Understanding Geometric Pattern and its Geometry Part 9 – On walking pentagons and Isfahani inflation

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Abstract: *We introduce Isfahani inflation, a simple inflation technique for decagonal tessellations. We discuss the creation process of geometric patterns using Isfahani inflation. Using examples from Bukhara and Iran, we show how one can reconstruct them with this inflation and produce numerous variants. Some extensions of Isfahani inflation will also be mentioned.*

Introduction

This paper is the 9th paper in the series of articles discussing the geometric features of geometric patterns. All the previous papers were published in eJMT and ATCM proceedings. Thus we will follow the concepts described in these papers without significant repetitions. A complete list of papers in the mentioned series is included in the references section (see [5] ... [12]).

While discussing geometric patterns, some authors claim that they are using, or rediscovered, traditional methods for pattern design, or even better – they discovered secret methods of medieval artists. Here comes a very important question about the word 'traditional.' What does it mean? Can the method used by a parent or grandparent of an author be considered traditional? No one ever agreed on how old a method should be to be treated as traditional. Are 100 years enough? We do not have much information on how in the past, patterns were created. We have scraps of knowledge, and it is incomplete. The famous Topkapi scroll contains drawings, some red lines underlying the geometric structure of the pattern, and no comments. The other existing pattern scrolls are even less informative. Thus whatever we write about them may not be the truth.

We get only speculations from some authors, often supported by doubtful arguments. It is often supposed that mathematicians created geometric patterns. That may not be the truth also. In old Persian and Central Asian mathematical papers and books, we find a continuation of Greek, Persian, or Indian mathematics with theorems, proofs, etc. There are very few medieval documents showing some patterns. We see some patterns in the two books by Abolvafa Mohammad ibn Mohammad Albuzjani. But these are intended as textbooks on geometry for craftsmen. These are not research papers or research books of the mentioned author[1]. It seems that most of the pattern design development was carried on by artists or architects but not by mathematicians.

This paper describes the foundations of a pattern design method called Isfahani inflation. No documents state that such a method was ever used in the past. The author invented the method in 2019 after analyzing several geometric patterns from Isfahan. Thus comes the name. However, all decagonal door patterns from Bukhara can be constructed using this method. The method is very flexible, allowing us to produce incredibly complex geometric patterns. We show here some of them. We do not claim that Isfahani inflation is a traditional method, but probably something similar could exist in the past.

Geometric inflation

In modern mathematics, geometric inflation emerged since aperiodic tessellations, and quasicrystals theory was invented. On the mathematical side, we can find several books and research papers discussing geometric inflation and aperiodic tilings (see [1] and [2]). The mathematics in these books is tough for a non-mathematician. On the popular side, many internet articles and papers are trying to introduce these topics to a wider audience. In this paper, we are not interested in aperiodic tessellations. We are interested exclusively in very simple inflation technique and how it affects geometric pattern design.

We know for sure that some geometric inflation techniques were known to medieval artists or architects. The Topkapi Scroll contains several images where one can decipher a few different inflation techniques (see [4]). Was geometric inflation known to medieval mathematicians? It seems they did not see this topic the same way we do it now. Simply outside of art, there was no serious need for such a mathematical theory. In our times, geometric inflation has many applications forcing the development of its theory.

Let us start by introducing some general notions.

> DEFINITION
>
> Let $\mathcal{A}$ be a set of tiles, and $\mathcal{T}$ a set of tessellations created using tiles from $\mathcal{A}$. An inflation technique is a transformation[2] I from $\mathcal{T}$ to $\mathcal{T}$, i.e., for a given tessellation t from $\mathcal{T}$, $I(t)$ is again an element of $\mathcal{T}$.

In practice, we have to define the set $\mathcal{A}$ and rules showing how we can proceed with the inflation of a given tessellation. Let us look at a very simple example.

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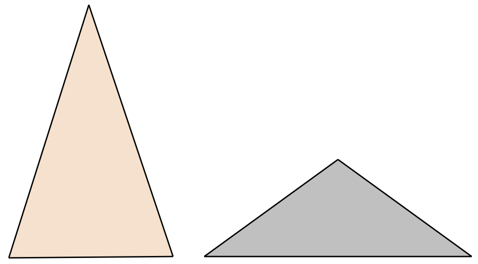
[1] There are a few translations of books by Abolvafa Mohammad ibn Mohammad Albuzjani. Each of them has its own drawbacks. The Iranian translation mixes both books together with some extra illustration. Some of them were copied from Kepler's work. Moreover, the translator does not show the original drawings. Instead of them he published his own drawings and some of them are incorrect. Another translation by Bulatov discusses only one of these books and there no original drawings. All constructions were done by the translator and again some are incorrect.

[2] In mathematical terms inflation should be considered as a '1 to many' relation, rather than a function.

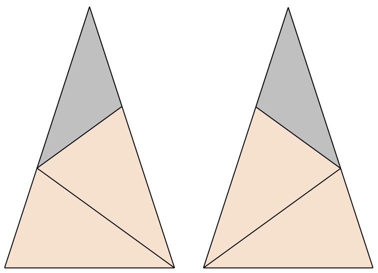
Penrose inflation for triangles



The set of tiles $\mathcal{A}$

We have only two tiles. Each is a triangle with angles of 72 and 36 degrees – the tall triangle and 108 and 36 degrees – the flat triangle.

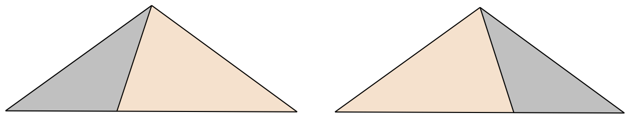
Both triangles frequently occur in tessellations for decagonal patterns. The tall triangle is 1/10 of a regular decagon. The flat triangle is half of a rhombus cut along its diagonal.



Inflation rules for the tall triangle

We have two rules for producing two symmetric images for the tall triangle. In mathematical terms, we call them chiral objects (asymmetric), meaning that any of them can be obtained from the other one only by reflections (no translations or rotations).

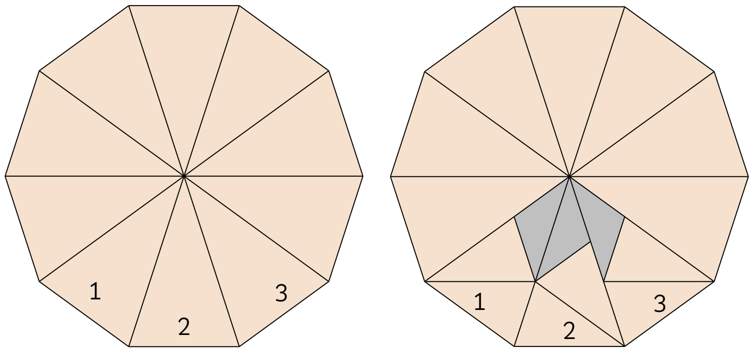
Later we will see why having two rules is essential.



Inflation rules for the flat triangle

Again we have two rules for producing two chiral images.

In the next few images, we will show how one can inflate a simple tessellation.

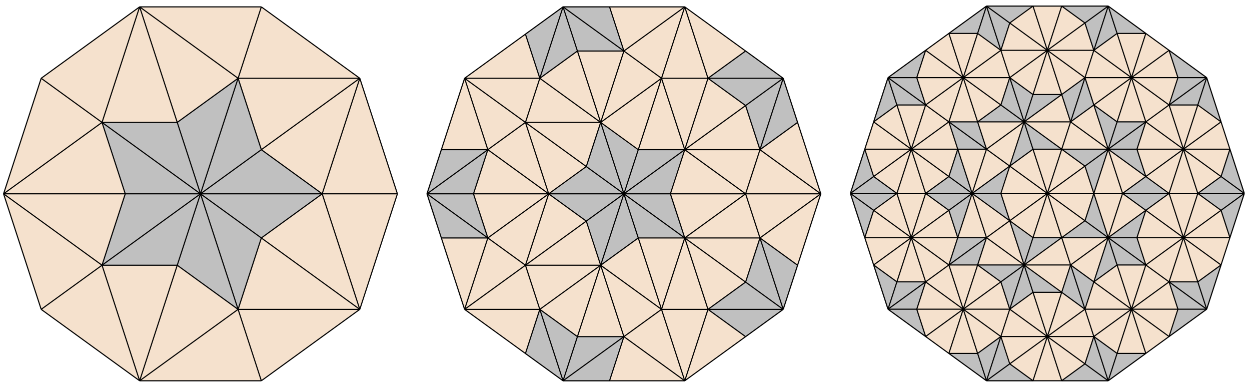


What can go wrong?

This image shows a regular decagon split into ten tall triangles. Here we can see why we need two rules for producing chiral objects.

