Education for the Future: Crafting 3D Geometric Models and Building Mathematics Knowledge with 3D Printing

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Abstract

This paper addresses the creation of 3D geometric models using 3D printers and introduces a newly designed and 3D-printed construction set of polygons for educational purposes. The process of creating these geometric models encompasses steps such as design, 3D computer modeling with Constructive Solid Geometry (CSG), 3D computer modeling of parametric surfaces using principles of differential geometry, 3D scanning of real objects, and the process of manufacturing itself. Students can be involved in the entire process of crafting models for 3D printers, and these resulting printed models can be utilized in geometry education at all levels (university and secondary school in our scenario) as instructional aids. We explore the potential methods to design geometric objects using 3D computer modeling software; this covers both commercial options and open-source software like Tinkercad, a free web application for 3D design, electronics, and coding. We present the fabrication of a new construction set of polygons which consists of different shapes of regular and irregular polygons and it is intended for use in mathematics teaching to study polygon properties and create diverse types of tessellations at the secondary school level. The effects of using these instructional aids were tested with several groups of students. All steps of the model design for 3D printing, in combination with the physical 3D printed models, shed new light on mathematics education and more broadly, to education as a whole. This process engages students in solving real-world problems and enhances their understanding of geometry, while familiarizing them with 3D computer modeling and 3D printing technologies. Both 3D virtual models and 3D printed models can act as manipulative instructional aids.

1 Introduction

The field of 3D printing technology continues to evolve, becoming increasingly user-friendly and reaching wider audiences. There is a significant growth in using 3D printers in various fields ranging from construction industries, medicine, cultural heritage, art designs to education [22].

Scholars and researchers in the field claim that 3D printing is a beneficial tool in facilitating learning, [6], enhancing skills, and increasing student engagement. This can be particularly applicable to areas such as STEM subjects, where 3D printing can provide tangible aids and together with the entire process of the model design including 3D computer modeling it can promote students' better and deeper understanding [3], [12], [19].

Learning geometry and mathematics is undoubtedly challenging at all stages of education. A variety of visual aids, such as virtual or concrete manipulatives, can significantly help in fostering an understanding of mathematical and geometrical concepts. Studies conducted worldwide have underscored the effectiveness of tangible instructional tools in mathematics education [17]. The use of virtual manipulatives also stimulates students' interest and enhances their performance [8].

Implementing 3D printing technology in education allows students to engage with both virtual and physical realities. Students have the opportunity to acquaint themselves with 3D computer modeling and design their own virtual models. Furthermore, this digital interaction is enriched by the ability to handle and explore 3D printed physical manipulatives. A significant portion of scientific literature indicates that 3D printing technology presents novel opportunities for innovative teaching practices by connecting engineering, technology, and the practical application of science subjects. International studies are focused on understanding the impact of integrating 3D printing technology within educational environments.

In their systematic review of literature, Novak and colleagues identified 78 publications focused on learning with 3D printing technologies [11]. Their conclusions indicated the positive influence of 3D printing on students' learning, its capacity to engage students in solving real-world problems, and the potential it provides for interdisciplinary research. Herrera et al. [7] found that 3D tools (and also augmented reality and virtual environments) have a positive effect on the development of spatial mathematical skills. 3D printing also supports students' understanding in concrete mathematics topics such as surface area [5]. Action-oriented training methods utilizing real physical models have shown good results in improving spatial abilities.

However, it is worth noting that there are studies indicating that manipulatives may not impact student achievement in mathematics [4]. It is important to remember that the use of manipulatives does not ensure success or meaningful comprehension.

The integration of digital fabrication, such as 3D printing, in education enriches the learning experience by mapping concrete items to abstract concepts and fostering problem-solving skills in geometry. This effective application of multiple representations is promising in enhancing the learning process. Therefore, it is beneficial to explore 3D printing further in educational contexts. Accordingly, this paper focuses on the processes of designing and creating 3D models on 3D printers and discusses learning experiences with them in a classroom context. This paper is a follow-up article dealing with 3D printing technology [20] and it furthermore explores other possibilities of digital fabrication and highlights its practical applications in mathematics education.

The paper is structured as follows. We begin with a brief overview of 3D printing technology. This is followed by a presentation of various methods for designing virtual geometric models emphasizing 3D computer modeling using constructive solid geometry. Subsequently, we introduce an educational initiative involving a newly created set of polygons produced through 3D printing. Finally, a gallery of selected models of tessellations is presented.

2 3D Printing Technology

Essentially, 3D printing transforms a digital model into a physical object, through an additive process where material is layered to form the final product. This technology, a multi-step process, combines designing, 3D computer modeling using CAD or other software platforms, and the actual fabrication.

There are numerous types of 3D printers that create objects through additive processes. In our research, we specifically utilize 3D printers that construct objects layer by layer, predominantly using a biodegradable plastic material known as Polylactic acid (PLA). We work with Original Prusa i3 MK3S+ 3D printer, which was awarded to us as a prize for the educational project that will be presented later in this article.

