# Arts and Maths: a STEAM introduction to envelopes with automated methods[*]

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## Abstract

Starting from a piece of string art, we propose a STEAM approach to motivate activities about envelopes of parametric families of surfaces. The examples are provided by 1-parameter families of planes and yield ruled surfaces with cuspidal edges. The topology of the surface obtained by geometric and algebraic work can be compared to the shape of well-known monuments (whence an incitement to outdoor mathematics). Later, we discuss the transition from 2D to 3D: regarding the automated methods, it is non trivial as commands available in a 2D setting may not be available for working in 3D. Nevertheless, the algebraic manipulations are similar in 2D and 3D, based on the same packages. Here, the main example is offered by an astroid in the plane, and its 3D generalization as an astroidal surface. Finally we discuss the examples according to the STEAM approach, and to Balacheff's computer transition.

## 1 Introduction

### 1.1 String art and beyond

A popular activity consists in creating pieces of art using a wooden plate, nails and threads, what is called string art. Figure 1(a) shows an astroid created with string art. Note that actually an astroid is a plane curve, but the concrete creation provides a kind of surface in 3-dimensional space, because of the thickness of the threads. This will be an incitement to extend the activity to 3D.

This piece of string art can be modelled using a Dynamic Geometry System (DGS). Figure 1(b) shows a screen snapshot of a GeoGebra applet which creates such an astroid dynamically. The concrete basis of the applet is the envelope of positions of a given ladder whose foot is on the $x-$axis and whose head is on the $y-$axis; remember that the ladder's length is constant. Various methods can be used for the construction, depending on the mathematical background and the
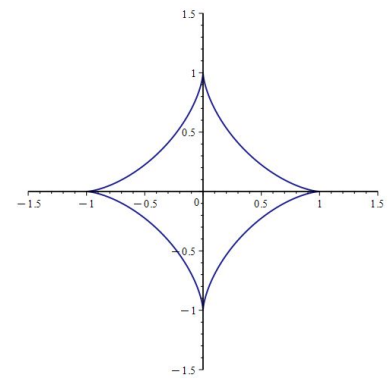
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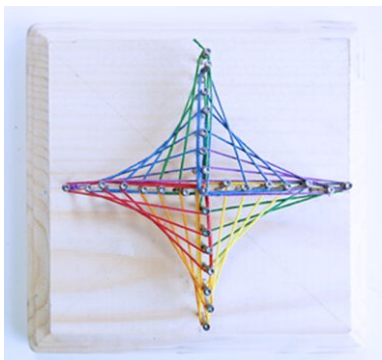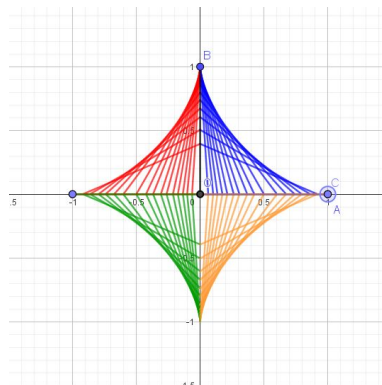
(a) Concrete realization  (b) Model with a DGS  (c) A plot with CAS

Figure 1: Astroid as string art and modelling

software literacy of the user. For example, the applet by Z. Kovács[1] shows a construction and provides an implicit equation for an astroid, using the automated command **Envelope**. Another applet provides a full construction of the astroid (it is analyzed in Section 3). Here, in Figure 1, the plots have been created in the first quadrant and reflected into the other quadrants. Then colors have been changed. This provides the dynamic effect of simultaneous creation in all the quadrants, moving one point with the mouse. The slower the moves, the higher the number of plotted segments. We may mention that sometimes, an illusion of 3D may be experienced, but not more than an illusion.
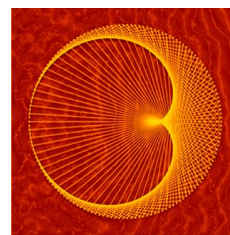
On a sunny day, the surface of a cup of coffee (Figure 2(a)) or a table where you forgot a brass ring (Figure 2(b) offer other examples. The curve appearing on the coffee is called a cardioid; it is the envelope of the solar rays reflected by the cup. See also Appendix A.3. Figure 2(c) shows a model with string art.



(a) Sun on a cup of coffee  (b) Reflection on a copper ring  (c) A model with threads

Figure 2: Playing with light

String art is not limited, neither to plane constructions nor to the classroom. Outdoor mathematics give access to numerous string constructs. A long time ago, engineers conceived hanging bridges for utility reasons, among them to be able to cross very wide rivers without pillars in the middle. Figure 3 shows (a) an old chain bridge at Chaksam (Tibet), built in 1430,

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[1]There are two mathematicians with this name, working in closely related areas. In this paper,we relate to the colleague from Linz, Austria, `https://matek.hu/zoltan`.

and (b) a drawing by Fausto Veranzio in 16th century[2]. In recent years, this kind of bridges are
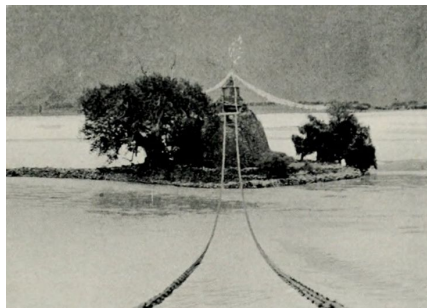


(a) An old bridge in Tibet
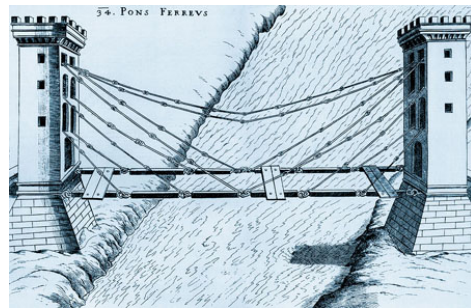


(b) A Renaissance drawing

Figure 3: Old hanging bridges

also conceived as pieces of art, actual realizations of ruled surfaces. Figure 4 shows two views of hanging bridges: (a) a bridge over the sea in Greece, and (b) a bridge for the light train in Jerusalem, over one of the busiest road crossings. The needed curvature of the tracks provided an opportunity for a wing design.



