Mathematics and Technology: Does it work?

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Abstract: *The objective of this paper is to show that technologies have a great impact on the teaching and learning of mathematics. Technologies can be programs, applications, or even automatic deduction systems that can be used to teach one or several mathematical concepts. The use of learning environments with or without tutors, which integrate a dynamic geometry system or another, can be used in the teaching and learning of mathematics. Introducing new methodological practices, combined with technologies, in learning, allows a differentiated teaching. Active methodologies that involve students in the learning process have as their fundamental objective to develop mathematical skills that allow them to know how to solve problems. Some tasks that were developed will be presented, for secondary education students and for higher education students, either with the use of programs or teaching and learning environments, online or offline. The results have been favourable in the teaching and learning of mathematics, with the use of technological tools and involving active methodologies.*

1. Introduction

Mashaal, in his book 'Bourbaki', refers that "mathematics is everywhere" [19, p. 139]. The basis of the idea was that mathematics was a universal language meant to be used in all disciplines and in all sciences, even including the humanities and social sciences [18].

Mathematics is a very rich area that, in a changing world, encompasses ideas as disparate as those used in everyday life. In most professions, in numerous scientific and technological areas that are more mathematised, and, at the same time, it is an area that has generated significant contributions to human knowledge throughout history.

The development of digital skills is part of the essential competences of the 21st century and should be considered as an objective universal education [13].

Information and Communication Technologies (ICT) and the internet, as a new pedagogical space, offer great possibilities and challenges for the cognitive, affective, and social activity of students and teachers at all levels of education, from kindergarten to university. But for this to happen, it is necessary to look at them from a new perspective.

The practice of using ICT in teaching also requires the acquisition of new skills in computer manipulation. Teaching with ICTs will be a revolution if we simultaneously change the conventional teaching paradigms, which keep teachers and students apart, otherwise we will be able to give a veneer of modernity, without mess with the essentials. It is necessary to look for innovative didactic forms, because the educational field becomes increasingly complex, requiring constant improvement from the educator, which will effectively improve teaching within the classroom.

The use of technological tools makes learning a dynamic process in which, the experimentation, the raising of hypotheses, the search for conjectures and the validation of what is perceived can lead the student to build a way of thinking about mathematics that is meaningful to him.

The new scenarios brought by technological tools require new teaching methodologies. It becomes increasingly imperative to look for methodologies that involve the student as the protagonist of their learning, allowing them to develop a critical sense of what is learned, as well as skills to relate this knowledge to the real world.

Paper overview. The paper is organised as follows: first, in Section 2, some technological tools, automatic deduction systems and learning environments are presented with examples of tasks performed by students; secondary school geometry problems and proof development with the help of ICT tools. In Section 3, teaching and learning strategies involving active methodologies are discussed. In Section 4, competences inherent in mathematical activities are mentioned, whether with or without the use of technologies. Section 5 talks about the articulation of technologies, methodologies and mathematics. In Section 6 final conclusions are drawn.

2. Technologies and Environments

The use of computers in teaching represents a technological support for the visualization of abstract concepts, allowing the production of mental models of the concept. With computer programs, students interact with educational material to develop the skills needed to solve problems using mathematics as a basis. The increase in the availability of ICT has fundamentally transformed the ways in which we access information and how we communicate with each other. The practice of using ICT in teaching also requires the acquisition of new skills in computer manipulation. The use of ICT in education improves the effectiveness of traditional education, providing greater freedom to develop learning activities in the classroom, or at a distance, without limitations of space and time.

It becomes almost essential to underline the importance of technology in science and in today's society and, therefore, obviously, in scientific education [1]. There is no doubt that ICT have a particular importance in our times. The horizons that, through ICT, were torn between us more than a decade ago[1] and open to education are constantly being updated, although they are far from being fully implemented on the ground. ICTs bring complex challenges to education and finding the best way to use them in this environment is a search that is as necessary and urgent as it is intrinsically unfinished.

A new challenge is emerging technologies, which offer a range of opportunities for teachers and students to achieve the outcomes of a learning paradigm [23]. When analysing from a mathematics education perspective, technologies involve interactions more than their use in education. The use of tools in mathematics education can be considered at three levels of interactions: between tool and knowledge; between knowledge, tool, and student; and a tool in the mathematics curriculum and classroom. These three levels highlight the four components that can be distinguished in a system involving any technological tool, that is, the tool, knowledge, student(s) and teacher, and the inevitable relationships between these 'vertices' [37]. Such a system can be represented by a didactic tetrahedron (Figure 1.1).

