How to prepare a digital geometric model which is enclosed by an assembly of surfaces for 3D printing

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Abstract

In this article we address the process of manufacturing 3D geometric models on 3D printers for educational purposes. The manufacturing of the geometric model includes the design, 3D computer modeling with Constructive Solid Geometry (CSG), 3D computer modeling of parametric surfaces based on differential geometry, 3D scanning of real objects, and the process of fabrication itself. Students can join the whole process of designing the models for 3D printers or its parts and the printed models can be included in teaching and learning geometry at every stage of education (at the university and the secondary school in our case). We present the possibilities how to model geometric objects in 3D computer modeling software; including commercial ones and open-source software like the free dynamic geometry system, Geogebra. We present particular constructions of selected geometric models whose boundary consists of parametric surfaces. All examples of printed geometric models presented in this article are intended to be used in mathematics instructions at the secondary school and in the undergraduate courses on descriptive geometry at Charles University (mandatory courses for secondary pre-service mathematics teachers who study teaching mathematics and descriptive geometry). All phases of designing the models for 3D printing together with physical 3D printed models bring new opportunities for teaching and learning geometry. It engages students in real-world problem solving and promotes students' knowledge in geometry while introducing them into 3D computer modeling and 3D printing technologies. 3D virtual models and 3D printed models themselves can serve as the educational manipulative aids.

Keywords: 3D printing, 3D computer modeling, 3D scanning, 3D printed models, GeoGebra dynamic geometry system, descriptive geometry, teaching and learning geometry

1 Introduction

Computer graphics, 3D computer modeling, and geometry are the bases of many modern applications and disciplines in industry and production. Computer-aided architectural and industrial design, technical drawings of buildings or of mechanical parts, construction fields [13], computer graphics and computer geometry algorithms [5], planning [9], 3D scanning of real surfaces [2], 3D printing [10] are just some of the applications that require knowledge of mathematics and geometry. Many fields of geometry are of immediate relevance to the challenges facing our modern globalized society. Nowadays, mathematics and geometry permeate many fields, and if we do not want to be just the unacquainted users but we want to understand these technologies more deeply or even contribute to their development, we have to master mathematics and geometry.

Geometry and mathematics are undoubtedly demanding topics to learn at any level of the educational process. In promoting an understanding of mathematics and geometry concepts different visual representations such as virtual or concrete manipulatives can be used with a potentially great benefit in education. The international researches showed the effectiveness of concrete manipulatives in mathematics instruction [3], [16]. Also using the virtual manipulatives encourages students and improves their achievements [8]. The use of 3D printing technology in the educational process combines the interaction with the both worlds, i.e. virtual and physical one. Students can familiarize with 3D computer modeling, create virtual models by themselves and in addition to this virtual environments, students can also benefit from using the physical manipulatives [4].

Nowadays, the 3D printing technology becomes more and more accessible to the public and it is also possible to work with 3D printers in the classrooms at secondary schools and universities. There exist the programs for supporting the 3D printers for schools in the Czech Republic too.

3D printing technology enables pupils and students to introduce them into modern applications of mathematics and geometry in practice. As has been already mentioned, products from a 3D printer can even serve as aids for teaching geometry and help to better understand geometric concepts [17].

This article addresses the particular constructions of selected geometric models whose boundary consists of parametric surfaces. We broaden the design of the geometric models in 3D computer modeling software with the process of their fabrication on 3D printers. All examples of printed geometric models presented in this article are intended to be used for educational purposes. First, solid figures can be used in mathematics instructions at secondary schools. Second, more complex geometric models which are enclosed by an assembly of surfaces (not only parts of the plane but also parts of another types of parametric surfaces) are planned to be used in the undergraduate courses on descriptive geometry at Charles University (mandatory courses for secondary pre-service mathematics teachers who study teaching mathematics and descriptive geometry).

Printed geometric models are the physical representation of some geometry, i.e. solid figures or some types of parametric surfaces. It means that the models serve as the real visualization of some abstract mathematics (or geometric) concepts. Using the models one can easily imagine and understand the properties of such geometric concepts. In addition to the pure visualization; the creation of the models stands on the other geometric aspects such as knowledge of differential geometry. Moreover, geometric models can work well as scaled physical models of real objects for example of architectural design and show the usage of mathematical surfaces in practice.

The paper is organized as follows. First, the 3D printing technology is briefly described. Second, the various ways how to design a virtual geometric model are presented - based on 3D scanning of real objects, 3D computer modeling with constructive solid geometry and various transformations, or 3D computer modeling using parametric equations. Third, the suggestions of using presented geometric models in education are provided. The gallery of selected models is given lastly.