(a) The Rion-Antirion bridge, Greece



(b) The chords bridge in Jerusalem, Israel

Figure 4: Modern hanging bridges

## 1.2 STEAM Education

STEAM is an acrostics for Science, Mathematics, Technology, Engineering, Arts and Mathematics. It represents an educational approach that aims to spark an interest and lifelong love of the arts and sciences in children from an early age, but not only. The first example above is doable by young children, the next one probably not. What we will show later is suitable for undergraduates, or at least for gifted High-School students having access to an extension of the curriculum. Until quite recently, a STEM approach was defined and analyzed, the A for Arts joined in the last years. STEM literacy has been defined as the ability to identify, apply, and integrate science, technology, and mathematical concepts to understand complex problems and innovate in their solution" [6]. The inclusion of the artistic disciplines is not

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[2]Both images are available in Wikipedia.

arbitrary. It broadens the interdisciplinary perspective by incorporating an external area into the scientific field. This is true for various artistic domains; here we deal with some visual arts, including architecture (what doubles the meaning of A). In fact, this A incites to use the cultural background of the students in order for them to develop new knowledge and new skills [20, 42].

Actually, the influence is multidirectional. Science, Technology, Engineering, the Arts and Mathematics are similar fields of study; they involve creative processes and none uses just one method for inquiry and investigation. Moreover, "since the European Community has recognized the role of life-long learning and its impact on the education of aware citizens , non-formal contexts started deserving the attention of scientists dealing with education and in particular science education" [43].

In all the components of STEAM, the name of the game is given by the 4 C's of 21st education: Collaboration, Communication, Critical Thinking and Creativity [39]. Important skills are included in these 4 C's: creativity and productivity are a must for innovation, information and media literacy are part of the skills for communication and collaboration, together with leadership and responsibility, initiative. In previous works, we mention the rich influence of using the cultural background of the student, to attract them to maths and other scientific domains [20]. Actually a 5th "C" should be added, namely Curiosity, without which Creativity may be limited. Curiosity will lead to and will be nurtured by exploration of general media, and to developing cross-cultural skills, which may be an important component of Communication and Collaboration..

For the examples shown in this paper, we use GeoGebra, and the Maple 2023 Computer Algebra System, both for its 3D plotting features and its algebraic abilities, working with polynomial rings.

## 1.3   New knowledge developed in a technology-rich environment

Artigue [4] notes that "the development of mathematics has always been dependent upon the material and symbolic tools available for mathematics computations". One of the most famous events is the transition from writing with a feather to writing with an iron quill. This has been noted by P. Lavoie [36], following whom Guin and Trouche write [32] "Lavoie, for instance, shows the consequences of the introduction of the iron quill (instead of the goose quill) for the learning of arithmetic in the 19th century (writing more easily allowed pupils to do longer computations by hand and this lead to an earlier introduction of arithmetic in curricula)."

More recently, in the 2nd half of the 20th century, hand held numerical calculators made the usage of numerical tables obsolete. Then new algorithms have been written, enabling the development of symbolic calculators and packages for personal computers. The era of Computer algebra Systems (CAS) and of Dynamic Geometry Systems (DGS) began. In some cases, these CAS were devoted to a specific mathematical domain, but general purposes CAS have also been developed; see [18]. An interesting comparison of the effects of different kinds of software for the same kind of problems is given in [33]. Nowadays, general purpose systems are ubiquitous. They provide an environment including Geometry, Algebra, one-variable and multi-variable Calculus, complex variables, etc. In what follows, we explore algebraic curves and surfaces. Therefore, we use mostly the geometric features and the algebraic packages, mostly the Polynomial Ideals and Gröbner packages.

The geometric objects of our study, envelopes of 1-parameter families of curves or of surfaces, appear first via a parametric presentation, obtained as the solution of a system of equations (see Definition 3 in next Section). This may be enough to explore their topology, but implicitization may be useful. A drawback is that implicitization can be heavy, as we will see in Section 3. Most CAS have a command for elimination, but computations may be very long (see [25]). Only a more profound insight into the algorithm may provide a better understanding, in particular an answer to the question "why do superfluous components appear?". This is due to the fact that elimination of variables actually computes the projection of a certain variety onto a space of smaller dimension and provides the closure of the desired variety, in Zariski topology [13, 38]. We experienced this phenomenon, not only for the determination of envelopes, but also for the determination of isoptics; see [22].

# 2 Envelopes of families of surfaces: mathematical treatment and automated methods

We consider a 1-parameter family of surfaces $\mathcal{F} = \{\mathcal{S}_u\}$ in 3-dimensional space, given by the equation $F(x, y, z, u) = 0$, where $u$ is a real parameter. We use the following definition of an envelope:

**Definition 1** *A family of surfaces is given by an equation of teh form $F(x, y, z, u) = 0$, where $u$ is a real parameter. An envelope of the family $\mathcal{S}_u$ is the image in 3-dimensional space of the solution set of the following system of equations:*

$$\begin{cases} F(x, y, z, u) = 0 \\ \frac{\partial}{\partial u} F(x, y, z, u) = 0 \end{cases}$$

This is one of the 4 definitions given in [12], and the only one given in [7].

We consider the family of planes whose equation is given by $F(x, y, z, u) = 0$, where $F(x, y, z, u) = x + uy - 2uz^2 - 3u^3$ and $u$ is a real parameter. According to Definition 3, we can compute a parametric presentation of an envelope of this family. We obtain:

$$(x, y, z) = (-6u^3 + 2u^2v, 9u^2 - 4uv, v). \tag{1}$$

Before we proceed further, we wish to mention that the output on the screen is actually $(x, y, z) = (-6u^3 + 2u^2z, 9u^2 - 4uz, z)$ and it is necessary to substitute another variable instead of $z$ (here we chose $v$). Otherwise, what follows is meaningless. Now, using elimination methods, as in [29], an implicit equation is obtained:

$$32xz^3 - 4y^2z^2 + 108xyz - 12y^3 + 243x^2 = 0. \tag{2}$$

Figure 5 shows two views of this envelope.