The 3D printer transforms digital 3D models into physical objects using 3D printing files. There are several 3D printing file formats, for example, STL (a widely used 3D printing file type containing the surface geometry of a virtual 3D object), OBJ (which entails high-quality geometry, texture information, and full color), 3MF (which contains definitions for colors, materials, and precise shapes that are not present in STL files), G-Code (the format of output from slicer software used to describe the path of the tool in the 3D Cartesian coordinate system). We create digital models in STL format and convert them into G-Code format.

There are several 3D printer slicer software which can be used as control interfaces and most of them are free (for example *PrusaSlicer*, *Repetier-Host* [15], [16]). The software is used for *slicing*, i.e. converting the STL format, for instance, in which the virtual model is saved, into the G-Code format.

During the process of creating 3D models for printing, it is crucial to consider the concept of a "manufacturable object". In geometry, we deal with abstract concepts such as dimensionless points, infinitely thin lines, and surfaces without thickness. However, these abstractions present challenges from a practical standpoint. Consequently, in practical applications, we work with manifold objects, which are characterized by edges shared by exactly two faces. We craft digital models that can be fabricated from specific materials. In the case of a parametric surface, for example, we need to print both the surface and the offset surface, considering the volume in between these two.

3D printers construct three-dimensional objects by laying layer over layer of plastic. Every newly deposited layer relies on the support of the one underneath. If a portion of the model begins in mid-air without any underlying support, additional support structures must be implemented to guarantee a successful print. These parts are then removed after printing. The object can be also divided into parts and these parts are printed separately.

The geometric model placed in the interface of PrusaSlicer software is shown in Figure 1. It is the model of the helical surface, which already requires the printing of supports (shown in green color).

Each 3D printer is designed to print within a specified footprint size, and the precision of the particular printer must be considered. The size of the model determines the printing time and amount of material consumed. Unfortunately, even relatively small 3D models can take several hours per geometric model to complete.

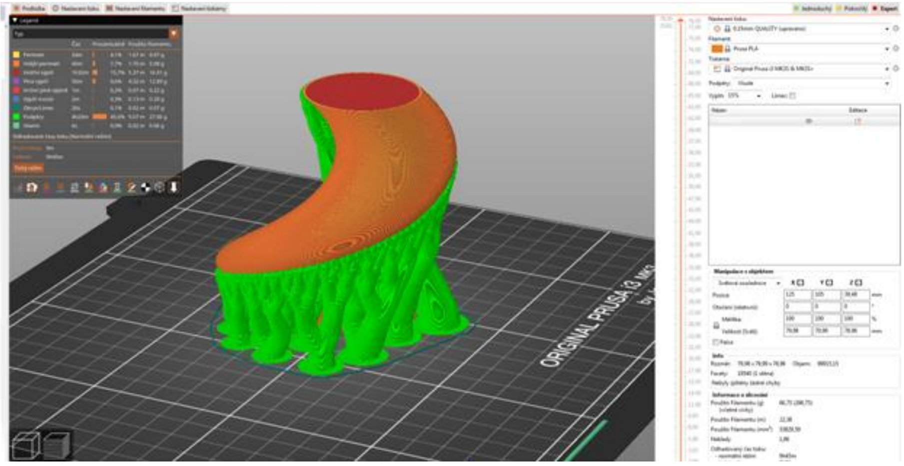

Figure 1: The sliced geometric model of a helical surface placed in the interface of PrusaSlicer and generated with organic supports (shown in green color).

3 The Journey from Abstract to Concrete with 3D Printing

We primarily use 3D printers to create geometric models that can be applied in mathematics education at all levels, at universities and secondary schools in our case. For example, 3D printed models of polygons and solid figures can be used in mathematics instruction at secondary schools. More complex geometric models, which are enclosed by an assembly of surfaces (not only parts of planes but also parts of other types of parametric surfaces), are intended for use in undergraduate geometry courses at universities.

In any case, the digital model for 3D printing has to be firstly created. There is always the possibility to download pre-made models on the Internet. There exist plenty of websites and the models are generally free to download [14]. For the certain educational purposes, it is always better to know how to create the geometric model. As previously mentioned, our plan is not only use the final products from 3D printers as instructional aids but we also want to encourage students (at any level of education) to participate in the process of digital fabrication.

The development of a geometric model for 3D printing requires an understanding of its definition and properties. There are various methods available to prepare a digital model for 3D printing. For example, 3D scanning can be an option, particularly when there is a need to digitize real objects. A mathematical description of a digital model is also acceptable. If we consider the digital model which is enclosed by an assembly of surfaces, these separate surfaces can be described in parametric forms.

A brief explanation of the 3D scanning process for real objects, as well as the use of parametric forms to define the closed volume of a solid, can be found in the previous paper [20].