After using one rule for tile 1, and another one for tile 2, we obtained an edge-to-edge tessellation.

In tiles 2 and 3, we applied the same rule to both triangles. Here edges of polygons do not match each other. The new tessellation is not edge-to-edge, and it is useless from the pattern design point of view.



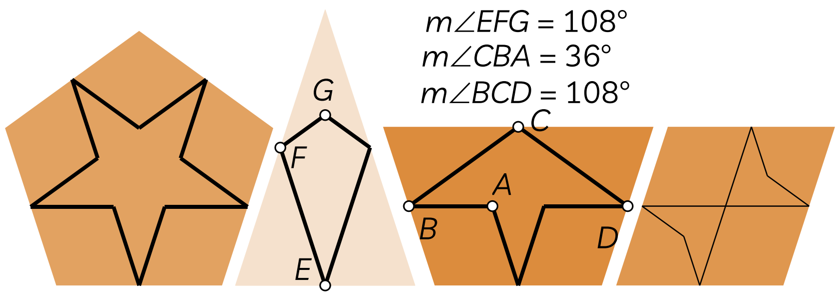
Three steps of inflation of a regular decagon

The three images show what we get after three consecutive steps of inflation applied to the triangles of a regular decagon. It is important to notice that we have a few options for applying the inflation rules in the second and next steps. This may produce several different tessellations for a given number of steps.

This simple example shows one important thing. Each step of inflation produces a tessellation more complex than the previous one. This way, we can produce more and more complex geometric patterns.

Tessellation tiles and pattern elements

In the remaining pages of this paper, we will deal only with one particular type of decagonal geometric pattern, the so-called Kukeldash Madrasah style, and tessellations suitable to produce such patterns. A specific feature of this type of pattern are angles of 36 and 108 degrees on crossings of pattern lines.



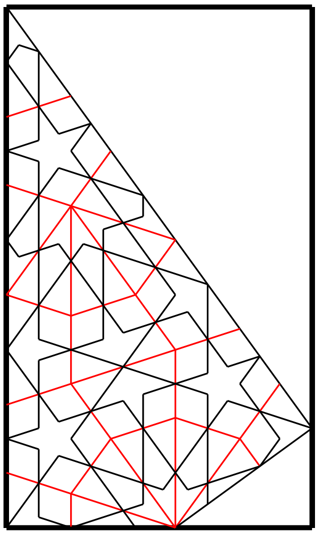
Selected tiles used in this paper

The three tiles – pentagon, trapezium, and long triangle - are most frequently seen in Kukeldash Madrasah style patterns. We added the rhombus, which is rarely used in Persian and Ottoman designs. But it is sometimes seen in Moroccan decorations.

The rhombus can be present in Kukeldash Madrasah patterns tessellations. But in some cases, it leads to some strange asymmetric shapes. These shapes are not acceptable. Thus patterns with them are considered incorrect (see [11]).

Isfahani inflation is much simpler than Penrose inflation for triangles. It was invented by the author while analyzing several geometric patterns from Bukhara and Isfahan. The method is surprisingly flexible, and almost each Kukeldash Madrasah pattern can be created with it. We can also produce some variants for a given pattern. Before making a formal definition let us examine two simple examples.

A design from the Topkapi scroll



Design 90 from the Topkapi scroll

This design is a template for a large decagonal medallion. It uses shapes typical for the Kukeldash Madrasah style (see the previous section):

star with sharp angles inscribed in a regular pentagon,

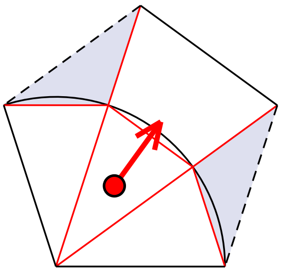
an arrow shape inscribed in a trapezium,

a long kite inscribed in a long triangle.

The author's previous papers described each of these shapes in detail.

More shapes may occur in the Kukeldash Madrasah style, but they are not present here.

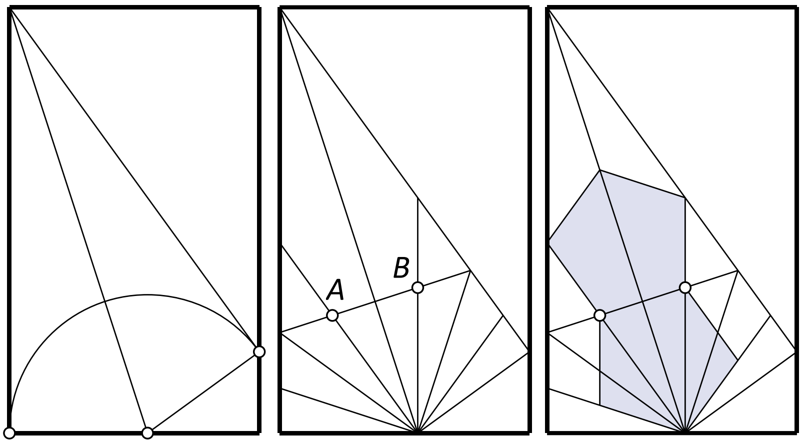
For this design, we will need only one inflation rule—the rule for the regular pentagon tile.



Inflation rule for the regular pentagon

A regular pentagon can be split into three long triangles, one trapezium, and two flat triangles (shaded). The shaded triangles should be appended to the shapes on the other side of the dashed lines, and the dashed lines should be removed. The continuous black segments may stay as a part of the second tessellation, or they will be removed if they touch a shaded triangle. The arrow points towards the trapezium. We will call it the direction of inflation for the pentagon.

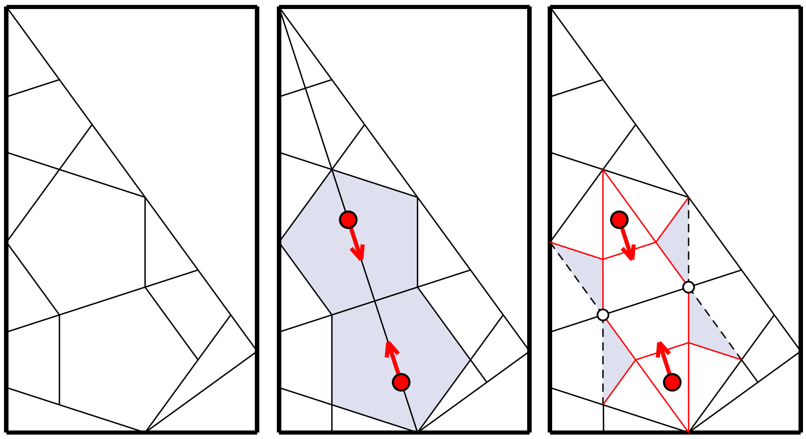
The detailed construction of this design and some interesting conclusions are presented in the next pages. Later we will show how one can create a large decagonal medallion using this design as a template.



Contour and first pentagons

The contour for this design is a long kite with 36 degrees at its narrow end and 144 degrees at its wide end.

After dividing the wide angle into 18-degree clusters, we get points A and B. We use segment AB to create two regular pentagons with AB as one of their edges.

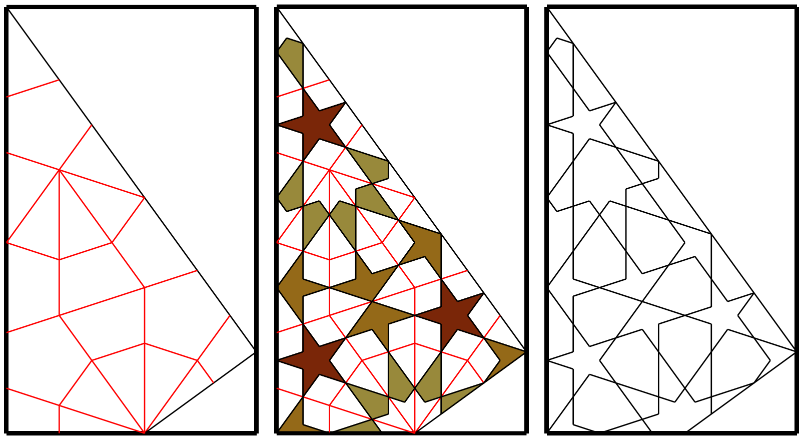


Inflation of pentagons

Having the two pentagons, we can finish the first tessellation.

We apply the inflation procedure to both large pentagons.

The gray triangles cannot be used for creating a pattern. We must augment them to the neighboring shapes – long triangles or trapeziums.

