A Topological View of Curves and Surfaces Inspired by 2D and 3D Locus Problems

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Abstract

Motivated by some locus problems discussed in [8], and further in [9] and [10] for 2*D and* 3*D cases, we explore scenarios in which if two topological manifolds are topologically equivalent with technological tools. We derive some interesting facts about topological equivalence that could be used in the classroom.*

1 Introduction

In a series of papers, [11, 8, 9, 10], we explored problems based on characterizing the loci, in 2D and 3D, generated by moving points that satisfied certain restraining conditions. The problem was originated from the practising exercises of college entrance exam from China.

For example, in [9], we discussed the following 3D locus problem:
Suppose we are given a point $A = (a_1, a_2, a_3) \in \mathbb{R}^3$ *and a generic point* $C = (c_1, c_2, c_3)$ *on a surface* $\Sigma \subset \mathbb{R}^3$*. Let* l *be the line in* $\mathbb{R}^3$ *passing through* A *and* C*. Suppose the line* l *intersects a well-defined point* $D = (d_1, d_2, d_3)$ *on the surface* Σ*. Thus, we want to determine the locus surface generated by a point* $E = (e_1, e_2, e_3) \in \mathbb{R}^3$*, lying on the line segment* CD*, and satisfying the condition* $\overrightarrow{ED} = s\,\overrightarrow{CD}$*, for a parameter* $s \in \mathbb{R}$*, when*

1. *The point* A *moves in a certain permissible manner.*
2. *The points* C*, or* D *on* Σ *are moved in a definite manner.*
3. *The parameter* s *changes continuously in some interval of real numbers.*

Remark that moving point A *is equivalent to moving point* $D \in \Sigma$ *since this last point is determined by* A *and* C*, through line* l*. Thus, in some situations we can define point* D *as the "antipodal" point of* C*, relative to point* A*. We will illustrate this situation with several examples.*

Notice that if we denote by O the origin in $\mathbb{R}^3$, then, from the parallelogram law, we conclude that this problem is equivalent to that of determining the locus generated by the point E satisfying

$$\overrightarrow{OE} = s\,\overrightarrow{OC} + (1-s)\,\overrightarrow{OD}. \tag{1}$$

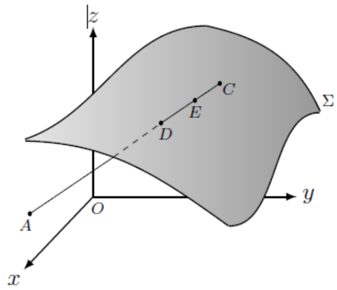

Figure 1. A 3D locus problem

See Figure 1, in order to have a better understanding of this problem.

In this paper, we continue exploring some scenarios related to this 3D locus problem, as described, but with an additional concern, namely, we are interested in determining the topological equivalence for different geometric loci generated by varying the parameter s in equation (1). Needless to say that using technological tools is a paramount part of our explorations and give rise to some interesting situations, worthy of investigation.

Thus, the plan for the paper is the following: In Section 2 we review some of the possible scenarios related to the settings that can be derived from equation (1) for different values of parameter s, and present some interesting examples. In Section 3, we introduce some topological concepts in order discuss the topological equivalence of some of the generated loci, both in 2D and 3D. We also present some examples. In Section 4 we continue the discussion of topological equivalence, but for some concrete examples.

It is important to remark that the use of technological tools is a relevant part in all the explorations and investigations presented in the paper. Thus, we provide all the computer programs and activities in the section of Supplementary Electronic Materials. The software used here correspond to Computer Algebra Systems such as Maple ([2]), alongside with Dynamic Geometry Software such as GeoGebra ([1]). All of the electronic materials provided are intended to give the reader a deeper insight and understanding of the different scenarios addressed in the paper.

2 Some curves and surfaces derived from locus problems

Equation (1) is a good start to explore some scenarios that can occur, and are directly related, to the locus problem just described above, however we must define some terminology, specifically antipodal point, and the types of surfaces we will be dealing with.

2.1 Some definitions

Definition 1 *Let* $R = \{(u, v) \mid u, v \in \mathbb{R}\} \subset \mathbb{R}^2$ *be a domain. The* parametric surface Σ *defined by the injective, real-valued coordinates functions* $x = x(u, v)$*,* $y = y(u, v)$*, and* $z = z(u, v)$ *is the set*

$$\Sigma = \{(x(u, v), y(u, v), z(u, v)) \mid (u, v) \in R\} \subset \mathbb{R}^3.$$

Usually, the coordinate functions in Definition 1 are assumed to be sufficiently smooth (in the sense of differentiability), so the parametrized surface is a smooth one. Moreover, most of the surfaces in the examples in this paper are *closed surfaces*, that is, compact and without boundary, examples of which are the sphere, the torus, the Klein bottle, etcetera. In addition, if the tangent space of every point lies the same side of Σ, we call Σ convex surface. Unless otherwise specified in this paper, we assume that the surface $\Sigma \subset \mathbb{R}^3$ is closed and convex.