[1] http://eden.dei.uc.pt/~adf/Forest95.htm (Figueiredo, A. D. (1995). O futuro da educação perante as novas tecnologias)

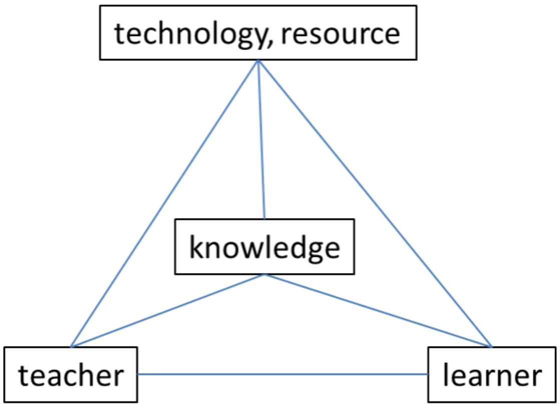

Figure 1.1 Didactic tetrahedron [37]

2.1 DGS Tools

The use of technological tools assumes a fundamental role because technology gives meaning to mathematics and mathematics justifies the use of technology [7, 14].

In the area of geometry, there is today a large number of computational tools that can be used to perform the most diverse tasks, dynamic geometry systems (DGS), computer algebra systems (CAS) (LCAS[2]), automatic geometry theorem provers (GATP), among others [26]. The DGS offer the student the possibility to experiment, create strategies, make conjectures, argue and deduce mathematical properties [35]. As support for learning geometry, we can highlight several dynamic geometry programs, such as: Cabri [15], C.a.R. [5], Cinderella [27], GCLC [12], GeoGebra [8], GeometerSketchpad [11] and JGEX [39, 40]. The DGS allows concrete manipulation of objects and abstract manipulation, thus making more rigorous deductions, which lead to the development of mathematical reasoning.

DGS are computer programs that allow the creation and manipulation of geometric constructions, mainly in the plane. One of the characteristics of DGS resides in the fact that they allow the construction of a geometric model of objects, such as points, lines, circles, etc., at the same time, the dependencies between these objects are created. The user has the possibility to move some parts of the geometric constructions preserving their properties.

DGS have stimulated investigations into students' conceptions of mathematical demonstrations, on the one hand, they provide environments in which students experiment freely, easily verifying their intuitions and conjectures in the process of looking for patterns, general properties, etc. On the other hand, they provide non-traditional ways for students to learn and understand mathematical concepts and methods. One of the advantages of dynamic geometry programs consists in carrying out tasks, not only for exploring geometric situations, but also for investigating situations that the tool itself promotes when moving objects, providing valuable support, both for students and for students. Teachers.

DGS are used to visualise complex geometric data, to perform calculations (including symbolic calculus), to construct and test geometric hypotheses, or to create geometrically accurate illustrations for later use in print documents or on the Web.

[2] List of interactive geometry software (LCAS) (2022, 1 July). In *Wikipedia*. https://en.wikipedia.org/wiki/List_of_computer_algebra_systems.

Most of DGS refer to plane geometry, but there are some that also allow constructions in three-dimensional Euclidean space, and some have the ability to recognize theorems. In addition to the DGS referred to, it is possible to consult an extensive list at LIGS[3].

In Figure 2.1 we can see two different constructions of a square, they were required to build using only the ruler and compass at the DGS, in the end different constructions of the square were evident (in this case without success).

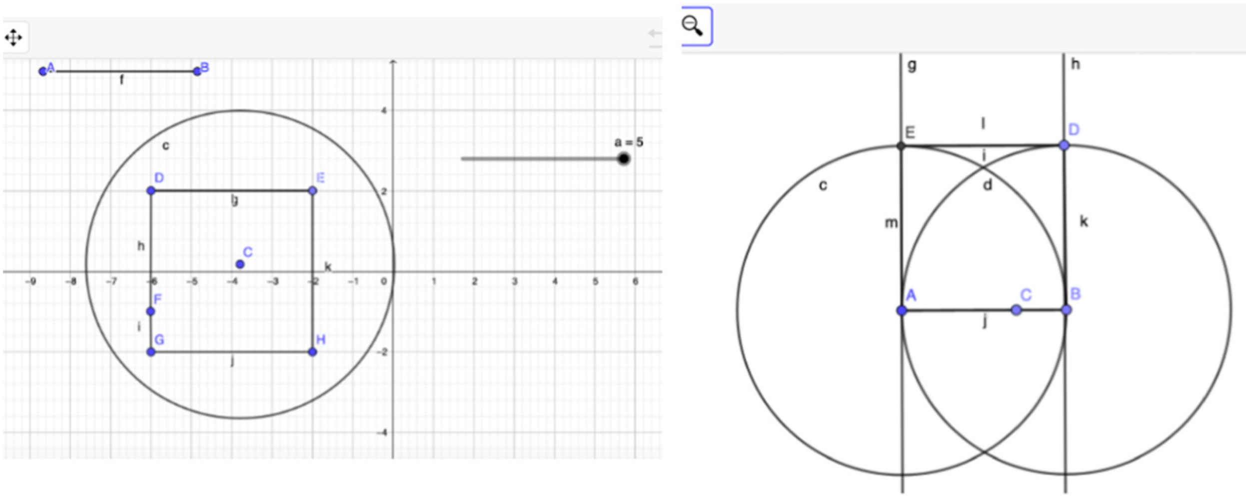

Figure 2.1 Construction by two students of a square [31]