2 3D Printing Technology

Using 3D printing technology we can manufacture a physical object from a digital 3D model. There exist a variety of 3D printers and processes in which material is added successively together, usually layer by layer. In our research we work exclusively with 3D printers which print the object layer by layer and we use typically plastic material (PLA - Polylactic acid which is biologically degradable).

For the communication with the 3D printer we use control interfaces which are freely available (for example *PrusaSlicer*, *Repetier-Host* [14], [15],). The control interfaces are used for the so-called *slicing*, i.e. converting the STL format in which the virtual model is saved into the gcode format (used to describe the path of the tool in the 3D Cartesian coordinate system). The software also enables the direct control of the 3D printer, it is possible to influence the speed, quality of printing, temperature, changing of colors and materials, and it also allows the basic editing of 3D models for printing such as placing the models on the heatbed (the heated surface being printed on by the 3D printer).

The geometric model placed in the interface of Repetier-Host software and of PrusaSlicer software is shown in Figure 1.

When modeling 3D objects for printing, one important thing we have to keep in mind is that we want to design a "manufacturable object". In geometry, we work with an abstraction – a point has no dimension, a straight line is infinitely thin, a surface has no thickness. However, this is a problem from a practical point of view. It is therefore common to work with manifold objects in practical applications. Such an object has edges that are shared by exactly two faces. Simply said, we prepare digital models which can be made in real from some material.

Every 3D printer is capable of printing a footprint of a certain width. The accuracy of the given printer has to be taken into account. Printing time and material consumption depends on the size of the given model. The printed models presented in this article were created in sizes up to 10 cm (in the longest size). Printing lasted from about 5 to 15 hours for one geometric model.

We work with the 3D printers that print objects layer by layer which also represent some limitations. This means that the next layer lies on top of the previous layer. If we want to print a more complex object that has, for example, some holes or we want to span some space, additional supporting parts of the model need to be printed too. These parts are additionally removed after printing. The object can be also divided into parts and these parts and these parts are printed separately.

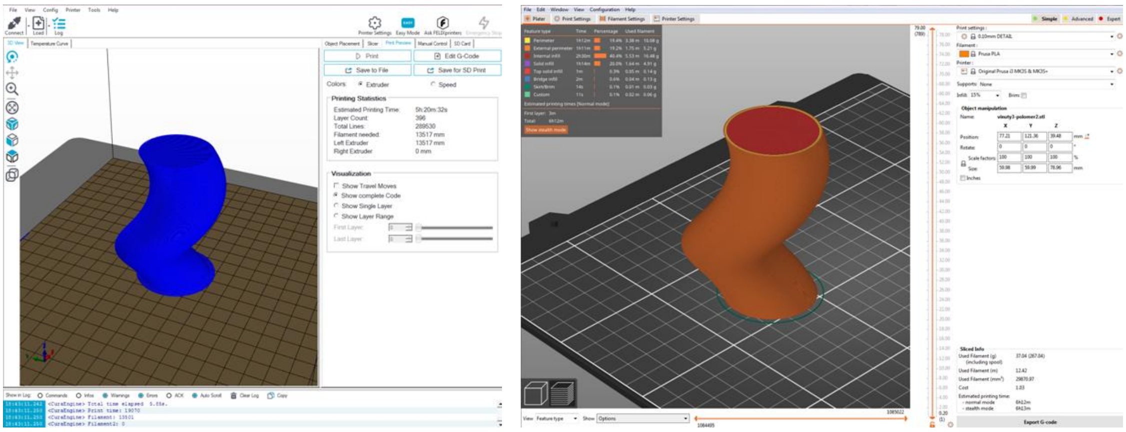

Figure 1: The geometric model of a helical surface placed in the interface of Repetier-Host software (on the left) and of PrusaSlicer (on the right).

3 The Process of Designing a Digital Geometric Model for 3D Printing

3D printable virtual models have to be firstly created. This can be done using various approaches - using 3D scanning, 3D computer modeling, or mathematical description. Another possibility is to use online libraries of digital data for 3D printing but here the disadvantage is obvious, the digital data does not have to meet our criteria.

3.1 3D Scanning of Real Objects

Digital data for 3D printing can be obtained by 3D scanning of real surfaces. A very interesting 3D printing project from real scanned data is the creation of a replica of Michelangelo's David marble sculpture. A real-size printed sculpture of David was placed at EXPO2020 in Dubai [1].

