# Developing Computational Thinking in Portuguese Mathematics Curricula with Collatz Conjecture

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**Abstract**

Several countries have recently adopted the inclusion of Computational Thinking (CT) in compulsory education curricula. In Europe, this movement has been fostered by a decision taken by European Union arguing that citizens' technological capacity must be deeper than their sole uses so that CT must be fostered already in schools. Thus, certain European countries integrated CT in their curricula. In Denmark, for example, a pilot project was established to investigate the possibility of incorporating CT into Mathematics or Information Communication Technologies (ICT) subjects. Also, Portugal was one of the first countries that implemented the strategy of incorporating CT into the mathematics curriculum. In our paper, we intend to examine some opportunities to include Computational Thinking (CT) as a skill in the mathematics curriculum, specifically those connected to the kinds of mathematical activities usually done in regular classrooms. The Collatz Conjecture (CC) will serve as the basis for us to demonstrate how we might engage with an open mathematical problem while developing Mathematical Thinking and other CT concepts and processes for students. As a classroom task, the CC approach may demonstrate how a contemporary mathematical subject might benefit from computing through emphasizing the usage of various technological tools or programming languages depending on the age of the student group.

## 1 Introduction

Computational Thinking (CT) has become a hot issue in education during the past years. The development of Artificial Intelligence and the digitization of society have contributed to the reflection of the role that technology plays in society, particularly in education. The term CT was first used by Seymour Papert [1, 2]. Later, Jeannette Wing [3] developed this theme and its applications arguing that CT involves the structuring of reasoning of human behavior for problem-solving actions. CT utilises concepts used in Computer Science to education and aims to systematize the steps of problem-solving using algorithms, by taking advantage of technology. Other authors argued that CT includes a broad set of thinking strategies and is oriented towards problem solving, including concepts such as abstraction, decomposition, evaluation, pattern recognition, use of logic and algorithm design [4, p.19]. Brennan and Resnick [5] consider three dimensions for the development of CT: 1) concepts used in programming; 2) problem-solving practices that occur in the programming processes; and 3) perspectives on working in computing. As we live in an era of greater integration of ICT, in general, particularly in education, the aim of CT implementation is to take advantage of the added value of technology, freeing, whenever possible, the routine tasks for machines.

We are interested in analysing how we can use a problem posed by Lothar Collatz in 1937 to develop CT. CC states that the sequence defined by

$$a_n = \begin{cases} \frac{1}{2}a_{n-1} & \text{for } a_{n-1} \text{ even} \\ 3a_{n-1} + 1 & \text{for } a_{n-1} \text{ odd} \end{cases}$$

always returns to 1 for a positive integer $a_0$. Until today, this conjecture remains to be proved and research-based in the use of computers has brought about instruments to get experimental evidence. The conjecture has been checked on a computer for all starting values up to $2^{68} \approx 2.95 \times 10^{20}$. All initial values tested so far eventually end in the repeating cycle $(4; 2; 1)$ of period 3 [6].

Our work consists of the algorithmic verification of the *Collatz Succession*, and has been publicly presented in May 2022 [7, p. 14], with a proposed task to be analyzed at various levels of schooling. This proposal consists of different types of approaches and depth suitable for each age. The present work is also relevant because, in June 2022, the Essential Mathematics Learning (EML) for the 11th grade [8, p. 25], included the *Collatz Succession* as an example of a problem to be addressed within the scope of CT. It should be noted that the EML propose the use of different programming languages and software to develop CT, as well as the use of calculators in classes, homework and exams.

The development of CT in non-higher education is on the agenda of many European curricula, which will be discussed in Section 2. In the Portuguese case, the option of including CT in Mathematics curriculum was made, which will be discussed in Section 2.1. The different approaches to CC at different levels of education will be presented in Section 3. Finally, we will present our conclusions in Section 4.

# 2 Computational Thinking in the curriculum of European Countries

For the Council of the European Union and the European Commission "Learning and acquiring digital competences go beyond pure ICT skills and involve the safe, collaborative and creative use of ICT, including coding" [9], which may explain the inclusion of CT development in many of the non-higher education curricula in several European countries, therefore, the involvement of students in activities that include CT emerges as a way to deepen citizens' skills in the use of technologies.