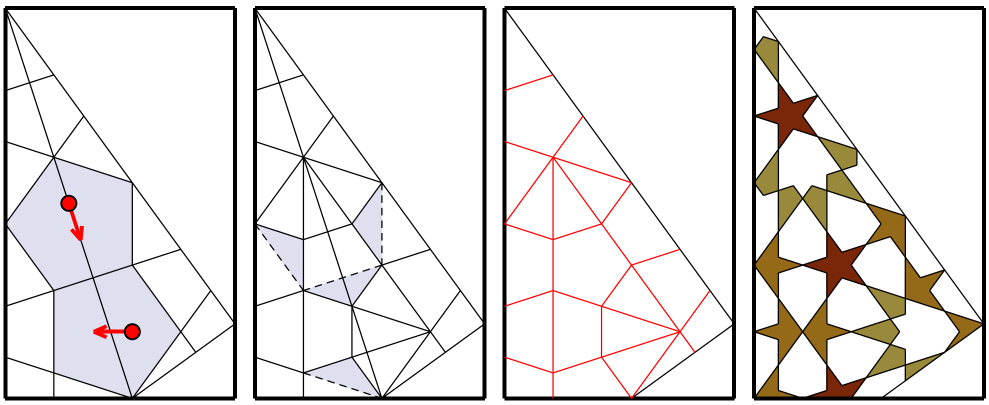


Second tessellation and the final pattern

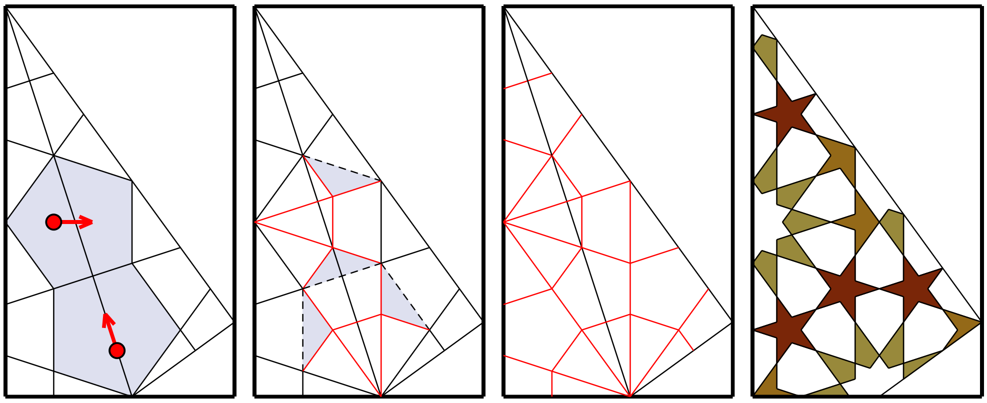
After removing the unnecessary edges of the first tessellation, and combining the shaded triangles with adjacent tiles, we get a complete tessellation for the second pattern.

The last drawing shows how the complete template will look. Later we will use it to create a decagonal medallion.

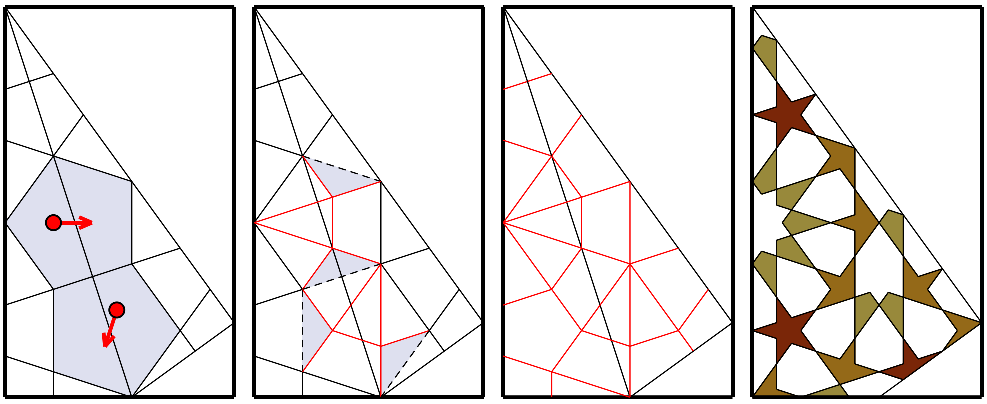
Design 90 from the Topkapi scroll can be constructed without the inflation of pentagons. However, Isfahani inflation gives us a good opportunity to go far beyond design 90. In fact, we can develop several different patterns by using different inflation directions for the two gray pentagons (drawing in the first row). Let us investigate this opportunity. The following drawings show four variants of this design.



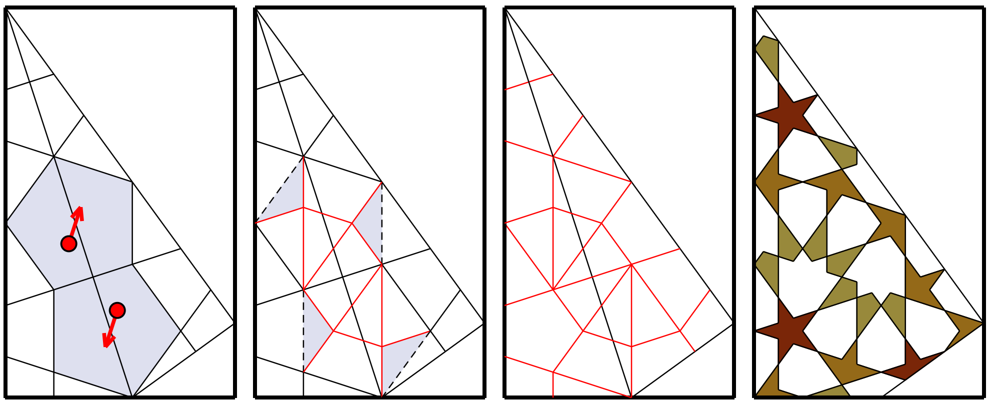
Variant 1



Variant 2

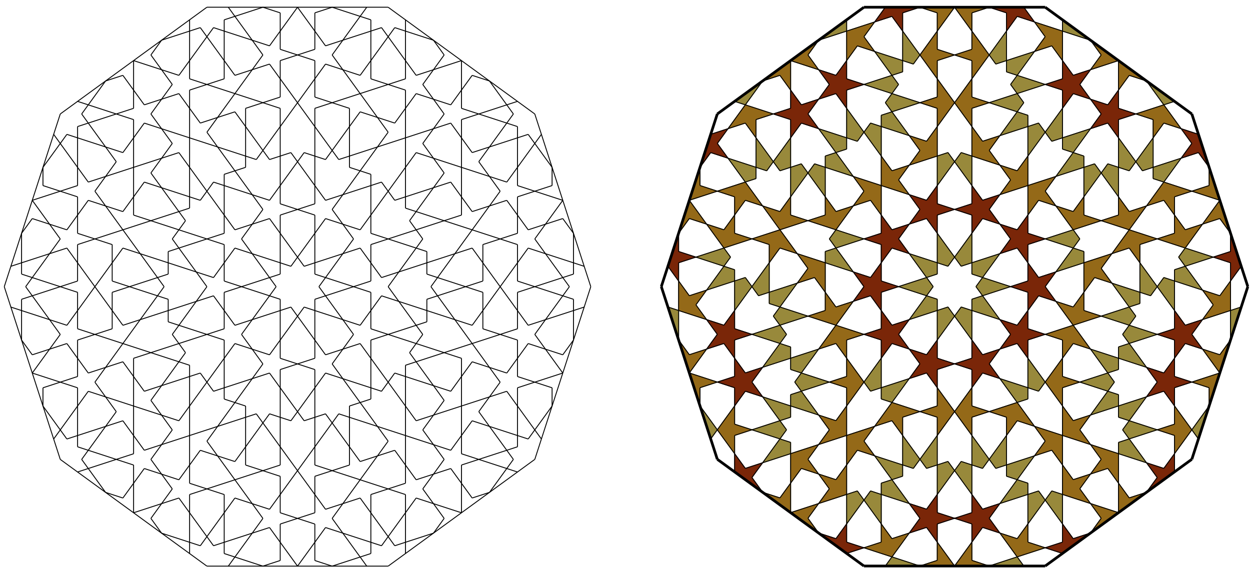


Variant 3



Variant 4

The enclosed drawings show only some possible modifications of the template from design 90. There are 25 of them, and I leave these investigations to the readers of this paper.