When a 1-parameter families of planes has an envelope $\mathcal{E}$, this has interesting general properties[3] :

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[3]A family of planes having a common line has no envelope, but in general the family has an envelope, and it verifies the properties mentioned here; see the Mathcurve website.
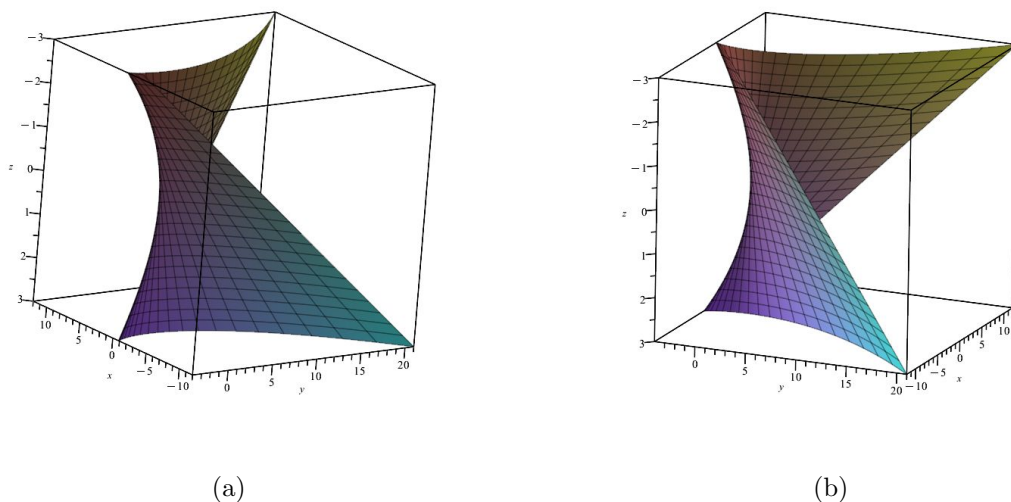
<div align="center">(a)               (b)</div>

Figure 5: Two views of the envelope of a 1-parameter family of planes

(i) The envelope $\mathcal{E}$ is a ruled surface;

(ii) It has a cuspidal edge $\mathcal{C}$;

(iii) The tangent to $\mathcal{C}$ are generators of $\mathcal{E}$.

The first two properties are apparent on Figure 5. The 3rd property is less visible. All the properties can be proven using suitable commands of a CAS.

In order to determine the cuspidal edge $\mathcal{C}$, two methods can be applied:

(i) Looking for singular points of the surface determined by the parametric presentation (1)

(ii) Looking for these points using the implicit equation (2) and computing a gradient.

Maple offers two packages to compute a gradient. We used the *Student[MultivariateCalculus]* package, the second one seems to interfere with the *PolynomialIdeals* package and to make the algebraic computations unreachable. We display here the Maple session:

```
restart:with(plots):
with(PolynomialIdeals):with(Student[MultivariateCalculus]):
F := -3*t^3 - 2*t^2*z + t*y + x:
derF := diff(F, t):
env := solve({F = 0, derF = 0}, {x, y, z}):
envplot1 := plot3d([rhs(env[1]), rhs(env[2]), rhs(env[3])], t = -1 .. 1, z = -3 .. 3,
          labels = [x, y, z]):
envplot2 := plot3d([rhs(env[1]), rhs(env[2]), rhs(env[3])], t = -1 .. 1, z = -3 .. 3,
          transparency = 0.4, labels = [x, y, z]):
p1 := x - subs(z = u, rhs(env[1]));
p2 := y - subs(z = u, rhs(env[2]));
p3 := z - subs(z = u, rhs(env[3]));
```

```
K := <p1, p2, p3>:
KE := EliminationIdeal(K, {x, y, z})
FF := Generators(KE)[1]:
G := Gradient(FF, [x, y, z]):
cuspedge := solve({G[1] = 0, G[2] = 0, G[3] = 0}, {x, y, z}):
cuspedgeplot := plot3d([subs(z = u, rhs(cuspedge[1])), subs(z = u, rhs(cuspedge[2])),
               subs(z = u, rhs(cuspedge[3]))], u = -3 .. 3, thickness = 5):
display(envplot2, cuspedgeplot):
```

The variable $FF$ denotes the polynomial displayed in Equation (2). The output of the **Gradient** command is in matricial form, namely

$$G = \begin{bmatrix} 32z^3 + 108yz - 486x \\ 8yz^2 + 108xz + 36y^2 \\ 96xz^2 + 8y^2z + 108xy \end{bmatrix} \tag{3}$$

whence the format of the **solve** command to determine singular points. The solution set of this command determines a curve with the following parametrization:

$$(x, y, z) = \left( -\frac{8}{243}t^3, \ -\frac{4}{9}t^2, \ t \right). \tag{4}$$

The previous remark about the need to substitute a new variable instead of $z$ is still valid here. Figure 6 shows the envelope with the cuspidal edge enhanced. For this purpose, the envelope has been plotted with some transparency (defined as *envplot2* in the above session).
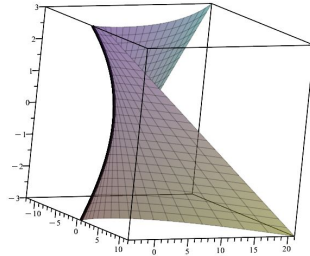


Figure 6: A view of the envelope with its cuspidal edge

# 3    A 3D version of an astroid

WLOG we can define an astroid as the envelope of a segment of length 1 whose end points are on the $x-$axis and on the $y-$axis respectively. It has the following parametric equation:

$$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \tag{5}$$

where $a$ is a positive parameter and $t \in [0, 2\pi]$. This astroid is also defined by the implicit equation

$$x^{2/3} + y^{2/3} = a^{2/3}. \tag{6}$$

Figure 1(c) shows the astroid for $a = 1$. Note that, on the one hand, this does not define an astroid as an algebraic curve, and on the other hand, that it request to be careful with powers of negative reals. In the present case, problems will be avoided if Equation (6) is written in the following form:

$$(x^2)^{1/3} + (y^2)^{1/3} = (a^2)^{1/3}. \tag{7}$$

Algebraic manipulation leads to the equivalent equation of degree 6:

$$(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0, \tag{8}$$

showing that an astroid is actually an algebraic curve. For a concrete construction as an envelope, we refer to the applets mentioned previously.

