Understanding Geometric Pattern and its Geometry
Part 10 – Geometry lesson from Paigah Tombs

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Abstract: *Symmetry groups are a fairly recent development in modern geometry. Their origins can be traced from a paper by M. J. Buerger and J. S. Lukesh (see* [1] *). A very solid mathematical foundation of them can be found in Conway's "Symmetries of Things." We can also find there his Magic Theorem for plane symmetry groups, the so-called wallpaper groups, with a complete proof. Some mathematicians believe this theorem can be used to create any plane geometric pattern. Unfortunately, the Magic Theorem is not enough. It can help to determine the overall geometric structure of the pattern, but it does not handle what is happening inside the fundamental region of it. Thus we may have an infinity of geometric patterns within the same symmetry group. However, in many cases, symmetry groups can help us reconstruct an existing geometric design or create a new design.*

This paper discusses a selection of patterns found in Paigah Tombs, or the Maqhbara Shams al-Umara, in Hyderabad, India. We will limit our discussion to a selection of hexagonal designs. Following this discussion, we will show how to analyze and reconstruct these patterns.

Introduction

Different classes of geometric patterns require different approaches and different methods. This paper discusses a specific selection of geometric patterns that we will refer to as hexagonal. Various authors provide different definitions of hexagonal patterns. Thus we need to specify what we mean by this term and how one can distinguish a hexagonal pattern from various other types.

For the purpose of this paper and any further investigations, we define hexagonal patterns by using plane symmetry groups and local symmetries of a pattern.

Plane symmetry groups

Plane symmetry groups (in the following pages called wallpaper groups) allow us to classify all patterns by their transformation properties, i.e., reflections, rotations, translations, and glide reflections. Conway's Magic Theorem states that there are seventeen wallpaper groups. Thus, according to it, we may classify all patterns into 17 disjoint classes. From a geometric pattern design point of view, this information is not very useful. For example, all decagonal patterns can be split between two wallpaper groups only. But the same two groups also contain other types of geometric patterns, e.g., octagonal, dodecagonal, etc. However, wallpaper groups allow us to determine the primary cell[1] of any pattern. In some cases, this is enough to proceed with pattern design. Unfortunately, the most difficult task is still to be done in other cases.

***Definition*[2]** *– a geometric pattern is considered as hexagonal if it was created on a tessellation of regular hexagons or polygons (equilateral or equiangular), derived from a regular hexagon, and its highest local symmetry is less or equal D6 or C6.*

Hexagonal patterns can be located in a few wallpaper groups. The majority of them belong to groups *632 and 632. However, we can see them also in groups 333, *333, 3*3, and in other groups. All patterns discussed in this paper belong to wallpaper groups *632 and 632. Thus we will limit further discussion to these two groups only[3].

The wallpaper groups *632 and 632 are very large and contain many patterns that we do not want to consider as hexagonal, i.e., patterns with dodecagonal stars or rosettes, patterns with nonagonal stars or rosettes, patterns with mixed local symmetries larger than D6. Thus the D6 and C6 requirement is necessary to remove all unwanted geometric designs. Let us briefly discuss the two wallpaper groups *632 and 632.

Wallpaper group *632

For a start, suppose that a pattern belongs to the wallpaper group *632. This means that we have in it points (kaleidoscopes) with 6 mirrors passing through them, kaleidoscopes with 3 mirrors passing through them, and kaleidoscopes with 2 mirrors passing through them. We will represent kaleidoscopes using the star symbol: *6, *3, and *2. The only possible configuration of mirrors and kaleidoscopes in the wallpaper group *632 can be illustrated as follows.

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[1] Some authors call it a fundamental region of the pattern.

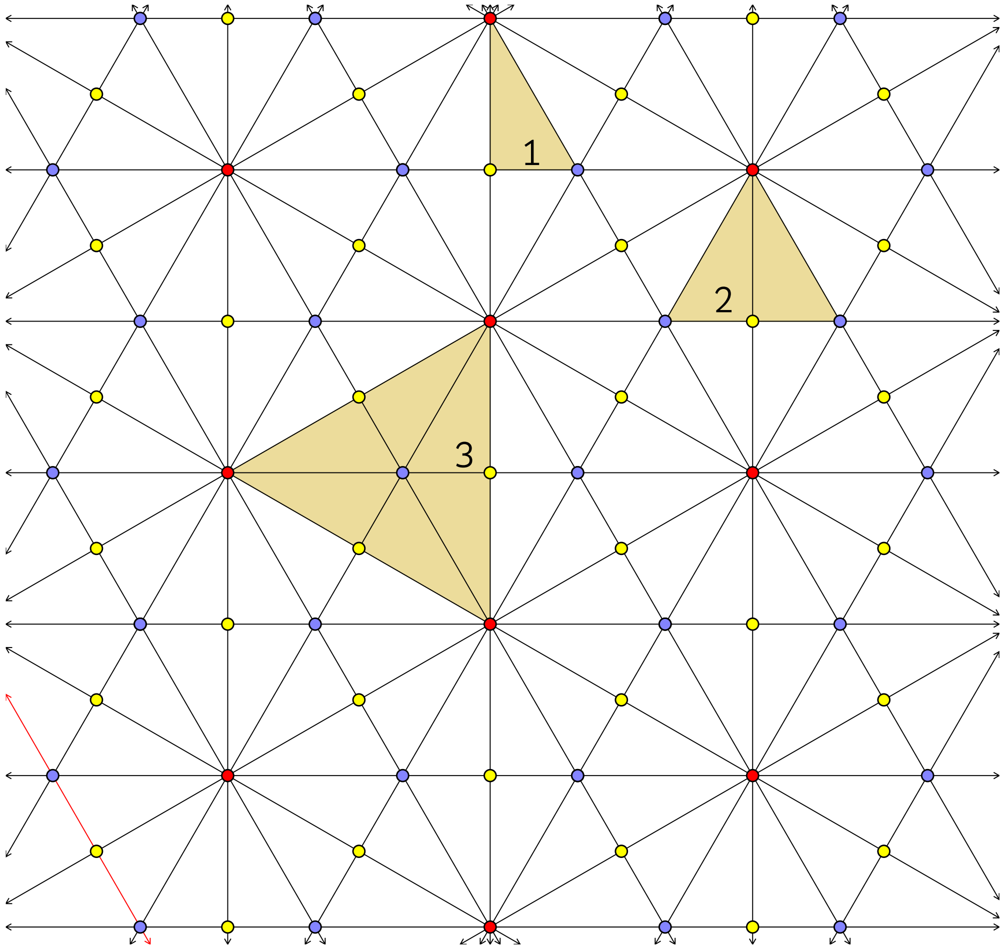
[2] Deciding whether a pattern is hexagonal, octagonal, or decagonal is more complex than many people think or teach. In fact, there are no precise mathematical definitions of these terms. Thus this definition should be treated as provisional, and further discussion is needed.