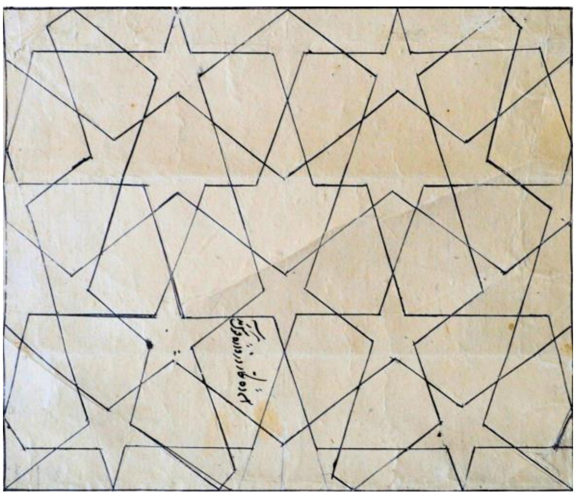
Two decagonal medallions created from design 90 and its modification

Left drawing - by taking ten copies of the pattern from design 90, we produce a large Zizi roundel. Usually, such a roundel is placed in a mosque on the side of the minbar.

Right drawing – design obtained from the last drawing on the previous page (variant 4). These two designs have two different symmetries. The left design has D10 symmetry, i.e., it has ten mirror lines passing through its center. The right design has only D5 symmetry, i.e., it has five mirror lines passing through its center. Why?

A design in a rectangular template

In old scrolls, most pattern templates are created in the form of a rectangle. Thus we face the problem of crossing inflated pentagons by the edges of the template. In this design, we will look into this issue.



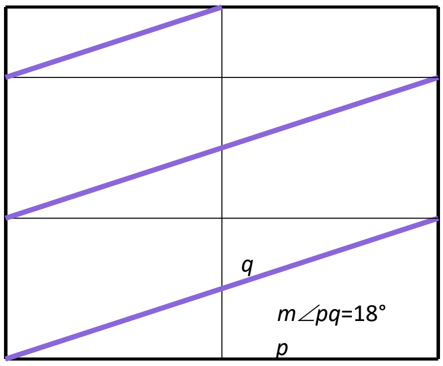
A pattern from one of the Bukhara scrolls

The drawing is part of one of the scrolls from Bukhara in Uzbekistan. It has a few interesting features – a very unusual proportion of the contour, shapes that look like a platanus leaf and an inscription in Bukhara Persian.

According to this inscription, the pattern was intended as a decoration for the arc of a madrasa. But we do not know which madrasa and where. We also have no date when this pattern was created. But we still can attempt to redraw this pattern and see how it will look in real artwork.

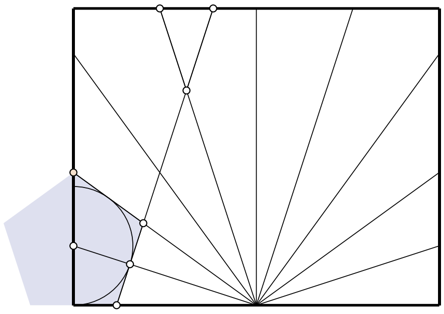
Let us start drawing the contour first. Then we will construct pentagons, the two tessellations, and finally, the resulting pattern.

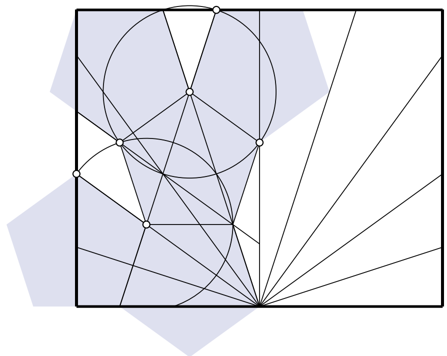
Three stages of creating the first tessellation



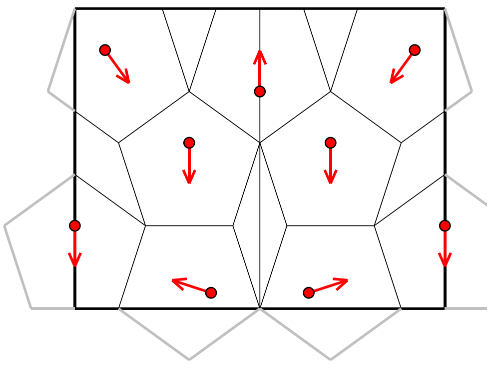
The contour is created from two long contours determined by the 18-degree angle and a combination of two smaller contours determined by the same angle. There are many designs in Iran where the contour was made as a similar combination of long or narrow contours.

Below left, we see how the very first pentagon was constructed. In fact, we created only half of a pentagon. Then two large circles in the drawing below are enough to get all pentagons on the left side of the tessellation. The right side will be a mirror reflection of tiles from the left side.

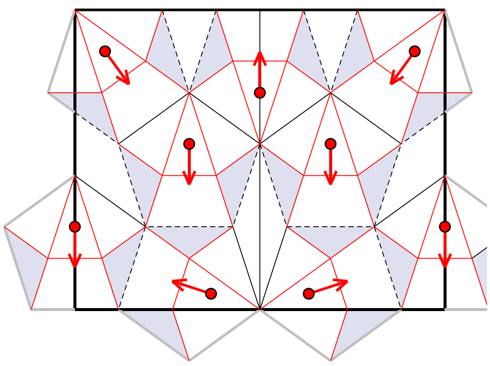




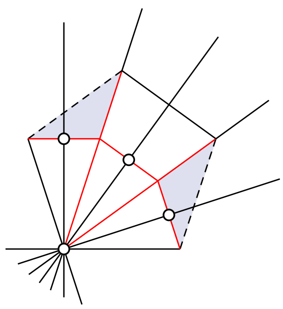
All pentagons for the first tessellation

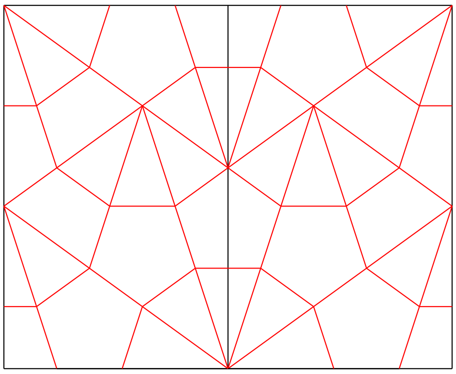


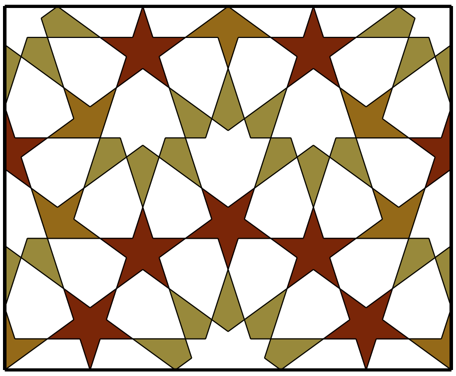
The first tessellation is a tiling of identical pentagons with some gaps between them. The gaps between pentagons can also be considered as tiles of this tessellation. Therefore we can consider it as proper edge-to-edge tessellation. The arrows show the directions of the Isfahani inflation in the original design. However, some of the pentagons can be inflated in a few other directions. Only pentagons crossed by the edge of the contour should be inflated according to one of their possible crossing lines (see the drawing below).

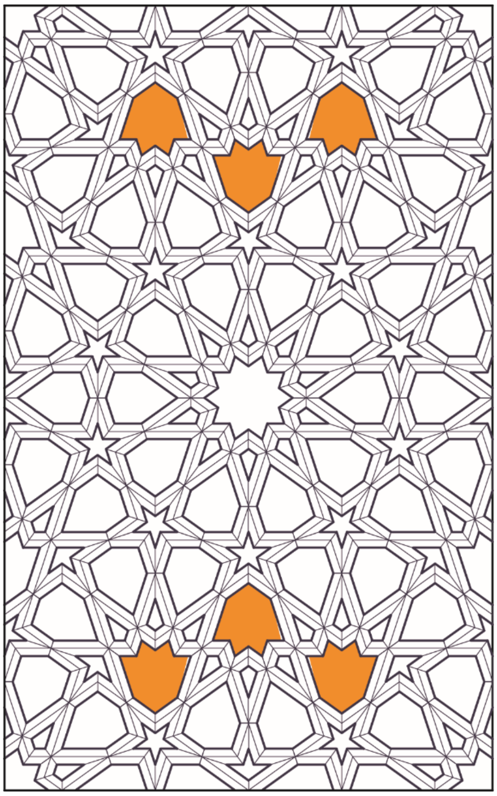


Below - possible ways of crossing the inflated pentagon by the edge of the contour (thin black lines)









From a tessellation to a real design

On the drawings, we have:

Top left – the tessellation for the pattern obtained by removing the dashed lines of the preliminary tessellation

