

Limit calculation outside the domain of definition of real functions using computer algebra systems: an educational panoramic view

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Abstract

We recently realized that one of the best known pieces of mathematical software (*GeoGebra*) evaluated the limits of certain functions at a point outside their domain of definition in different ways according to the so called “View” used (*GeoGebra* is a very powerful dynamic geometry system with algebraic capabilities). The examples used would classically be classified as “removable discontinuities” or “jump discontinuities”. In this paper we have checked the output of some of the best known computer algebra systems in different examples, providing a panoramic view of the situation. Clearly, the behaviour of the pieces of software depend on the decisions made during their design step, and condition their applications on mathematics teaching.

1 Introduction

We recently found that *GeoGebra* [1], a well-known dynamic geometry system (DGS) with algebraic capabilities, evaluated the limits of some of these functions at a point outside their domain of definition in different ways according to the so called “View” used. This fact and a comparison with what the computer algebra system (CAS) *Maple*¹ [2, 3, 4] does, was described in the previous article [5]. In this new paper we have checked the output of some of the best known computer algebra systems in different examples. In both works we have considered functions whose domain and range are both subsets of the real numbers, and will denote them “real-valued function of one real variable”.

The chosen examples in [5] would classically be classified as “removable discontinuities” or “jump discontinuities” (the range was always $\mathbb{R} \setminus \{0\}$). Now we have added to the examples chosen for the test a function defined piecewise and an oscillating function, providing a panoramic view of the present situation.

Clearly, the behaviour of the pieces of software depends on the decisions made during the design step and condition their applications on mathematics teaching. After the pandemic, distance learning and self learning have become much more popular. The diversity of results shown below makes it clear that the standalone use of a complex piece of software such as a CAS can cause problems.

2 The objective of this work. Related works

We will focus on real-valued functions of one real variable and will analyze how the different CAS chosen:

- evaluate the functions at a point outside the domain of definition of the functions², and
- compute the limits of the functions at a point outside their domain of definition (valued in $\mathbb{R} \cup \{-\infty, +\infty\}$, although symbols ∞ and $\pm\infty$ are sometimes also used).

It is important to note that we do not enter into the discussion of the adequacy of saying, as is very usual, for example, in Secondary Education in Spain, that a function like $\frac{1}{x}$ has a discontinuity at 0 [6, 7], as this doesn’t make sense at a point outside its domain of definition (what is what is usually done in upper educational levels [8, 9]). Perhaps to speak of a “conflicting point”, encompassing both the points of the domain border where the function is not defined and the discontinuities in the formal sense (point of the domain of the function where the usual $\epsilon - \delta$ definition of a continuous function does not hold), could be a solution.

[10] is an encyclopedic work comparing several CAS, pushing limits computations to the limit. There are more specific articles, comparing the capabilities of different CAS with respect to certain tasks, like solving algebraic systems [11]. There are also works about the construction of the limit concept with a CAS [12], that studies the problem from the students’ side.

¹All product names, trademarks and registered trademarks are property of their respective owners.

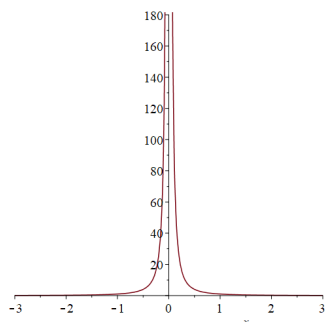
²Evaluating a function outside its domain makes no sense from the mathematical point of view. However, a function like, for example, $\frac{1}{x}$ is usually given to the CAS carelessly, without specifying that $x \neq 0$, so the different CAS initially consider the function $\mathbb{R} \rightarrow \mathbb{R}$, and try to evaluate it at 0. We check if the different CAS realize that there are problems at that point (the value of the function is not finite or the function is undefined) and the clarity of the output.

Meanwhile, we focus on the computational drawbacks from an educational point of view, considering the problems from the computational side. More precisely, we focus on the evaluation of the function by the CAS outside the domain of the function and on the limit computations, and how the correctness of the output and the appropriateness of the notation used by the CAS may interfere with the presentation of a Secondary Education student to the concept of limit.