2.2 GATP Tools

The subject of automated deduction has been progressing for many years, but there are many obstacles when we talk about the use of geometry automated theorem provers (GATP) to be used in secondary school classes and their use and results are designed to be understood by experts, not by a secondary teacher/student. On the one hand, some of the methods (for example, algebraic methods) do not produce geometric proofs, on the other hand, many of the synthetic methods produce a proof, but in an axiomatic system different from the one commonly used by secondary school teachers/students. The issues related to the use of automated deduction methods and tools are discussed in the book edited by Gila Hanna, David Reid and Michael de Villiers, 'Proof Technology in Mathematics Research and Teaching' [6]. In its four parts: *Strengths and limitations of automatic theorem provers*; *Theoretical perspectives on computer-assisted proving*; *Suggestions for the use of proof software in the classroom*; *Classroom experience with proof software*, we have a comprehensive coverage of the area [25].

DGS and GATP are tools that allow teachers and students to explore existing knowledge, create new constructs, and conjecture about new properties. The DGS allow to build geometric constructions from free objects and elementary constructions, it is possible to manipulate the free objects, preserving the geometric properties of the constructions. Although these manipulations are not formal proofs because only a finite set of positions are being considered and because the visualization can

[3] List of interactive geometry software (LIGS) (2022, 1 July). In *Wikipedia*. https://en.wikipedia.org/wiki/List_of_interactive_geometry_software.

be misleading, they do provide a first clue to the veracity of a given geometric conjecture. On the other hand, GATP allow the development of formal proofs. Based on different approaches (e.g. algebraic, synthetic) they allow their users to verify the solidity of a construction and also, in some cases, create formal proofs for a given geometric conjecture [25].

Formal tests, made with the help of ICT tools (GeoGebra, JGEX, GCLC, Prove9[4]), were used in problems 1 and 2 (Figure 2.2 and Figure 2.3). In Figure 2.2, the Problem 1, is to show that for any convex quadrilateral, [ABCD], that [EFGH], where each of the points is the midpoint of a segment in [ABCD], is a parallelogram, i.e., it is to show Varignon's theorem. The Problem 2 in the Figure 2.3 is to show $|BD| > |AC|$ considering the convex quadrilateral [ABCD] and the conditions |BD|> |BC| and $\alpha = \measuredangle CAB > \measuredangle ABC = \beta$.

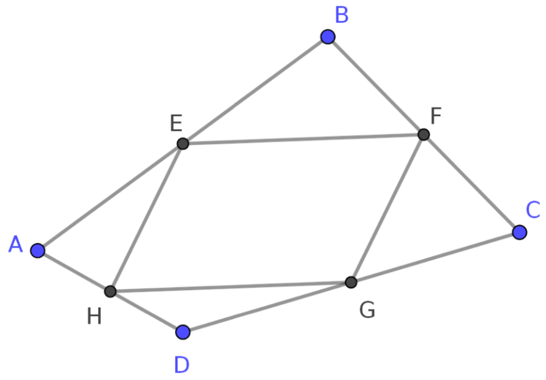

Figure 2.2 Problem 1 [25]

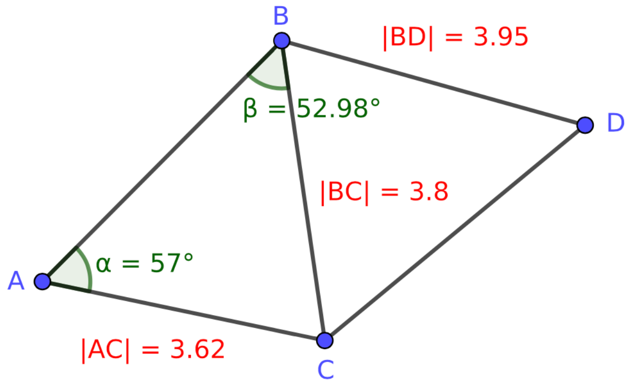

Figure 2.3 Problem 2 [25]

Different formal proof approaches were used in the two Problems, although in Problem 2 the different formal proof approaches are not possible (apart from the new algebraic approach built into GeoGebra) as the current axiomatic systems/methods do not deal with inequalities [25].

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4 https://www.cs.unm.edu/~mccune/prover9/

2.3 Learning Environments

The mediation function of a computer is related to the possibility of creating a communication channel between the teacher and the student based on a common language. The computer language opens up communication between the user and the machine, but at the same time, it opens up a channel of communication between the student and the teacher.

Computer-supported learning environments are considered an important way of learning for non-face-to-face teaching. They can be described in a context in which the computer facilitates interaction between students for the acquisition of knowledge, skills and attitudes. With the development of technologies, learning environments are enriched with tools that allow sharing, communication and collaboration with others.