Above – a template for a final pattern

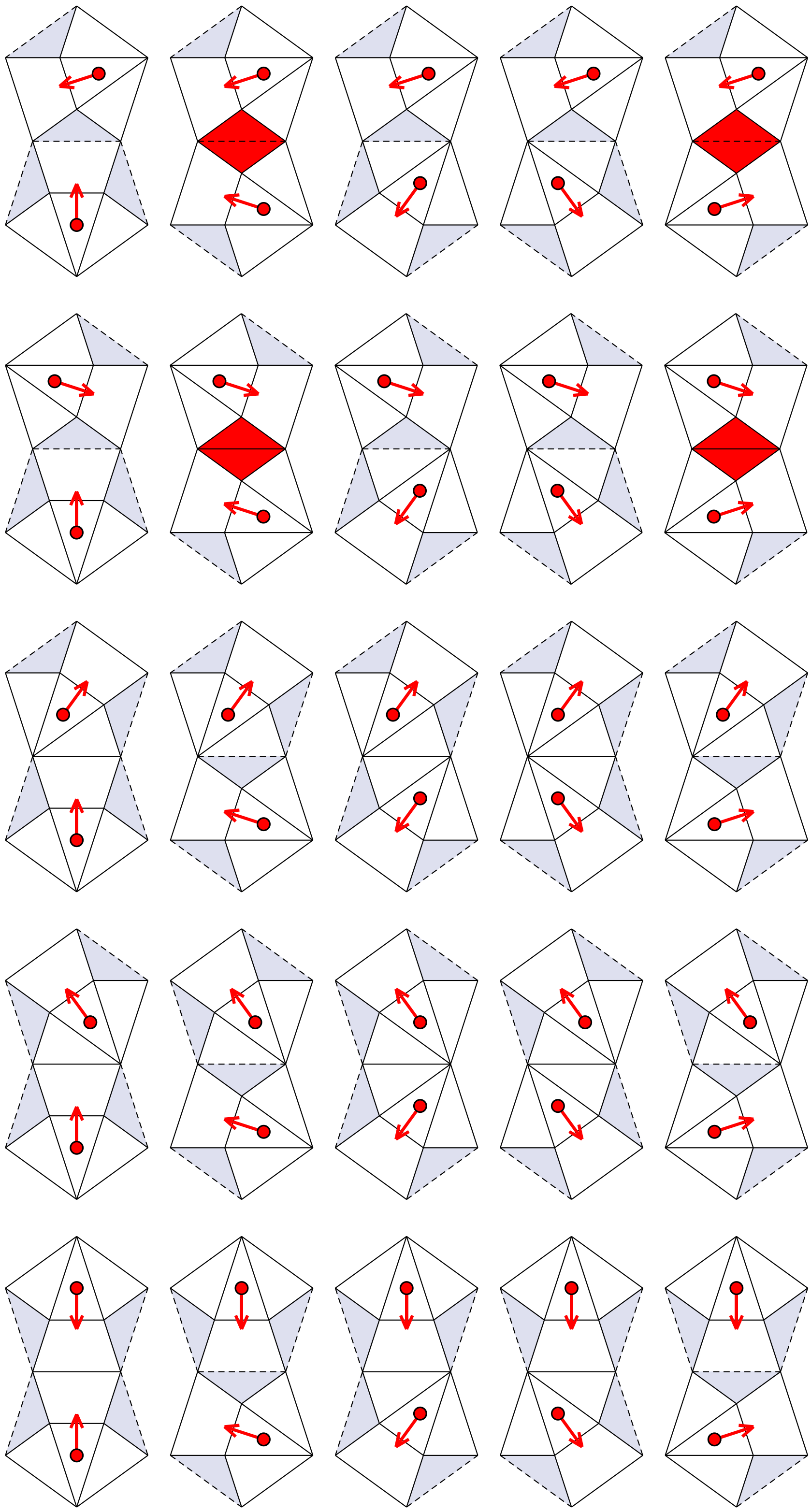
Left – a design for a kundekari door.

The orange shapes have an old name – barg-e-chanar, i.e., the leaf of the Platanus tree. Some pattern designers call them duck or goose feet, and others devil feet.

Another pattern from Isfahan

There are many geometric patterns all over Iran, Syria, and Turkey using a similar set of shapes. These shapes usually have interesting names, like barg-e-chanar (Platanus leaf), daane baloot (Oak seed), toranj (orange), shamse (sun shape), paa bozi (goat foot), panj-e-tond (sharp five), shesh (six or hexagon), etc. Some of them are presented in the enclosed photo.

Variants



Some arithmetic

In the process of creating a pattern for both examples, we could place inflation arrows in any direction possible. This means that a single pentagon can be inflated in 5 different directions.

With two free pentagons, i.e., not restricted by the edges of the whole design (see next example), we expect to have 25 tessellations. Some of them may be eliminated due to symmetries or wrong resulting shapes.

With three free pentagons, we may have 125 tessellations.

For n free pentagons, we may have 5^n tessellations. These numbers can go quite high, e.g. $5^6 = 15625$ tessellations.

Two tangent pentagons inflated

The drawing shows all possible tessellations for two tangent pentagons.

Note the red rhombi. We can fill them with pattern, but they are usually avoided in Persian and Ottoman designs. As we will see later, their presence in a tessellation may lead to some troubles.

On free and inflation-fixed pentagons

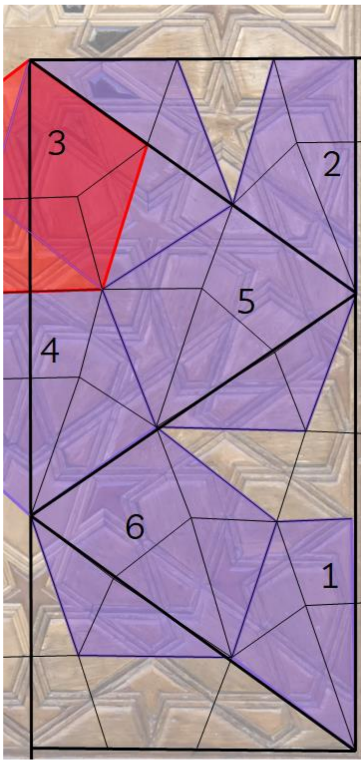
Let us examine a more complex example. We will go step-by-step through the whole process from drawing the contour, designing the first pentagon, preliminary tessellation, final tessellation, and the geometric pattern.

Analysis of the pattern from Hakim Mosque in Isfahan

The photograph below shows one of the two patterns from the doors in Hakim Mosque in Isfahan. Copies of this pattern occur in many other places, e.g., Bukhara (Kukeldash Madrasah) and Tashkent. It was also copied by a few present-day masters in Turkey and Iran.

A quarter of the pattern from Hakim Mosque in Isfahan

A proposal for a tessellation

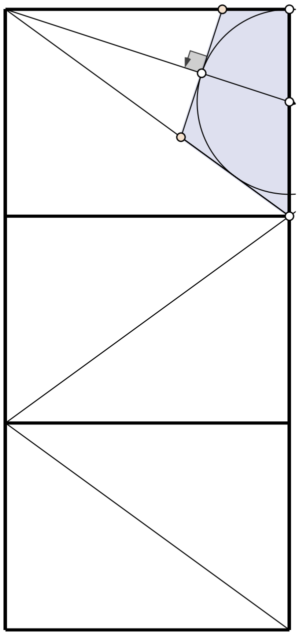


A proposal of a preliminary tessellation

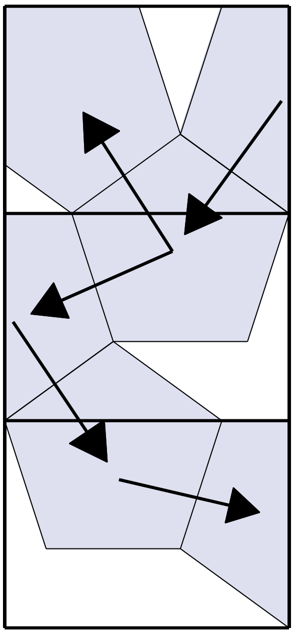
The proposed preliminary tessellation may have three slightly different versions – pentagon 3 can be drawn with a symmetry line along the left edge of the contour or tangent to the top edge of it. It is shown in the last drawing.

NOTE – pentagons 1, 2, 3, and 4 are inflation-fixed. We cannot inflate them in any other way than the one shown here. Pentagons 5 and 6 are free. We can inflate each of them in five directions. Thus, we have two inflation-free tangent pentagons, as in design 90 from the Topkapi scroll.