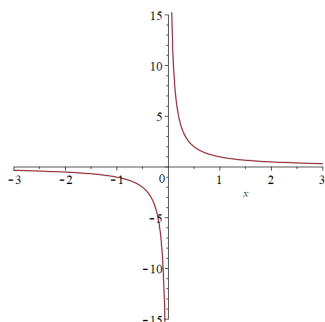
3 About the functions in the test

We have chosen the following functions $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ for the test:

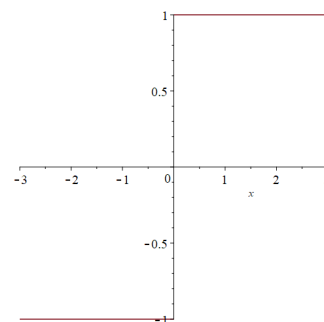
- $m(x) = \frac{1}{x^2}$ ($\lim_{x \rightarrow 0^-} m(x) = +\infty$, $\lim_{x \rightarrow 0^+} m(x) = +\infty$, so: $\lim_{x \rightarrow 0} m(x) = +\infty$ –see Figure 1a)
- $f(x) = \frac{1}{x}$ ($\lim_{x \rightarrow 0^-} f(x) = -\infty$, $\lim_{x \rightarrow 0^+} f(x) = +\infty$ –see Figure 1b)
- $w(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$
(the function is defined piecewise, and $\lim_{x \rightarrow 0^-} w(x) = -1$, $\lim_{x \rightarrow 0^+} w(x) = 1$ –see Figure 1c)



(a) *Maple* plot of function $m(x) = \frac{1}{x^2}$ obtained typing: `m:=x->1/x^2; plot(m(x),x=-3..3);`



(b) *Maple* plot of $f(x) = \frac{1}{x}$. *Maple* has to be informed that there could be a “discontinuity” in order to avoid a spurious vertical line by typing: `f:=x->1/x; plot(f(x),x=-3..3, discont=true);`



(c) *Maple* plot of $w(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$

Figure 1: *Maple* plots of the first three functions chosen.

- $g(x) = e^{\frac{1}{x}}$ ($\lim_{x \rightarrow 0^-} g(x) = 0$, $\lim_{x \rightarrow 0^+} g(x) = +\infty$ –see Figure 2a)
- $u(x) = \sin(1/x)$ (the function is undefined in 0 and it has no side limits in 0, and, consequently, no limit in 0, as it oscillates faster and faster between -1 and 1 as it approaches 0 –see Figure 2b)

- $h(x) = \frac{x}{x}$ (All these functions are considered $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ and therefore $h(x) = 1$. However, this is a conflicting example if considered as an function $\mathbb{R} \rightarrow \mathbb{R}$ because it depends on whether the simplification $\frac{x}{x} = 1$ is firstly carried out or not -see Figure 2c).

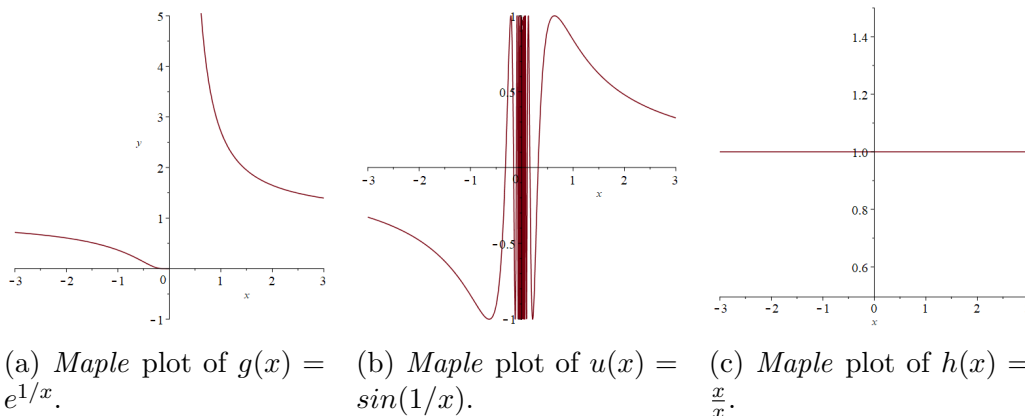


Figure 2: Maple plots of the last three functions chosen.

