Mathematics teacher training from the perspective of STEM – a particular case

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Abstract

The ability to respond creatively and effectively to new challenges, whether it is acquiring new knowledge, solving problems, or the teaching process, is a key factor in determining the success of today's teachers. An important task of teacher training schools is therefore to create a suitable environment for the provision of education, as well as impulses for the development of the necessary knowledge and skills. This paper presents a specific project and corresponding activities for students implemented within mathematics teaching courses at the University of South Bohemia. Historical sources, both classical and local, from the area of present-day Czechia, are used. Emphasis is placed on the use of computers, especially dynamic geometry software, to model problems and their effective solution.

1 Introduction

The content and methods of teacher training should reflect the complexity of knowledge and competences that current practice demands from teachers. The basic categories of teacher knowledge, subject matter content knowledge, pedagogical content knowledge and curricular knowledge, were identified by Shulman in his well-known study [18]. In addition, at the present time, with the increasing opportunities to learn about real-world phenomena in which the boundaries between educational disciplines overlap, the importance of skills that allow a teacher to interpret such phenomena to students through a STEM (Science, Technology, Engineering, and Mathematics) lens, of which creativity and flexibility are key, is becoming more and more important [15]. Recent research has also proved the importance of learning experience for future teachers [11]. In this paper, we present a specific case of a topic that provides opportunities to apply the above principles in mathematics teacher training.

The central topic of the article is the solution of the angle trisection problem. It is used as a supporting medium for illustrating the possibility of creative education of mathematics teachers. Historical materials devoted to this problem, both classical and local, from the area of present-day Czechia, are used as an impetus for the creative and research work of prospective mathematics teachers. Although the problem of trisection is currently perceived as a historical problem that is not the focus of attention in contemporary geometry, the paper brings a new discovery in this field, obtained and proved using tools offered by the GeoGebra software environment [5].

As is widely known, the trisection of an angle, i.e. division of an arbitrary angle into three equal parts, together with squaring a circle and doubling a cube, belongs to the classical problems of Greek mathematics, that were proved to be impossible constructions using just a straightedge and compass. The impossibility of this so-called Euclidean construction of a trisection of an angle was proved by French mathematician Pierre Laurent Wantzel in 1873. For more information see [10, 14].

2 The forgotten trisection method

In 1881, J. R. Vaňaus, a Czech grammar school teacher of mathematics and physics, presented in an article entitled *Trisektorie* (Czech term for trisectrix), which was published in the Czech *Journal for the Cultivation of Mathematics and Physics* [19], a previously unknown method of using an oblique strophoid [16] for the trisection of an angle. This article offers interesting possibilities to modern researchers.

2.1 Vaňaus' trisectrix

From the text of the article [19] it is apparent that Vaňaus probably did not know that the curve in question is a strophoid. He arrived at it when investigating the properties of algebraic curves given by the equation

$$Ax^3 + By^3 + Cxy^2 + Dx^2y + Ex^2 + Fy^2 + Gxy = 0, \tag{1}$$

which have a double point in the origin of the related Cartesian coordinate system. For a certain class of these curves he identified that they are sets of points equidistant from a circle and some of its secants. As we know today, this feature indicates a strophoid [12, 17]. Since he discovered that such curves were suitable for trisecting an angle due to this property, Vaňaus decided to call them *trisektorie*.

Besides determining the conditions for the values of the coefficients of equation (1), he defines it as a locus of points as follows (see Fig. 1): *Given a circle c of arbitrary radius r with diameter OA, where O is the origin of the Cartesian coordinate system and A lies on the $x-$axis, draw line l, the secant line of circle c, through A at any direction angle α and place point B on it. Then draw line OB and mark its intersection with circle c as point D. Finally, place M on line OB so that $|MD| = |DB|$. The locus of points M for B moving along line l is the curve in question.* Based on this definition, Vaňaus derives the Cartesian equation of the resulting locus curve, which is

$$a(y^2(2r + x) - x^2(2r - x)) = y(y^2 + x^2 - 4rx), \tag{2}$$

where r is the radius of the circle c and a is the slope of the line l, i.e. $a = \tan\alpha$. Because for him it was a hitherto unknown curve, which he discovered through his research, the author