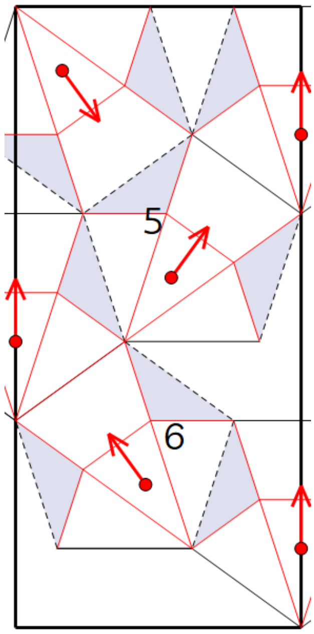
In the next series of drawings, we will show the complete construction of the original geometric pattern, and then we will discuss its variants.



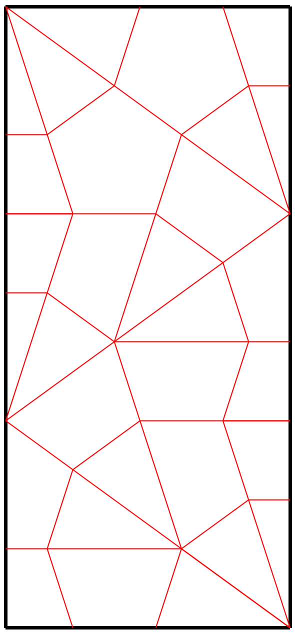
The contour consists of three identical rectangles. The first pentagon is built on a circle inscribed in the triangle between the edges of the rectangle and its diagonal.



As soon as we get the first pentagon, we can start **walking with pentagons** according to the arrows shown here. This way, we get a preliminary tessellation with pentagons and gaps between them.

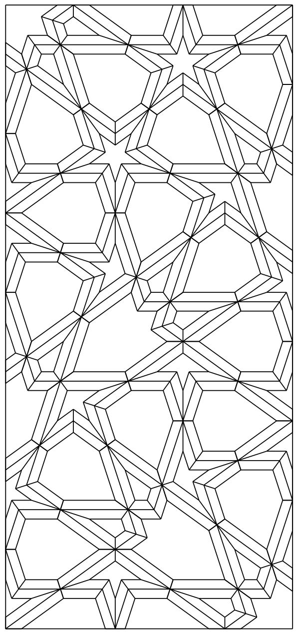


We are applying Isfahani inflation following the original pattern.

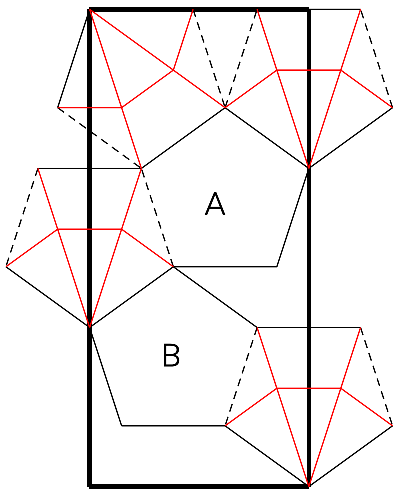


The final tessellation

We can see clearly how these three rectangles are filled with tessellation tiles.



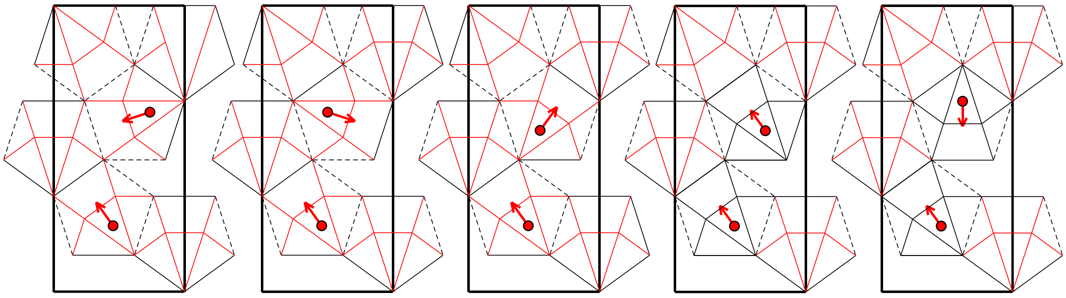
A sketch of the pattern prepared for a craftsmen's woodwork. The complete design will consist of four copies of the rectangular fragment shown here.



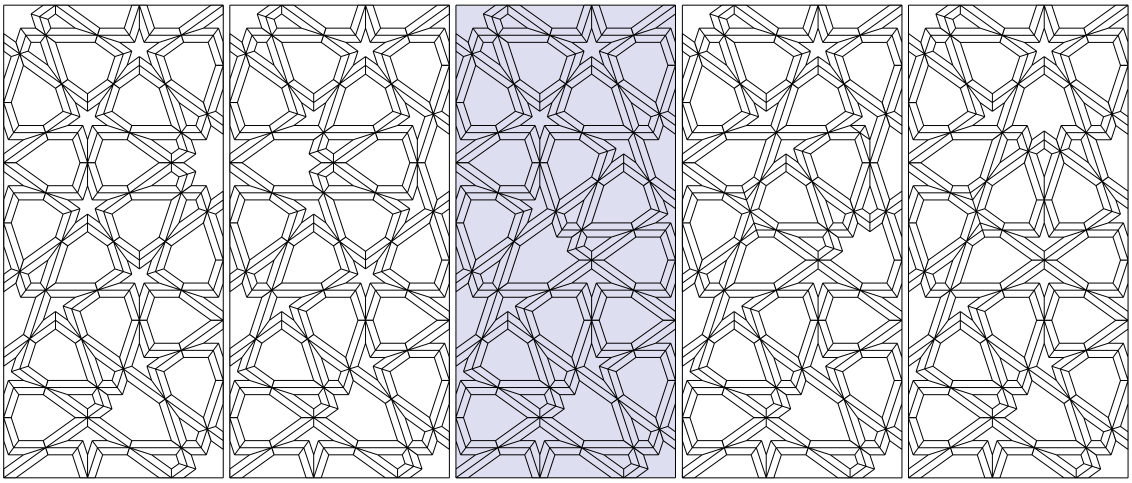
Tessellation showing inflation-fixed and inflation-free pentagons

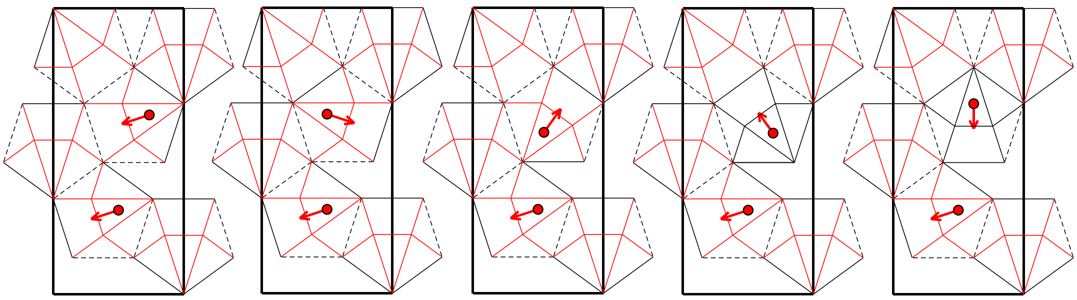
Pentagons A and B are free. All other pentagons are inflation-fixed. We can inflate pentagons A and B in any of the five directions. This way, we will get 25 tessellations.

Variants of the Hakim Mosque design



set 1





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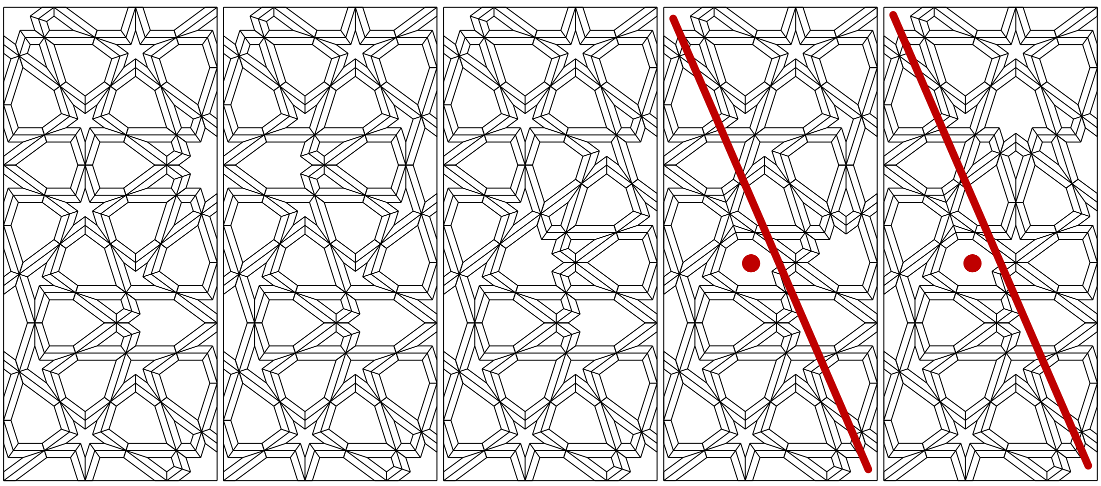