An astroid can be studied with automated methods. Here we use GeoGebra's **Envelope** command. Figure 7 shows a snapshot of a devoted applet. The syntax of the command is **Envelope**(<path>,<point>), therefore the applet has been built as follows:

i. A segment $AB$ is defined on the $x-$axis. It is denoted by $f$, which in GeoGebra denotes also its length.

ii. A point $C$ is attached to the segment. This will be the variable point.

iii. Denote by $E$ the point of intersection of the $y-$axis with the circle centered at $C$ with radius $f$.

iv. The segment $CE$ is denoted by $g$.

v. Apply the command **Envelope**($g$,$C$).

The construction refers to objects in the first quadrant only, but as in Figure 7:

- The output on the graphical window is an (almost) entire astroid. The plot is inaccurate in a small neighborhood of each vertex, as the vertices are singular points. A better plot could be afforded by a parametric representation of the astroid, which is not provided by the command. The inaccuracy of plots around singular points is discussed in [44] for any CAS.

- The astroid's equation, given by a quartic polynomial, is displayed in the algebraic window.

- This polynomial equation defines the entire astroid, not only an arc in 1st quadrant. The reason is that the computation, based on Gröbner bases and elimination, computes a Zariski closure.

We wish now to make a transition towards 3D, i.e., to define a surface generalizing the astroid to 3D. As the current version of GeoGebra does not offer a 3D version of the **Envelope**
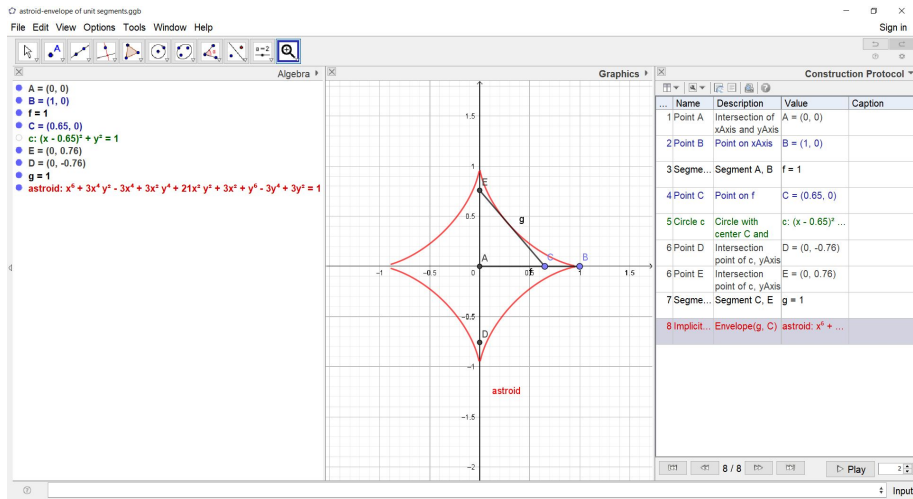
Figure 7: Snapshot of the construction of an astroid with the Envelope command

command (neither does Maple anyway), we adopt another method. Following the Mathcurve page, we define an *astroidal ellipsoid* $\mathcal{A}$ by the parametric presentation
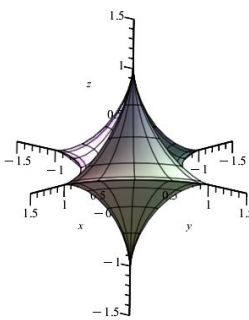
$$\begin{cases} x = a \ \cos^3 u \cdot \cos^3 v \\ y = b \ \cos^3 u \cdot \sin^3 v \\ z = c \ \sin^3 v \end{cases} \qquad u \in [0, 2\pi], \ v \in [0, \pi]. \tag{9}$$

where $a > 0, b > 0, c > 0$. If $a = b = c$, the surface is called a called an *astroidal surface*; it can be also defined by the equation:

$$(x^2)^{1/3} + (y^2)^{1/3} + (z^2)^{1/3} = (a^2)^{1/3}. \tag{10}$$
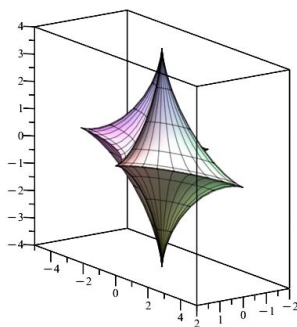


(a) $(a, b, c) = (2, 5, 4)$     (b) $a = b = c = 1$, general view     (c) $a = b = c = 1$, from above
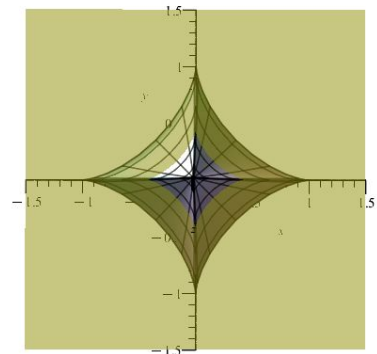
Figure 8: Astroidal ellipsoids

In the present paper, we deal with the astroidal surface $\mathcal{A}$ given by $a = 1$. The intersections of the surface $\mathcal{A}$ with the coordinate planes are astroids (see Figure 8(c)). In the literature, this surface is also called *hyperbolic octahedron* or *astroidoctahedron*.

From Equation (10), by algebraic manipulations (i.e. moving terms from side to side, raising both sides to 3rd power, and iterating the process several times, it can be shown that an astroidal surface is an algebraic surface given by a polynomial of degree 18 in 3 variables[4]. Another way to derive an implicit equation for $\mathcal{A}$ is to use the same method as in [23, 24], substituting cosines and sines with rational expressions in Equations (9). We only sketch the method:

$$\begin{cases} \cos u = \frac{1-r^2}{1+r^2} \\ \sin u = \frac{2r}{1+r^2} \end{cases} \quad \text{and} \quad \begin{cases} \cos v = \frac{1-s^2}{1+s^2} \\ \sin v = \frac{2s}{1+s^2} \end{cases} .$$

and from there defining 3 polynomials in 5 variables $x, y, z, r, s$. They generate an ideal $K$ in $\mathbb{R}[x, y, z, r, s]$. Elimination of the parameters $r, s$ provides a polynomial $F(x, y, z) \in \mathbb{R}[x, y, z]$ whose vanishing set is the Zariski closure of the required envelope.