Definition 2 *Let* $\Sigma \subset \mathbb{R}^3$ *be a closed and convex surface such that* A *is a point in the interior of* Σ*. Let* C *be an arbitrary point on the surface* Σ *and let us denote by* l *the straight line through points* A *and* C*. We say that a point* D *and* $D \neq C$ *in* Σ *is the* antipodal point *of* C *relative to* A *if* D *is the intersection of* l *and* Σ*.*

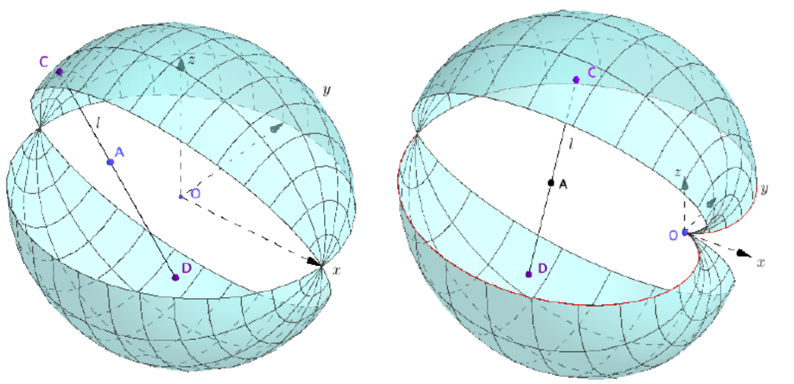

Figure 2(a). Antipodal points on a sphere

Figure 2(b). Antipodal points on a surface of revolution

Figure 2(a) illustrates the situation described in Definition 2. Notice that when the surface Σ is symmetrical about the origin and $A = (0, 0, 0)$, then antipodal points are easy to calculate. Moreover, even when the surface is not a convex one, but it is non-self intersecting, antipodal points can also be calculated because Definition 2 is relative to a given point, inside the surface, as shown in Figure 2(b).

Theorem 3 *Let* $\Sigma \subset \mathbb{R}^3$ *be a parametrized, closed and convex surface and let* $A \in \mathbb{R}^3$ *be an interior point to* Σ*, that is,* A *is enclosed inside the surface. If* $C \in \Sigma$ *is an arbitrary point, then there exists a point* $D \in \Sigma$ *such that* D *is the antipodal point of* C *relative to* A*.*

Proof. Let $\varphi : R \to \mathbb{R}^3$ be a parametrization of surface Σ with $\varphi(R) = \Sigma$ and $\varphi(u,v) = (x(u,v), y(u,v), z(u,v)) \in \Sigma$, for all $(u,v) \in R$ in the sense of Definition 2. Since $C = (c_1, c_2, c_3) \in \Sigma$, there exists a point $(u_0, v_0) \in R$ such that $\varphi(u_0, v_0) = C$, and thus

$$x(u_0, v_0) = c_1, \qquad y(u_0, v_0) = c_2, \qquad z(u_0, v_0) = c_3.$$

On the other hand, the straight line connecting points C and A is given, in parametric form, by $l : C + t(C - A)$, for $t \in \mathbb{R}$. Thus, if $A = (a_1, a_2, a_3)$, then we need to find a value of the parameter t, say $t_0 \neq 0$, such that

$$\big((1+t_0)c_1 - t_0 a_1, (1+t_0)c_2 - t_0 a_2, (1+t_0)c_3 - t_0 a_3\big) \in \Sigma.$$

Now, consider the plane Π_{AOC} passing by the three points C, A, and the origin $O = (0,0,0)$. We can calculate the equation for this plane in the form

$$ax + by + cz + d = 0.$$

The plane Π_{AOC} contains line l and defines a closed curve on the surface Σ, say $\alpha = \alpha(s)$, which is the intersection of Σ with Π_{AOC}. Here $s \in \mathbb{R}$ is the parameter that defines the curve $\alpha : I \to \mathbb{R}^3$ for some interval $I \subset \mathbb{R}$. Therefore, there exist values of the parameter s, say s_1 and s_2 such that $\alpha(s_1) = C$ and $\alpha(s_2)$ is a second point of intersection of line l, plane Π_{AOC} and the surface Σ. We denote this point by D: $\alpha(s_2) = D = (d_1, d_2, d_3) \in \Sigma$. We also have

$$ad_1 + bd_2 + cd_3 + d = 0,$$

and there exists $(u_1, v_1) \in D$ such that $\varphi(u_1, v_1) = (d_1, d_2, d_3)$ and we can solve for t_0 in any of the three equations

$$\begin{aligned}(1+t_0)c_1 - t_0 a_1 &= d_1,\\ (1+t_0)c_2 - t_0 a_2 &= d_2,\\ (1+t_0)c_3 - t_0 a_3 &= d_3.\end{aligned}$$

We illustrate some examples of this demonstration with a DGS (see [S1],[S2], and [S3]) in the Supplementary Electronic Materials. ■

2.2 Several scenarios for study

Following the above cited papers [6, 7, 8, 9, 10, 11], the locus constructions for surfaces are natural generalizations of those corresponding to curves in the plane. Thus, we also will be making use of several $2D$ examples in order to illustrate some of the concepts we will be dealing with. In particular, we will talk about antipodal points on curves and loci in $2D$.

Now, let us return back to the situation of equation (1) in order to depict some scenarios that are worthy to explore, which are the main motivations for this paper, and that will be addressed in what follows.

1. Suppose we fix both, the point $C \in \Sigma$ and the parameter s in (1). Let D be the antipodal point of C, relative to the fixed point A, when connected by a line l through points A and C. Thus, when we consider the locus of E satisfying $\overrightarrow{ED} = s\,\overrightarrow{CD}$, which is equivalent to (1), it is not difficult to figure out that the locus E represents a shifted and scaled surface from Σ. In fact, think that the only point freely to move around the whole surface is D, and it is joined with E through a line segment.