3.1 3D Computer Modeling

It is not necessary to only use 3D scanning or describe a digital model through mathematical equations. One can take advantage of the modeling features of 3D computer modeling software.

There are numerous free software options available for creating 3D printable digital models. For example GeoGebra 6 is a free dynamic application including free tools for geometry, and it is user-friendly for both students and pupils. In GeoGebra 6, we can model solid figures or parametric surfaces (or parts thereof, to be precise), and this model can be directly exported to STL format. However, GeoGebra's built-in functions are limited. There are also professional CAD systems and 3D computer modeling software that include advanced modeling features. For example, Rhinoceros (a commercial NURBS-based 3D modeling tool commonly used in design processes), SketchUp, Blender (free), or FreeCAD (free). Arguably the easiest way to create a model is to use the online editor Tinkercad, a free web application for 3D design, electronics, and coding [2]. It is easy to use, and many video tutorials support users. Fusion 360, a more advanced tool, follows up Tinkercad and is free for educational use [1].

A simple geometric model of a mechanical component created in Tinkercad can be seen in Figure 2. It was modeled using the concepts of primitives which are basic geometric solid figures such as a cube (or cuboid), cone, sphere, cylinder and pyramid. These primitives can be combined and modified by using Boolean operations into more complex shapes. This technique for solid modeling is known as *Constructive Solid Geometry.* The process of creating a mechanical component using Constructive Solid Geometry is captured in Figure 3.

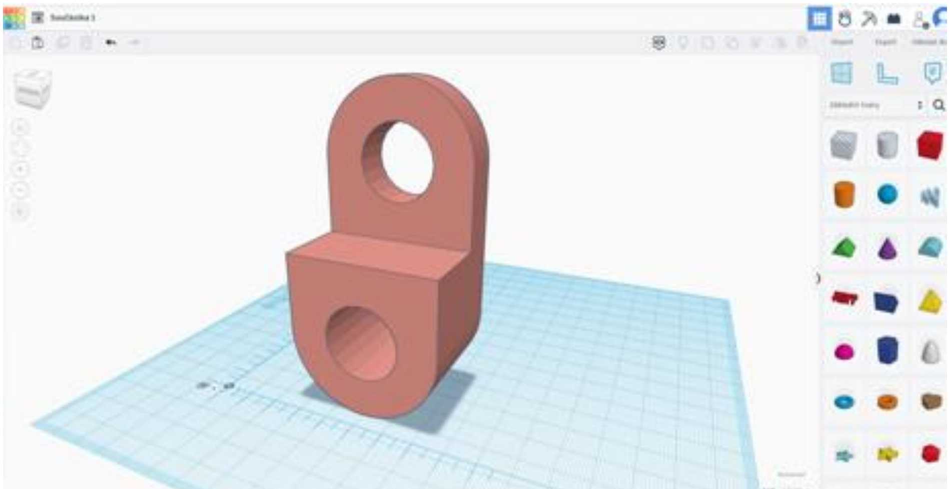

Figure 2: The geometric model of a mechanical component created in Tinkercad.

We have found that pupils at secondary schools and university students can easily use Tinkercad and create impressive models without any intervention from a teacher. Any digital model which is created in Tinkercad can be directly exported to STL format and is ready for slicing and 3D printing. Once again we would like to emphasize that working with any 3D computer modeling software requires familiarity with the specific tool, but also a thorough understanding of the geometric principles and properties that define the objects being modeled.

4 Educational Project with a 3D Printer

Programs supporting the use of 3D printers in schools exist in the Czech Republic. We also took advantage of this opportunity at the Grammar School where the author teaches. After