[3] According to our intuitions, we should expect a large number of hexagonal patterns in symmetry groups 333, *333, and 3*3. It is incredibly easy to develop geometric patterns with these three signatures. But some statistics show very few known hexagonal patterns with these signatures.

In the book „Islamic Design a Mathematical Approach" Brian Wichmann published the frequency of patterns in each symmetry group. According to his statistics in a collection of 1500 patterns (as for 2017), there are 198 patterns in group *632, 26 in group 632, 9 in group 3*3, and 1 in each group 333 and ∗333. We assume that since 2017 his collection grew up, and now these numbers are slightly different. But, still we may not expect many additions to groups 333, ∗333, and 3∗3.

In some other symmetry groups, there are also patterns that we would count as hexagonal. But there is no good way to extract them from these groups.

All known statistics do not reflect the coloring of patterns. They deal with segments only, ignoring colors of particular shapes. Thus a pattern that was considered as 632 with colors may change to 333.



Network of mirrors and kaleidoscopes in any pattern with wallpaper group *632

One can easily prove that this is the only possible configuration of kaleidoscopes where six mirrors intersect, three mirrors intersect, and two mirrors intersect.

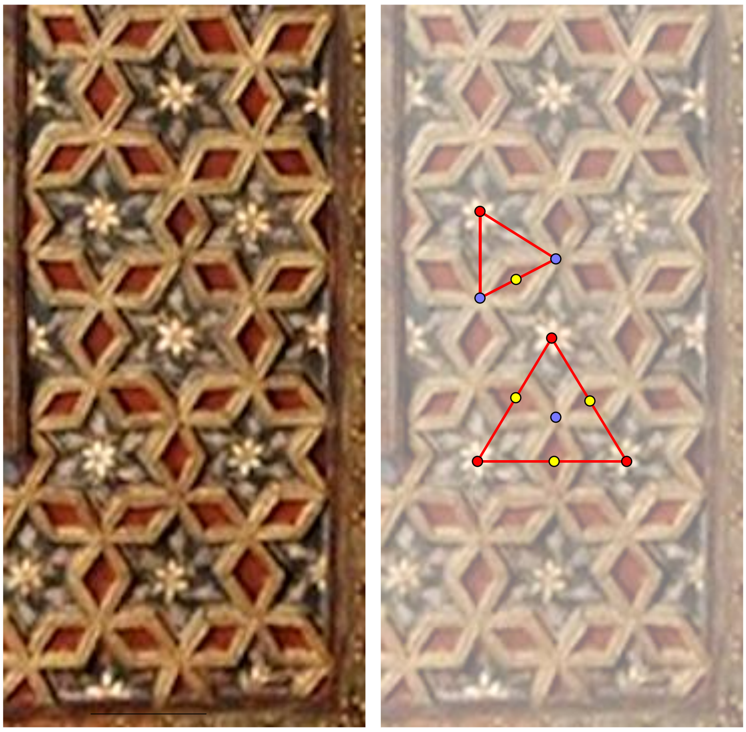
The **triangle 1** contains the smallest part of the pattern necessary to produce a larger design by mirroring it multiple times. We will call it the **primary cell of a pattern**.

Triangle 2 was created using two copies of the fundamental region. We will call it a **small triangle** or **s-triangle**.

Triangle 3 was created using six copies of the fundamental region. We will call it a **large triangle** or **L-triangle**.

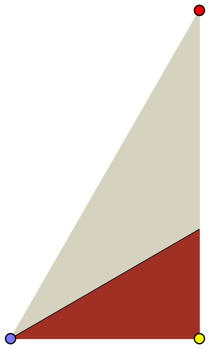
Although the primary cell contains the smallest part of the pattern, necessary to proceed with the pattern design; usually, it is more convenient to design patterns using larger structures. In practice depending on the pattern, one has to decide which triangle is more suitable to create his construction. In many cases, we can use hexagonal tiles built out of one of these triangles. In some cases, it is more convenient to construct a pattern using two or more different hexagonal tiles. Especially designing hexagonal patterns with two or more types of regular hexagonal tiles can be very helpful.

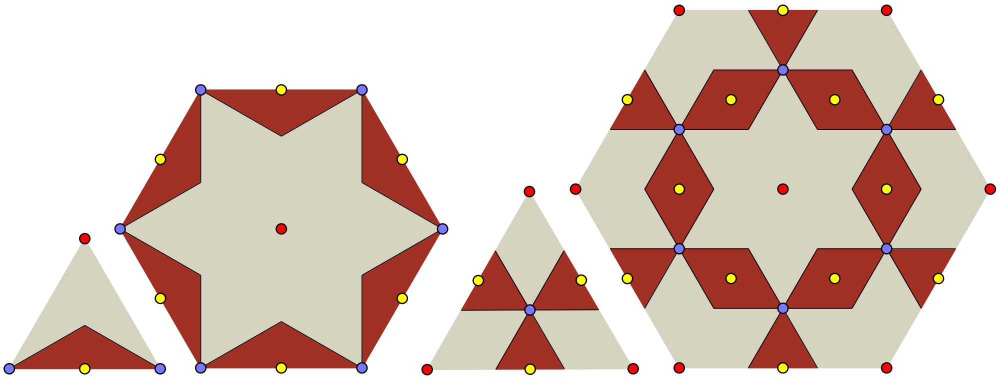
Example 1. Pattern with wallpaper group *632



The drawing shows a typical pattern with wallpaper group *632. Red points are points with six mirrors, i.e., kaleidoscopes *6, blue with three mirrors (*3), and yellow with two mirrors only (*2).