2. Clearly, from the above description of the problem, there is a direct dependence of the locus generated by E in (1) on the points C and D, even when the parameter s varies. We shall see how a curve or surface determined by the point C is transformed to another curve or surface determined by D.

In what follows, we will have opportunity for exploring each one of the situations described above. Nevertheless, as a motivation of what we intend to do, we provide an illustrative $2D$ example in the following subsection.

2.3 Scaled and shifted locus

Suppose $\Sigma \subset \mathbb{R}^3$ is a regular surface, and let C be a fixed point on Σ. For any point $A \neq C$, let D be the antipodal point of C on the surface Σ when connected by a line l through points A and C. We notice that locus of the point E satisfying $\overrightarrow{OE} = s\,\overrightarrow{OC} + (1-s)\,\overrightarrow{OD}$ can be characterized as:

1. A dilation carried out by the point D for $1 - s > 0$. When this number is negative, the locus is a reflection of point $|1 - s|D$ by the origin.

2. It is a translation defined by the vector $s\,\overrightarrow{OC}$.

We remark that a transformation of a curve c to any other curve is not unique. Thus, we have the opportunity to talk about topological transformations of curves and surfaces, a topic that we address in the following section.

3 Locus and topological transformations

Recall that a topological space is a non-empty set together with a topological structure that characterizes the open sets of the topology. A non-empty set X can be equipped with different topological structures, and any subset $A \subset X$ can also be given a topological structure, namely, the *relative topology*: If $O \subset X$ is any open set in the topology of X, then $A \cap O$ will be the corresponding open sets in the relative topology of A.

For the spaces $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, the usual topological structure is that of the open balls, which is defined by the metric structure of $\mathbb{R}^n$. Recall that the *metric function* $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ makes $(\mathbb{R}^n, d)$ into a *metric space*. This metric characterizes the open balls $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, for any $\varepsilon \in \mathbb{R}$, with $\varepsilon > 0$, which are the basic open sets for the usual topological structure of $\mathbb{R}^n$.

Thus, any curve or any hyper-surface in $\mathbb{R}^n$ can be viewed as topological spaces with their respective relative topologies, acquired from their environment space. For the cases we are interested on, the environment spaces will be simply $\mathbb{R}^2$ or $\mathbb{R}^3$; thus, we can view these curves and surfaces as topological spaces on their own, for which we can apply the topological tools that are introduced in what follows.

Now, let us address a fundamental concept in topology, namely that of *continuity*. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, with τ_X and τ_Y their respective topologies. A function $f : X \to Y$ is *continuous* if for any open set $O \subset Y$, then $f^{-1}(O) \subset X$ is open. Here, the notation $f^{-1}(U)$ denotes the inverse image, in X, of $U \subset Y$.

When f is a one-to-one and an onto correspondence between X and Y, then it has an inverse, $f^{-1} : Y \to X$ defined, for any $y \in Y$ as $f^{-1}(y) = x \in X$ if and only if $f(x) = y$. We say that f is a *homeomorphism* if it is continuos, bijective and with f^{-1} also continuos. In this situation, we say that X and Y are *homeomorphic* topological spaces.

Homeomorphisms between topological spaces are also called *continuous transformations*, or *topological transformations*. Intuitively, these transformations can be thought of as functions that transforms points of one space, that are arbitrarily close to each other, onto points of another space that are also arbitrarily close. Spaces that are related in this way are said to be *topologically equivalent*. In a technical sense, we have to be careful with the notion of closeness since it has only sense when we can measure distances in a space, so we need to have a metric structure. Nevertheless, that is what we meant by "intuitively".

If a space or some region in a space is transformed into another equivalent space or region by bending, stretching, etc., the change is a special type of topological transformation called a *continuous deformation*. Two figures (e.g., certain types of knots) may be topologically equivalent, however, without being changeable into one another by a continuous deformation.

It is intuitively evident that all simple closed curves in the plane and all closed polygons are topologically equivalent to a circle. Similarly, all closed cylinders, cones, convex polyhedra, and other simple closed surfaces are equivalent to a sphere. On the other hand, a closed surface such as a torus (a doughnut) is not equivalent to a sphere, since no amount of bending or stretching will make it into a sphere, nor is a surface with a boundary equivalent to a sphere, e.g., a cylinder with an open top, which may be stretched into a disk (a circle plus its interior).

In view of the locus generated in the settings of our problem by a point E satisfying 1: For a given fixed A, we can find the maximum value of s such that the locus is a simple closed curve in $2D$ or a surface that is topologically equivalent to a sphere in $3D$. Subsequently, we extend the idea of finding a new surface generated by two or more surfaces.

3.1 Transforming topologically one simple closed curve into another

In the settings of our exploratory investigations, we will show how a circle or a sphere can be transformed to another simple closed curve or into another surface that is topologically equivalent to the sphere. First, we consider the following elementary $2D$ scenarios.