The choice of functions hasn't been made trying to test the capabilities of the CAS but trying to clarify their behaviour when facing functions that can arise at Secondary Education level. They reflect the main situations we have guessed regarding asking the CAS to try to compute the value of the function outside its domain of definition and computing limits at points outside the domain of definition. We have ordered the functions from the simplest to the most complex.

The reasons for choosing these functions are detailed afterwards. $m(x) = \frac{1}{x^2}$ is probably the simplest function with an infinite limit. $f(x) = \frac{1}{x}$ is similar and has infinite side limits of different signs. The simple piecewise function $w(x)$ has different side limits. $g(x) = e^{\frac{1}{x}}$ is not so simple as the previous ones, but has one finite side limit and one infinite side limit. $u(x) = \sin(1/x)$ is an interesting case: $|u(x)| \leq 1$, but has no limit at 0. Lastly, $h(x) = \frac{x}{x}$ has been include for reasons already mentioned above.

4 Results

We have chosen the following CAS: *Maple 2022* [2, 3, 4], *DERIVE 6.1* [13], *Maxima 5.46.0* [14], *Mathematica 13.1* [15], *REDUCE CSL 2022* [16, 17], *SymPy 1.10.1* (a *Python* library), *SageMath 9.6* [19], *Xcas 1.7.0* [20] and *GeoGebra 6* [1] (both "CAS View" and "Algebra View"). They are all well known general purpose CAS, but it is not an exhaustive list, as there other ones like *Axiom*, *YACAS*, etc.

Moreover it has to be taken into account that there are also specific purpose CAS like *CoCoA*, *Risa/Asir*, *Singular*, *Macaulay*, *PoluyBoRi*, etc., many of which cannot calculate limits, and have been ignored.

The code used for defining the chosen functions and for evaluating them at 0 is detailed in Table 1. Moreover, the code used for computing the limit and one-sided limits at 0 for $w(x)$ is included in Table 2 as illustration. This way the reader can reproduce the authors' results.

	$m(x) = \frac{1}{x^2}$	$m(0)$	e^x	$w(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$
<i>Maple</i> Input:	<code>m:=x->1/x^2;</code>	<code>m(0);</code>	<code>exp(x)</code>	<code>w:=x->piecewise(x<0,-1,x=0,undefined,x>0,1);</code>
<i>DERIVE</i> Input:	<code>m:=1/x^2</code>	<code>m(0)=</code>	<code>EXP(x)</code>	<code>w(x):=IF(x < 0, -1, IF(x > 0, 1))</code>
<i>Maxima</i> Input:	<code>m(x):=1/x^2;</code>	<code>m(0);</code>	<code>exp(x);</code>	<code>w(x):=if x<0 then -1 else (if x>0 then 1);</code>
<i>Mathematica</i> Input:	<code>m[x_]:=1/x^2;</code>	<code>m[0];</code>	<code>Exp[x];</code>	<code>w[x_]:=Piecewise[{{1,x>0},{-1,x<0}}];</code>
<i>REDUCE</i> Input:	<code>m(x):=1/x^2;</code>	<code>sub(x=0,m(x));</code>	<code>exp(x);</code>	<code>let w(~x)=>-1 when numberp x and x<0, w(~x)=>1 when numberp x and x>0;</code>
<i>SymPy</i> Input:	<code>def m(x):</code> <code>return 1/x**2</code>	<code>m(0)</code>	<code>sympy.exp(x)</code>	<code>def w(x):</code> <code>if x<0:</code> <code>return -1</code> <code>if x>0:</code> <code>return 1</code>
<i>SageMath</i> Input:	<code>m(x)=1/x^2</code>	<code>m(0)</code>	<code>exp(x)</code>	<code>w = piecewise([((-Infinity,0),-1),((0,+Infinity),1)],var=x)</code>
<i>Xcas</i> Input:	<code>m(x):=1/x**2</code>	<code>m(0)</code>	<code>exp(x)</code>	<code>w(x):=piecewise(x<0,-1,x>0,1)</code>
<i>GG (CAS V.)</i> Input:	<code>m(x):=1/x^2</code>	<code>m(0)</code>	<code>exp(x)</code>	<code>w(x):=If(x<0,-1,x>0,1))</code>
<i>GG (Alg. V.)</i> Input:	<code>m(x)=1/x^2</code>	<code>m(0)</code>	<code>e^x</code>	<code>w(x)=If(x<0,-1,x>0,1))</code>