Thus the pattern can be created by using the following primary cell:





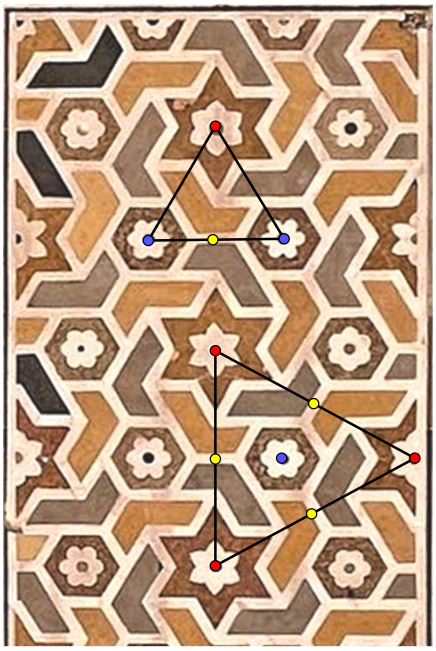
Small and large triangles and hexagonal tiles for the pattern

The pattern used in this example is often seen in many countries and places. The one presented here comes from Sabil, located at the intersection of al-Muizz and Tambakshiya streets in Cairo.

Wallpaper Group 632

Patterns in wallpaper group 632 do not have mirrors but still have some characteristic points that we will call rotation points or gyrations (we mark them using the • symbol). These points form an identical network as the one for the *632 group. Let us examine one of such patterns. In this example, we will discuss a geometric design from Itimad-ud-Daula's tomb in Agra. The pattern treated as a construction of segments only has 632 signature. The situation changes when we start distinguishing tiles with the same shape but different colors. We no longer have gyrations •2, and the gyration •6 changes into a gyration •3.

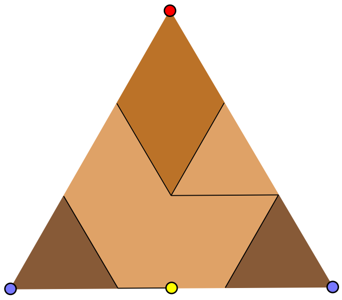
Example 2. Pattern with wallpaper group 632



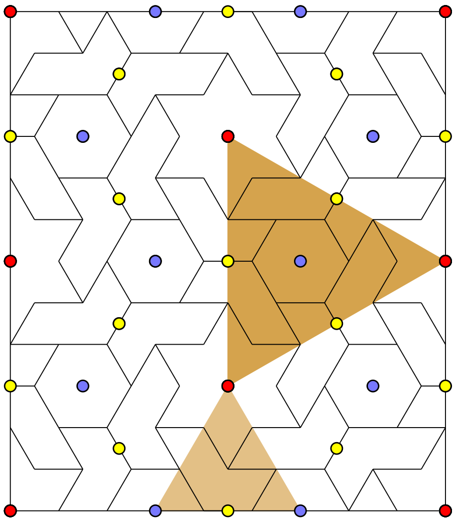
The pattern shown here can be seen in Agra at the Itimad-ud-Daula's Tomb. In this analysis, we ignore the colors of shapes.

The red points in this pattern are points where the pattern can be rotated 6 times 60 degrees (gyrations ●6), blue points where we can rotate it 3 times each time 120 degrees (gyrations ●3), and yellow points, mark rotations 180 degrees (gyrations ●2). Thus the signature of the pattern is 632.

Note – in this pattern, we do not have mirrors. We have only rotation points.



The primary cell for this pattern is an equilateral triangle, which is rotated 6 times around the red point. This way, we obtain a hexagonal tile filled with a pattern. A larger design can be obtained by translating the hexagonal tile.



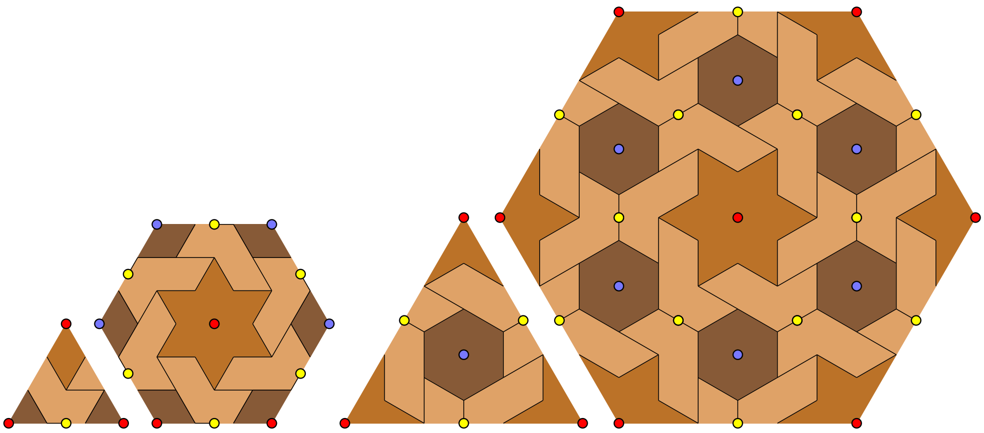
Small and large triangles for the pattern

If we want to construct a large instance of the pattern, we need to create one of these two triangles and rotate it 6 times around the red point (rotation 60 degrees). This way, we will get a hexagonal tile, small or large. The whole pattern can be produced by translating this tile.

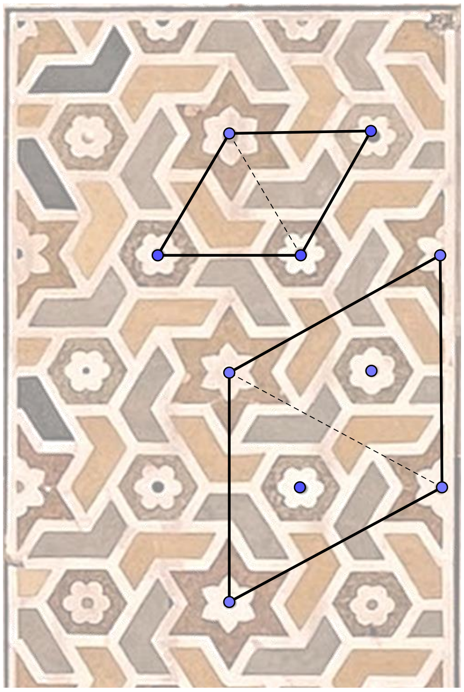
Depending on our needs, we can use small or large hexagonal tiles.

In the following construction, we ignore the colors of elements of the pattern. Therefore we treat the orange and gray V-shapes as identical.

Below: small and large triangles for the pattern and small and large hexagonal tiles



Assuming that shapes with different colors are different, we end up with a pattern with signature 333. This fact is presented in the following illustrations.