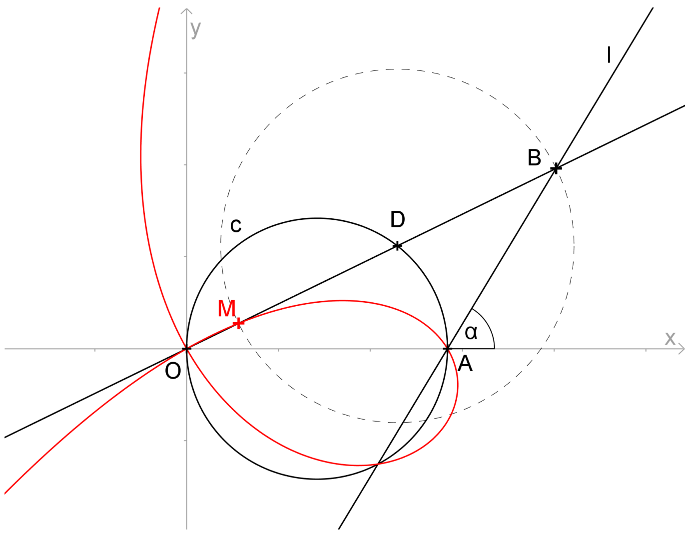

Figure 1: Vaňaus' trisectrix as the locus curve

was looking for analogies with already known curves. For the similarity of the definition as the locus of points he mentions the *cissoid of Diocles* [2], and for the similarity of the shapes, the *folium of Descartes* [4].

In the remainder of his article, Vaňaus presents the use of the introduced curve, which we will hereinafter call an *oblique strophoid*, for trisecting an angle. He proved both his knowledge of the theoretical impossibility of solving this problem with a straightedge and compass alone, as well as his wide overview of other non-Euclidean ways of doing so, adding the confident statement that the method based on the curve introduced by him is the least complex.

2.2 Vaňaus' trisection

Vaňaus' angle trisection method is described in detail in [6], as is proof of its correctness. In addition, the reader is confronted with an interesting problem that needs to be solved through the trisection of an angle. Vaňaus assigned this problem to the readers, grammar school students, of the *Journal for the Cultivation of Mathematics and Physics* in the problem section [20]. The methods of solving this problem are discussed both from the point of view of a student at that time and the present day in [6]. We will therefore present this method only briefly here for the purposes of the further direction of this paper.

In Fig. 2, the same objects are shown as in Fig. 1, but with point B in a different position on line l, it lies on the chord AF, and with the ray OA' and the circle m, with centre O passing through A, added.

Vaňaus proved that for the position of point B on line l, at which point M coincides with the intersection of the strophoid and the circle m, the angle $A'OB$ is a third of the angle $A'OA$ [19, 6]. The proof is not complicated at all. The application of the basic properties of angles in a triangle and a circle is enough, see Fig. 3. A detailed description is given in [6].

The method used by Vaňaus for trisecting angle u is clear, see Fig. 3. The rays OA', OA and OM have a common point O, the vertex of the angle u, while points A', A and M lie on the circle m. The angle $A'OM$ is a third of the angle $A'OA$ if, and only if, point B lies on the perpendicular line from A to OA', D being the midpoint of the segment MB.

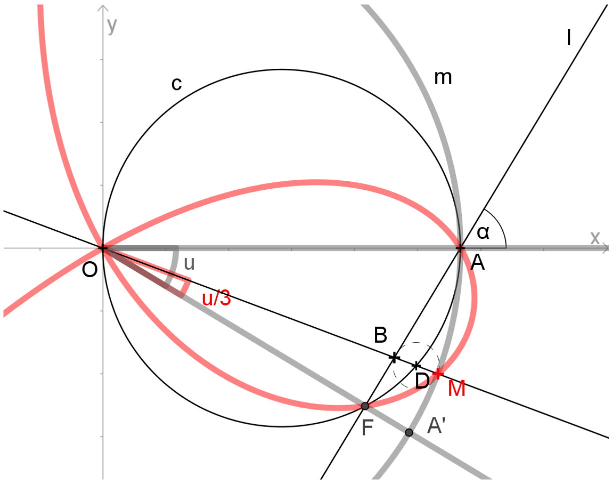

Figure 2: Configuration for trisecting an angle

Although the theoretical basis of the method is therefore well proven and quite understandable, how do we perform this trisection in practice? In this regard, Vaňaus left his successors with a mysterious task, the correct solution of which we will probably never know. At the end of his article, he states: *"In order to correctly draw the part of the trisectrix AMF and, above all, to place the intersection point M precisely, I assembled a very simple device, where the intersection of arc AA' at point M is created by means of a double forced movement."* [19].