Accordingly to Pacansky-Brock an LMS is a digital version of a classroom, where we can deliver resources, it is possible to communicate with students, and the student-student interactions can be done [23].

In this section, three learning environments will be referenced, such as the GeoGebra Classroom[5], Desmos[6] and the Web Geometry Laboratory[7] (WGL). GeoGebra Classroom is a collaboration platform where student progress can be monitored in real time. Desmos an online platform that allows teachers to develop remote lessons and activities for students to complete online. WGL is a collaborative blended-learning Web-environment for geometry that incorporates a DGS.

The GeoGebra Classroom was created as a resource to promote active and inclusive student participation [41]. It is a virtual platform through which teachers can assign interactive tasks to groups of students, viewing the progress of their work in real time, seeing all student responses instantly.

In Figures 2.4 and 2.5 we can observe the students' task about Thales' theorem (congruent angles of perpendicular sides; congruent triangles) in the GeoGebra Classroom. In which we have the student's view on the Figure 2.4 and on the Figure 2.5 we have the teacher's view (for example, we can see the student 2 did not solve the task).

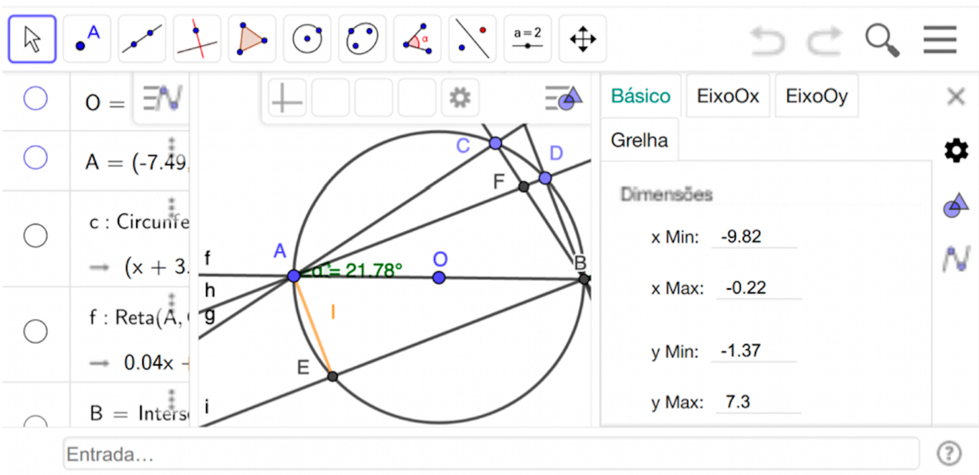

Figure 2.4 Student view [28]

[5] https://www.geogebra.org/m/hncrgruu
[6] https://www.desmos.com/?lang=en
[7] http://hilbert.mat.uc.pt/WebGeometryLab/

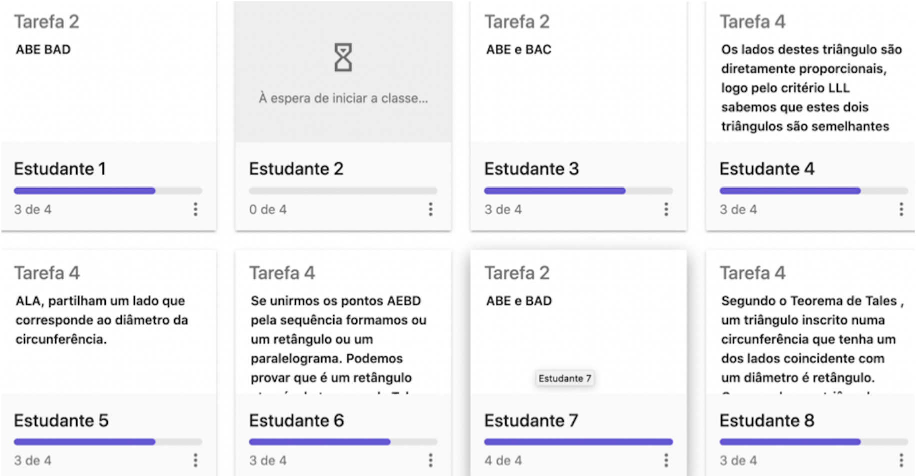

Figure 2.5 Teacher view [28] (Caption: “Estudante” is Student)

Desmos is an online platform that allows teachers to develop their classes and activities remotely for students to complete online. Desmos is intuitive, easy to use and brings a number of possibilities to work with algebraic and graphical representations of different functions and add an extra layer of interactivity and feedback to activities with the computation layer [29].

In Figure 2.6 were asked to fill in the table with the length and height values of a rectangle knowing that the area is 48 and the perimeter is 28. In Figure 2.7, it is an attempt with the wrong answer, and this can be seen in the fact that that the shaded area exceeds the height of the rectangle, and the length of the black line exceeds the size of the perimeter.