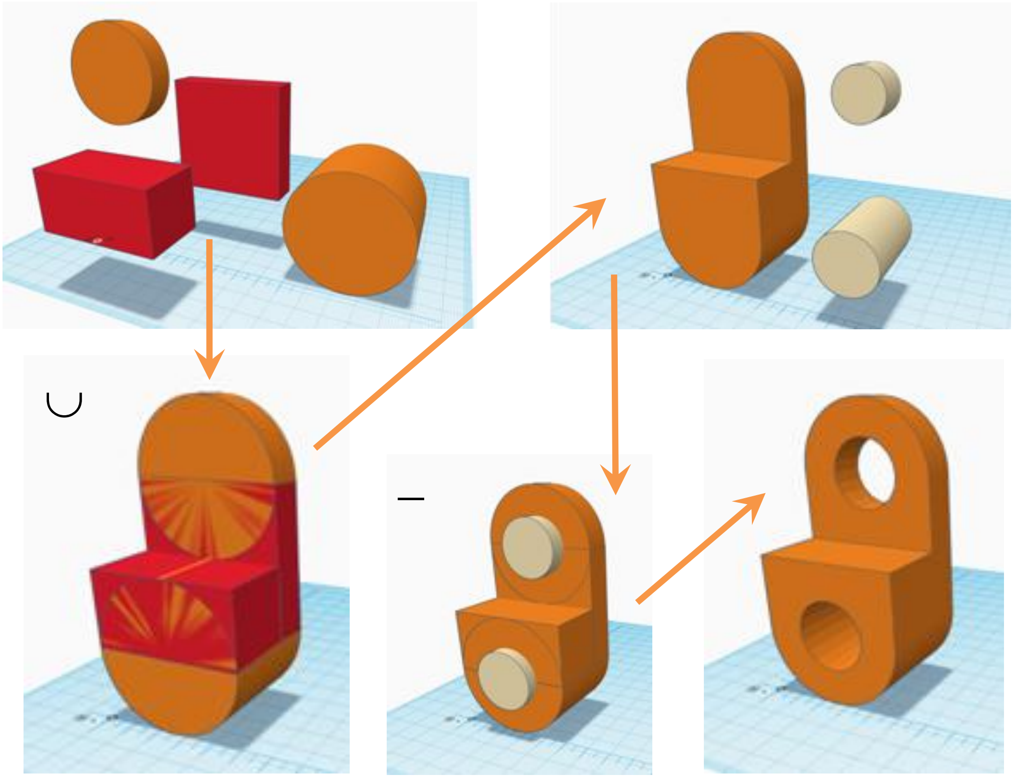

Figure 3: Generating a geometric model of a mechanical component with Constructive Solid Geometry.

proposing and defending the educational project aimed at enhancing the understanding of geometry, our school received a 3D printer at no cost. Together with the students, we designed and manufactured a construction set of polygons using the 3D printer as the educational project [18].

4.1 Polygons, Congruence Transformations, and Tessellations

A *polygon* is a plane figure defined by straight line segments acting as its boundaries. Typically, additional conditions are added to get only a simple polygon, which does not self-intersect. This geometric topic is standard in the curricula of elementary and secondary schools in the Czech Republic. The subject of polygons includes the exploration of their various types and properties. Other topics of planar geometry which are taught in the Czech Republic include geometric transformations such as translations, rotations, and axial or central symmetries, which are congruent, meaning they preserve shapes and distances.

A *tessellation* (or tiling), refers to the coverage of a plane using one or multiple types of tiles, which can be either regular or irregular polygons, leaving no overlaps or gaps. From a geometric perspective, tessellation is an interesting subject due to its reliance on the geometric properties of the polygons used in its construction, as well as the application of congruence geometric transformations. Tessellations can take various forms, for example, there are regular, semi-regular, or aperiodic tilings. There exist only three *regular tessellations*, consisting of equilateral triangles, squares, and regular hexagons. Moreover, an infinite number of tessellations can be generated from these three regular forms using congruence transformations. Figure 4 showcases several rules applied to a hexagonal tiling.