An astroidal surface is also an envelope of a family of planes. As this is a really non-trivial statement, we emphasize it:

**Proposition 2** *An astroidal surface is the envelope of the planes intersecting the 3 axes at the 3 vertices of a triangle, for which the distance between the center of gravity and the origin $O$ is constant, equal to $a/3$.*

We will only sketch the proof. Consider 3 points $A(\alpha, 0, 0), B(0, \beta, 0), C(0, 0, \gamma)$, on the axes in the 1st octant, i.e. $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. The center of gravity $G$ of the triangle $ABC$ has coordinates $\frac{1}{3}(\alpha, \beta, \gamma)$, thus we have $OG = \frac{1}{3}\sqrt{\alpha^2 + \beta^2 + \gamma^2}$. We have:

$$OG = \frac{a}{3} \qquad \Leftrightarrow \gamma = \sqrt{a^2 - 9\alpha^2 - 9\beta^2}. \tag{11}$$

The plane of the triangle $ABC$ has equation

$$-\gamma\beta x - \alpha\gamma y \alpha\beta z + \alpha\beta\gamma = 0. \tag{12}$$

Substitution of the formula (11) into Equation (12) yields an equation of the plane $ABC$ depending on 2 parameters only. The envelope of the 2-parameter family of planes is computed by standard methods, as in [21, 17], according to the following definition:

**Definition 3** *A family of surfaces $\mathcal{S}_{u,v}$ is given by an equation $F(x, y, z, u, v) = 0$, where $u, v$ are real parameters. An envelope of the family $\mathcal{S}_{u,v}$ is the image in 3-dimensional space of the solution set of the following system of equations:*

$$\begin{cases} F(x, y, z, u, v) = 0 \\ \frac{\partial}{\partial u} F(x, y, z, u, v) = 0 \\ \frac{\partial}{\partial v} F(x, y, z, u, v) = 0 \end{cases}$$

The geometric construction that we performed is valid in the first octant. What happens in the seven other octants is obtained by reflections about the coordinate planes.

As mentioned in the previous section, the envelope of a 1-parameter family of planes has a cuspidal edge. In [21], an example of a 2-parameter family with the same property has been given. We can prove this here too. In the previous section, we could use the implicit

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[4]The computation takes a very long time.

equation and gradients to find the cuspidal edge. Here, the implicit equation is too complicated (therefore, we did not display it - the interested reader can derive it using a CAS). We can show that the same situation appears here, using partial derivatives of Equations (9). We have:

$$
\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} -3\cos^2 u \cdot \cos^3 v \cdot \sin u & -3\cos^3 u \cdot \cos^2 v \cdot \sin v \\ -3\cos^2 u \cdot \sin^3 v \cdot \sin u & 3\cos^3 u \cdot \sin^2 v \cdot \cos(v) \\ 3\sin^2 u \cdot \cos u & 0 \end{pmatrix} \tag{13}
$$

Singular points correspond to values of the parameters $u$ and $v$ for which the matrix vanishes. These points are on the coordinate planes. The corresponding sets are enhanced in Figure 9.
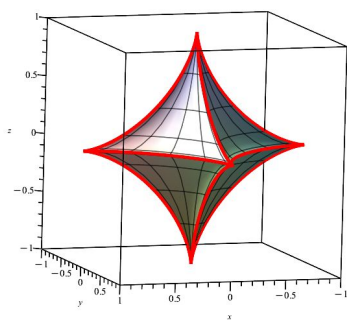


Figure 9: An astroidal surface with enhanced cuspidal edges

As visualization of a 3D object on a screen may be difficult to understand (it has to be rotated in several ways to have enough points of view), we can use a quite new technology[5],we mean 3D printing. Figure 10 shows such a 3D printed astroidal surface. Note that for technical reasons, it has been 3D-printed in 3 steps: the first 2 steps were identical, namely 3D-printing twice half of the object, let's say the part of the surface above the $xy-$plane, then gluing them together (not really high-tech).
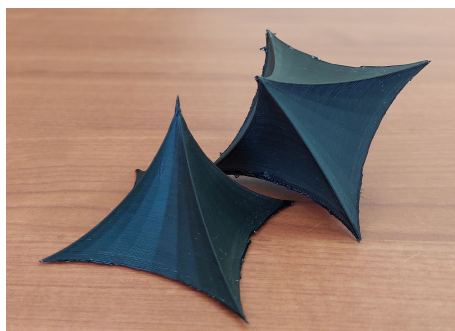


Figure 10: A 3D printed astroidal surface

[5]At least, new in educative environments.

# 4 Discussion

## 4.1 The needed dialog between technologies

The different affordances of a CAS and a DGS complete each other. Generally one is stronger for interactive display of curves and surfaces, and the other for algebraic computations. Actually most CAS contain plotting features, but require a priori programming. Then some intervention is possible during the animation, but no interactive "essential" changes. On the other hand, a software such as GeoGebra, began as a DGS, but has been supplemented with several other components, in particular a CAS (called Giac [35]). A new version of this CAS has been released recently. The need of an automated CAS-DGS dialog has been explained a long time ago (see [41]) and has still to be developed. In many cases, the dialog is performed by Copy-Paste only; see [23].

Development of automated dialog between CAS and DGS will strongly ease the exploration and discovery of new properties. Among the 4 C's of 21st Century Education, are mentioned Collaboration and Communication. If, in the original OECD documents [39], they were meant between humans, we advocate for these two C's also between man and machine, a domain which has already seen numerous studies, but also now between machines.