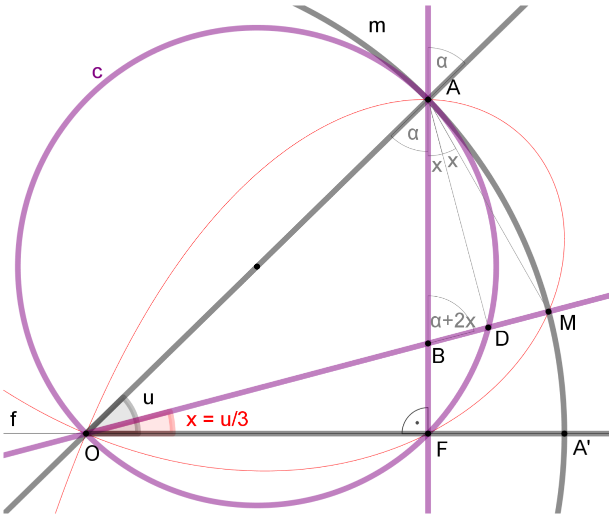

Figure 3: Vaňaus' method of trisection

Unfortunately, the device that Vaňaus mentioned has not survived, as have any written documents or sketches for its construction. One of Vaňaus' legacies, in addition to his written work, is therefore a challenge to us and our students to replace this lost artifact with our own solution. In the following passages of this paper, two intriguing and original results are presented. These arose from attempts to design and build a mechanism that could possibly correspond to the device that Vaňaus mentioned.

The first, is the design and production of a possible Vaňaus trisector by a student teacher of mathematics and technical education. The process ended with his successful participation in the national *Olympiad of Technology 2023* (`http://olympiadatechniky.cz`).

The second, is the discovery of an unsuspected connection between Vaňaus' method and *Ceva's trisectrix*, a curve described by Tomas Ceva in 1699 [1], which led to the discovery of a new and very simple application of this curve to angle trisection.

3 Possible Vaňaus trisector

The conditions defined by the essence of Vaňaus' method of trisection using an oblique strophoid, which we noted in the comment on Fig. 3 above, must be met by the device that we want to design for the purpose of performing the trisection. We can therefore try to construct a mechanism to comply with these conditions. With this task, student teachers of mathematics, prospective teachers at lower secondary school, were approached. One of them, Tomáš Randa, studying teaching of technical education as a second major, took this task to the stage we see in Fig. 4.

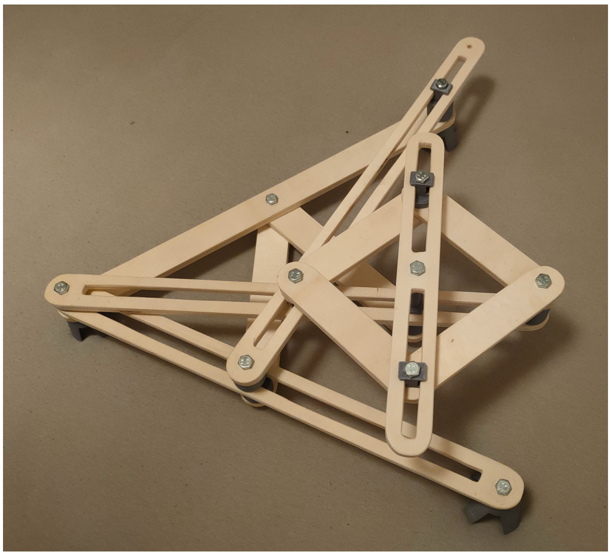

Figure 4: Vaňaus trisector – student work, T. Randa, 2023

It is a mechanism made by hand from poplar wood, with joints printed on a 3D printer. As stated by Vaňaus, the mechanism performs an angle trisection through a double forced movement. The original plan was to print the entire mechanism on a 3D printer, which would have been much easier. However, the student decided to enter the *Olympiad of Technology*, one of the conditions of which is that a significant part of the submitted object must be made by hand. His efforts paid off in the end, with him taking third place in the category "Didactic technical works" among students from all over the Czech Republic. The work was awarded not only for its technical implementation, but also for its geometric essence and the potential to recall, in a new way, one of the classic problems of geometry and bring the history of efforts to solve it closer to students.