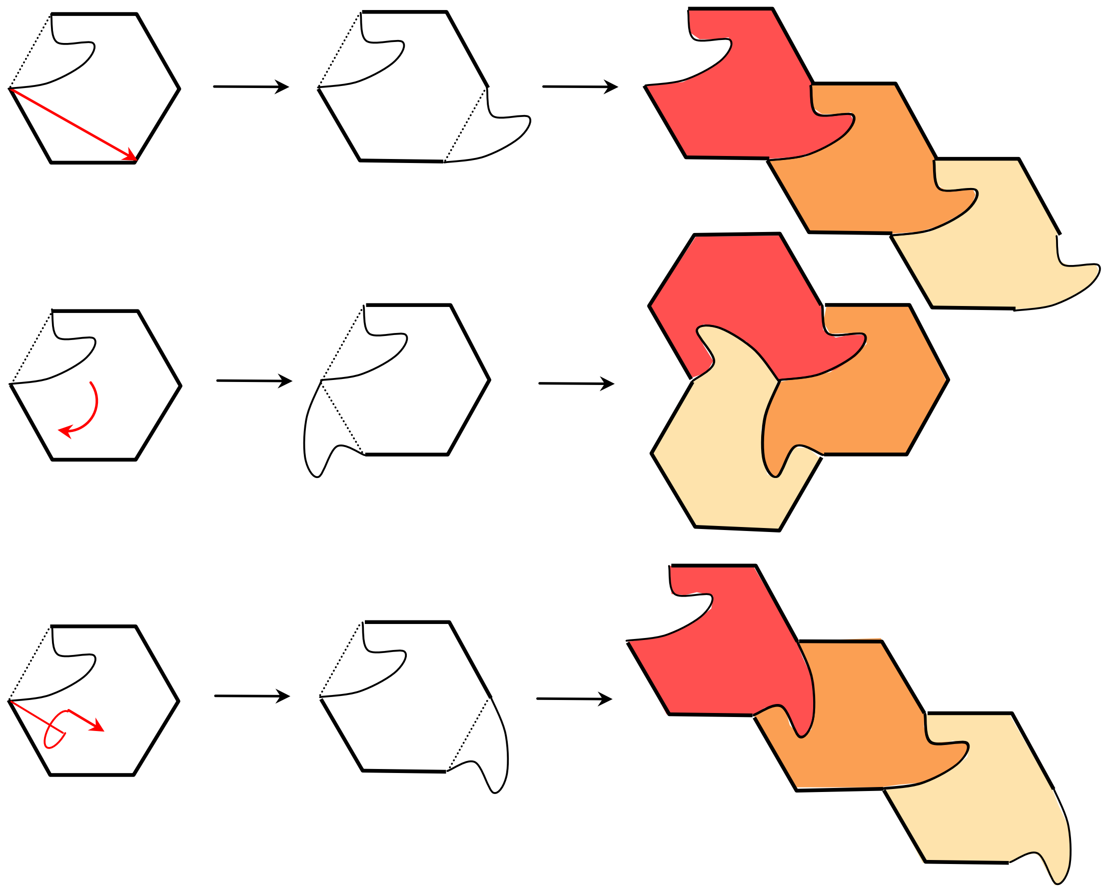

Figure 4: Three types of tessellations created from hexagonal tiling, based on geometric principles of translations, rotations, and glide reflections.

A tessellation created from hexagonal tiling and inspired by Escher's work [10] can be seen in Figure 5.

Semiregular tessellations (or Archimedean tessellations) refer to tessellations of the plane composed of two or more types of convex regular polygons. In these tessellations, each polygon vertex is surrounded by the same polygons in the same order. There are eight such tessellations in the plane.

Next to the various tessellations created by regular polygons, tilings by other polygons have also been studied. Any triangle or quadrilateral (even non-convex) can be used as a tile to form a tessellation using only one type of a tile. Such a tessellation is called *monohedral tessellation*. Figure 6 shows a monohedral tessellation made up of arbitrary triangles and a monohedral tessellation made up of arbitrary convex quadrilaterals.

Fifteen types of convex pentagons are known to form a monohedral tessellation of the plane.

Tessellations can also be formed by convex regular polygons that are not edge-to-edge. Such tilings can be considered edge-to-edge as nonregular polygons with adjacent collinear edges.

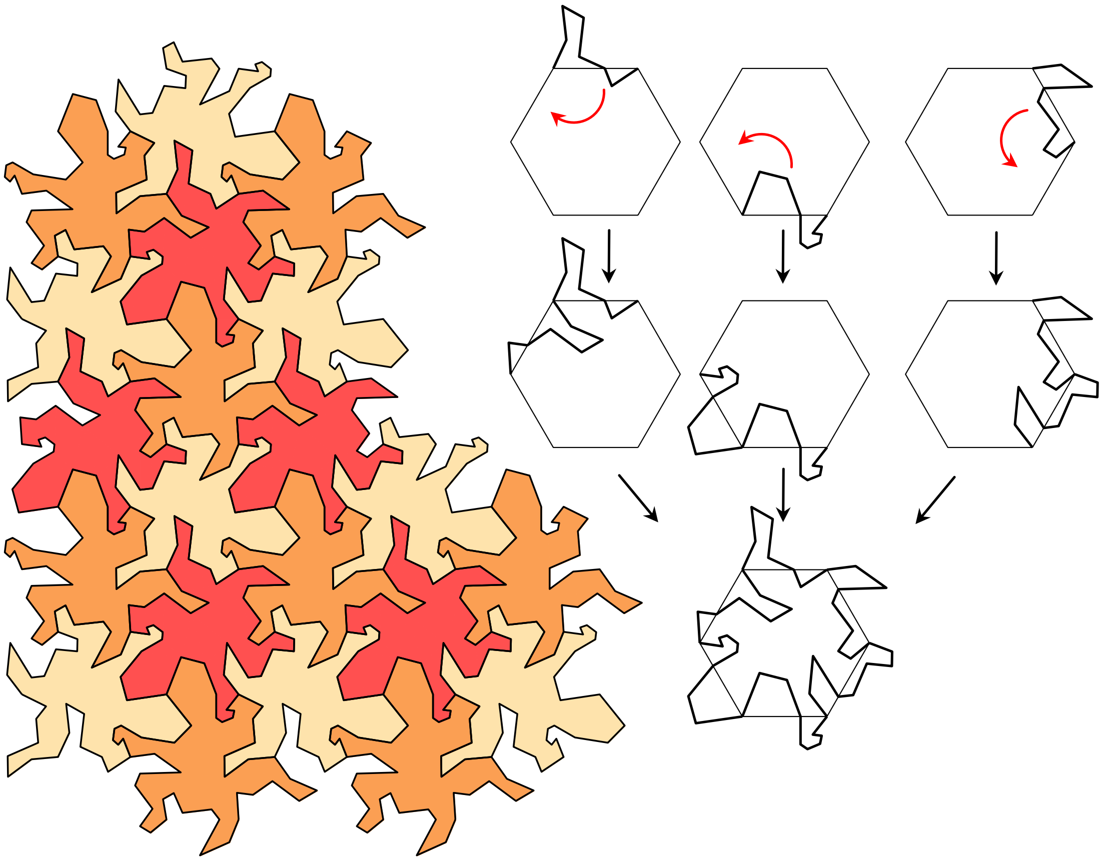

Figure 5: A tessellation inspired by M. C. Escher.

