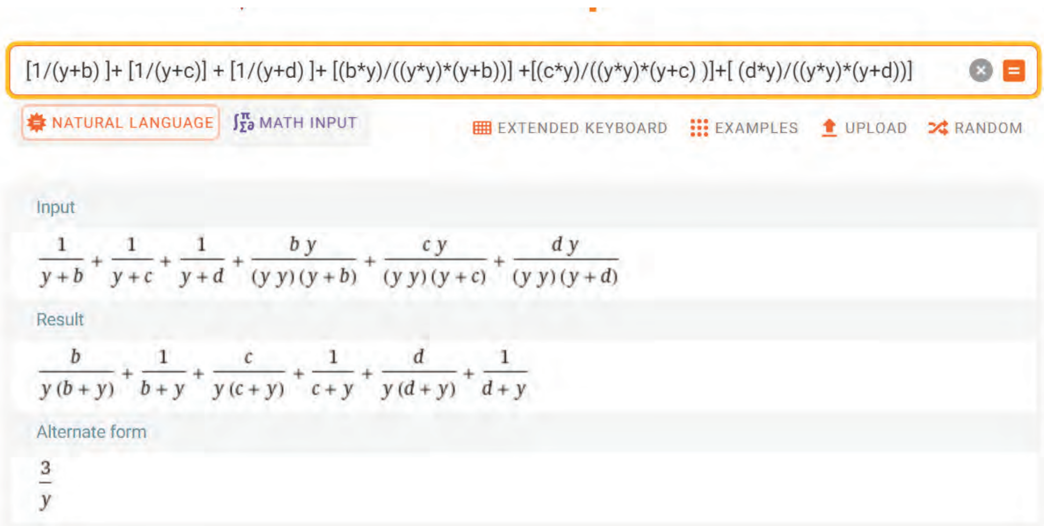
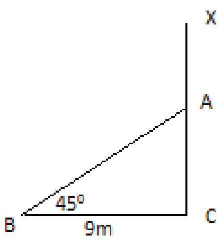



Figure 4: Simplification by Wolfram Alpha

Furthermore, these chatbots might not always offer the best solution to a problem. Next, since logical reasoning is still one of the weaknesses of chatbots like Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha, these chatbots might not always be able to solve issues requiring real-world contextual knowledge or context. However, chatbots are likely to be capable of solving problems in linear algebra, calculus, and statistics by providing step-by-step solutions. So, human intelligence is still essential for tasks that cannot be accomplished through statistical big data approaches. AI has not yet reached that level of capability. As such, teachers and students should adapt to the age of AI by employing new training methods. See below to distinguish the solutions done by researchers and ChatGPT of the question—the upper part of a straight tree broken by the wind makes an angle of $45^0$ with the plane surface at a point 9 m from the foot of the tree. Find the height of the tree before it was broken.




Correct figure and description

In rt. angled Δ ABC

tan $45^0 = \frac{AC}{9}$

Or, AC = 9m

Also, In rt. angled Δ ABC

cos $45^0 = \frac{9}{AB}$

Or, AB = $9\sqrt{2}$ m

∴ CX= AB +AC = 21.72 m

Figure 5: Solution by Researchers

**Prompt to ChatGPT**: The upper part of a straight tree broken by the wind makes an angle of $45^0$ with the plane surface at a point 9 m from the foot of the tree. Find the height of the tree before it was broken.
**Response in embed code**: To find the height of the tree before it was broken, we can use trigonometric functions. Specifically, we will use the tangent function.
The tangent of an angle \( \theta \) in a right triangle is defined as the ratio of the opposite side to the adjacent side.
For a 45-degree angle, \[ \tan(45^\circ) = 1 \]
In the case of the broken tree, let \( h \) be the original height of the tree before it was broken. The part of the tree that is broken and is leaning makes an angle of 45 degrees with the ground.
If the tree is broken at a point that is 9 m from its base, then the length of the part of the tree that is leaning would also be 9 m (since the angle is 45 degrees and \(\tan(45^\circ) = 1\)).
Thus, the original height of the tree would be \( h = 9\, \text{m (height of unbroken part)} + 9\, \text{m (height of broken part)} \).
\[ h = 9\, \text{m} + 9\, \text{m} \]
\[ h = 18\, \text{m} \]
So, the height of the tree before it was broken was 18 meters.

On the other hand, Google Bard, ChatGPT, and Bing Chat's capabilities will continue to grow and advance. Chatbots like Google Bard, ChatGPT, and Bing Chat are evolving to a multimodal model that outperforms these chatbots in many ways, including its capacity to solve mathematical problems as the successor emerges. These chatbots still perform at a high level of difficulty, but future updates to the GPT model will only improve their math skills and make them more intelligent.