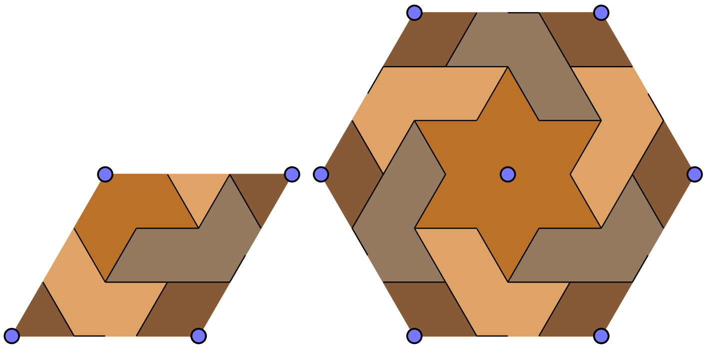


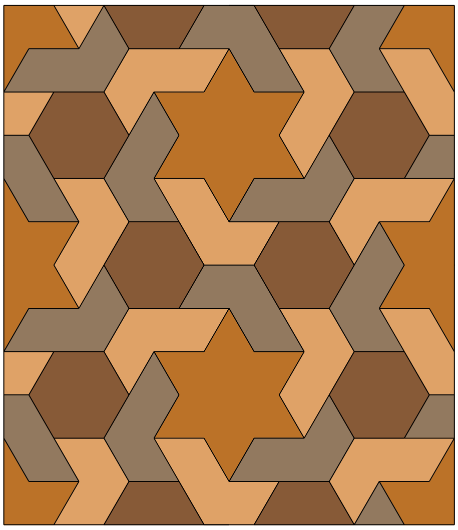
Example 3. Pattern with wallpaper group 333

The illustration shows the same pattern as in the previous example. But this time, we consider shapes with different colors as different. Thus we lost all yellow points. We do not have rotations about 180 degrees. Red points turned into blue. There are no longer rotations of about 60 degrees. All points marked on the photo are gyrations •3.

Important – the pattern can be reconstructed using two different triangles (here separated with dashed lines) or by constructing a pattern for one of the rhombi and rotating it 120 degrees around the gyration point near the wide angle of the rhombus. This way, we will produce two hexagonal tiles, small and large, and then a large pattern can be constructed by translating one of them.

The small rhombus is the primary cell of the pattern.





Up: Small rhombus tile and small hexagon tile

Left: Fragment of a large pattern created using the small hexagonal tile

The third example shows that the number of patterns with signature 333 can be significantly larger than provided by Brian Wichmann in his statistics. The same argument can also be used for patterns from the *632 wallpaper group.

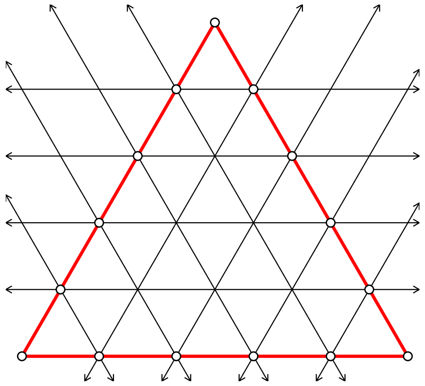
On Paigah Tombs' patterns

"Paigah Tombs, or Maqhbara Shams al-Umara, are the tombs belonging to the nobility of the Paigah family, who were fierce loyalists of the Nizams, served as states people, philanthropists, and generals under and alongside them. The Paigah tombs are among the major wonders of Hyderabad State, which are

known for their architectural excellence, as shown in their laid mosaic tiles and craftsmanship work. The Paigah's necropolis is located in a quiet neighborhood 4 km southeast of Charminar Hyderabad, in the Pisal Banda suburb, down a small lane across from Owasi Hospital near Santosh Nagar. These tombs are made out of lime and mortar with beautiful inlaid marble carvings. These tombs are 200 years old and represent the final resting places of several generations of the Paigah Nobles." (Copied from Wikipedia[4]).

In Paigah tombs, we can find a large collection of geometric patterns carved in lime. The majority of them are hexagonal patterns. Almost all of them were created using a specific technique – triangular grids built on equilateral triangles. We will split them into three groups and discuss each group separately.

Patterns created on equilateral triangles with parallel grids



Parallel grid network

The concept is simple and frequently used for patterns in Paigah tombs. It was also used in Iran and Anatolia.

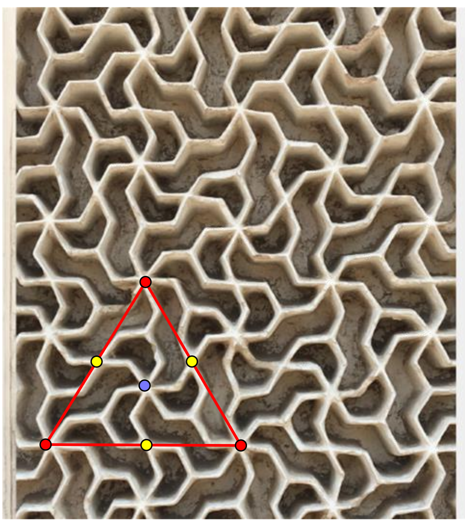
Take an equilateral triangle. Split each edge into n equal parts (here, five parts). Draw lines passing through these points and parallel to one of the edges of the triangle.

Now we can use this network to draw a pattern with edges along grid lines.

This procedure can be used to create an s-triangle or L-triangle of the future pattern.

Below we discuss one of the most complex patterns from the Paigah Tombs using parallel grids.

Example 4. One of many patterns from Paigah Tombs



The photo presents a pattern with a large equilateral triangle. All its vertices are points where the pattern can be rotated six times 60 degrees (gyrations ●6). The center of the triangle is ●3 gyration. Finally, the midpoints of the edges of the triangle are ●2 gyrations.