4 New use of Ceva trisectrix

An unexpected consequence of both the analysis of the above mentioned Vaňaus' method of angle trisection and the effort to design the simplest possible device for its implementation, was the discovery of a new and simpler use of the Ceva trisectrix for the trisection of an angle.

4.1 Ceva trisectrix

In 1699, *Tommaso Ceva*, the younger brother of *Giovanni Ceva*, published the Latin book *Opuscula mathematica*, in which he presents, among other things, the curve he calls *cycloidum anomalarum* [1]. Among the properties of this curve, he mentions its suitability for angle trisection.

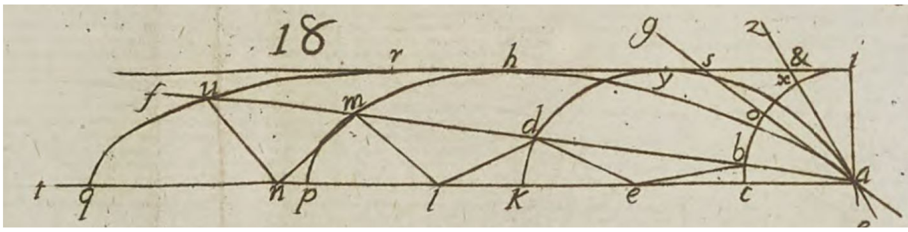

Figure 5: Cycloidas anomalas, T. Ceva, 1699 [1]

Ceva defines his cycloid as a curve that is described by the endpoint of a polyline with an odd number of segments of equal lengths, whose vertices lie alternately on two rays with a common point of origin, which is also the initial vertex of the polyline. In Fig. 5, which is Figure No. 18 from Ceva's book, we can see for example a curve drawn by point d, the endpoint of polyline *abed* linking rays af and at, when af rotates around a. The other two curves in the figure are those drawn in the same way by points m and u, endpoints of polylines *abedlm* and *abedlmnu*, respectively.

An elementary knowledge of the relations between interior and exterior angles of a triangle is sufficient to prove that if $|\measuredangle cab| = \varphi$, then $|\measuredangle led| = 3\varphi$, $|\measuredangle nlm| = 5\varphi$ and $|\measuredangle qnu| = 7\varphi$, see also Fig. 6, where the relationships between angles essential to the proof are indicated. In what follows, we are only interested in the trisection of an angle, so we will focus mainly on the first of the curves, drawn by point d. This curve is referred to in available sources as the *Cycloid of Ceva* [21] or *Ceva trisectrix* [3]. It is a sextic, the algebraic curve of order 6, with the equation

$$(x^2 + y^2)^3 - r^2(3x^2 - y^2)^2 = 0, \tag{3}$$

where r is the length of a segment of the polyline forming it, i.e. the radius of the circle (hereinafter k) along which point b moves in Fig. 5. Under the same definition of r, the polar equation of Ceva trisectrix is

$$\rho(\varphi) = r + 2r\cos 2\varphi. \tag{4}$$

For the sake of completeness, the algebraic equations of the other two cycloids, the parts of which Ceva sketched in his picture are as follows

$$(x^2 + y^2)^5 - r^2(5(x^2 - y^2)^2 - 4y^4)^2 = 0,$$
$$(x^2 + y^2)^7 - r^2(7(x^2 - y^2)^3 - 2y^2(7x^4 - 3y^4))^2 = 0.$$

As is evident, only parts of the curves in question are presented in Fig. 5. The entire Ceva trisectrix is shown in Fig. 6. To verify that the curve is really drawn by a fixed point (point D) on a rolling circle (the green circle), the reader is invited to play the animation in the applet *Ceva trisectrix* [7].