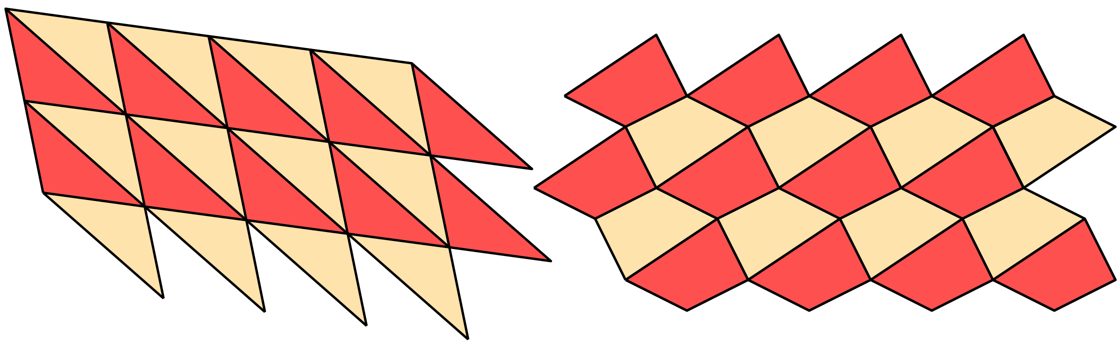

Figure 6: Monohedral tessellations: on the left, made up of arbitrary triangles, and on the right, made up of arbitrary convex quadrilaterals.

A tessellation is not typically included in the curriculum of elementary and secondary schools in the Czech Republic but it can serve as a subject for evaluating students' conceptual understanding of polygons, their properties, and congruence geometric transformations.

Further information on planar geometry, specifically regarding polygons, congruence transformations, and tilings, can be found in relevant literature [13].

4.2 A Construction Set of Polygons

The set printed on a 3D printer consists of different shapes of regular and irregular polygons and it is designed for use in mathematics teaching to study polygon properties and create diverse types of tessellations. As previously mentioned, for 3D printing we need to work with printable objects. Thus, we added some height to the polygons to create solid figures representing these polygons. We explained this modification to the students, who quickly understood and accepted it.

4.3 An Educational Experiment with 3D Printed Shapes

After creating a construction set of polygons, we also investigated the effects of using these instructional aids in students' understanding of geometry with several groups of students at secondary schools in the Czech Republic [21].

Our aim was to examine students' conceptual understanding of polygons through collaborative activities and how physical 3D printed manipulatives can help in solving geometry planar problems.

The experiment was repeated four times with four groups of students of different ages. In total 25 elementary school students and 34 secondary school students took part in the experiment. Elementary school students from one class (age 12-13 years) were one group (within the experiment they were divided into subgroups of 8, 8, and 9 students), three groups were secondary school students always approximately one half from one class – 14 students (age 15 years), 9 students (age 16 years), and 11 students (age 17 years). Since mathematics instructions are taught one a week divided into two. The experiments always took two mathematics instructions, i.e. 90 min, and were conducted in the school year 2021/2022 in the Czech Republic.

The experiment was conducted four times with students of varied ages: 25 from elementary school (ages 12-13) and 34 from secondary school (ages 15-17). Each session spanned 90-minute math lessons, held weekly, during the 2021/2022 school year in the Czech Republic.

Students were given tasks on polygon types, properties, and tiling applications. All students had prior knowledge on polygons from their previous studies. The study aimed to compare their success in solving geometric tasks with and without concrete manipulatives and to observe their collaborative activities. The experiment had four phases: (1) inquiring about polygon types (regular and irregular) and their properties, (2) identifying tessellations without any visual aids, (3) using drawing images, and finally (4) using concrete manipulatives from 3D printer. First and third tasks were solved individually of each student on a sheet of paper. Second and fourth tasks were discussions among students with the guidance by the author.

In the first phase, results indicated students had a strong factual knowledge of polygons and their characteristics. Almost all of them could identify and describe basic polygonal properties (number of sides, sizes of interior angles, regular and irregular, convex and concave, area of polygons ...), with older students outperforming the rest.

In the second phase, students immediately knew that tessellations can be made from equilateral triangles, squares, and rectangles. But they had uncertainty regarding other regular polygons, with some incorrectly suggesting tessellations from regular pentagons. A few students recalled seeing natural tessellations from regular hexagons. Irregular polygon tessellation concepts, however, remained elusive.