3D scanning is a method of data acquisition on a surface of a real object, creating a digital model based on it. The method resulted in a finite set of points (point clouds) in the three-dimensional space which usually requires further processing to create a reconstructed surface such that the input points lie on or near the original surface of a real object. The process leading from an unorganized set of points in three-dimensional space to a reconstructed surface in a compact description is the subject of a research area on surface reconstruction from point clouds [7].

The process of surface reconstruction usually consists of several phases and there exist algorithms which approximate the obtained point cloud with a polygonal (triangle) mesh. Most software for 3D scanning offer the alignment of scans such as smoothing and mesh simplification. Nowadays, meshes are the main outputs of 3D scanners created by an integrated software. Further modification of a scanned object represented as a mesh can be done in a CAD software by converting the mesh to a solid CAD drawing [6]. The process of scanning a real object with an optical or contact 3D scanner, an obtained point cloud and approximating that point cloud with a triangular mesh is shown in Figure 2.

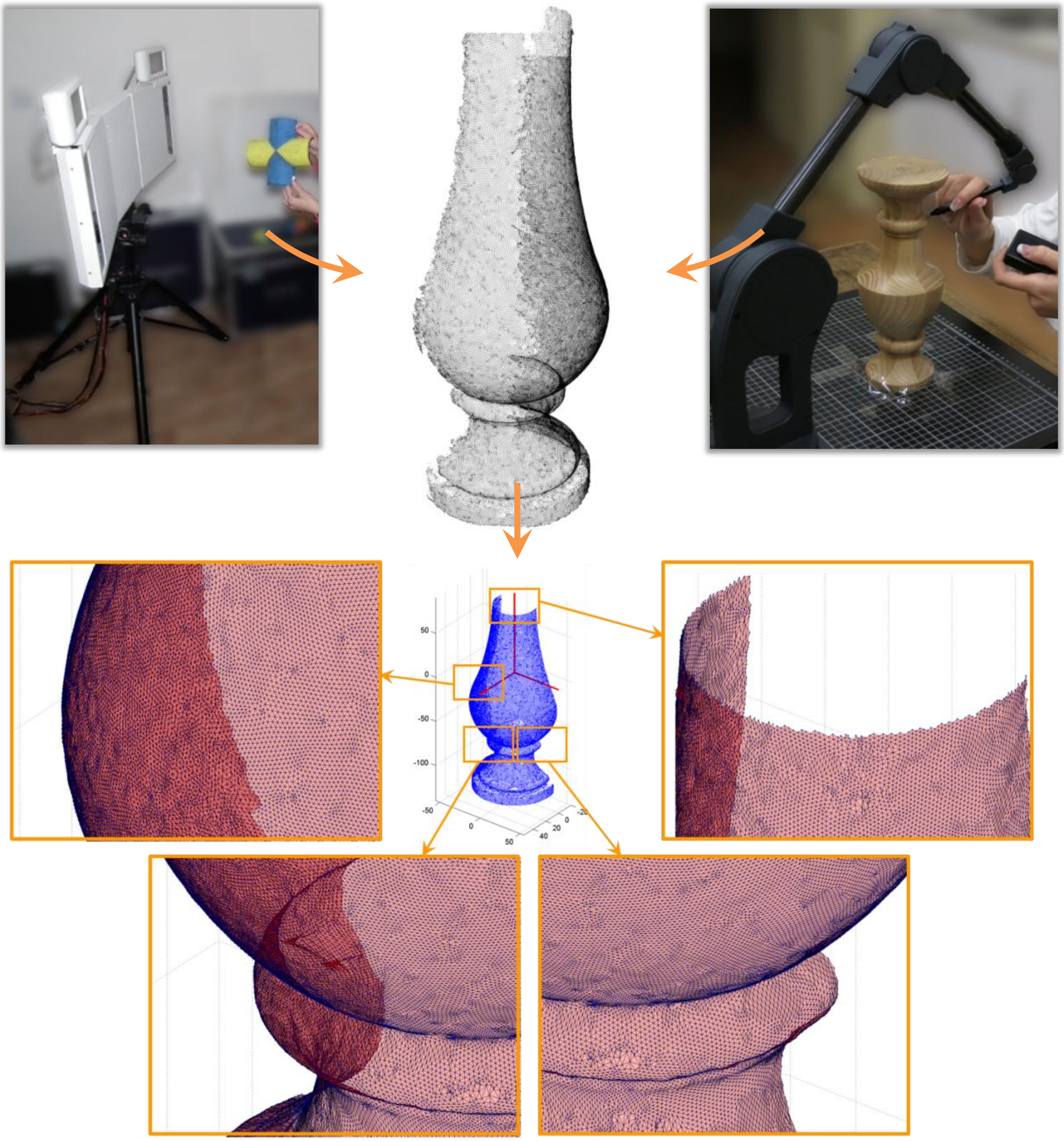

Figure 2: The process of scanning a real object with an optical or contact 3D scanner, an obtained point cloud and approximating that point cloud with a triangular mesh.