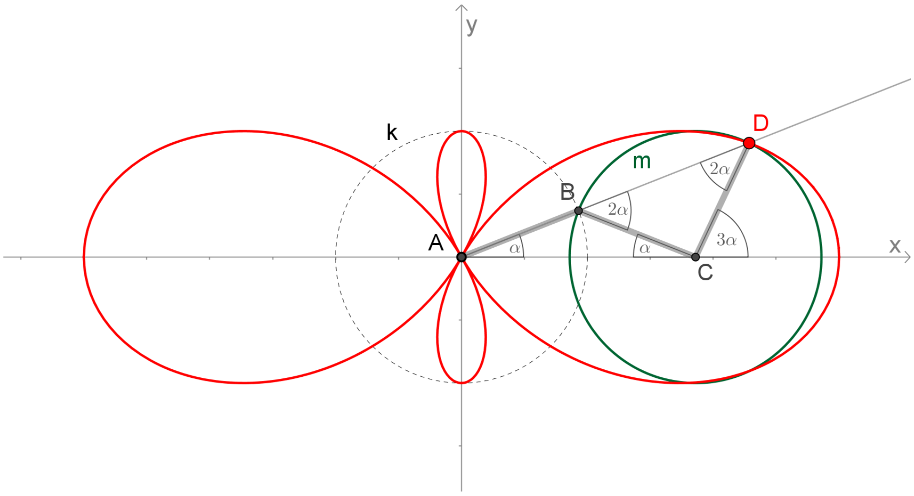

Figure 6: Ceva trisectrix

4.2 New way of use

As we already know, Ceva's curve is the closed curve of the sixth order, with two mutually perpendicular axes of symmetry, having four loops, of which the opposite are always congruent, one pair larger, one pair smaller, pairs not similar to each other, see Fig. 6.

The traditional trisection method based on this curve works with the larger of the loops, as shown in Fig. 6. In contrast, a new method, which we aim to introduce in the following text, works with the smaller of the loops, as shown in Fig. 7.

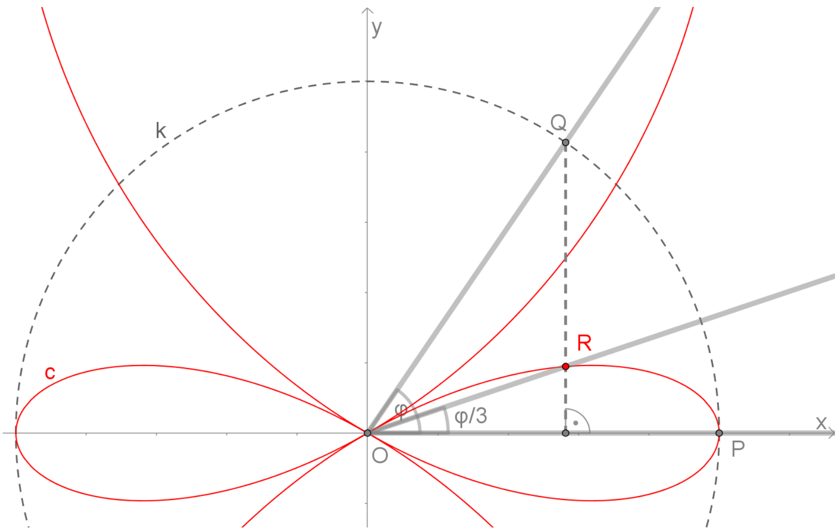

Figure 7: New use of Ceva's curve to trisect an angle

The step-by-step procedure for this new use of the Ceva curve for the trisection of angle φ is as follows:

1. Place the curve in the Cartesian coordinate system, as shown in Fig. 7. Use the polar equation $\rho(\varphi) = r - 2r\cos 2\varphi$.

2. Draw a circle k, the defining circle of the Ceva curve, the radius of r which is equal to the length of the shorter loop.

3. Construct the angle φ, whose third you are interested in, into the first quadrant, so that its first arm merges with the positive semi-axis x, and its second arm intersects k at Q, within this quadrant. From Q, draw a perpendicular line to x, and mark its intersection with Ceva curve c as R. Then the ray OR is the arm of the angle $\measuredangle POR$, the third of $\measuredangle POQ$.

4.3 Genesis of the method

This new method of angle trisection appeared as an unexpected result of the analysis of the trisection method developed by Vaňaus. See section 2.2, or for a more detailed introduction [6]. The main impetus for this was the geometric simplification of the Vaňaus method, which would allow the design of a sufficiently simple device for its implementation.