There exist cases where communication between technologies exist, such as export images as jpg or other format, export a session as a LaTeX file, etc. This is useful, but does not fit what we mean by a dialog. Here, we can 3D print some objects by exporting the output of Maple plot commands to STL, the language used by the printer (without a human having to translate). A similar situation prevails with GeoGebra.

This dialog is not intended only for easing the technical work. It enables to concentrate on understanding the situation and the proposed task. A DGS and a CAS provide several registers of representation (in the sense of Duval [30, 31]): numerical, graphical, symbolic, etc. A central issue is switching between registers, most CAS enabling to switch from symbolic to graphical (generally via the numerical, after some discretization of the data because of hardware constraints, i.e. the pixel definition - see [5], but this may be transparent to the user), sometimes in reversed direction, from graphical to numerical, more rarely from graphical to symbolic. This last switch is proposed by GeoGebra Discovery[6], in particular with the commands used in the present paper. 3D printing adds a new register, making mathematical objects more present, concrete. This fills a gap, mentioned by Duval [30, 31]: unlike Physics, Chemistry and other scientific domains, the objects under study in mathematics are abstract, and not graspable with hands. Therefore, they need these registers of representation. 3D printing offers a new register, more concrete.

## 4.2 3D printing

Visualization of a 3D object on a screen is not easy, sometimes an obstacle. New technologies may help, in particular 3D printing. Figure 12 enables a comparison between visualization on the screen and concrete realization. Of course, 3D printing is in itself a collaboration between CAS-DGS and 3D printer, the last one having its own language. For this kind of

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[6]GeoGebra Discovery is an experimental version of GeoGebra. It offers both extended versions of existing commands and new commands. For existing commands, the improvement is to offer a symbolic proof, when the orginal one is numerical.

communication, we need to make the following remark, already addressed in [17]: the data transfer from the DGS to the 3D printer requires the object to be recognized by the DGS as a geometric construct. For a reader who did not experience this issue, we explain it on a 2D example in Appendix A.1.

All this discussion does not apply to a 3D problem. In GeoGebra, no equivalent to the 2D-automated commands **Locus**, **LocusEquation** and **Envelope** exist (at least to the day this paper is written). Therefore, other tools have to be used. Algebraic methods of a CAS, namely commands such as **Solve** exist no matter the dimension, and when Gröbner methods have to be used, they work. It should be noted that the algebraic engine at work for these commands (via Giac in GeoGebra's background, or visible when working with Maple, is mostly Gröbner packages; see for example [24, 18]). The differences are transparent to the user. An important difference lies in visualization, a topic which has been addressed widely in the past. This issue is partly answered by 3D printing: the objects are not represented on a screen (a 2D representation of a 3D object, but can be manipulated by hand, offering all the possible points of view.

This has already been illustrated by the author regarding canal surfaces [17]. Figure 11 shows a model of a twisted lemniscate, defined by parametric equations[7]. It had to be thickened and the equations to be prepared to be understood by the software as a geometric object. The original data, written with Maple, defined a curve. The thickness option of the **spacecurve** command was not enough for that.

Moreover, a mathematical activity involving 3D printing enables a strong communication between participants. See [2, 3].



Figure 11: A twisted lemniscate

Despite the astroidal surface $\mathcal{A}$ seeming simpler than the canal surface evoked above, this example reveals another issue: in order to 3D print $\mathcal{A}$, it is necessary to understand its symmetries. To achieve this understanding, the teacher and the students may have a dialog[8] around the properties of the defining functions of the surface, such as parity, and what are their consequences. Figure 8 illustrates the nonintuitive situation, whence the necessity to deal with general properties of the functions.This enables to decompose the printing into 2 parts, which is compulsory to print on the machine table; see Figure 12. Afterwards, a low level task has still to be done: to glue the two parts together.

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[7]This, and what is shown in Figure 12, have been 3D printed by E. Ulbrich, at Johannes Kepler University (Linz, Austria).

[8]Students, to whom the study of such surfaces is proposed, are beyond the need of a strong scaffolding around functions.
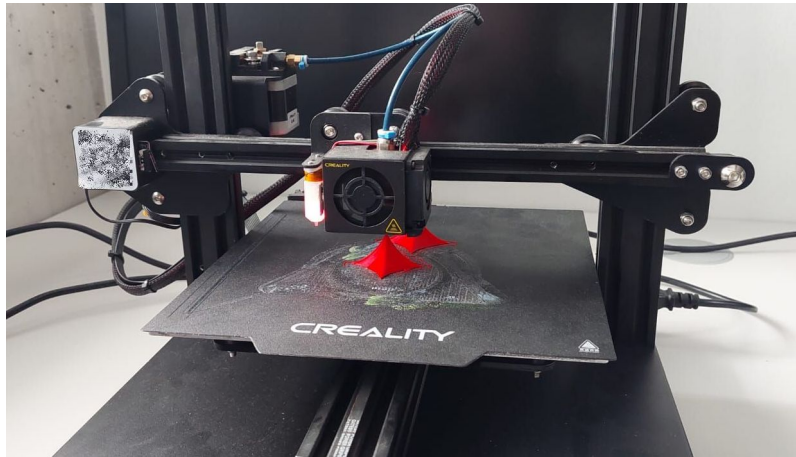
Figure 12: Printing in two parts

## 4.3 Back to Arts

As mentioned above, the A of STEAM is intended for Arts, but sometimes it can be reduced to Architecture. In subsection 1.1, we wrote that the exploration of envelopes of 1-parameter families of planes provides ruled surfaces, and we could compare this with the construction of hanging bridges. We may have another connection with architecture. Compare the topology of the envelope described in Section 2 with the monuments (to call them buildings does not seem to me enough) shown in Figure 13.
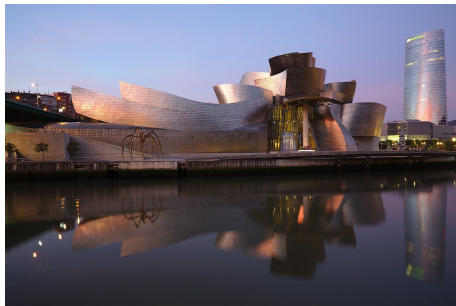


(a) Guggenheim Museum, Bilbao, Spain



(b) Walt Disney Hall, Los Angeles, USA

Figure 13: Two buildings whose geometry evokes the envelopes under study.