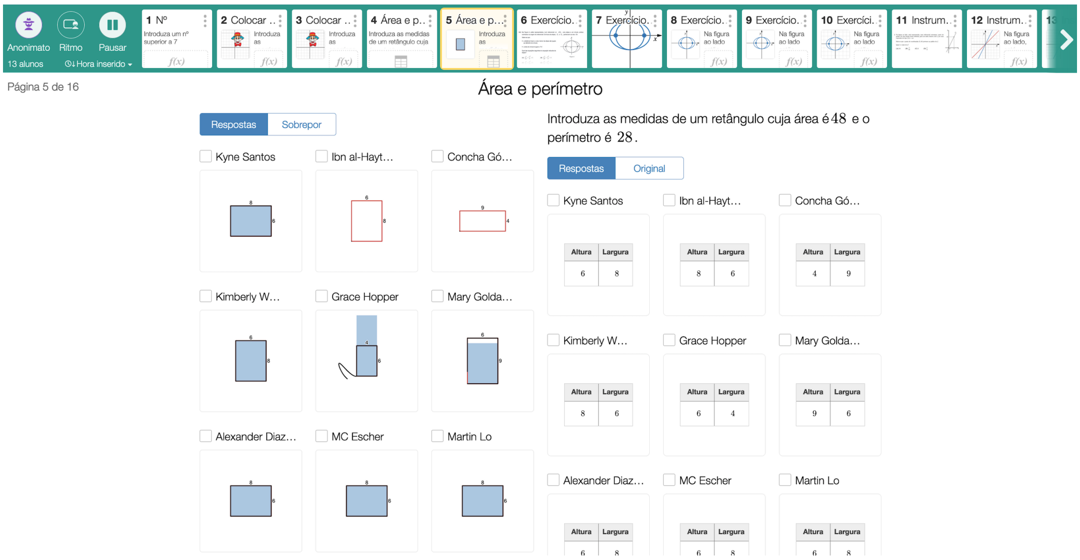

Figure 2.6 Teachers’ view [29]

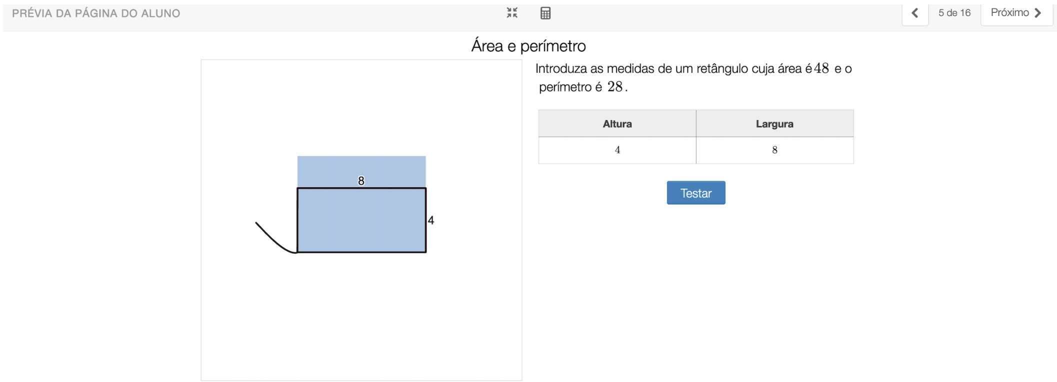

Figure 2.7 Students' view [29]

The Web Geometry Laboratory (WGL) is a collaborative blended learning web environment for geometry that incorporates a DGS. It can be used in the classroom or remotely. During a WGL collaborative session, students can exchange geometric and textual information, producing the geometric constructions collaboratively. The advantages of DGS in a geometry learning environment are manifold: it is easy to use, it stimulates creativity and the discovery process. GeoGebra was chosen as an applet with a suitable JavaScript application programming interface and its large base of users worldwide [32, 33, 34].

In this learning environment (WGL), any student has access to two windows of the DGS workspace, in which the DGS workspace on the left is the group workspace, that is, the construction that develops in this DGS is shared by all the students. members of a particular group. At the same time, the student has his/her workspace; this can be used to develop your own construction, following the work being done by the group representative, or to anticipate the group construction, or to develop an auxiliary construction (Figure 2.8).