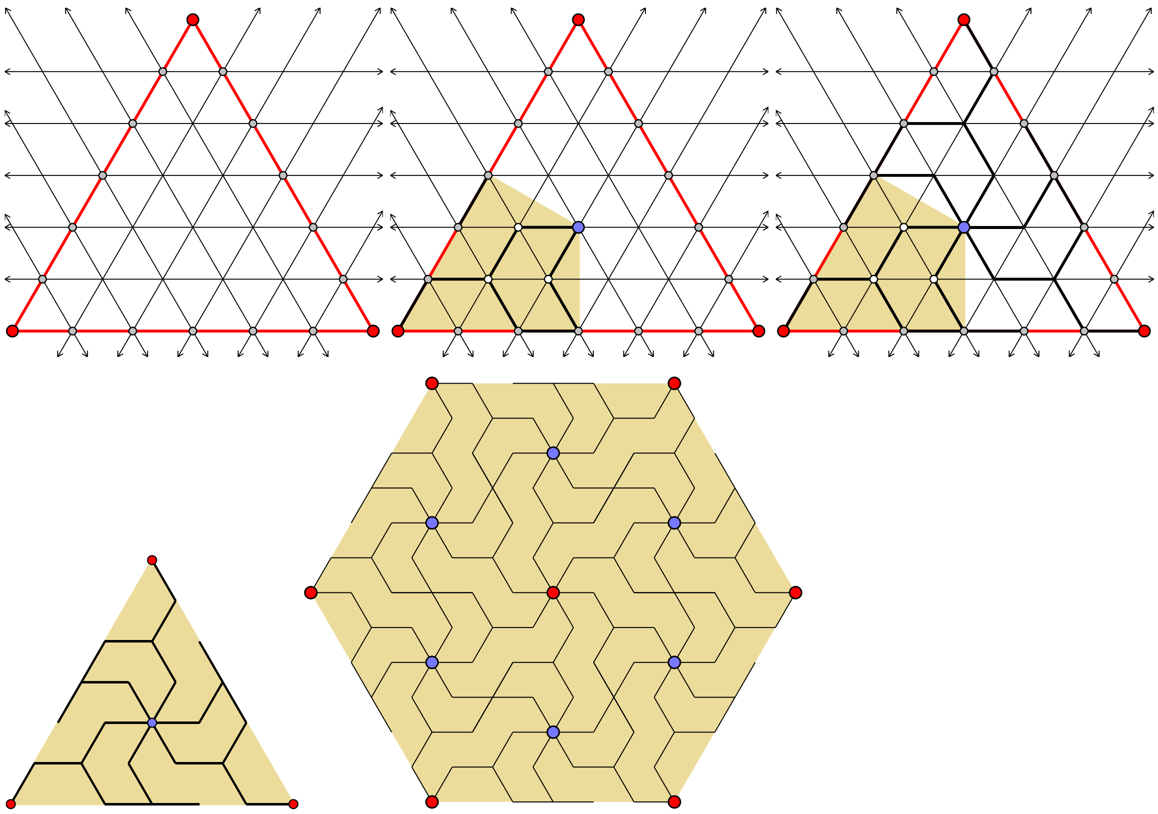
Consequently, we do not have mirrors. We have rotations only. Thus we have a pattern with 632 wallpaper signature.

One can easily notice that the pattern follows the parallel grid network with 6 lines parallel to each edge of the triangle.

This pattern has a few interesting properties that we will discuss later.

[4] https://en.wikipedia.org/wiki/Paigah_Tombs

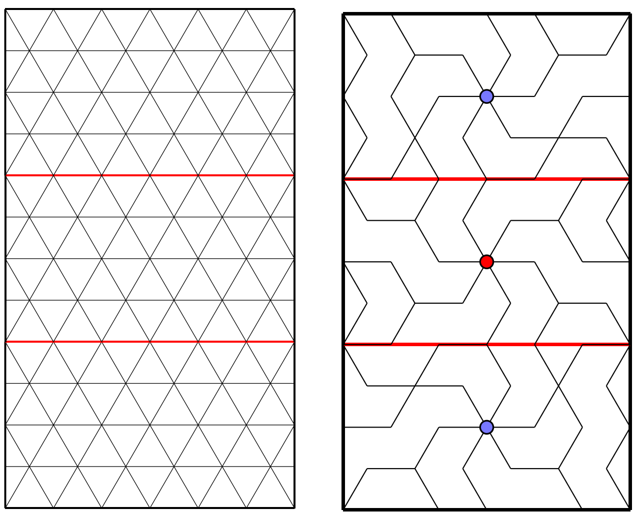
Construction of the pattern



By dividing each edge of the triangle into 6 equal parts, we produce a dense triangular grid. Then we design pattern in the shaded kite only. Finally, we rotated the pattern from the shaded area twice around the triangle center 120 degrees. This way, we get the large triangle of the pattern.

A large hexagonal tile is the last step in reconstructing the pattern.

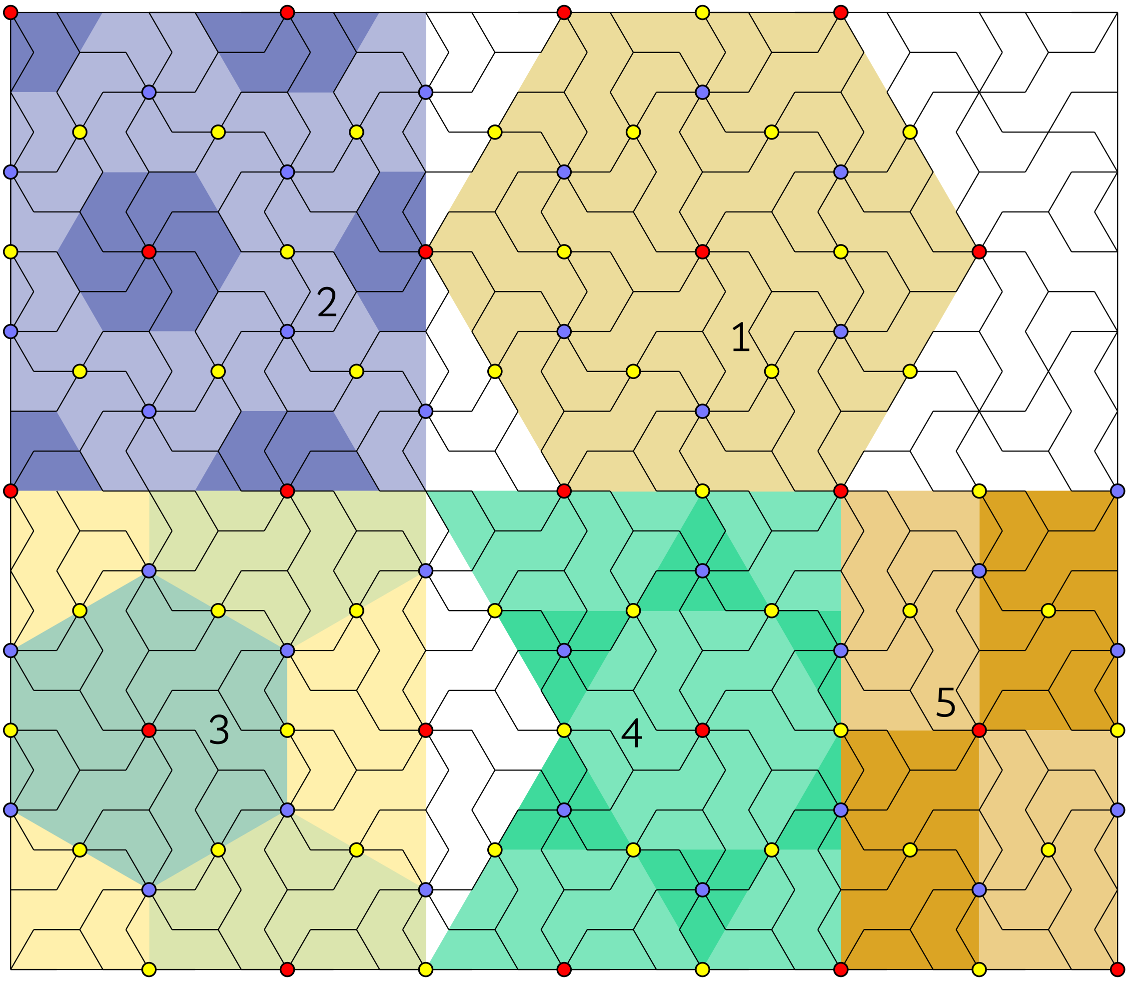
The pattern presented here is one of the most interesting patterns seen in Paigah tombs. It can be created in many other ways.