3.2 3D Computer Modeling

There are a variety of free software which can be used for modeling 3D printable virtual models. GeoGebra 6 dynamic geometry system is a suitable environment which can be used for modeling simple geometric models at secondary schools and it can be used even by students. In GeoGebra 6, the solid figure can be modeled using built-in functions and it can be directly exported to STL format. The export from GeoGebra 6 can be seen in Figure 3.

The functions of GeoGebra 6 are limited and there are no advanced features for modeling like in professional CAD systems and 3D computer modeling software. That is why we also use Rhinoceros (NURBS Modeling for Windows) software which is a commercial NURBS-based 3D

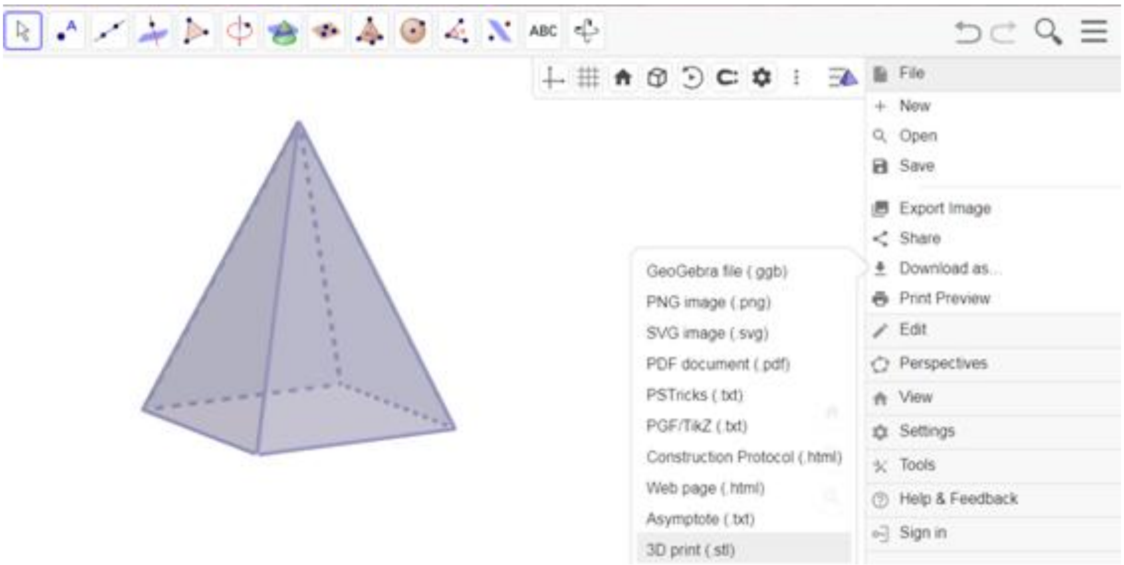

Figure 3: Export of the model in GeoGebra 6 to STL format.