Table 1: Input used for defining the chosen functions in the different CAS and for evaluating them at 0. ‘GG (CAS V.)’ stands for ‘GeoGebra CAS View’. ‘GG (Alg. V.)’ stands for ‘GeoGebra Algebra View’.

	$\lim_{x \rightarrow 0} m(x)$	$\lim_{x \rightarrow 0^+} m(x)$	$\lim_{x \rightarrow 0^-} m(x)$
<i>Maple</i> Input:	<code>limit(m(x),x=0);</code>	<code>limit(m(x),x=0,right);</code>	<code>limit(m(x),x=0,left);</code>
<i>DERIVE</i> Input:	<code>LIM(w(x),x,0)=</code>	<code>LIM(w(x),x,0,+1)=</code>	<code>LIM(w(x),x,0,-1)=</code>
<i>Maxima</i> Input:	<code>limit(m(x),x,0);</code>	<code>limit(m(x),x,0,plus);</code>	<code>limit(m(x),x,0,minus);</code>
<i>Mathematica</i> Input:	<code>Limit[m[x],x->0];</code>	<code>Limit[m[x],x->0,Direction->-1];</code>	<code>Limit[m[x],x->0,Direction->1];</code>
<i>REDUCE</i> Input:	<code>limit(m(x),x,0);</code>	<code>limit!+(m(x),x,0);</code>	<code>limit!-(m(x),x,0);</code>
<i>SymPy</i> Input:	<code>sympy.limit(m,x,0)</code>	<code>sympy.limit(m,x,0,'+')</code>	<code>sympy.limit(m,x,0,'-')</code>
<i>SageMath</i> Input:	<code>limit(m(x),x=0)</code>	<code>limit(m(x),x=0,dir='+')</code>	<code>limit(m(x),x=0,dir='-')</code>
<i>Xcas</i> Input:	<code>limit(m(x),x,0)</code>	<code>limit(m(x),x,0,'+')</code>	<code>limit(m(x),x,0,'-')</code>
<i>GG (CAS V.)</i> Input:	<code>Limit(w,0)</code>	<code>LimitAbove(w,0)</code>	<code>LimitBelow(w,0)</code>
<i>GG (Alg. V.)</i> Input:	<code>Limit(w,0)</code>	<code>LimitAbove(w,0)</code>	<code>LimitBelow(w,0)</code>

Table 2: Input for computing the limit and one-sided limits at 0 for $w(x)$.

Some comments regarding the input/output:

- Note the use of **undefined** for defining piecewise functions (*Maple* considers that 0 is the value of the piecewise function where no value is specified). Note that $+\infty$ is represented by ∞ . Observe that if the side limits are different, it answers “undefined” in all cases.
- Note that in *Maxima* it is not possible to calculate limits of piecewise functions. Note that $+\infty$ is represented by ∞ and ∞ is represented by **“infinity”**.
- Note that *Mathematica* considers that 0 is the value of the piecewise function where no value is specified (that’s why there is a blank cell in Table 3).
- Note that both ∞ and $+\infty$ are represented by ∞ in *REDUCE* (we write a ‘Wrong’ in the cell concerned in Table 4).
- Apparently, *SymPy* computes the limit as x approaches 0 as the right limit (see Table 4) when both side limits are different.
- It is remarkable that what is notated ∞ in *Xcas* when valuating a function is notated $+\infty$ when computing limits.
- It is strange that the input in *GeoGebra 6* “CAS View” and “Algebra View” is not always identical (see the two last rows of Table 1). There are also differences between output (see the two last rows of Table 3).