Example 4 *Let us denote by* $c = c(u) = \big[x(u), y(u))\big] = \big[(1-\cos u)\cos u, (1-\cos u)\sin u\big] \in \mathbb{R}^2$ *the parametrized cardioid curve, with* $u \in [0, 2\pi]$*. Next, denote by* $d = d(u) = [x'(u), y'(u)] = \big[\cos u, \sin u\big] \in \mathbb{R}^2$ *the curve that parametrizes the unit circle.*

Now let us pick up any arbitrary point C *on the curve* c*, and similarly, let us select any* *point D on the unit circle d. For a parameter $s \in [0,1]$, define the transformation T*

$$\begin{pmatrix} s \\ u \end{pmatrix} \xrightarrow{T} \begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix}$$

where

$$\begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix} = s \begin{bmatrix} x(u) \\ y(u) \end{bmatrix} + (1-s) \begin{bmatrix} x'(u) \\ y'(u) \end{bmatrix}. \tag{2}$$

Notice that if $O = (0,0)$ denotes the origin in $\mathbb{R}^2$, and setting

$$OE = \begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix}, \qquad OC = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix}, \quad \text{and} \quad OD = \begin{bmatrix} x'(u) \\ y'(u) \end{bmatrix}$$

then (2) is nothing but our well known relationship (1):

$$OE = s\,OC + (1-s)\,OD,$$

which allow us to express (2) in explicit terms as

$$OE = \left[\begin{array}{c} F(s,u) \\ G(s,u) \end{array} \right] = s\,OC + (1-s)\,OD = s \left[\begin{array}{c} (1-\cos u)\cos u \\ (1-\cos u)\sin u \end{array} \right] + (1-s) \left[\begin{array}{c} \cos u \\ \sin u \end{array} \right]. \tag{3}$$

Clearly, for $s = 1$ we have that the image of transformation T is the cardioid c and for $s = 0$ the image of T is the unit circle d. Also, it is clear that T is a continuous transformation that transforms, point by point, the cardioid into the unit circle, as s varies from 0 to 1. Accordingly, T^{-1} is a continuous transformation from the unit circle to the cardioid. Therefore, we can say that a simple closed cardioid is topologically equivalent to the unit circle.

In Figure 3, we can appreciate how the cardioid is transformed, according to (2) and (3), into the unit circle, for different values of the parameter s. Remark that the symmetry with respect to the x-axis is conserved. [See [S4] for exploration]

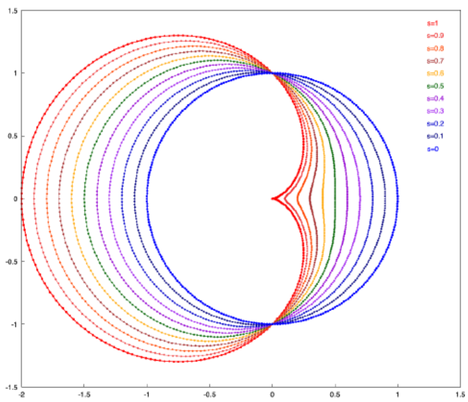

Figure 3. Continuos deformation of a cardioid into a circle

Exercise: We leave it to the reader the exercise of exploring how to find an appropriate transformation between a circle and a cardioid that has being shifted to the right by one unit from the original one, given by the parametric functions $x(u) = (1 - \cos u)\cos u$ and $y(u) = (1 - \cos u)\sin u$. Readers would find that the correct circle to be chosen for this transformation should be the one shifted to the right by one unit from the original unit circle.

Example 5 *For this example we interchange the role of the unit circle and its equivalent topological curve, which in this case, is a* 3*-cusp epicycloid or trefoil [4]. Thus, let* $c = c(u) = [\cos u, \sin u]$ *be the unit circle, and let* $d = d(u) = [8\cos u - 2\cos 4u, 8\sin u - 2\sin 4u]$*, where* $u \in [0, 2\pi]$*, be a parametrization for the epicycloid, see Figure 4.*

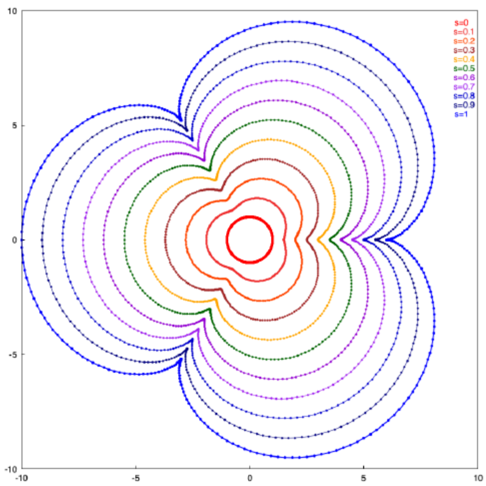

Figure 4. The unit circle is topologically equivalento to a 3-cusp epicycloid

Again, our main tool is equation (1): $OE = s\,OC + (1-s)OD$*, with* C*, and* D*, arbitrary points on curves* c *and* d*, respectively. Therefore, the locus generated by point* E *is satisfies the following relation,*

$$OE = \begin{bmatrix} F(u,s) \\ G(u,s) \end{bmatrix} = s\,OC + (1-s)OD = s\begin{bmatrix} \cos u \\ \sin u \end{bmatrix} + (1-s)\begin{bmatrix} 8\cos u - 2\cos 4u \\ 8\sin u - 2\sin 4u \end{bmatrix}. \qquad (4)$$

It is clear, from (4), that for $s = 1$ *we have the unit circle, and for* $s = 0$ *we get the* 3*-cusp epicycloid (shown in red) in Figures 5(a)-(c).*