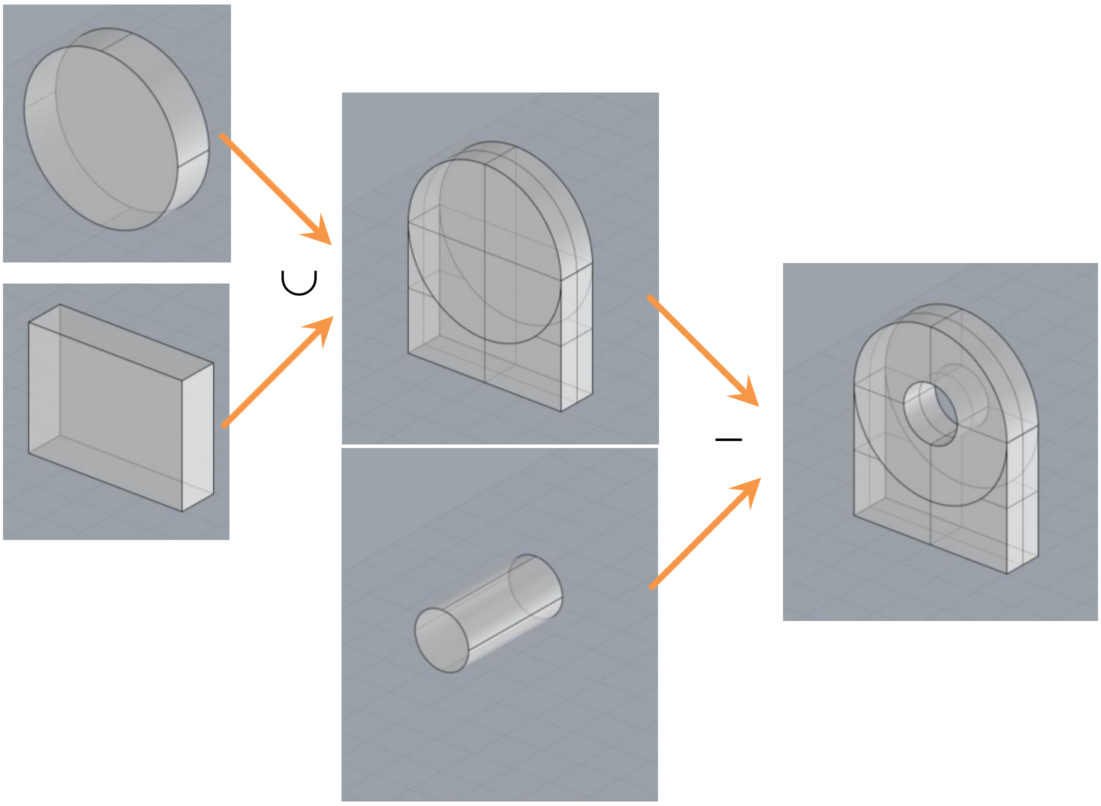

Figure 4: Generating a geometric model with Constructive Solid Geometry.

modeling tool commonly used in the process of design [11].

Rhinoceros software is widely used in various areas of production and industrial design, as well as in education and amateur hobby creation. Rhinoceros is primarily a freeform surface modeler that utilizes the NURBS mathematical model. A virtual model can be modeled using various functions. There is the wide range of various types of digital geometry such as solids and NURBS. Rhinoceros offers compatibility with other software as it supports also CAD file formats for importing and exporting. One of the formats to which a model can be exported in Rhino software is the STL format.

A simple concept of using primitives as basis for modeling can be employed for generating a design for 3D printing. Primitives are basic geometric solid figures such as a cube (or cuboid), cone, sphere, cylinder and pyramid. These primitives can be combined and modified by using Boolean operations into more complex shapes. This technique for solid modeling is called *Constructive Solid Geometry*. The process of solid modeling is captured in Figure 4.

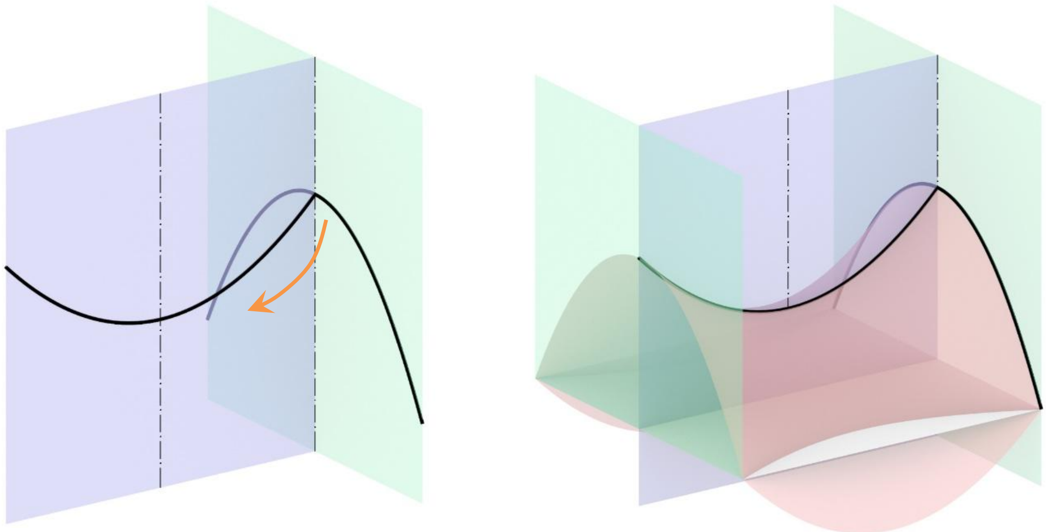

Figure 5: The process of sweeping a parabola along another parabola (on the left) and creating of the boundary to get a solid model of the hyperbolic paraboloid.