The development of CT in the mathematics curriculum is one way to develop the EU strategies related to digital competencies, and this issue is one reality in Finland and Portugal, also all the EU members point to CT to develop problem-solving and logical thinking skills [10].

In Portugal, the reason for the inclusion of CT in the Mathematics curriculum is discussed, ICT teachers claim that it is a subject that they should deal with, taking time away from the subject of Mathematics. However, the designers of the mathematics curriculum in Portugal claim that CT is a way of solving problems, ie that it should be used in Mathematics classes. In fact, this discussion takes place in numerous countries among professionals in Computer Science and Mathematics.

While in Portugal the integration of CT in the subject of Mathematics has already been assumed, in Denmark the option was to create a pilot project to study the effects of the inclusion of CT in the subjects of ICT or Mathematics. In the case of Denmark the introduction of technologies competences in the curriculum are in a pilot project, but in terms of Mathematics refer to the fact that students need to act with judgment concerning the use of digital technologies, by working with open problems from the surrounding world. Furthermore, the ICT Dansk curriculum refers:

> "Students are introduced to data and data processes (typically from other subjects such as Mathematics or Science / Technology or from problem areas for digital artifacts) and practice recognizing patterns in these. Against this background, students modify or construct corresponding data and algorithm structures for subsequent computer programming." [11, p.37]

The Dansk pilot project includes two groups of schools, in one of them ICT is developed in a separate subject, in the other the ICT skills are merged in the whole curriculum [12]. Actually, the literature also discusses the concept of Computational Thinking *versus* the concept of Programming and Computational thinking (PCT), for Helenius and Misfeldt [13]. This concept is closely related to the scientific domain of computer science, and it sometimes also draws on other fields, such as Sociology and Philosophy, beyond Math.

In Mathematics research practice, diagrams and various forms of pictorial representations can have a generative power in mathematical practice and representations, suggesting new proof strategies, playing a generative role in the development of mathematical concepts and informing new connections [14], moreover in some fields of contemporary Mathematics the use of algorithms and their computational implementation is a way to get new results, in a certain way, some research in Mathematics use PCT. In this regard, Conrad Wolfram notes that Mathematics presented in schools does not allow computer-based Math's,

> "It arbitrarily cuts out many real-life problems, randomizes ordering of conceptual complexity and muddles up broader Maths concepts with calculating techniques. It causes unnecessary confusion and boredom, and worst of all it often programs everyone with the wrong approach to problems, led by an easier-to-compute method first rather than a more balanced view of all available methods that might be applicable." [15, p.140]"

This way, it will be necessary to develop new practices and approaches in Mathematics education that provide students with rich Mathematics experience in class. Next we will see in more detail what the options in the introduction of the CT in Portugal were.

## 2.1 Computational Thinking in Portuguese Math Curriculum

In the case of Portugal, we have ICT as an independent subject in some years of compulsory education, but it is also transversal to the entire curriculum. Recently, CT was proposed as a transversal content of the Mathematics curriculum in basic education [16] and it is now in public discussion. The curricular documents EML, in articulation with the - Students' Profile, are relevant to tackle the idea that it will be necessary to develop new practices and approaches in Mathematics education that provide a rich mathematics experience in class to all students, relating these practices with the profile of students at the end of 12 years of compulsory education. Only in June of 2022, was the EML, for secondary education [8], presented and it will be in discussion until September of 2022. Even though Portugal has twelve documents to define the Mathematics curriculum, all of them were prepared by the same team, considering CT as a way to solve problems, giving suggestions for tasks to develop CT in class, for the different mathematical topics for each grade, and also propose the use of program languages as Scratch, and Python, applets and the use of GeoGebra to develop CT in Mathematics.

The introduction of CT development in the Mathematics curriculum rekindled the discussion about computer science in compulsory education. Objections by teachers to the introduction of programming were also raised, however the suggestions contained in EML for the use of programming are based on simple samples of use of Scratch or Python. Nevertheless, in our opinion, the development of automation procedures, in addition to allowing the debugging, is essential to the development of the CT, allowing us to accomplish mathematical tasks whose long routines or demanding complex calculation processes could be easily executed by the machine.