Thus, equation (4) is an example of an application that transforms the unit circle into an epicycloid and it is also an example of a homeomorphism. Therefore this shows that the unit circle is topologically equivalent to the 3*-cusp epicycloid. The animations by using the parameter* s *(See Figures 5 below). Also, we refer to [S5] for animation and [S6] of using the CAS [2] for* *exploration. It is clear to see that the transformation is a homeomorphism.*

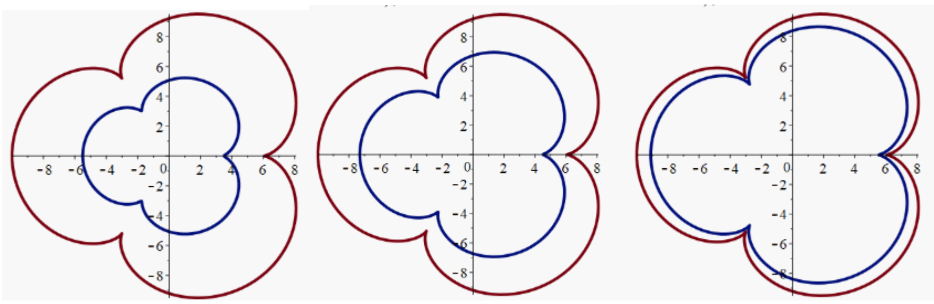

Figure 5(a). When $s = 0.5$ *and* $s = 0$ *Figure 5(b). When* $s = 0.7$ *and* $s = 0$ *Figure 5(c). When* $s = 0.9$ *and* $s = 0$

3.2 Maximum value of parameter s for topological equivalence

As it is discussed in [9] and [10], the locus generated by the point E on an ellipsoid depends on the location of the fixed point A and the parameter s. In [10], we have proved that when the fixed A is at an infinity, and $s > 0$ is a finite real number, the locus is an ellipsoid, which we denote it by $E(s)$, and is topologically equivalent to a sphere. However, when A is at an infinity and $s \to \infty$, then the locus becomes an elliptical cylinder, say we denote it by $E_c(s)$. Therefore, there exists an s^* so that it makes the ellipsoid $E(s)$ when $s < s^*$ topologically unequivalent to elliptical cylinder $E_c(s)$ when $s > s^*$. However, finding such maximum value of s^* will need further investigation.

In the remaining paper, we shall address the situation for a given fixed point A, **not at an infinity**.

4 Surfaces parametrized by longitude and colatitude

Recall that any surface is a topological space whose topological structure is acquired from the ambient space $\mathbb{R}^3$ it lives on. Also, two surfaces are topologically equivalent if there exists a homeomorphism f between them, viewed as two topological spaces. Such a homeomorphism f is a bijection with the property that both f and its inverse f^{-1} preserve open sets. If a certain geometric object is transformed into another topologically equivalent by bending, stretching, etcetera, this process entails a special type of topological transformation called a *continuous deformation.*

Actually, we could refer to the surfaces we are working with in this paper as *topological manifolds* since, as subsets of $\mathbb{R}^3$, they comply with all the conditions for that kind of manifolds: being Hausdorff topological spaces, with a covering of open sets, each one homeomorphic to an open set in $\mathbb{R}^3$ and in the intersections of the sets of the covering, the *transition functions* are also homeomorphisms from a subset of the surface onto another subset of the surface. Thus, on what follows, we will be dealing with surfaces that are topological manifolds and we will refer to them with such a terminology.

4.1 New surfaces from old ones

Suppose S is a given parametrized surface, say

$$S = \left\{ \left(f(u,v), g(u,v), h(u,v) \right) \mid u, v \in R \right\}, \tag{5}$$

for a domain $R \subset \mathbb{R}^2$, and real valued continuos functions f, g, and h defined on R. Usually the values of the parameters will be $0 \leq u \leq 2\pi$, and $0 \leq v \leq \pi$, hence we will refer to them as *longitude* and *colatitude*, respectively.

Now, with the same parametrization (5) for S we can consider some other surfaces, namely

$$S_1 = \left\{ \left(f(u+\alpha_1, v), g(u+\alpha_2, v), h(u+\alpha_3, v) \right) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\},$$

or

$$S_2 = \left\{ \left(f(u, v+\beta_1), g(u, v+\beta_2), h(u, v+\beta_3) \right) \mid \beta_1, \beta_2, \beta_3 \in \mathbb{R} \right\}.$$

We remark how the new parameters α_i and β_i, for $i = 1, 2, 3$, change the longitude u and colatitude v. It is not difficult to get aware that the new surface S_1 is topologically equivalent to S for $u \in [0, 2\pi]$ since the change of the coordinate of latitude in f, g and h is a continuos operation, and these functions are also continuos.

Remark: Although the new surface S_2 is a rotation of S by the colatitude v, we shall see that surfaces S_1 and S_2 may or may not be topologically equivalent to the surface S, which depends on if we choose the range for $v \in [0, \pi]$ or $v \in [0, 2\pi]$, respectively.

We recall that our initial equation (1), that was also applied in (3) and (4), now we explore another surface resembling the processes described above for curves. In fact, we ask for the properties of a new surface defined as the locus generated by points in the following way:

$$\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix} = s \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + (1-s) \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix} \tag{6}$$

Here, and only for more clarity, we have denoted the points in $\mathbb{R}^3$ in vertical format. We will continue using this notation when it becomes necessary in what follows.