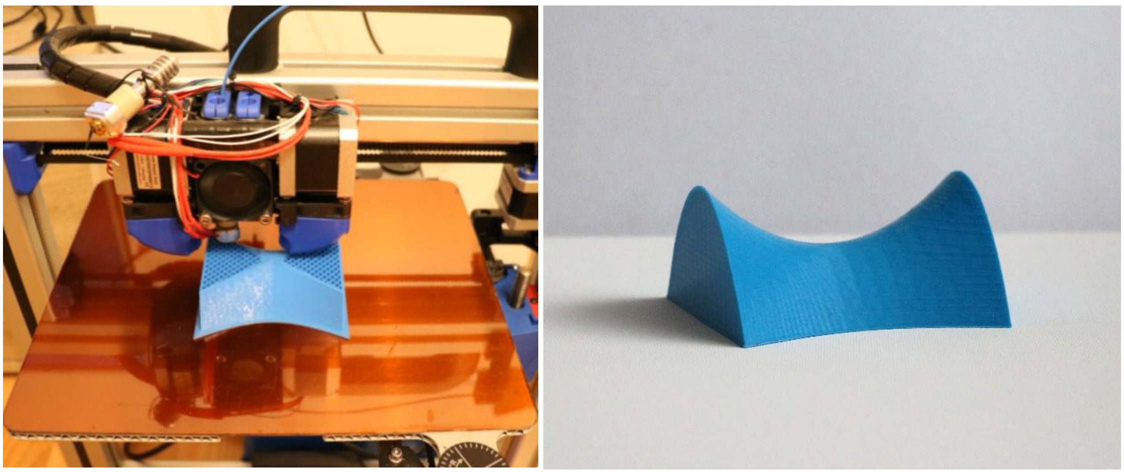

Figure 6: Printing of a geometric model and the final physical printed model of the hyperbolic paraboloid.

Another option how to define a solid in Rhinoceros software is to combine a set of surfaces to describe boundary of a single closed volume. There are traditional ways how to generate the surface in computer graphics and these techniques are available here too. Among these options belong to extrude a curve, to screw a curve, to revolve a curve, to sweep a curve along a rail, or to loft. All these methods can be further modified.

Let us present an example of a hyperbolic paraboloid to show the process of creating the solid model for 3D printing. A hyperbolic paraboloid is an infinite surface so we can model just a part of it. It can be generated using various ways, in this case we swept a parabola along another parabola. These two parabolas lie in perpendicular planes, have parallel axes and are oppositely oriented, see Figure 5 on the left. To get a solid model we used parts of perpendicular and parallel planes to cut the hyperbolic paraboloid and get closed volume. The hyperbolic paraboloid is the part of the boundary of this obtained digital geometric model. Interesting thing is that the edges of the model are hyperbola and two parabolas. The process of sweeping and creating of the boundary of a solid are shown in Figure 5. Printing of a geometric model and the final physical printed model are presented in Figure 6.

3.3 Parametric Surfaces

A digital geometric model which is enclosed by an assembly of surfaces can be described in different ways too. Separate surfaces which are combined to define a closed volume of a solid model can be given in a parametric form. Also Rhinoceros software provide *the Math plug-in* modeling a surface using the parametric equation.

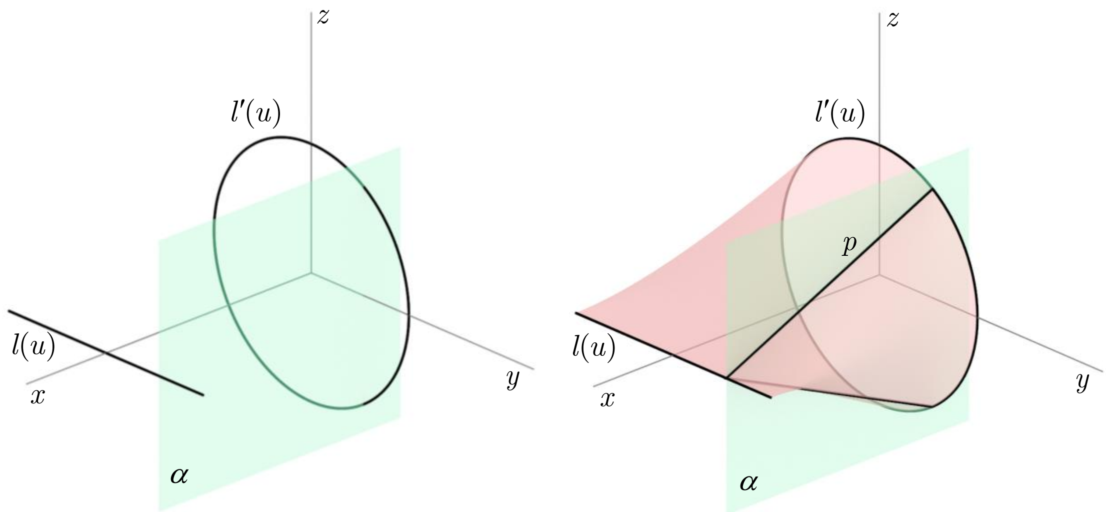

Figure 7: The determination of the conoid.