We mentioned visual arts and architecture, but many other examples can be used, coming from the everyday life of the students, or from their centers of interest. Figure 14 show two such examples. The boat bow shows a cuspidal edge which is itself singular (ask civil engineers why this shape has been chosen); actually the upper part (the blue one) and the lower part (the red one) show different examples of the same remark. The snow plow shows a truncated envelope with a cuspidal edge; the surface has been cut by a plane, probably to allow the plow driver to see where the train is going. Here too, we have examples for outdoor mathematical activities, and an incitement to learn mathematics based on everyday life and the cultural background of

students (of course, not in every location on Earth, students may have the same experiences, such as a train with a snow plow and a big cargo boat).



(a) A boat bow

(b) A snow plow

Figure 14: Other concrete realizations of such envelopes

STEAM combines "arts" with STEM. This learning is aimed at improving student engagement, creativity, innovation, problem-solving skills, and other cognitive benefits [37]. With a STEAM approach involving the 4C's, students' motivation increases and their learning may be more effective; see [42]. We wrote already that Curiosity should be added, and is enhanced with ipout from the so-called "real world".

## 4.4 Modelling and Computer transposition.

Let us look again at Figure 1. An astroid is a plane curve. It has been modelled using string art, whose constraints were visible: instead of a plane curve, the output is a surface dependent on the threads' thickness. Then a computerized model has been constructed with a DGS. Jablonski [34] says that "Mathematical modelling is characterized through its interplay of reality and mathematics. It offers a way to integrate references to reality into the classroom and shows students where in everyday life their mathematical knowledge can be applied." Here we see that interplay at work. The process is structured as follows:

i. A real-world problem is given and analyzed.

ii. It is translated into a mathematical setting (Descartes claimed that every problem has to be transformed into a system of equations).

iii. The mathematical problem is solved.

iv. The solutions have to be interpreted and validated with respect to reality.

The first 3 points have been processed using software, in particular automated methods. Then, we replaced the last point by an extension in another direction, namely generalizing to 3D. The possible paths for the usage of modelling are described in Figure 15: back to reality with a more profound understanding, or extending the reality towards other creations. The paths
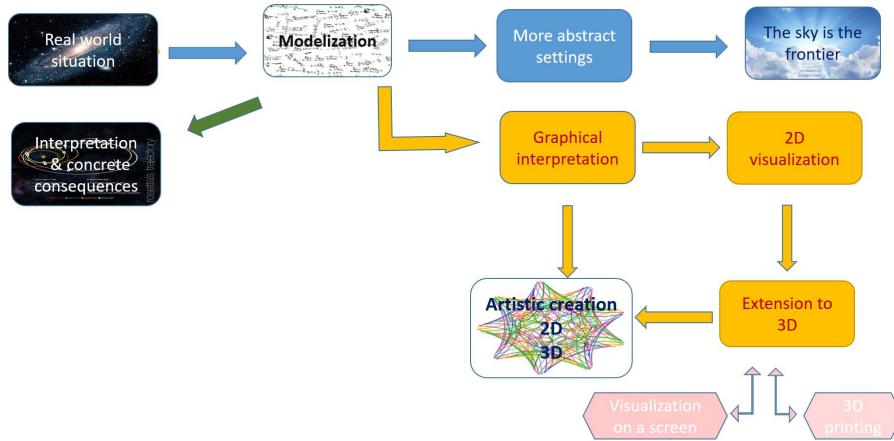
Figure 15: Different paths from modelling onwards

leading to "Artistic Creation" starting from parametric curves (modelling trajectories in space) are the topic of an ongoing work [28].

A computable model (in the sense of Balacheff [5]) was easy to obtain from the literature, a CAS or a DGS provided a good representation on the screen. And then, back to reality with a 3D printing, revealing once again that "the choice of representations implies a structuring of the data which determines processing constraints" [5]. Constraints, either of the hardware or of the software, when working with a CAS, have been analyzed by Guin and Trouche [32]. An additional constraint, called motivating constraint, is mentioned in [14]. Such a constraint provides a motivation to achieve a more profound understanding. We met such a constraint when analyzing the symmetries of an astroidal ellipsoid. In our work, we see two new appearances of constraints:

 i. The recognition by the software of and object as a geometric object (v.i. Appendix A.1)

 ii. More concrete: the impossibility to 3D print the astroidal surface in one shot, and the need to 3D print it in two parts, and then glue them.

Note that the example of an astroidal surface is quite easy to analyze prior the 3D printing. The symmetries of the surface are a direct consequence of parity of the functions involved in the parametric presentation of the surface. Other examples may be more difficult to analyze for symmetry: the difficulties of the computer transition are met early. The 3D printed object may provide confirmation a posteriori that the choices were accurate. Hopefully, the 3D printed object will enable discovery of new properties, maybe even a shorter way to have the object 3D printed.

# A  Appendix

## A.1  Recognition of an object as a geometric construct.

We mentioned above that for certain GeoGebra commands, the object appearing on the screen has to be recognized by the software as a geometric object. We illustrate this claim in different

situations, namely the usage of the **Trace On** feature, the usage of the **LocusEquation** command, and of the **Envelope** command, all relying on examples.

## A.2  Determination of a envelope and of a geometric locus

The automated determination of an envelope and of a geometric locus rely on the same algebraic machinery, no matter which software is used. Pech [40] works mostly with CoCoA[10], and in this paper we work with GeoGebra and Maple. The representations of the results may be different, but the mathematical meaning is the same. In this paper, we deal more with envelopes than with geometric loci; as, in the literature, there are more occurrences of computations of geometric loci, and also as GeoGebra has more commands for this determination than for envelopes, we discuss here the case of loci, whose automated methods have seen important developments along the years; see for example [10, 1, 9, 27]. Envelopes encounter similar issues.