Chatbots such as Google Bard, ChatGPT, and Bing Chat lack a comprehensive understanding of geometry and are unable to correct misunderstandings effectively. The accuracy and effectiveness of the solutions provided by Chatbots such as Google Bard, ChatGPT, and Bing Chat may depend on the complexity of the equation, the input data, and the instructions provided to Chatbots such as Google Bard, ChatGPT, and Bing Chat. It is anticipated that chatbots such as Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha will become more effective at solving increasingly complex mathematical problems (Wardat et al., 2023). On the whole, AI chatbots have the potential to revolutionize mathematics education. They can be used to provide personalized instruction, help students learn at their own pace, and make math more accessible to everyone. AI has the potential to revolutionize the way we solve mathematical problems. Even more, AI math solvers can quickly and accurately solve complex equations and problems, even those that would be difficult or impossible for humans to solve accurately in a short interval of time. Chatbots such as Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha can also provide step-by-step explanations of their solutions, which can help students learn and understand math concepts. Speed and accuracy, step-by-step explanations, and access to difficult problems are some potential benefits of using AI math solvers. And cost, accuracy, and creativity are limitations to AI math solvers.

On the contrary, Lee and Yeo described developing and evaluating an AI-based chatbot that preservice teachers can use to practice responsive teaching skills [5]. Responsive teaching is a teaching approach that involves using questioning and other strategies to adapt instruction to the needs of individual students. The chatbot was engineered to function as a simulated student exhibiting misunderstandings about fractions. PSTs were able to interact with the chatbot in real time, asking questions and providing feedback.

In addition to the potential benefits and limitations mentioned above, there are a few other things to consider when using AI chatbots as math solvers. First, it is important to make sure that the AI math solver is reliable and accurate. There are a number of different AI math solvers as chatbots available, so it is important to do some research to find one that has a good reputation and accuracy. Second, it is important to use AI math solvers as chatbots in a way that is beneficial to learning. AI math solvers can be a great way to get help with difficult problems, but they should not be used as a substitute for learning the material. Finally, it is important to remember that AI math solvers on chatbots are still under development. As AI technology continues to improve, AI math solvers on chatbots will become even more powerful and versatile.

## 6. Conclusions

Math problem-solving chatbots like Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha are effective tools. These chatbots can explain concepts and formulas, provide step-by-step solutions, and solve simple and complex math problems in seconds. Likewise, AI chatbots are powerful tools that can be used to solve math problems and help students learn mathematics at their own pace and speed with accuracy and step-by-step explanations by accessing difficult problems. However, it is important to be aware of their limitations and use them in the most beneficial way for learners. Next, AI chatbots have the potential to be effective tools for tackling algorithms math problems starting from arithmetic to statistics from school to university level. They can be used to create practice problems, explain the underlying ideas and formulas, and offer step-by-step solutions to problems. This can be useful for anyone needing math help, including students, teachers, teacher educators, among others. However, AI chatbots have some limitations when it comes to solving algorithms and math problems. Even more, chatbots such as Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha cannot always understand the subtleties of human language to start with. This may result in misinterpretations and mistakes in the solutions they offer. Furthermore, they are only as good as the training data. They might not be able to solve problems outside of their training set if they are not trained in a diverse enough range of problems. Despite these drawbacks, AI chatbots have the potential to be a useful resource for resolving issues with algorithms and math problems at the basic to university level. On the whole, AI-based chatbots, such as Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha, have shown significant potential in solving mathematical algorithm problems. They can understand and generate mathematical expressions, provide step-by-step solutions, and enhance students' learning experiences. However, challenges such as limited understanding of natural language models, inability to solve complex problems, and potential bias shall emerge. Future research should focus on improving these areas to increase their accuracy and reliability. Despite these advancements, it is crucial for students to continue interacting with human teachers to develop their cognitive math skills and conceptual understanding.

To sum up, chatbots such as Google Bard, ChatGPT, Bing Chat, and Wolfram Alpha function through the extensive training of large language models (LLMs), natural language models (NLMs), and computational intelligence using enormous datasets. Despite this, there remain specific areas where human intelligence continues to outperform AI, necessitating a nuanced strategy for educating both math teachers and students in the age of artificial intelligence. By blending the strengths of humans and machines, we can forge a more effective educational approach. Given their current capabilities and limitations, we hope that AI chatbots will increasingly serve as a standard resource for math education and problem-solving in the future.

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