Let us present an example of ruled surfaces. Ruled surfaces contain a continuous family of straight lines called *generators*. For the demonstration of a parametric description of ruled surfaces we chose a concrete example the so-called *conoid* [13]. In our case the generators of a conoid are defined in every position using three objects – the director plane α, the straight line l', and the circle l. All generators (straight lines on the conoid) are parallel to this director plane α and intersect the straight line l' and the circle l. See Figure 7 on the left where all three geometric objects are determined in the 3D Cartesian coordinate system. The director plane is parallel to the coordinate plane (x, z), the straight line l' is parallel to y-axis and lies in the coordinate plane (x, y), and the circle l has its center at the origin and lies in the coordinate plane (y, z).

The parametric equation of the circle l is:

$$l\,(u) = [0, r\cos u, r\sin u]\,, u \in [0, 2\pi)\,, r > 0 \tag{1}$$

and the parametric equation of the straight line l' is:

$$l'\,(u) = [a, r\cos u, 0]\,, u \in [0, 2\pi)\,, r > 0, a \neq 0 \tag{2}$$

The generator p of the conoid connect the points of the circle $l(u)$ and the points of the straight line l' which have the same value of the y coordinate. We thus obtain a parametric equation of the conoid:

$$\begin{aligned} Q\,(u, v) = [av, r\cos u, r\,(1-v)\sin u]\,, \\ u \in [0, 2\pi)\,, v \in \langle 0, 1\rangle\,, r > 0 \end{aligned} \tag{3}$$

The parametrized conoid is shown in Figure 7 on the right. Again it is an infinite surface, to control that only part of the conoid is displayed, the parameter v runs in the closed interval. The final printed model of the conoid is presented in Figure 8.

Figure 8: The final physical printed model of the conoid.

4 Printed Geometric Models in Education

3D printed models can be successfully used in secondary school mathematics, for example in the topic of solid geometry. This traditional geometric topic covers the study of two-dimensional and three-dimensional Euclidean space. It includes the study of properties of and relationships between geometrical objects in the plane and in the three-dimensional space. It includes the measurements of volumes of various solid figures such as pyramids, prisms, polyhedrons, cylinders, cones, truncated cones, or spheres, cross-section of solids, transformations of two- and three-dimensional shapes in the plane and the three-dimensional space. A typical task can be to determine the relative positions between two geometrical figures – straight lines and planes. It is necessary to understand the spatial situations when they are depicted in two dimensions using some projections. Here 3D printed models can be used as concrete manipulatives to enhance understanding of spatial situations. In Figure 9 we can see the usage of 3D printed models for modeling the relative positions of two straight lines. Printed basic geometric solid figures is possible to use for the demonstration of their properties.

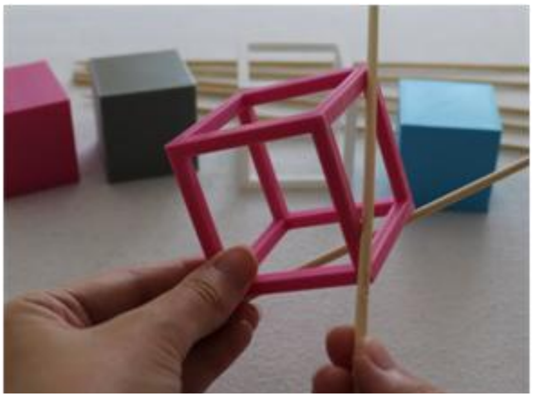

Figure 9: The 3D printed model of a "cube" with auxiliary sticks to show the relative positions of two straight lines.

The printed models which are enclosed by an assembly of parametric surfaces can be used in mathematics and geometry courses at universities. The models serve as the real visualization of abstract geometric concepts.

Much of scientific literature shows the usefulness of 3D printing in education [4], [12]. At every stage of educational process, students can join the process of the whole fabrication of geometric models.

5 Gallery of Geometric Models

Geometric models of solid figures are presented in Figure 10 and geometric models with parametric surfaces as their boundaries are shown in Figures 11, 12, 13, 14.

Figure 10: Printed models of prisms as concrete manipulatives for teaching geometry at secondary schools.