Construction of the pattern from Example 4 using a rectangular template

This is one of the most popular methods for creating complex patterns. We create a rectangle with a grid shown in the left picture. Then we follow the grid lines to get the pattern (right drawing). This way, we produce a rectangular template.

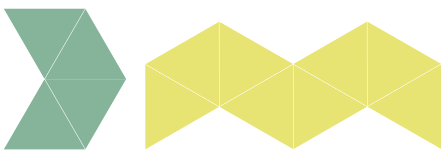
Using this template, we can create any larger design. Note – we should translate this template along its edges. No reflections should be used.



Some other ways of creating pattern from example 4

The drawing illustrates a few other ways of creating this pattern.

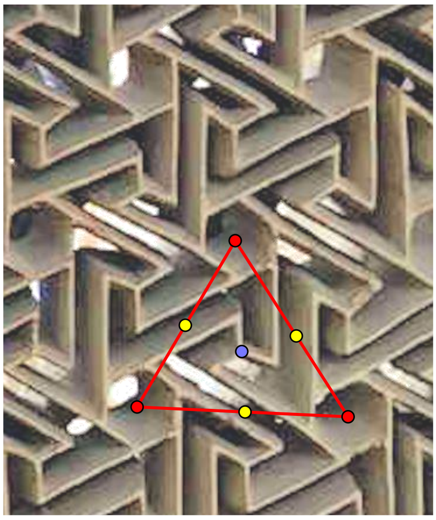
1. This is the method that we used in our solution. We created the L-triangle and then L-hexagon.
2. We use two hexagons with different rotations of the patterns around their centers. Note how different is the role of each type of hexagon. Hexagons with centers ●6 are surrounded by hexagons with centers ●3.
3. In this method, we create an s-triangle and then s-hexagon. Each of the hexagons shown here is the same. We use different colors to separate them.
4. In method 4, we use two different shapes a hexagon with center ●6 and vertices ●2 and an equilateral triangle with center ●3 and vertices ●2.
5. In this method, we use two rectangles of the same size but with different orientations of patterns inside them. Both rectangles should be placed on the plane, forming a chessboard-like structure.
6. The whole pattern is a mosaic built out of two shapes, and each is assembled from equilateral triangles. How many other patterns can be created using these two shapes?



Patterns created on triangles and hexagons with perpendicular grids

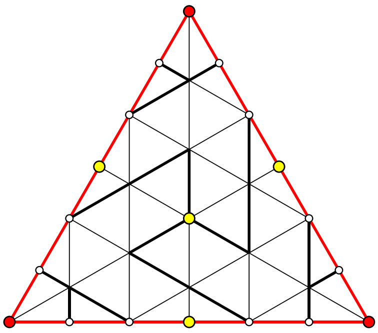
In the same way, we can use grids with lines perpendicular to the edges of the triangle. The following drawings show a pattern created using a perpendicular grid.

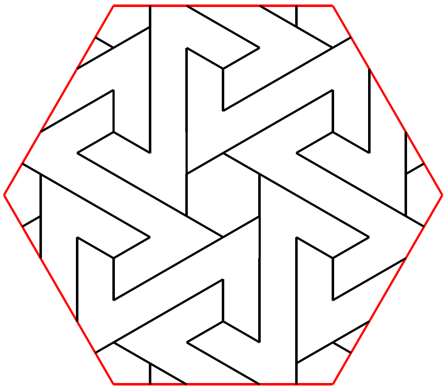
Example 5. Another pattern from Paigah Tombs



In this photo, we have another hexagonal pattern from Paigah Tombs. Although the pattern looks very complicated, its construction is very simple.

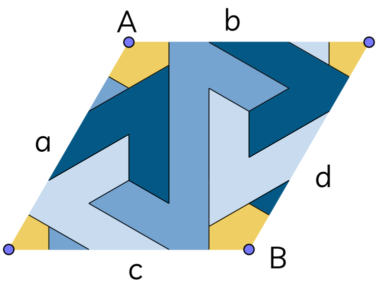
The L-triangle shown in the photo demonstrates how one can reconstruct the pattern. We use a triangular grid with lines perpendicular to the edges of the triangle.

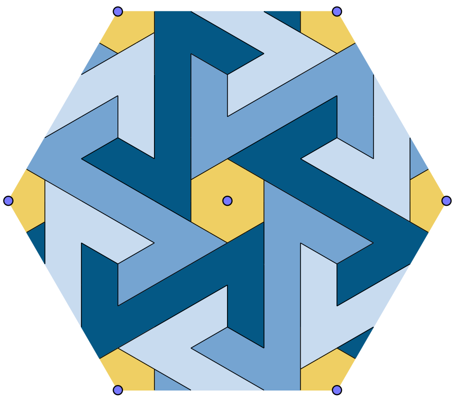




This pattern has signature 632, i.e., we have three gyrations: •6, •3, •2.

We can easily turn the pattern into a pattern with signature 333 by using two different colors for Z-like shapes.





Left: the rhombus created from the original T-triangle and its rotated 180 degrees copy. The coloring was applied in such a way that colors along the edge *a* match colors along edge *b*.