Nevertheless, we will have the opportunity to see that the scaling factor s plays a crucial role in (6). in fact, the following example shows that it is important to choose proper ranges for u and v, as well as specific angles of α_i, β_i with $i = 1, 2, 3$ before determining if two surfaces in (5) and (6) are topologically equivalent, since s also plays its role.

Example 6 *Let us start with the surface of a parametrized ellipsoid*

$$S = \left\{ \left(f(u,v), g(u,v), h(u,v) \right) \right\} = \left\{ \begin{bmatrix} a\sin(v)\cos(u) \\ b\sin(v)\sin(u) \\ c\cos(v) \end{bmatrix} \mid u \in [0, 2\pi], v \in [0, \pi], a, b, c > 0 \right\}. \tag{7}$$

By selecting particular angles α_i and β_i, with $i = 1, 2, 3$, respectively, we form the new generated surface

$$S_1 = \left\{ \left(F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s) \right) \right\},$$

via the transformation (6)

$$\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix} = s \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + (1-s) \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}, \tag{8}$$

where $s \in \mathbb{R}$.
Notice that for $s = 1$ *in (8) we have the original ellipsoid (7), and for* $s = 0$ *in (8) we get the surface defined by the points of* $\mathbb{R}^3$ *whose coordinates are of the form*

$$\begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}.$$

We shall discuss some topological properties for the surface defined through (8), since the selection of the values of the parameters involved is quite important. We do so in the following subsection.

5 Discussions

The settings in this subsection are those of Example 6, where the surface S is a parametrized ellipsoid, with the parametrization given by (7). For demonstration purpose, we choose $a = 5, b = 4$ and $c = 3$. Furthermore, we assume $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, and $\beta_3 = \frac{\pi}{4}$.

5.1 When considering $u \in [0, 2\pi]$, and $v \in [0, \pi]$

1. If we let $u \in [0, 2\pi]$, and $v \in [0, \pi]$ with $s = 0$, according to (8), we obtain the surface generated by points of $\mathbb{R}^3$ that have coordinates given by

$$\begin{bmatrix} F(u,v,\pi/4,\pi/2,0) \\ G(u,v,\pi/3,\pi/3,0) \\ H(u,v,\pi/6,\pi/4,0) \end{bmatrix} = \begin{bmatrix} f(u+\pi/4, v+\pi/2) \\ g(u+\pi/3, v+\pi/3) \\ h(u+\pi/6, v+\pi/4) \end{bmatrix} = \begin{bmatrix} 5\cos(u+\pi/4)\sin(v+\pi/2) \\ 4\sin(u+\pi/3)\sin(v+\pi/3) \\ 3\cos(v+\pi/4) \end{bmatrix}.$$

   This surface is a continuous deformation of the original ellipsoid, and it is topologically equivalent to the ellipsoid.

2. If we consider $u \in [0, 2\pi]$ and $v \in [0, \pi]$ for the surface $\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix}$, we leave it to the readers to explore that it will not be topologically equivalent to the original surface S unless $s = 1$ or $\beta_i = 0$ for all $i = 1, 2, 3,$. We depict pictures with various values of s

in Figures 6(a)(b) below. See [S8] or [S9] for explorations.

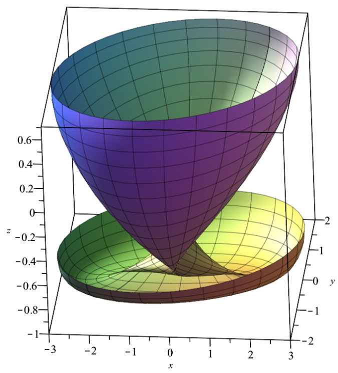

Figure 6(a). The surface when $s = 0$.

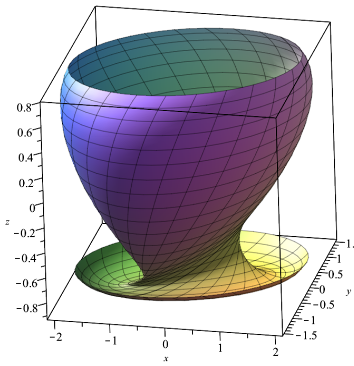

Figure 6(b). Transformation when $s = \frac{1}{3}$

5.2 Two copies of surfaces when considering $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$

1. If we consider the parameters $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$, we can think the surface $\begin{bmatrix} F(u, v, \alpha_1, \beta_1, 1) \\ G(u, v, \alpha_2, \beta_2, 1) \\ H(u, v, \alpha_3, \beta_3, 1) \end{bmatrix} = \begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} 5\sin(v)\cos(u); \\ 4\sin(v)\sin(u) \\ 3\cos(v) \end{bmatrix}$ to be two copies of the ellipsoid S, which we denote it by S^*.

2. However, when we vary the scalar s, we shall investigate the respective surfaces further. For example, when $s = \frac{1}{6}$, the surface $\begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ G(u, v, \alpha_2, \beta_2, \frac{1}{6}) \\ H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \end{bmatrix}$, $u, v \in [0, 2\pi]$, shown the Figure 7(a) is a surface that may not be topologically equivalent to S^*. Thus, we investigate the surface closer as follows:

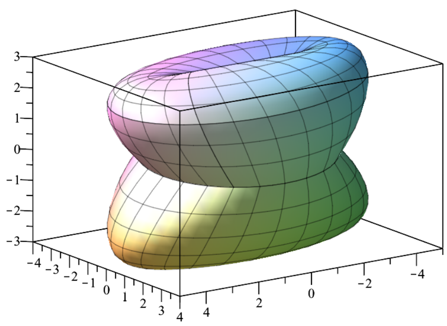

Figure 7(a). A closed surface but may not be topologically equivalent to S^* for $s = \frac{1}{6}$.