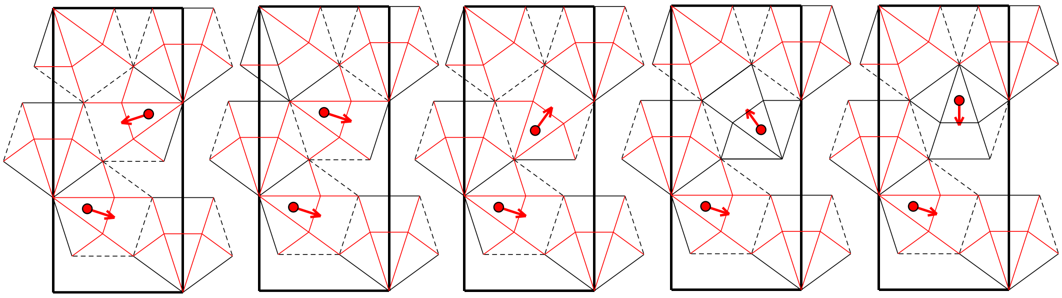

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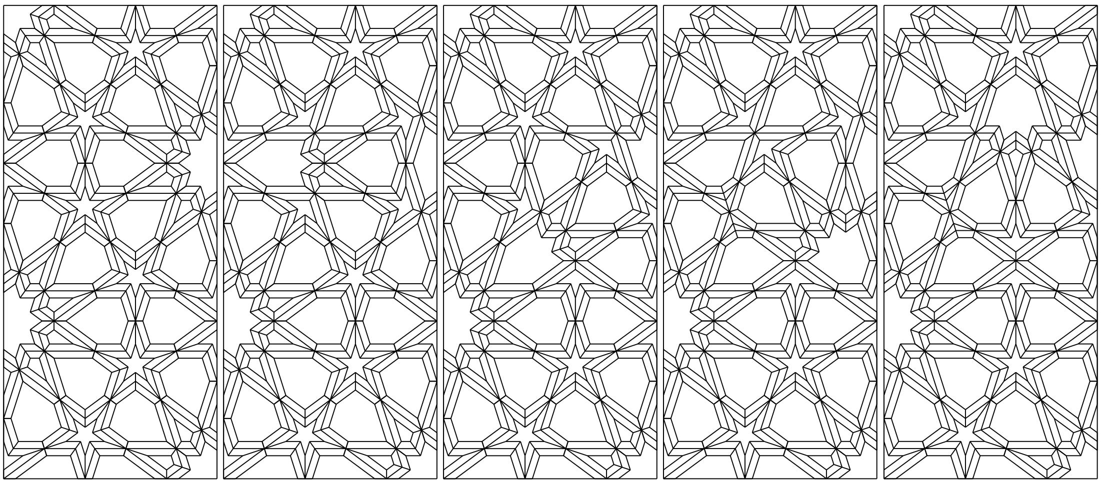

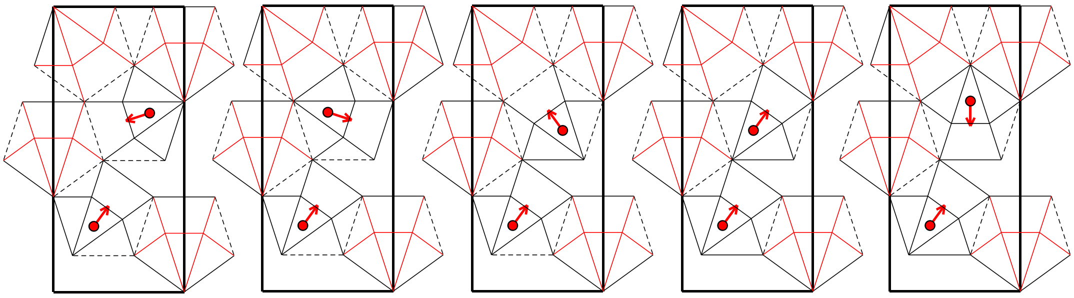

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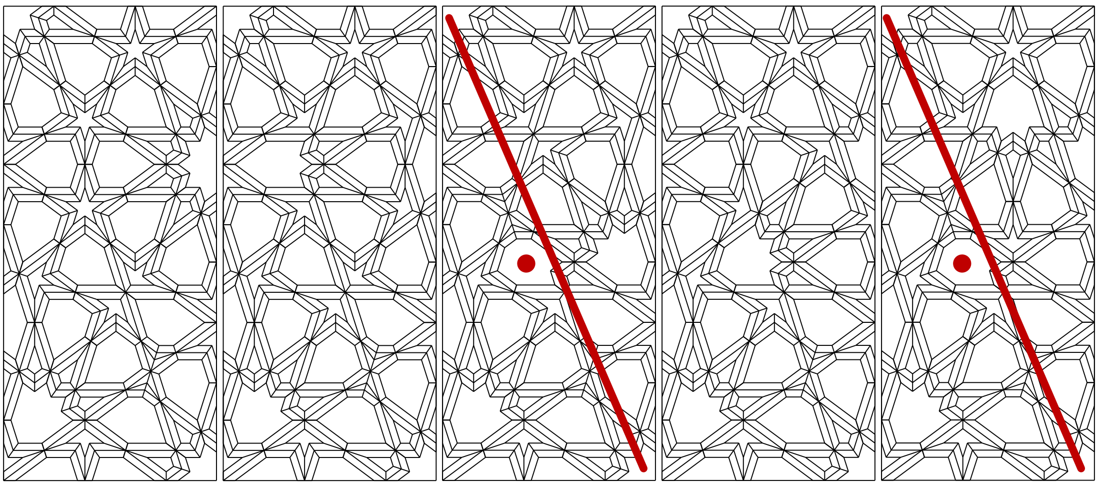

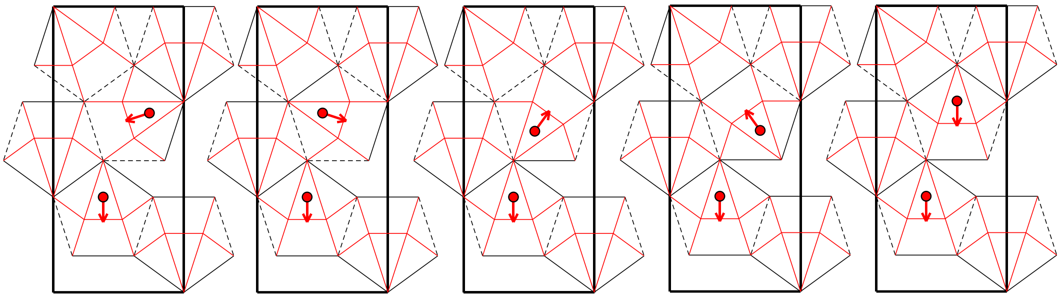

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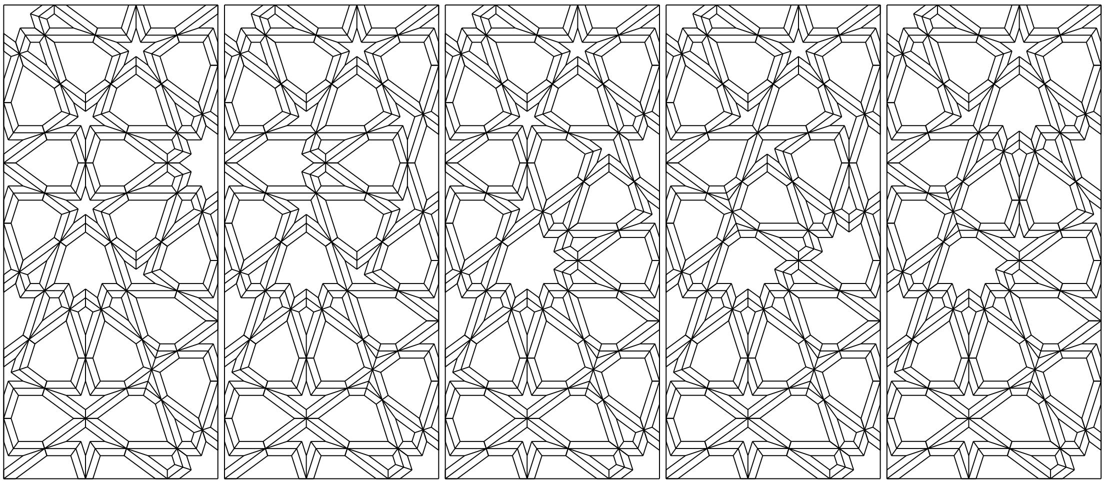


Comments for the variations

As a result of different directions for inflation of free pentagons, we obtained 25 designs. All of them are different. Four of these designs are unacceptable due to a strange asymmetric shape obtained around the rhombus in the tessellation (marked red circle). This is probably why we rarely see rhombus in tessellations for the Kukeldash Madrasah style. But still, we have got 21 patterns that can be used for large woodworking or ceramic project.