   We illustrate this remark with the following example (for the sake of simplicity, in 2D). Recall the bifocal definition of a Cassini oval:

**Definition 4** *Let be given two points $F_1$ and $F_2$ in the plane, and a positive real number $f$. The geometric locus of points $M$ in the plane such that $MF_1 \cdot MF_2 = f$ is called a Cassini oval. The points $F_1$ and $F_2$ are the foci of the oval.*

A plot and an equation of the curve can be obtained using GeoGebra's **LocusEquation** command, as shown in Figure 16. This is the equation appearing in the algebraic window labelled as *eq1*. The curve plotted on the screen is not a geometric construct recognized by the software; as a consequence, it is impossible to define a **Point on Object** (clicking with the mouse on the curve does not provide anything). Therefore, the equation has been copied (using the sequence Ctrl-C Ctrl-V). The new version is labelled by *eq2*. The new plotted curve looks identical to the previous one, but now it is a geometric object recognized by the software. It is now possible to define a Point on Object, and to continue with geometric constructs, such as the tangent and the normal to the curve at this point.
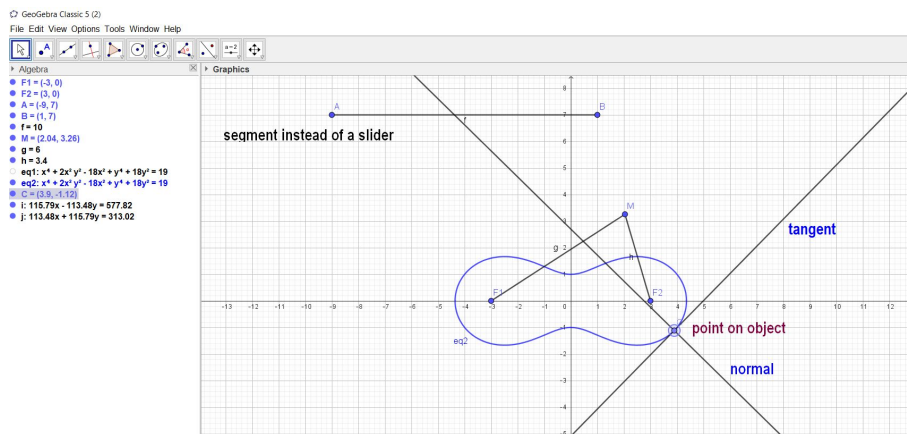


Figure 16: A Cassini oval as a geometric loci obtained with automated commands

   At this point, it is important to mention that, instead of a slider for the parameter $f$, we used a segment $AB$, this in order to define every object as a geometric construct. Otherwise, the

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[10]CoCoA stands for Computations in Commutative Algebra.

command **LocusEquation** may not work. Modifying the segment modifies the curve labelled *eq1*, but not the curve labelled *eq2*. This has still to be done by hand. Such issues have been met when studying offsets of Cassini ovals [24],and are under study in a subsequent ongoing work.

## A.3   Envelope

The same remark is generally valid for the output of the **Envelope** command, and nothing ensures that the **Point on Object** will provide the expected result. Actually, using the already mentioned GeoGebra applet for constructing an astroid as an envelope of unit segments, we could attach a point to the astroid. This is due to the fact that the construct is purely geometric.

We must mention a phenomenon which can often occur: the output on the screen may contain irrelevant components. This is the case with the following applet for the automated construction of a nephroid; see Figure 17, which displays a screenshot of the online applet. In this example, we obtain the nephroid as an envelope of the family of circle centered on the unit circle and tangent to the $y-$ axis. The construction follows steps similar to what is explained in
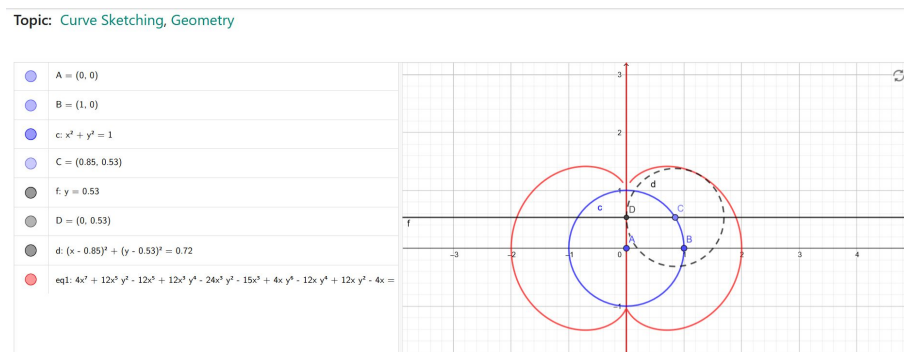


Figure 17: Automated determination of a nephroid

Section 3; in particular, the construction is fully geometric, i.e., does not use equations of lines and circles. The last command, namely **Envelope**($d$,$C$) (with the notations of the figure) yields both the plot (in red) and the corresponding equation (we reordered according to decreasing degree):

$$4x^7 + 12x^5y^2 + 4xy^6 + 12x^3y^4 - 12x^5 - 12xy^4 - 24x^3y^2 - 15x^3 + 12xy^2 - 4x = 0. \qquad (14)$$

A factor $x$ appears clearly, corresponding to he $y-$axis, the other factor (of degree 6) can be easily proven to be irreducible. It determines the nephroid. For geometric reasons, it is understood that only the segment of the $y-$axis inside the unit circle can be part of an envelope, but this cannot be determined only with the algebraic methods used by the **Envelope** command. These are based on computations of Gröbner bases and elimination, whence actually providing the Zariski closure of the requested envelope; see [13, 38]. More exotic examples can be found, they are beyond the scope of the present paper.

It is now time to recall the sunny cup of coffee from Figure 2. An automated construction is available as an applet by Z. Kovács. Other constructions of a cardioid as an envelope exist, but they cannot be treated with the automated tool used here. We could now be able to reproduce

part of the work in [15] for the determination of isoptics of an astroid. This will be done in a subsequent paper.

Another situation where recognition of a construct as a geometrical object is important is the determination of offsets of curves. This is often included in the general framework of envelopes (see [12]), but in fact requires different methods, belonging to the determination of a geometric locus; for example, see [16]. An ongoing project tries to bypass the problems posed by the lack of recognition by the software of a locus as a geometric object (in which case the involved polynomials are of high degree).

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