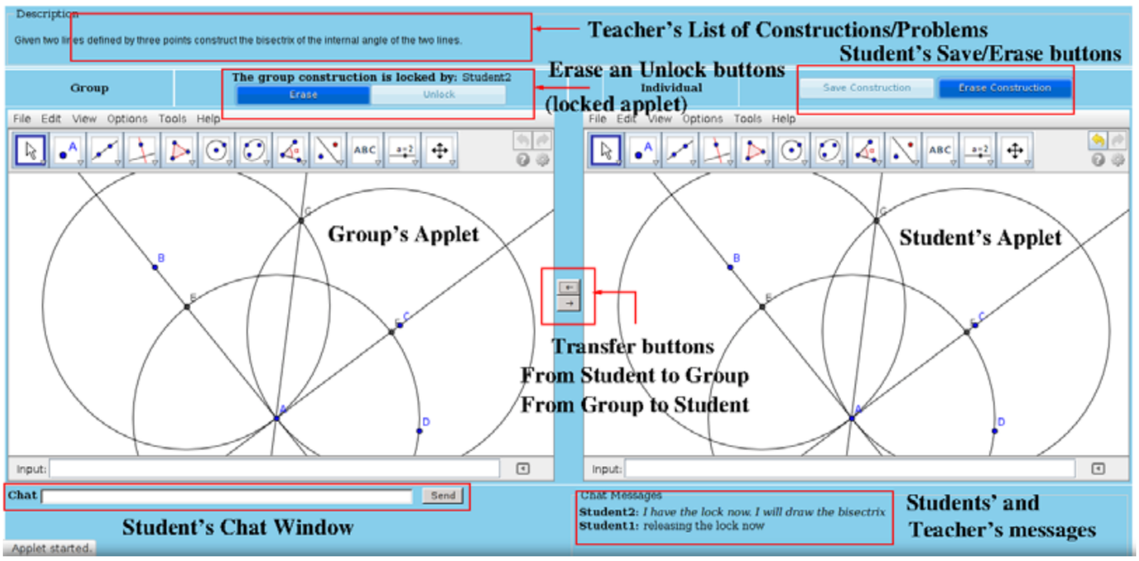

Figure 2.8 Collaborative Work – Students' View [32]

The teacher also has access to a work-space window where he/she can follow the work of all groups and of all students individually (Figure 2.9).

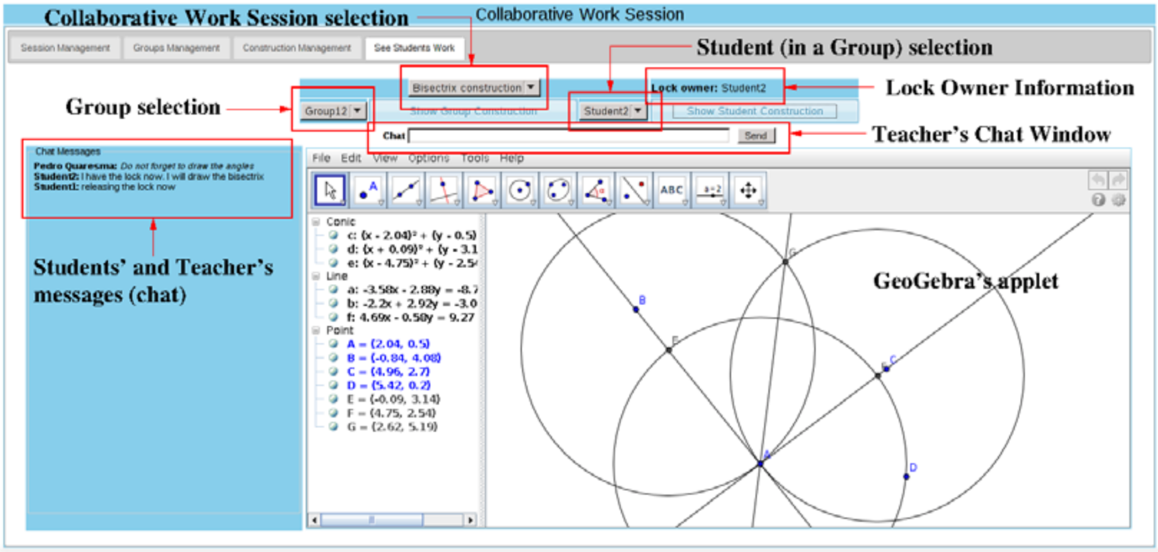

Figure 2.9 Collaborative Work – Teachers' View [32]

2.4 Adaptative Environments

Adaptive environments offer an advanced form of learning environment, as the objective is to adapt to the needs of each student. To build an individual student profile or individual learning paths, these environments collect information about student interactions when they are in self-paced work mode. Adaptive environments provide a personalisation of the needs of each student [4, 10, 16, 36], building a model of students with different goals and learning paths [3].

Adaptive learning is based on behaviour during the performance of tasks, knowledge and learning preferences of each student. Evidently, the tasks are defined by the teacher, taking into account the level of knowledge and learning preference of each student, their skills and their learning path. The student model is built according to the student's knowledge and behaviour at a given moment, allowing a description, as complete and accurate as possible, of all aspects related to the student's behaviour when using the system. With the student model, the most interesting information is adapted and presented, and the student is helped during the performance of his task. For a learning system to be "intelligent", it must be able to adapt to the student when using the system, which can only be achieved if the student model is known. The knowledge building base of student behaviour will help to define the characteristics of the students in order to make it easier to adapt the contents. Adaptive environments help users by adapting the content to them and adjusting their learning path. Depending on the adaptive system, one chooses from a variety of different learning strategies and technologies to predict and recommend learning content.