If we ignore all tessellations with the rhombus, we still get nine patterns acceptable for a woodworking project.

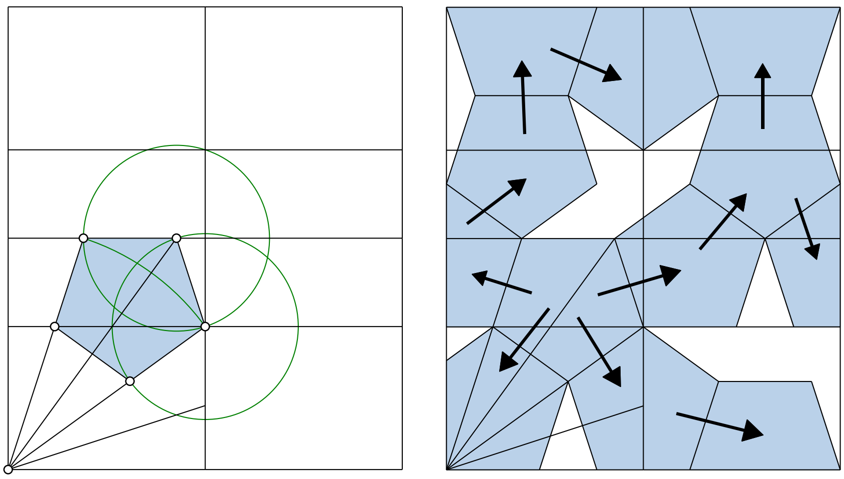
A summary of Isfahani inflation

First, we must notice that in constructing all patterns in this paper, we created two tessellations. The first one, the ‘preliminary tessellation,’ consists of a number of tangent pentagons and some gaps between them. The gaps are relatively small, and we cannot extend the network of tangent pentagons into them. In the next step, we transform the preliminary tessellation into a tessellation without gaps. In this step, the inflation rule is applied to each regular pentagon. Finally, we fill the second tessellation with the pattern.

Note – Isfahani inflation cannot be applied twice. We have only one rule for the pentagon, but in the second tessellation, we have many other polygons, and we do not have rules to inflate them. Thus Isfahani inflation is one step only, but this is still enough to produce a large number of Kukeldash Madrasah style patterns. Depending on our preferences, this number can be decreased, but we still get a large selection of good patterns.

Walking pentagons

In all examples in this text, we used a technique that we did call ‘walking pentagons.’ This is a powerful tool for creating tessellations of pentagons with gaps. The key to this technique is to draw a convenient contour based on decagonal geometry and draw the first regular pentagon. As soon as we have the first pentagon, we add new pentagons by try-and-error. We must consider how the contour edges cut these pentagons in this process. Let us look at a fairly complex example.



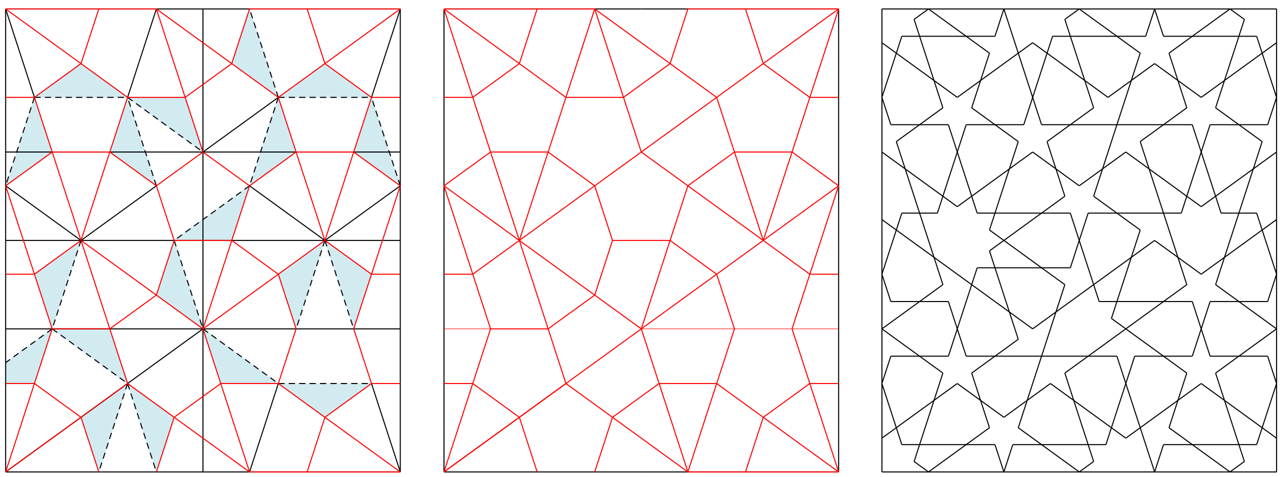
Walking with pentagons

The contour was built with four copies of the two rectangles in the bottom left corner.

The first pentagon was created in the bottom right corner of the contour.

Starting from the first pentagon, we add all other pentagons by making arbitrary decisions.

In this preliminary tessellation, we have six inflation-free pentagons. Thus we may produce $5^6 = 15625$ patterns. Some will be removed for using asymmetric shapes or a rhombus in their tessellation.



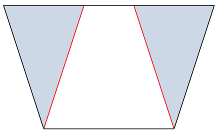
Isfahani inflation and final pattern

The Isfahani inflation process has two moments when we are free to make our own decisions (with some restrictions from geometry). First, we have to walk with pentagons so that the edges of the contour will correctly cross each pentagon. But we decide about the locations of the pentagons. The second moment when we make again our own decisions is the moment when we inflate pentagons. Here we have some restrictions from geometry, but we still have many ways to proceed.

The design shown here is slightly chaotic, but we did not care about any artistic result. We wanted to inflate pentagons. That was all. If we want a more artistic look, we should plan how we organize particular elements of the final pattern. This means locating stars, rosettes, and other nice shapes in planned places.

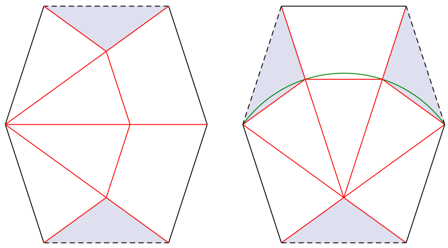
Extensions

Until now, we created preliminary tessellations with pentagons only and some gaps. We inflated pentagons only. We may produce inflation rules for some other tessellation tiles. We may still keep with symmetric inflation rules, or we may also add some chiral elements. The drawings below show only rules for producing symmetric objects.



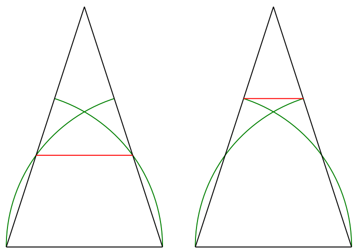
Inflation rule for a trapezium

This is a symmetric subdivision of a trapezium. There are also possible two chiral subdivisions.



Inflation rules for a double trapezium

Two symmetric rules for two trapeziums connected along the long edge (D-hexagon). Each of them has a different mirror line.



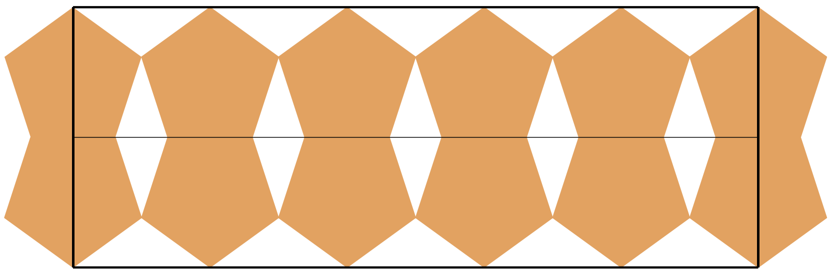
Inflation rules for a long triangle

A very simple way to subdivide a long triangle. We can use it to inflate a regular decagon.

Final comments

In this short overview of the Isfahani inflation, we could show some of its benefits. By adding some extra extensions, we may introduce the possibility of inflation of the second and next levels.

With Isfahani inflation, we can produce huge patterns with very complex structure. For example, a preliminary tessellation like the one shown below was used in a few of the most intricate designs in Isfahan mosques.



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