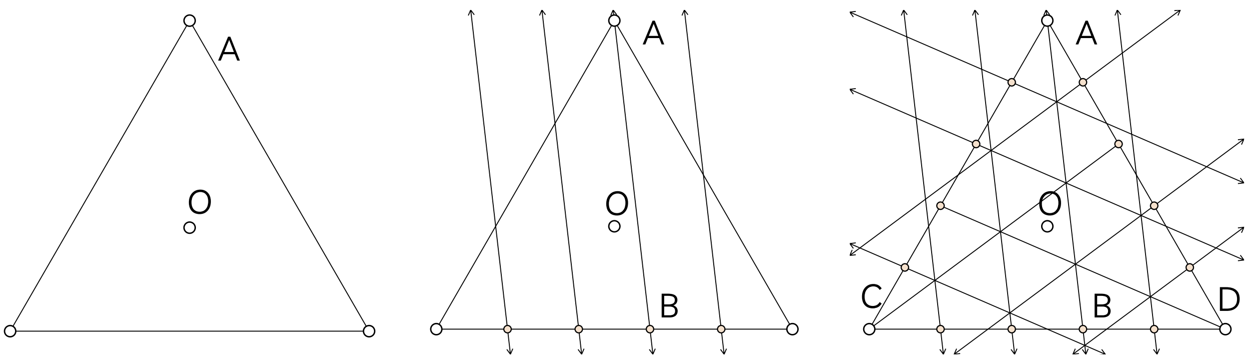
Right: The L-hexagon results from the rotation of the rhombus around point A three times 120 degrees.

One can easily notice that if an s-triangle of a pattern was created using a parallel grid, then its L-triangle can be created with a perpendicular grid. If an s-triangle of a pattern was created using a perpendicular grid, then its L-triangle can be created with a parallel grid. Thus these two methods are equivalent. One can choose an approach that is more convenient for him.

Patterns created on triangles and hexagons with twisted grids

Persian and Anatolian Seljuk craftsmen used the concept of twisted grids for centuries. In Seljuk architecture in Anatolia, we can find very complex examples following this method. In Paigah tombs, there are also designs using twisted grids.

The twisted grids method



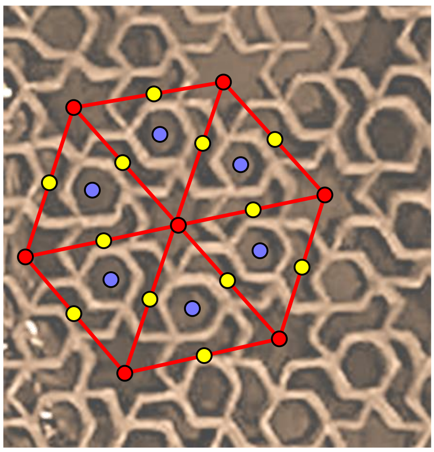
Description

1. Start with an equilateral triangle with its center shown.
2. Divide one of the triangle edges into any number of equal segments. Here we divided the bottom edge into five equal parts. Draw a line connecting the opposite vertex with one of the division points. Here it is, line AB. Draw lines parallel to it, passing through the remaining division points.
3. Rotate two times the set of parallel lines about the center of the triangle at 120 degrees. This way, we get the grid shown in the right drawing. We call it a twisted grid.
4. Use this grid to create any pattern. Note – the edge AC of the pattern should match the edge AD.

COMMENTS: To create a twisted grid, we must divide the edge of the triangle into at least three equal parts. Point B should not be a midpoint of the edge.

With twisted grids, we can construct patterns with signature 632. A colored version of such a design can often be turned into a pattern with signature 333. With a more complex grid, we can easily design patterns with signature 333 without using colored tiles.

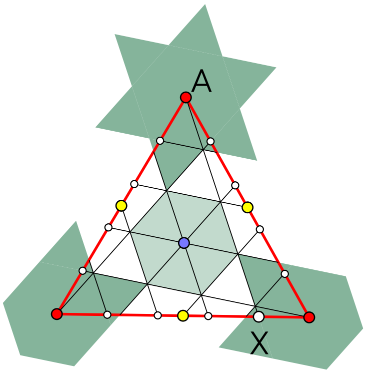
Example 6. Yet another pattern from Paigah Tombs

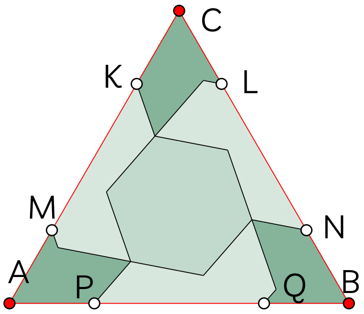


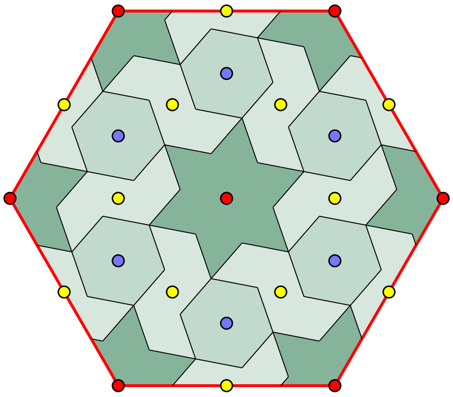
The pattern presented here uses a twisted grid. For creating an L-triangle, we will divide the edge of a triangle into five equal parts. We could also create the s-triangle, but such construction might be less intuitive.

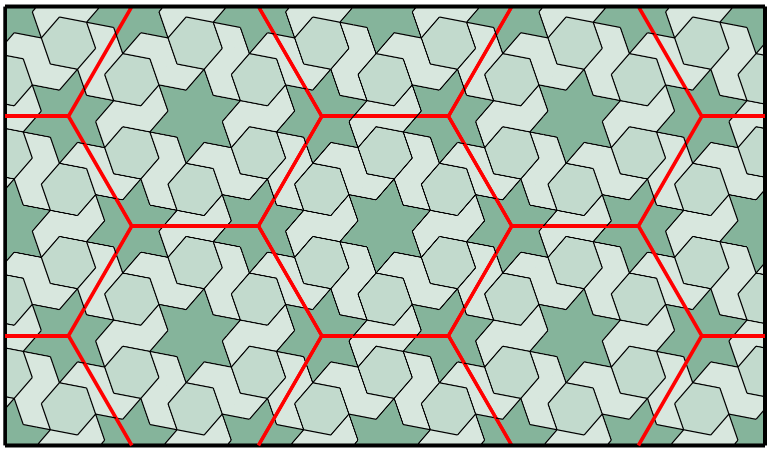
In the next drawings, we show a brief construction of this pattern.