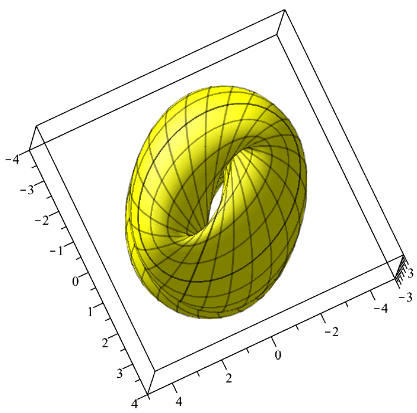

Figure 7(b). Not topologically equivalent to S^* for $s = \frac{1}{3}$.

(a) For $s = \frac{1}{6}$,

$$S_1 = \left\{ \left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_1, \beta_1, \frac{1}{6}), H(u, v, \alpha_1, \beta_1, \frac{1}{6}) \right] : u \in [0, 2\pi], v \in [0, 2\pi] \right\} \tag{9}$$

as the union of two copies of

$$S_1^t = \left\{ \left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_1, \beta_1, \frac{1}{6}), H(u, v, \alpha_1, \beta_1, \frac{1}{6}) \right] : u \in [0, 2\pi], v \in [0, \pi] \right\} \tag{10}$$

and

$$S_1^b = \left\{ \left[F(u, v, \alpha_1, \beta_1, \frac{1}{6}), G(u, v, \alpha_1, \beta_1, \frac{1}{6}), H(u, v, \alpha_1, \beta_1, \frac{1}{6}) \right] : u \in [0, 2\pi], v \in [\pi, 2\pi] \right\}. \tag{11}$$

In other words, we have

$$S_1 = S_1^t \cup S_1^b. \tag{12}$$

We can think of S_1 (See Figure 8(c)) as the union of two open tea-cups (S_1^t is the yellow in Figure 8(a), and S_1^b is the blue in Figure 8(b)).

(b) However, it is beyond the scope of this paper to determine if S_1^t or S_1^b is an orientable surface. Readers can explore further in [S8].

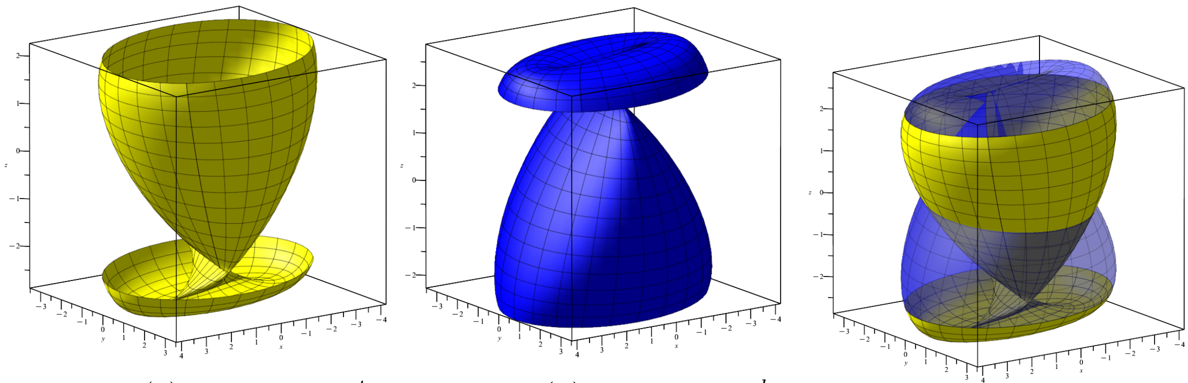

Figure 8(a). Graph of S_1^t Figure 8(b). Graph of S_1^b Figure 8(c). Graph of $S_1^t \cup S_1^b$

3. When $s = \frac{1}{3}$, the surface $\begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{3}) \\ G(u, v, \alpha_2, \beta_2, \frac{1}{3}) \\ H(u, v, \alpha_3, \beta_3, \frac{1}{3}) \end{bmatrix}$, $u, v \in [0, 2\pi]$, is a surface (shown the Figure 7(b)) that is not topologically equivalent to S^*. **Therefore, for a given set of values of α_i and β_i, one may ask how to find the maximum value of $s \in (0, 1)$ such that the surface $\begin{bmatrix} F(u, v, \alpha_1, \beta_1, s) \\ G(u, v, \alpha_2, \beta_2, s) \\ H(u, v, \alpha_3, \beta_3, s) \end{bmatrix}$ is still topologically equivalent to the original ellipsoid.** See [S7] or [S8] for explorations.

6 Future works

There are several instances one can investigate further, which we describe below:

1. Suppose we are given a surface Σ and a point $C \in \Sigma$, with D its the antipodal point when connected by a line l passing through point C and a fixed point A. In this situation, we shall explore the following scenario: For a given Σ we select three vectors AB, AC and AD, defined by the three points B, C and D, respectively, which lie on surface Σ. We will investigate the new generated affine surface

$$\Sigma' = r \cdot AB + s \cdot AC + t \cdot AD, \tag{13}$$

where $r, s, t \in \mathbb{R}$, satisfying $r + s + t = 1$.