The Portuguese EML for secondary education foresee the testing of conjectures as one of its objectives, as well as a systematic resource for the use of technology as:

> *"the use of spreadsheets, dynamic geometry environments, various digital applications, simulations, smartphones, graphing calculators and sensors, as well as other equipment and materials, must be done systematically. Programming activities must be integrated with progressive complexity, being relevant to the development of algorithmic processes, structured thinking and logical reasoning, providing a vast field of application of Mathematics and genuinely involving the formulation and resolution of problems, in addition to promote the development of computational thinking."*[8, p.6].

In the particular case of the 11th grade, in the *Succession* theme, the definition of succession by recurrence and the use of technology to generate them as well as to analyze their behavior is foreseen. In fact, in the 11th grade, the notion of sequence, already worked on in previous years, assumes the formalization as succession, that is, successions are defined as functions of a natural variable.

# 3 Different ways to explore Collatz Conjecture in school

In this section we will present several ways of working the CC and implementing CT in the Portuguese curriculum, promoting exploratory tasks in the classroom [17, p.21]. Although we use different technologies, we will discuss whenever possible the use of GeoGebra, as it is one of the software recommended by EML, and also because it is widely used by teachers in their classroom work in Portugal.

## 3.1 Unplugged tasks and using spreadsheets in K4

The work with the CC, can initially be done only with paper and pencil. A calculator could still be available to facilitate calculations. At this stage it is not intended for the process to be automated, being the student responsible for the execution of the entire verification process. This task could be performed with students in the 1st cycle of basic education in Portugal, developing the mathematical theme of *Numbers and Operations*, since they would only have to recognize the parity of a natural number, use the operations of addition and multiplication, and work the *Collatz Succession* as just a sequence.

This first attempt at the task allows the development of the theme *Problem Solving*, at the subtopic *Strategy*, once the student should "apply and adapt problem-solving strategies, in various contexts, including using technology." [16, p.13].

In relation to the theme *Mathematical Reasoning* this task is adjusted to the subtopic *Conjecture and generalize* because the student can "formulate and test conjectures/generalizations, from the identification of regularities common to objects under study, namely using technology." [16, p.14]. Thus, these two references to the use of technology, in EML, make us propose that spreadsheets may be used at the end of the 1st cycle of the elementary school in Portugal. In our opinion, the step of introducing the spreadsheet will be a greater challenge for the teacher than the student.

It should be noted that, unplugged activities do not allow the development of CT in all its procedures, therefore, the use of a spreadsheet is a first step in the automation process.

Using for example the GeoGebra spreadsheet, it is intended that the student through the conditional expression `IF`, frequent in programming algorithms, be able to apply a line of code for action if the condition is true and for the case where it is false. So, the student should be able to create a statement similar to:

$$\begin{aligned} &= \texttt{If}(\texttt{condition};\ \texttt{value if true};\ \texttt{value if false}) \\ &= \texttt{If}(\texttt{Mod}(B1,2) == 0, B1/2, 3B1+1) \end{aligned} \tag{1}$$

getting the result that can be seen in Figure 1.

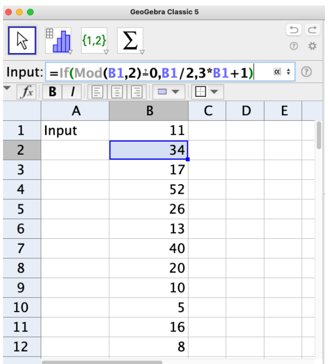


Figure 1: GeoGebra spreadsheets with some terms of Collatz sequence.

## 3.2 Using Scratch

Since Scratch is a programming language, it is desirable for students to anticipate an idea of the processes to be used. Teachers should encourage students to structure and write these ideas, which over time will be shaped in the form of schemes or pseudocode. To explore CC we could get a pseudocode similar to the presented in Algorithm 1.