Construction of the pattern from Example 6







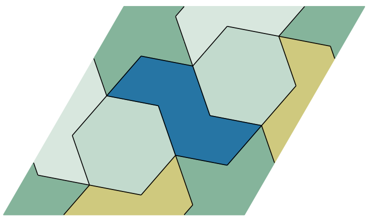


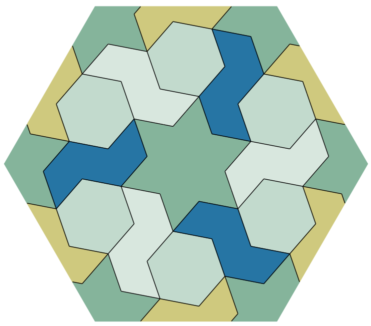
Description:

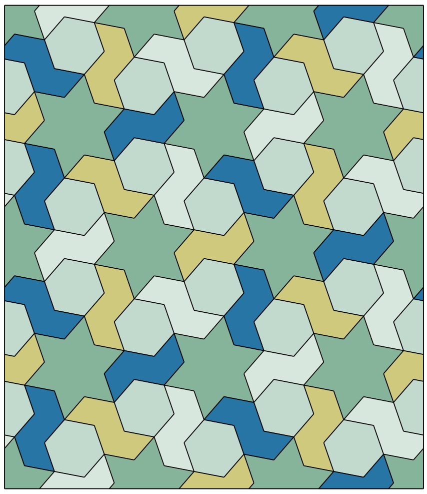
Divide the bottom edge of the triangle into 5 equal parts and draw a line through points A and X. Create a grid of lines parallel to AX, and rotate it two times 120 degrees about the center of the triangle.

Note how the pattern was created. Points K and L must be at the same distance from the vertex C and points M and N should be at the same distance from vertices A and B, respectively.

Points P and Q should be at the same distance from A and B, respectively. This is the only way we can rotate the L-triangle around point A, and lines will flow without breaks from the L-triangle to its copy.







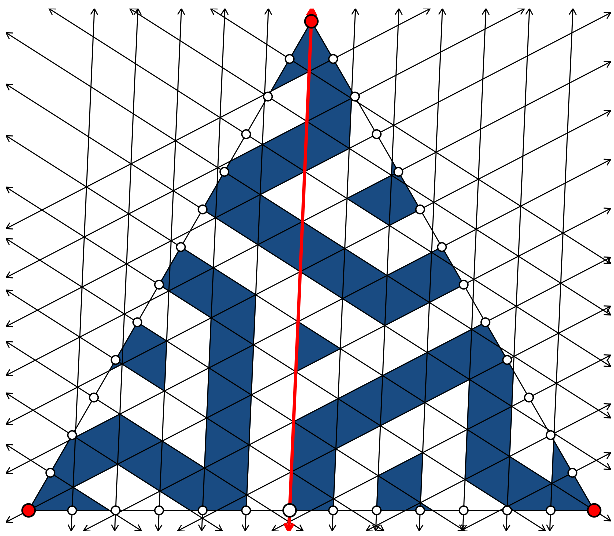
The drawings above show how we can modify the pattern from this example to obtain a pattern with signature 333.

Example 7. Pattern from Isfahan

With twisted grids, we can produce very complex geometric patterns. Here is one of them.

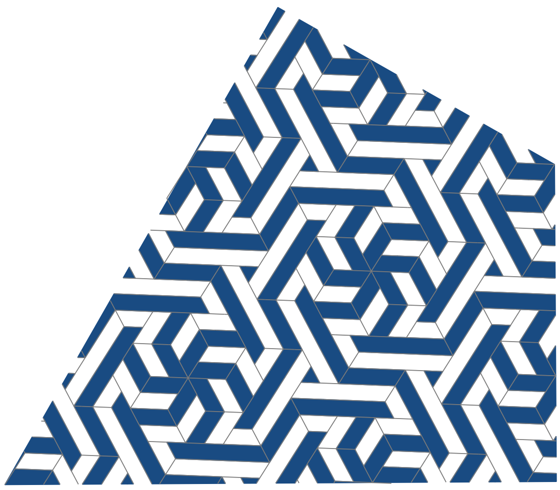
The photo shows a fragment of a pattern from Isfahan. The artwork is very rough, and it has many errors. Thus reconstructing this pattern is a challenging task.

In the photo, the L-triangle was displayed. Its vertices are gyrations ●6. The center of the triangle contains gyration ●3. Finally, gyrations ●2 are located in the middle of each edge of the triangle.



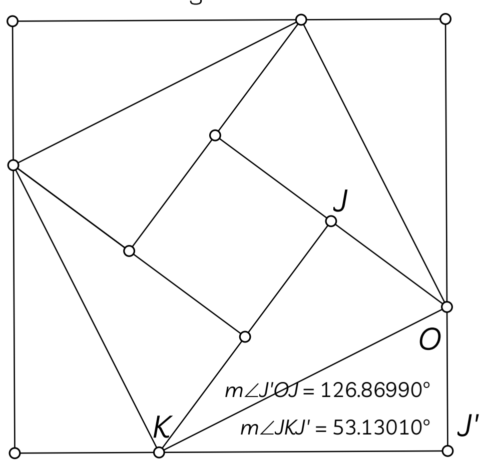
This drawing shows the twisted grid that was used to create the L-triangle. The bottom edge of the triangle was divided into 13 equal parts. The thick red line shows how the grid was started. The rest of this construction is easy to follow from the drawing.

Right: final design of the L-triangle.



A kite created using the L-triangle from our construction.

IMPORTANT: the kite presented here differs from the one in the photograph. Kite in our construction has angles of 60, 90, 90, and 120 degrees. Kite in the photograph has angles of 53.13010 deg, 90, 90, and 126.8699 degrees.



Summary

Various types of geometric grids were used for centuries for creating geometric art. We see them in Japanese Kumiko artworks, in European art, in the Middle East, Iran, and Turkey. We can also find them in tribal arts of Africa and South Pacific islands. In this paper, we discussed only a few of them and methods for designing them – parallel and perpendicular triangular grids, and twisted triangular grids. We did not discuss multicentered grids since there are very few geometric patterns using them.

Each type of triangular grid is a powerful tool for reconstructing old patterns and designing many new ones.

References

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[3] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020)

[4] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry (part 1)*, eJMT, vol. 14, Nr 2, pages 87-106.

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[5] This paper is available only in a few scientific libraries. No electronic version exists.