The basic source of the findings for the analysis was a dynamic geometric model of the Vaňaus method created in GeoGebra [5]. This model is shown in Fig. 8. It is available in interactive form in the applet *Vaňaus' trisection – trace of B* [9]. If we change angle φ by moving

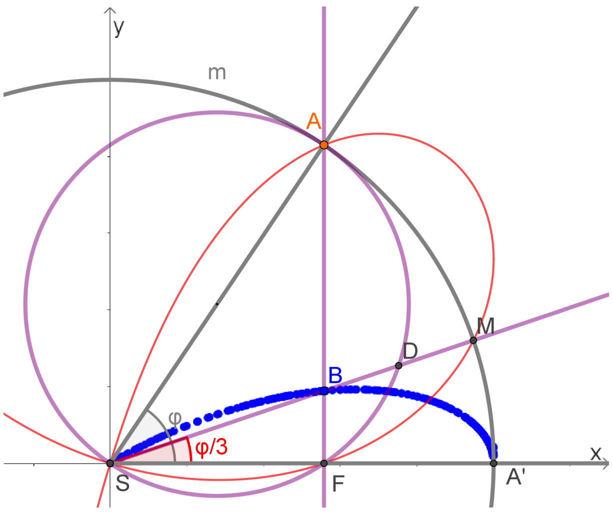

Figure 8: The locus of B when A moves along m

point A, the ray SM, where M is the intersection of the corresponding oblique strophoid (it is different for each angle φ) with circle m, is the arm of the angle of size $\varphi/3$. The question arises, what is the trajectory of point B when A moves along m? The answer to this question could bring about the sought-after simplification. To initially examine the curve of this trajectory, we use the *Trace on* setting for B in GeoGebra. It draws the blue trace, as can be seen in Fig. 8, or in the corresponding applet [9].

The structure shown in Fig. 8 defines the geometric relationship between A and B such that, for a particular position of A, the position of B can be described by a system of algebraic equations. Using the *Eliminate* function in GeoGebra CAS, we can then derive the general equation of the locus curve of B. To derive the desired system of equations, we first assign coordinates to the decisive points in Fig. 8 as follows: $A[a_1, a_2]$, $B[x, y]$, $M[m_1, m_2]$. Then, as a preparatory step, we consider the family of Vaňaus' strophoids with parameters a_1, a_2, the coordinates of A, and derive the equations

$$s_1 : 2\,R\,r - a_1^2 - a_2^2 = 0, \tag{5}$$

$$\begin{aligned} s_2 :\ & -2\,R\,a\,a_1^2\,m_1^2\,r + 2\,R\,a\,a_1^2\,r\,m_2^2 - 8\,R\,a\,a_1\,m_1\,r\,m_2\,a_2 + 2\,R\,a\,m_1^2\,r\,a_2^2 \\ & - 2\,R\,a\,r\,m_2^2\,a_2^2 + 4\,R\,a_1^2\,m_1\,r\,m_2 - 4\,R\,a_1\,m_1^2\,r\,a_2 + 4\,R\,a_1\,r\,m_2^2\,a_2 \\ & - 4\,R\,m_1\,r\,m_2\,a_2^2 + a\,a_1^3\,m_1^3 + a\,a_1^3\,m_1\,m_2^2 + a\,a_1^2\,m_1^2\,m_2\,a_2 + a\,a_1^2\,m_2^3\,a_2 \\ & + a\,a_1\,m_1^3\,a_2^2 + a\,a_1\,m_1\,m_2^2\,a_2^2 + a\,m_1^2\,m_2\,a_2^3 + a\,m_2^3\,a_2^3 - a_1^3\,m_1^2\,m_2 - a_1^3\,m_2^3 \\ & + a_1^2\,m_1^3\,a_2 + a_1^2\,m_1\,m_2^2\,a_2 - a_1\,m_1^2\,m_2\,a_2^2 - a_1\,m_2^3\,a_2^2 + m_1^3\,a_2^3 + m_1\,m_2^2\,a_2^3 = 0. \end{aligned} \tag{6}$$

defining A and M as the points of intersection of this family of curves with circle m, see lines 2 and 3, respectively, in the GeoGebra CAS code in Fig. 9. Subsequently, we apply the defining