**Algorithm 1:** Pseudocode

```
i ← 0                              ▷ the number of iterations has been initialized to 0
input value                        ▷ request for the value to be tested
while value > 1 do
    ▷ cycle that only ends when the value 1 is obtained
    i ← i + 1                      ▷ variable i is incremented by 1 value
    if value is even then
        ▷ logical test that allows checking if value is even
        value = value/2            ▷ if value is even will be replaced by half
    else
        value = 3 × value + 1      ▷ if is odd will be replaced by triple plus 1
    end
end
show i                             ▷ show the number of iterations until gets value 1
```

From the students' perception of the code's structure, its translation to any platform or programming language will be simple. By analyzing the code, we can verify that the proposed structure includes, in addition to the existing condition in the spreadsheet, the "while" cycle, which will be responsible for the total automation of the process. When using Scratch, the pseudocode from Algorithm 1, would give rise to the Scratch block structure as seen in Figure 2.

## 3.3 Using Python

While in the basic education Mathematics curriculum, the programming language indicated is Scratch, in secondary education it becomes Python. Let's see a program to explore CC in Python, for example, we could consider the pseudocode in Algorithm 1, resulting in the code presented in Listing 1, which corresponds to the proposal we indicated for Scratch.

```python
i = 0  #increment variable definition
number = int(input("What's the number?"))  #setting input value to integer
while number!=1: #start of cycle
    i+=1
    if number%2==0: #logical test
        print("It is an even number, so its half is computed")
        number=int(number/2)
        print(number)
    else: print("I is an odd number, so it is trippled and add 1")
        number=int(number*3+1)
        print(number)
    print("The number of iterations was",i) #show number of iterations
```

Listing 1: Python code

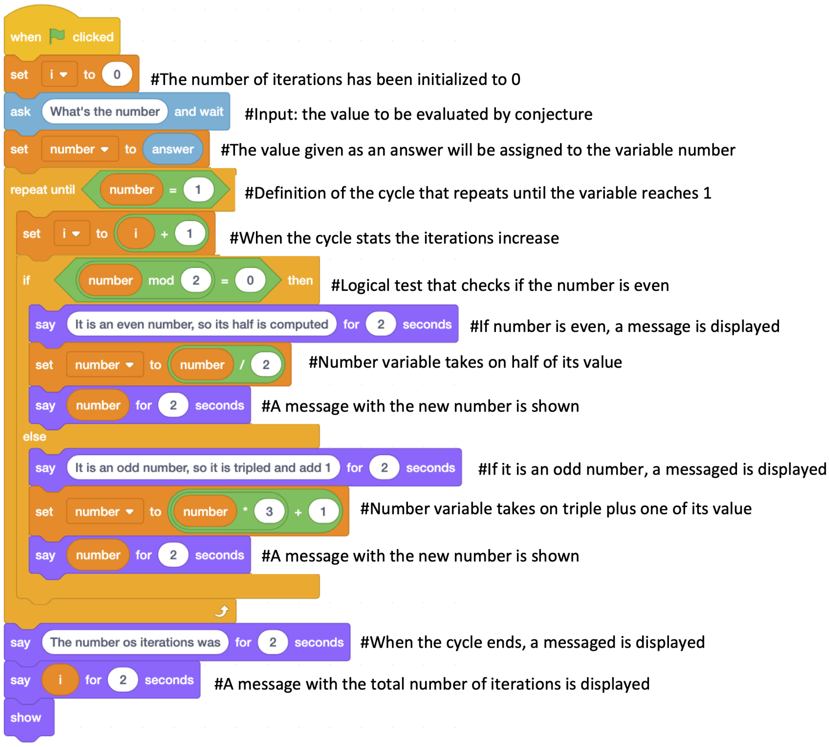


Figure 2: Scratch blocks code to generate the Collatz sequence.

In Portuguese high schools students should use graphical calculators, mandatory by the curriculum, and all recent and mostly used models, including Python programming language, as shown in two examples of Figure 3.

The current version of GeoGebra does not allow programming in Python, that is, the implementation of the CC algorithm must use the JavaScript language, so we have a new proposal for the implementation of CC in high school.