	$m(x) = \frac{1}{x^2}$	$f(x) = \frac{1}{x}$	$w(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$	$g(x) = e^{\frac{1}{x}}$	$u(x) = \sin(1/x)$	$h(x) = \frac{x}{x}$
<i>Maple</i> Output: Comment:	Error, (in m) numeric exception; division by zero Correct	Error, (in m) numeric exception; division by zero Correct	undefined Correct	Error, (in m) numeric exception; division by zero Correct	Error, (in m) numeric exception; division by zero Correct	Error, (in m) numeric exception; division by zero Correct
<i>DERIVE</i> Output: Comment:	∞ Correct in $\mathbb{R} \cup \{-\infty, \infty\}$	$\pm\infty$ Correct in $\mathbb{R} \cup \{-\infty, \infty\}$? Correct (it is undefined)	? Correct	$\sin(\infty)$ Strange way	? Correct
<i>Maxima</i> Output: Comment:	Undefined: 0 to a negative exponent Correct	Undefined: 0 to a negative exponent Correct	false (it is undefined) Strange way	Undefined: 0 to a negative exponent Correct	Undefined 0 to a negative exponent Correct	Undefined 0 to a negative exponent Correct
<i>Mathematica</i> Output: Comment:	Infinite expression $\frac{1}{0}$ encountered Correct	Infinite expression $\frac{1}{0}$ encountered Correct		Infinite expression $\frac{1}{0}$ encountered Correct	Infinite expression $\frac{1}{0}$ encountered Correct	Infinite expression $\frac{1}{0}$ encountered Correct
<i>REDUCE</i> Output: Comment:	Zero divisor Correct	Zero divisor Correct	$w(0)$ Strange way	Zero divisor Correct	Zero divisor Correct	1 Wrong
<i>SymPy</i> Output: Comment:	ZeroDivisionError: division by zero Correct	ZeroDivisionError: division by zero Correct	NaN (Not a Number) Correct	ZeroDivisionError: division by zero Correct	ZeroDivisionError: division by zero Correct	ZeroDivisionError: division by zero Correct
<i>SageMath</i> Output: Comment:	ValueError: power::eval(): division by zero Correct	ValueError: power::eval(): division by zero Correct	ValueError: point 0 is not in the domain Correct	ValueError: power::eval(): division by zero Correct	ValueError: power::eval(): division by zero Correct	1 Wrong
<i>Xcas</i> Output: Comment:	∞ Correct in $\mathbb{R} \cup \{-\infty, \infty\}$	∞ Correct in $\mathbb{R} \cup \{-\infty, \infty\}$	No case applies Correct	undef Correct	undef Correct	undef Correct
<i>GG (CAS V.)</i> Output: Comment:	? Correct	? Correct	? Correct	? Correct	? Correct	1 Wrong
<i>GG (Alg. V.)</i> Output: Comment:	∞ Correct in $\mathbb{R} \cup \{\infty\}$	∞ Correct in $\mathbb{R} \cup \{\infty\}$? Correct	∞ Wrong	? Correct	? Correct

Table 3: Evaluation at 0 of the chosen functions by the different CAS considered. ‘GG (CAS V.)’ stands for ‘GeoGebra CAS View’. ‘GG (Alg. V.)’ stands for ‘GeoGebra Algebra View’.