A task was made available to the students, which consisted of circumscribing a triangle. In the end, the teacher was able to visualise all the steps performed by the student (Figure 2.10). We can see the task performed by the student that shows all the steps performed by the student, that is, the teacher can see what the student did as if it were a "movie". The adaptive module has the ability to record all the geometric steps made by each student, during the sessions on the workbench, and the teacher's visualisation of the constructions, by his students, performed in these sessions. Geometric information is recorded when students are using the DGS applet in an individual session. In the sequence presented, the options taken by the student are observed, but we also verify that the student does not completely accomplish his task (circumscribe the triangle).

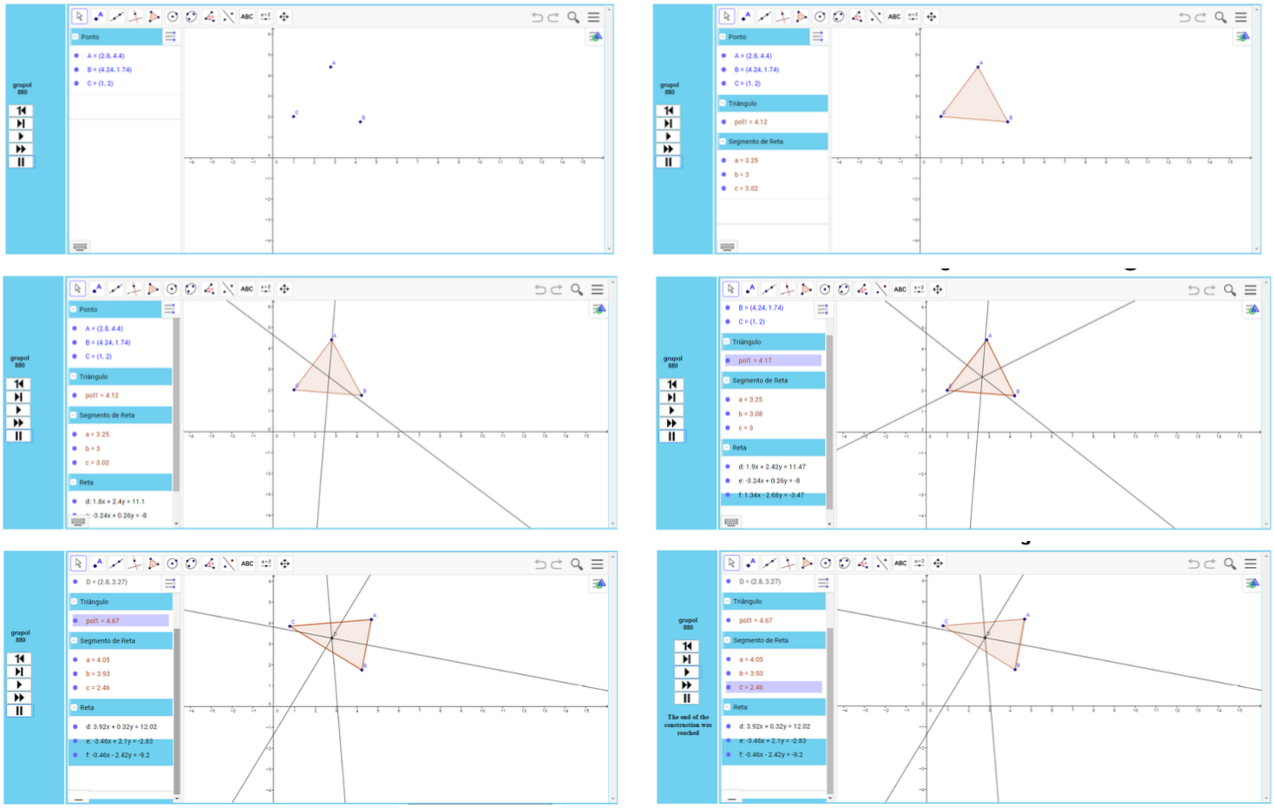

Figure 2.10 Student steps in the Adaptive Module – teacher's perspective

3. Teaching and Learning Strategies

Learning strategies are a topic present in educational research. In the investigations, the focus has been the concern with the quality of student learning, that is, how to make the student an active agent of his own learning process. Among the factors that most corroborate the improvement in the quality of the teaching process, are the choice of learning strategies capable to contemplate the different learning styles of students inserted in the school environment and the identification, by the student, of the way he learns [9].

When we are faced with a question such as: 'How to promote meaningful learning?', we will have to look at two sides, the technological and the technological. The technological facet includes different tools such as Audience Response Systems (ARS), Padlets, collaborative software (e.g., Miro), learning environments or other digital tools (section 2). The methodological facet includes different strategies, such as Project Based Learning (PBL), Team Based Learning (TBL), Flipped Classes or Think-Pair-Share, it is possible to consult an extensive list at CELT[8]. In practical terms, teachers have to think about the changes they want to promote in their teaching and learning practices [24].