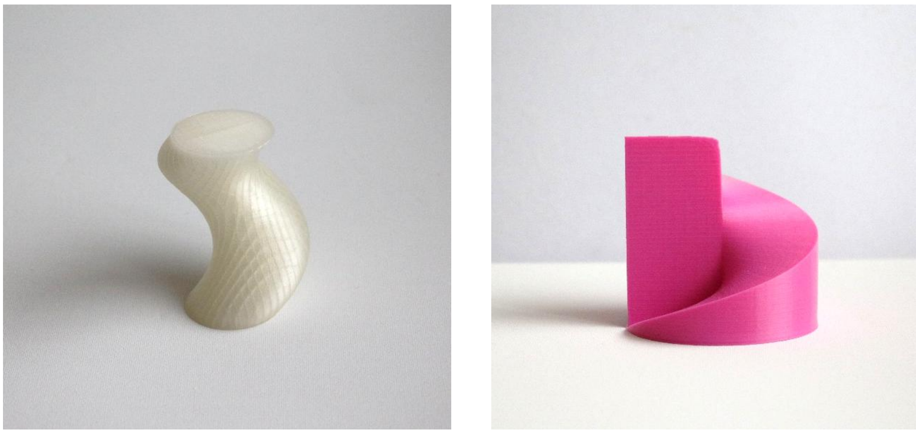

Figure 11: Printed models of helical surfaces.

Figure 12: Printed models of the one-sheet rotational hyperboloid and the one-sheet non-rotational hyperboloid.

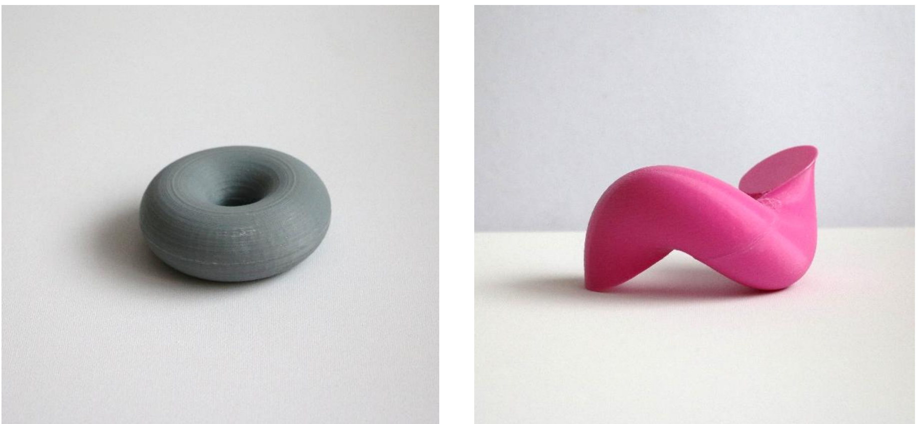

Figure 13: Printed models of the torus and the helical surface.

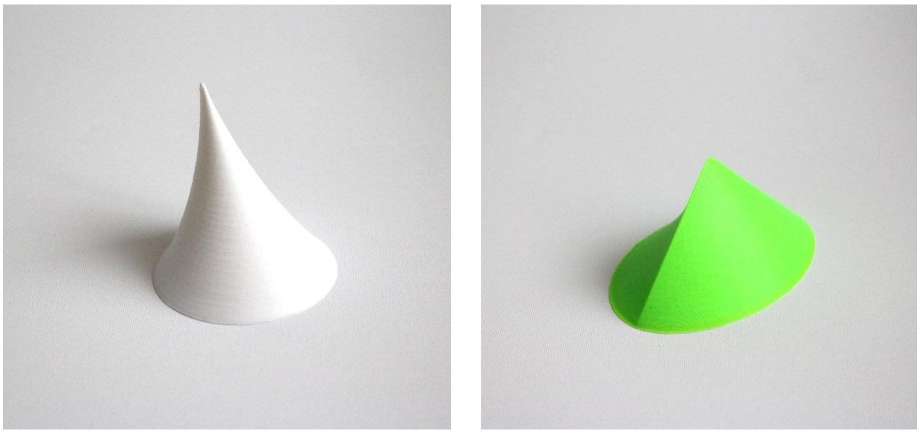

Figure 14: Printed models of conoids.

6 Conclusion and Future Work

We showed several ways how to design a virtual geometric model for 3D printing – based on 3D scanning of real objects, 3D computer modeling with constructive solid geometry and various transformations, or 3D computer modeling using parametric equations. We also presented examples of geometric models in education and provided a gallery of printed models.

Regarding the future work we plan to extend the database of printed geometric models. Our aim is also to conduct the research studies in which we would examine the effect of using of 3D printing technology and printed models on students' understanding of geometry.

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