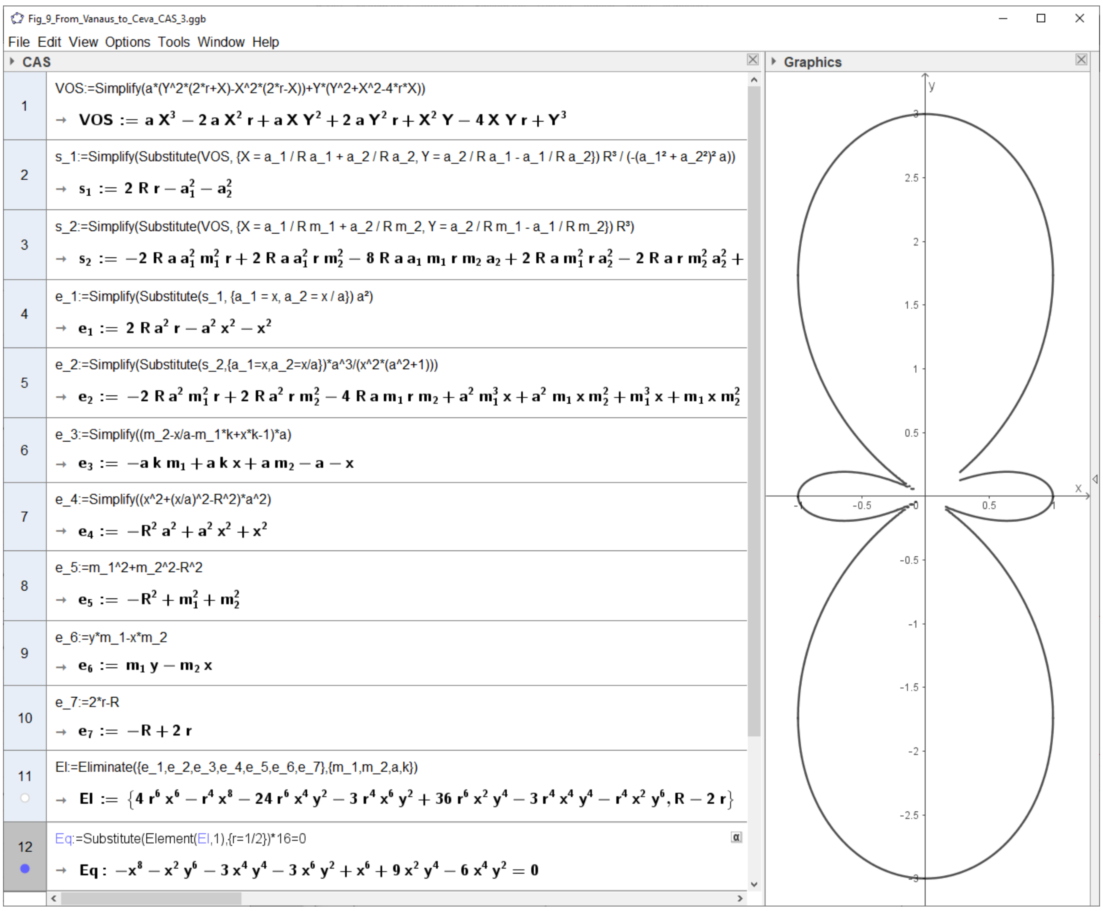

Figure 9: The locus curve of B derived in GeoGebra CAS, plotted for $r = 1/2$

relation $a = \tan\alpha = \dfrac{a_1}{a_2}$ for parameter a, which was introduced in (2), together with the fact that A and B lie on the perpendicular to x, and write $a_1 = x$, $a_2 = x/a$, i.e. $A[x, x/a]$. The equations (5),(6) are thereby simplified to the first two equations of the required system

$$e_1 : \ 2\,R\,a^2\,r - a^2\,x^2 - x^2 = 0, \tag{7}$$

$$\begin{aligned} e_2 : \ -2\,R\,a^2\,m_1^2\,r + 2\,R\,a^2\,r\,m_2^2 - 4\,R\,a\,m_1\,r\,m_2 \\ + a^2\,m_1^3\,x + a^2\,m_1\,x\,m_2^2 + m_1^3\,x + m_1\,x\,m_2^2 = 0. \end{aligned} \tag{8}$$

To avoid the situation where $M[m_1, m_2] \equiv A[a_1, a_2]$, we set the non-degenerate condition $(m_2 - a_2) - k(m_1 - a_1) - 1 = 0$, where k is an additional parameter, which leads to the third equation

$$e_3 : \ -a\,k\,m_1 + a\,k\,x + a\,m_2 - a - x = 0. \tag{9}$$

Then, we add two equations expressing that A and M lie on the same circle m with radius R

$$e_4 : \ -R^2\,a^2 + a^2\,x^2 + x^2 = 0, \tag{10}$$

$$e_5 : \ -R^2 + m_1^2 + m_2^2 = 0. \tag{11}$$

The following equation determines that B is the intersection of SM with the line passing through A perpendicular to the x-axis

$$e_6 : \ m_1\,y - m_2\,x = 0. \tag{12}$$

Finally, the last equation

$$e_7 : \ -R + 2\,r = 0 \tag{13}$$

expresses the relation between the radii r and R.