2. One can investigate a new surface that is generated by three vectors that satisfies an affine combination. We start with a surface S of $\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix}$, then by picking particular angles α and β respectively, we form the new generated surface via the following transformation:

$$\begin{bmatrix} F(u,v,\alpha,\beta) \\ G(u,v,\alpha,\beta) \\ H(u,v,\alpha,\beta) \end{bmatrix} = r \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + s \begin{bmatrix} f(u+\alpha,v) \\ g(u+\alpha,v) \\ h(u+\alpha,v) \end{bmatrix} + t \begin{bmatrix} f(u,v+\beta) \\ g(u,v+\beta) \\ h(u,v+\beta) \end{bmatrix}, \tag{14}$$

where $r, s, t \in \mathbb{R}$ with the **affine combination condition** of

$$r + s + t = 1. \tag{15}$$

We believe such transformation is valuable in computer graphics and other areas because of the scaling factors of r, s, t and rotation factors of angles α and β. We can also view the new generated surface as a linear combination of the vectors, $\overrightarrow{OP}, \overrightarrow{OQ}$ and $\overrightarrow{OR}$, where $\overrightarrow{OP} = [f(u,v), g(u,v), h(u,v)]$, $\overrightarrow{OQ} = [f(u+\alpha,v), g(u+\alpha,v), g(u+\alpha,v)]$, and $\overrightarrow{OR} = [f(u,v+\beta), g(u,v+\beta), h(u,v+\beta)]$.

3. We may investigate topological surfaces that involve a torus under the transformed surface defined by 14. For example, we consider the surface of a torus $\begin{bmatrix} (a+b\cos u)\cos v \\ (a+b\cos u)\sin v \\ c\sin u \end{bmatrix}$ for $u \in [0,2\pi), v \in [0,2\pi)$, and consider the generated surface $\begin{bmatrix} F(u,v,\alpha,\beta) \\ G(u,v,\alpha,\beta) \\ H(u,v,\alpha,\beta) \end{bmatrix} = r \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + s \begin{bmatrix} f(u+\alpha,v) \\ g(u+\alpha,v) \\ h(u+\alpha,v) \end{bmatrix} + t \begin{bmatrix} f(u,v+\beta) \\ g(u,v+\beta) \\ h(u,v+\beta) \end{bmatrix}$, where $\alpha \in (0,2\pi)$, $\beta \in \left(0,\frac{\pi}{2}\right)$, $r, s, t \in \mathbb{R}$ with $r + s + t = 1$.

7 Conclusions

It is exciting to see that a series of papers, [8], [9], [10], [11], and [12], which was originated from a simple college entrance exam from China, have utilized ideas from widely-ranging and of different fields such as algebraic geometry, projective geometry, differential geometry and etc. Many will agree that one objective of exploring mathematics with technological tools is to inspire undergraduate or graduate students to expand their content knowledge in broader fields.

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. In the meantime, we use a CAS, for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school, which are excellent examples for professional trainings for future teachers.

Evolving technological tools definitely make mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. Encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to explore challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

8 Supplementary Electronic Materials

[S1] GeoGebra worksheet, SuplMat2.ggb.

[S2] GeoGebra worksheet, SuplMat5.ggb.

[S3] GeoGebra worksheet, SuplMat6.ggb

[S4] GeoGebra worksheet, SuplMat1.ggb.

[S5] Animation for Example 5, Example5.gif.

[S6] Maple file for Example 5, Example5-cardioid.mw.

[S7] Animation for Section 5.2, Section5_2.gif.

[S8] Maple file for Section 5.2, Section5_2.mw.

[S9] Animation for Section 5.2 when $v \in [0, \pi]$, Section5_2_1.gif.

References

[1] GeoGebra (release 6.0.562 / October 2019), see https://www.geogebra.org/.

[2] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[3] Image of a sphere is an ellipsoid: https://sites.math.northwestern.edu/~clark/354/2002/ellipse.pdf.

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[5] Mathworld: https://mathworld.wolfram.com/QuadraticSurface.html

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[8] Yang, W.-C., *Locus Resulted From Lines Passing Through A Fixed Point And A Closed Curve*, the Electronic Journal of Mathematics and Technology, Volume 14, Number 1, 2020. ISSN 1933-2823, published by Mathematics and Technology, LLC.

[9] Yang, W.-C. & Morante, A., *3D Locus Problems of Lines Passing Through A Fixed Point*, The Electronic Journal of Mathematics and Technology, Volume 15, Number 1, 2021. ISSN 1933-2823, published by Mathematics and Technology, LLC.

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[11] Yang, W.-C., *Exploring Locus Surfaces Involving Pseudo Antipodal Point*, the Electronic Proceedings of the 25th Asian Technology Conference in Mathematics, Published by Mathematics and Technology, LLC, ISSN 1940-4204 (online version), see https://atcm.mathandtech.org/EP2020/abstracts.html#21829.

[12] Yang, W.-C. & Morante, A., *Locus of Antipodal Projection When Fixed Point is Outside a Curve or Surface*, the Electronic Proceedings of the 26th Asian Technology Conference in Mathematics. Published by Mathematics and Technology, LLC, ISSN 1940-4204 (online version).