	$m(x) = \frac{1}{x^2}$	$f(x) = \frac{1}{x}$	$w(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$	$g(x) = e^{\frac{1}{x}}$	$u(x) = \sin(1/x)$	$h(x) = \frac{x}{x}$
<i>Maple</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	undefined $\infty -\infty$ Correct	undefined 1 -1 Correct	undefined $\infty 0$ Correct	-1..1 -1..1 -1..1 Strange way	1 1 1 Correct
<i>DERIVE</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	$\pm \infty \infty -\infty$ Correct	? 1 -1 Correct	? $\infty 0$ Correct	$\sin(\infty) \sin(\infty) \sin(\infty)$ Strange way	1 1 1 Correct
<i>Maxima</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	infinity $\infty -\infty$ Correct	Does not calculate Strange way	und $\infty 0$ Correct	ind ind ind Correct	1 1 1 Correct
<i>Mathematica</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	Indeterminate $\infty -\infty$ Correct	Indeterminate 1 -1 Correct	Indeterminate $\infty 0$ Correct	Indeterminate Indeterminate Indeterminate Correct	1 1 1 Correct
<i>REDUCE</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	$\infty \infty -\infty$ Wrong	Does not calculate Strange way	$\infty \infty 0$ Wrong	Does not calculate Strange way	1 1 1 Correct
<i>SymPy</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	$\infty \infty -\infty$ Wrong	1 1 -1 Wrong	$\infty \infty 0$ Wrong	AccumBounds(-1, 1) AccumBounds(-1, 1) AccumBounds(-1, 1) Strange way	1 1 1 Correct
<i>SageMath</i> Output (limit - right - left); Comment:	$+\infty +\infty +\infty$ Correct	$\infty +\infty -\infty$ Correct		undefined $+\infty 0$ Correct	ind ind ind Correct	1 1 1 Correct
<i>Xcas</i> Output (limit - right - left); Comment:	$+\infty +\infty +\infty$ Correct	$\infty +\infty -\infty$ Correct	Unidirectional limits are distinct -1,1 Error: Bad Argument Value 1 -1 Correct	undefined $+\infty 0$ Correct	bounded_function(1) bounded_function(3) bounded_function(5) Strange way	1 1 1 Correct
<i>GeoGebra (CAS View)</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	? $\infty -\infty$ Correct	? 1 -1 Correct	? $\infty 0$ Correct	? ? ? Correct	1 1 1 Correct
<i>GeoGebra (Algebra View)</i> Output (limit - right - left); Comment:	$\infty \infty \infty$ Correct	? $\infty -\infty$ Correct	? 1 -1 Correct	? $\infty 0$ Correct	? ? ? Correct	1 1 1 Correct

Table 4: Output of the different CAS considered when computing the limit of the chosen functions as x approaches 0 as well as the side limits.

5 Conclusion

The developers of the CAS have to make decisions when designing their CAS. These are impressive pieces of software with thousands of lines of code. In most cases, all CAS provide correct answers (from a certain point of view). For mathematicians with a background in computing, who use the CAS as a tool, there is no problem with the different approaches chosen by the different developers (because the underlying mathematical idea is clear to these end users).

But the decisions made during the design stage of the software pieces condition their applications in fields such as teaching. Thus, when the CAS is used as a digital technology for acquiring knowledge, drawbacks may arise: there can be differences between the assumptions or notations of the textbook and the digital tool, different CAS with different assumptions or notations can be used by classmates that are in touch with each other, etc. These drawbacks are added to the difficulties, conflicts and obstacles that Secondary Education students usually have in learning mathematical notions such as one-sided limits and discontinuities. In fact, it is well known that there are students who often confuse the concepts of limit and continuity.

Although it is interesting to consider different representation registers in teaching, it is very important to know how to coordinate them. A proper coordination helps in the development of thinking and understanding of mathematical concepts. In this sense, this work shows the importance of knowing very well the behaviour of a complex piece of software such as a CAS for self-learning the concept of limit. Thus, a standalone use of CAS by a Secondary Education student can cause problems in the understanding of some mathematical concepts.

As a final conclusion, CAS are incredible digital technologies, so close to classic mathematics, but active guided instruction always seems preferable for learning mathematics.

Lastly, as a disclaimer, we would like to remark that we have summarized the results in the tables to the best of our knowledge. We sincerely apologize for any unwanted errors (due to our lack of expertise handling the CAS) or errata when transferring the data to the tables.

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