Then, by eliminating parameters m_1, m_2, a, k from the system of equations $e_1, \ldots, e_7$, utilising the *Eliminate* function, see line 11 in the GeoGebra CAS code in Fig. 9 (for the complete code from Fig. 9 visit [8]), we obtain

$$4\,r^6\,x^6 - r^4\,x^8 - 24\,r^6\,x^4\,y^2 - 3\,r^4\,x^6\,y^2 + 36\,r^6\,x^2\,y^4 - 3\,r^4\,x^4\,y^4 - r^4\,x^2\,y^6 = 0, \tag{14}$$

after factorisation

$$-\,r^4\,x^2\,\left(x^6 - 4\,x^4\,r^2 + 3\,x^4\,y^2 + 24\,x^2\,r^2\,y^2 + 3\,x^2\,y^4 - 36\,r^2\,y^4 + y^6\right) = 0, \tag{15}$$

where the third factor gives us the final equation of the locus curve of B

$$(x^2 + y^2)^3 - (2r)^2(x^2 - 3y^2) = 0, \tag{16}$$

which is the equation for Ceva's trisectrix, rotated by ninety degrees compared to (3), except that the value of radius r is now half the value considered in (3).

We can thus take it as proven that the sought-after locus curve of B is a specific part of Ceva's trisectrix with equation (16). On this basis, it is possible to propose a new trisection method, as is clearly presented in Fig. 10. To use it, it is sufficient to make, for example on a 3D printer, the relevant part of the Ceva curve. This artifact will then, together with a ruler and compass, form a complete set of tools for trisection.

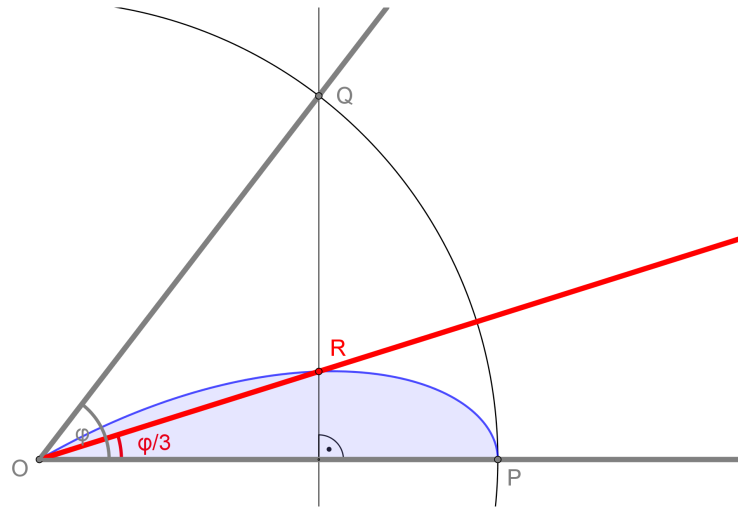

Figure 10: New method of trisection based on Ceva's trisectrix

5 Conclusions

As already stated, the skills and abilities that determine the success of a teacher include, among other things, creativity, problem-solving skills, and the ability to acquire new knowledge. We believe that with this paper we have succeeded in showing that this can be achieved through the effective use of suitable computer software, in this particular case, the dynamic mathematics program GeoGebra.

Current scientific studies into STEM education as a way of preparing students for an ever-changing world largely concur that concepts such as creativity, the ability to learn and apply new knowledge, problem solving skills, and the carrying out of authentic inquiry, play a significant role in its implementation [13, 15]. Among the ambitions of the paper was to show how these skills can be practiced and developed on the basis of a historical topic, namely solving a specific student project in which the basic mathematical curriculum is applied to solve problems based on appropriately collected and presented historical materials. It was shown that within the framework of such a project, students not only practice and develop the aforementioned skills, in conjunction with their digital competencies, but also carry out the technical design and production of the relevant artifact.

The ambition of the author was also to inspire colleagues to assign the presented problems to their students. Isn't it a captivating idea to collate as many ideas as possible to solve the Vaňaus trisector mystery?

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