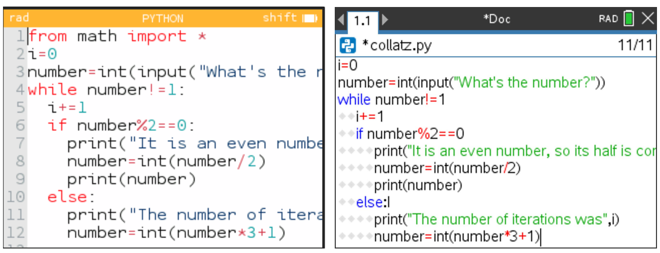


Figure 3: Screens of Python algorithm in Graphics Calculators

## 3.4 Using JavaScript in GeoGebra

GeoGebra was already used at the higher education level [18] to work the CC using the `IterationList` command. In this proposal, the `IterationList` command applies a recurrence law in a function, defined by branches, to the last value generated, and it is necessary to indicate the number of iterations to be applied, running the risk of process continuity if the value 1 is obtained or the defined number of iterations is not sufficient.

Since students also can have GeoGebra on their mobile phones, the teacher could choose to maintain the same work ecosystem in the math room. In this case, teachers would have to propose programming in the JavaScript language.

Implementing CC in GeoGebra using JavaScript can be done starting by defining a natural parameter `n` (value to be tested) and, within the scripting property, in the `On Update` option (see Figure 4), the following JavaScript code (in Listing 2) must be inserted.

With this code, when the value `n` is updated the "while" cycle starts and, when it ends, a text box with the entire sequence will be shown in the application window as well as the parameter `It` that contains the extension of the Collatz sequence.

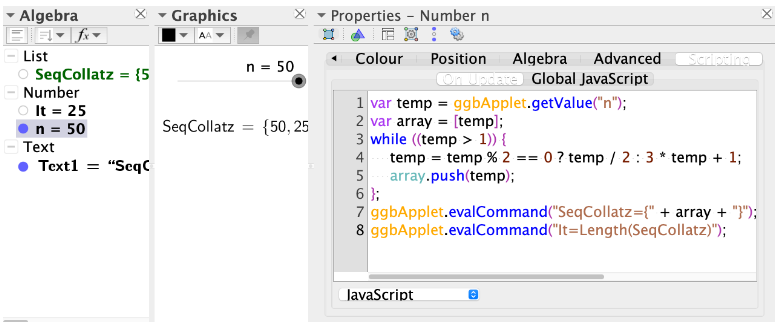


Figure 4: Global JavaScript GeoGebra applet.

```javascript
var temp = ggbApplet.getValue("n");  //input numerical value to be evaluated by the conjecture
var k = 0;   //the number of iterations has been initialized to 0
var array = [temp];  //creating a list that will start with the value of temp
while ((temp > 1)) {  //cycle that only ends when the value 1 is obtained
    k++;              //variable k is incremented by 1 value
    temp = temp % 2 == 0 ? temp / 2 : 3 * temp + 1;  //logic test
    array.push(temp);  //inclusion in the list of the obtained temp value};
ggbApplet.evalCommand("SeqCollatz={" + array + "}"); //inclusion of the list in the GGB application
ggbApplet.evalCommand("It=Length(SeqCollatz)"); //inclusion of It in the GGB application
```

Listing 2: JavaScript algorithm.

# 4 Conclusions

Throughout the previous sections our arguments show that it is feasible to implement, in the classroom, a chain of tasks on an open mathematical problem, as is the case of the CC using CT. This problem can be revisited throughout compulsory schooling, according to the Mathematics curriculum in Portugal, increasing the degree of programming complexity, allowing the progressive development of abstraction and automation inherent to CT. The different approaches, following our proposal, allow strategies in more advanced years, promoting the development of debugging, promoting the search for procedures increasingly efficient from the point of view of computing, namely when the algorithm is tested for very high numerical values.

From the point of view of the use of GeoGebra, it will be an added value if this software reintegrates the Python language. The approach with the "IterationList" command raises new challenges in the discussion of mathematical knowledge and evidences the technological limitation through the function that is defined.

As a support for this work, presented on May 13, 2022, it is possible to highlight the proposal in public consultation of EML for secondary education in Portugal, which was made public on June 7, 2022, and which provides an example of application of the CT to the approach to CC, in Python, only in the 11th year of schooling, about *Successions*. However, before formalizing the concept of succession, it is important that students have experience working with sequences including the use of some of the concepts and procedures of Computational Thinking.

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