In the third phase, most students sketched basic tessellations using equilateral triangles, squares, and regular hexagons. Few identified the rule concerning the sum of interior angles of polygons in vertices where polygons meet.

The fourth phase was particularly illuminating: despite initial reluctance from secondary school students toward using concrete manipulatives —deeming them 'childish'—they quickly engaged and found value in them. Students discovered even semi-regular tessellations previously unidentified. Significantly, without teacher guidance, they began formulating general rules and mathematical proofs during their tessellation explorations. Notably, the older students formulated a statement about the sum of interior angles in irregular quadrilaterals (based on the division of quadrilaterals into triangles) and drew a geometric proof for tessellations with any quadrilateral.

The experiment showed that students retain facts but lack a deeper understanding of polygons. Using 3D printed manipulatives effectively concretizes abstract mathematical concepts and helps in comprehension. This is in compliance with literature, for example [9].

Students' activities and their collaboration when they were trying to find various types of tessellations and were using the construction sets of polygons can be seen in Figure 7.

Figure 7: Students' activities with the construction sets of polygons at Grammar school.

5 Gallery of 3D Printed Polygons for Tessellations

Several types of tessellations made up of various shapes of regular and irregular polygons are presented in the following pictures. See Figures 8, 9, 10, 11, 12.

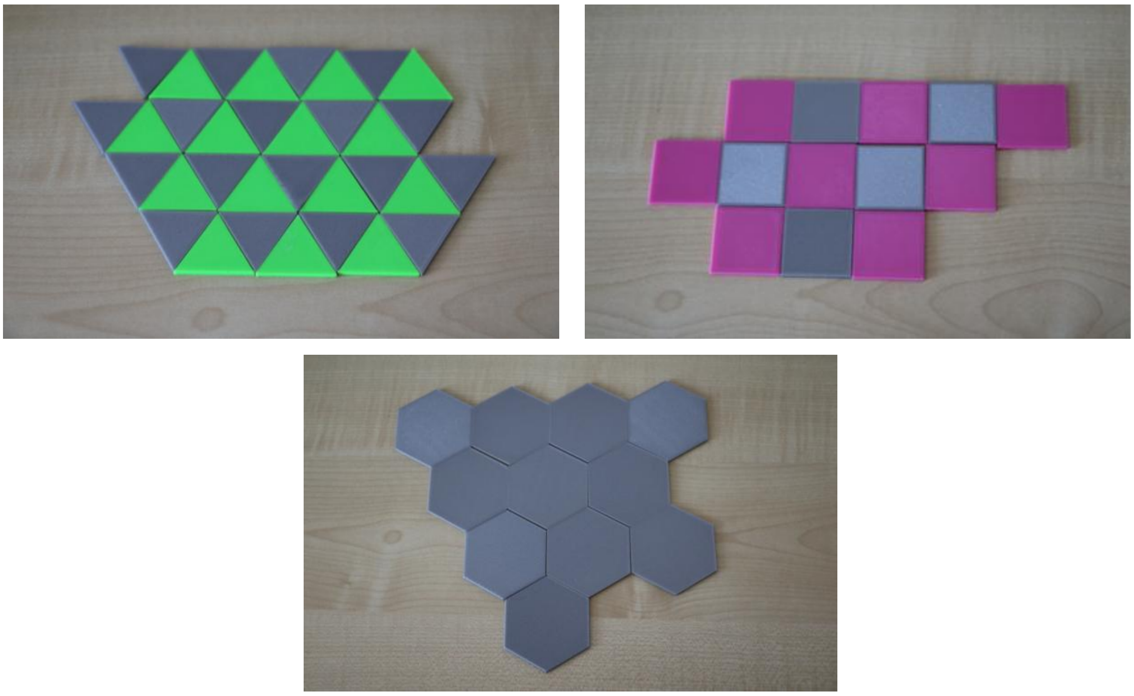

Figure 8: Three regular tessellations, consisting of equilateral triangles, squares, and regular hexagons (colors are used solely for clarity).

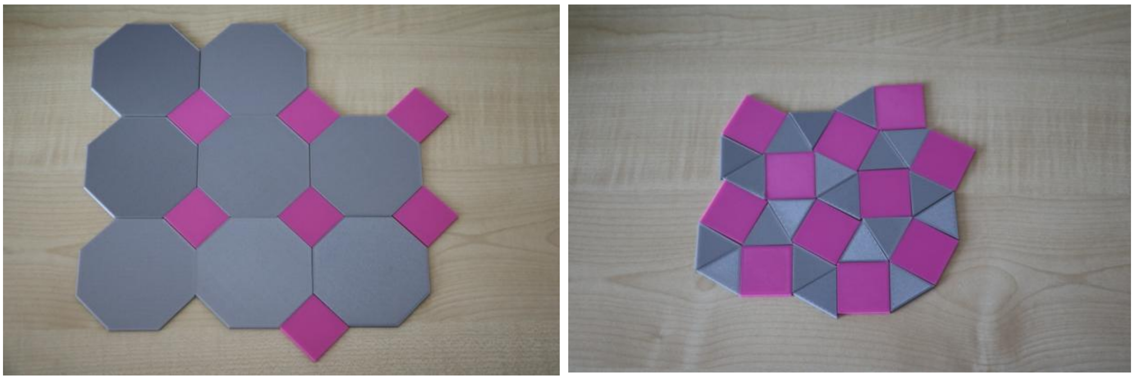

Figure 9: Two examples of semiregular tessellations: on the left, made up of regular hexagons and squares, on the right, made up of equilateral triangles and squares (colors are used to distinguish between different shapes).

6 Conclusion and Future Work

In this article, we have demonstrated various methods for creating a virtual geometric model suitable for 3D printing; we aimed especially at the simplest Tinkercad software. We have presented a new construction set of polygons produced through 3D printing. Furthermore, we have showcased examples of how these geometric models can be applied in educational contexts and have provided a collection of printed models for tessellations.

Regarding the future work, our aim is to expand our database of printed geometric models.

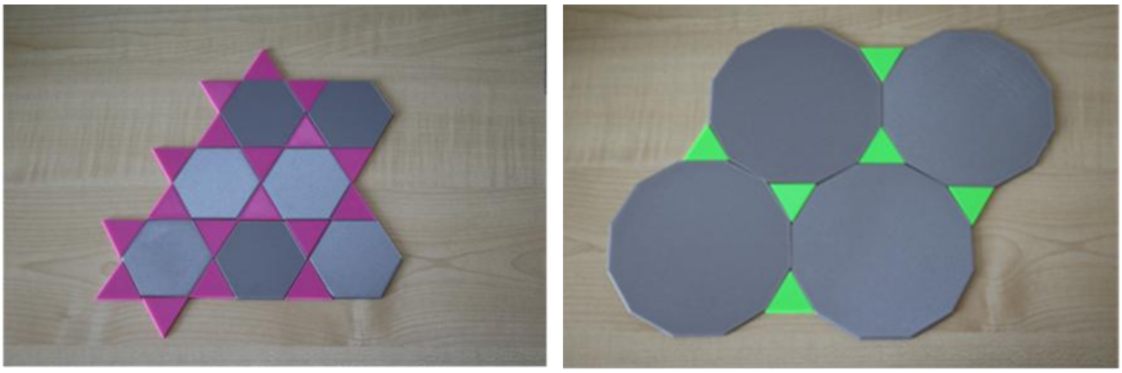

Figure 10: Two examples of semiregular tessellations: on the left, made up of regular hexagons and equilateral triangles, and on the right, made up of equilateral triangles and regular dodecagons (colors are used to distinguish between different shapes).

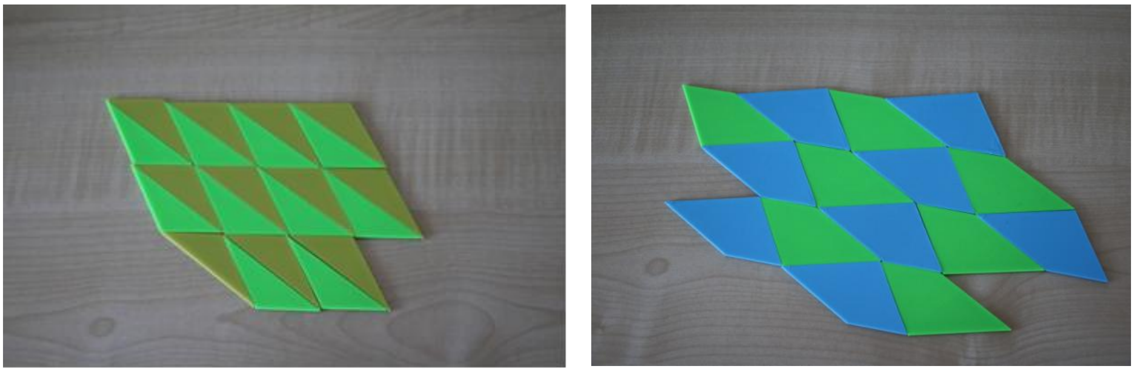

Figure 11: Monohedral tessellations: on the left, made up of arbitrary triangles, and on the right, made up of arbitrary convex quadrilaterals (colors are used solely for clarity).

Figure 12: A tessellation inspired by M. C. Escher.

We also intend to carry out research studies to investigate the impact of 3D printing technology and the utilization of printed models on students' geometric comprehension.

We plan to expand the use of 3D printing in education by creating models for other mathematical topics, such as solid volumes and deriving formulas for their calculations. We aim to explore the effectiveness of utilizing these 3D printed models in enhancing comprehension and engagement in these areas.

7 Acknowledgments

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