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8 https://www.celt.iastate.edu/wp-content/uploads/2017/03/CELT226activelearningtechniques.pdf

3.1 Active Methodologies

The definition of ‘active learning’ is not consensual, because for some the concept itself is redundant, as it is impossible learn everything passively; other authors consider the active learning as a teaching approach in which learners students participate in the material they study. This concept is also linked to the concept of student-centred teaching, which means that the student is active and responsible for the construction of knowledge [30].

Active learning is generally defined as any method of instruction that involves students in the learning process. In this type of method, students are required to do meaningful learning activities and think about what they are doing. It may include traditional activities such as homework, but active learning refers to activities that are introduced in the classroom. The core elements of active learning are student activity and involvement in the learning process. This method in pedagogical practices with the use of technology has been shown to be a more engaging way for students, as they become more active in the learning process [2, 24, 30].

The differences (Table 1) between the implementation of traditional methodologies and the development of competencies, skills and knowledge that will guarantee progress in the practical application of methodological strategies, in which it is a question of making a change in teaching methodology with a focus on student learning process [2, 20].

Table 1. Traditional model to active-participatory method

Traditional Method	Active-Participatory Method
Passive and receptive students	Active, constructive and critical students
Individual and lone work	Group and motivational work
Transmissive teachers	Teacher as guide and counsellor
Individualism of the teaching staff	Coordination of teaching teams
Assessment limited to the final result	Assessment of the process as an end in itself

4. Competences

The document ‘The future of education and skills. Education 2030: the future we want' [22], published by the OECD, takes a stand against the uncertainties of the future, in order to prepare the new generations and national education systems from the development of skills, attitudes and values to face the future. With the objective of building a better future and life in 2030, the document brings a new architecture in relation to the development of the necessary competences, based on three sets of learning: knowledge (disciplinary, interdisciplinary, epistemological); skills (cognitive and meta-cognitive, social and emotional, physical and practical) and attitudes and values (personal, local, societal and global).

Being mathematically competent today involves, in an integrated way, a set of attitudes, skills and knowledge related to mathematics. This mathematical competence involves several aspects, from understanding the notions of conjecture, theorem and proof, the exploration of problematic situations, mathematical reasoning to mathematical communication. These mathematical skills promote the mobilization of knowledge (cultural, scientific, and technological) to understand reality and to address situations and problems. At the same time, it provides instruments that favor the use of appropriate languages to express ideas. Indeed, mathematics distinguishes itself from all other sciences, especially in the way it approaches generalization and demonstration and in how it combines experimental work with inductive and deductive reasoning, offering a unique contribution as a means of thinking, of accessing the knowledge and to communicate.

Skills such as problem solving, critical thinking, cooperation, creativity, computational thinking and self-regulation are, more than ever, essential in a rapidly changing society like ours. These are the tools needed to translate what has been learned in real time, in order to generate new ideas, new theories, new products and new knowledge [21]. For example, critical thinking develops skills to identify or formulate problems and solve them, evaluate information and use it, test ideas against relevant criteria, recognize your own judgments and put them to the test of new arguments, and communicate effectively. with others [31]. Another example is computational thinking, in which this skill can be mobilized for problem solving sets through a set of concepts and skills such as abstraction, algorithmic thinking and structured problem. It's about solving problems completely and solving the problems of a solution (even an approximate one) and exploring the meaning of the solution, studying or coming to the event that led us to the solution and thinking about what lessons can be learned.

5. Articulation between Technologies, Mathematics and Methodologies

Along with pedagogical and methodological changes, the scientific and technological advances and economic and cultural globalisation emerge as global challenges that require a reflective and holistic approach. It is in this context that the need to provide individuals with skills that allow them to adapt to a complex and constantly changing world [24].

The possibility of associating active methodologies with technologies in the teaching and learning of mathematics is not just about bringing technologies to the classroom or adding a paraphernalia of technologies, but making this technology become a positive differential, which brings results.

The development of mathematical competence is fundamental for us to act, and proactively, in favor of a better and sustainable world. In the educational context, this process requires the assumption of teaching practices that favor active learning on the part of students. A structuring of learning moments in line with the exploratory approach and enriched with innovative didactic strategies can prove to be an interesting possibility, not in a radically disruptive logic but, rather, assuming a transforming rationality. And the appropriate technologies, namely digital, carefully selected, can be effective mediators of a solid education in mathematics.

6. Conclusions

In the 21st century, it is argued that formal education must be transformed to enable new forms of learning, so necessary to face the complex global challenges. To face these challenges, students must acquire skills that allow them to solve problems, in a context of critical thinking, and be able to communicate (effectively and efficiently) in collaborative and cooperative environments [24].

Quality education for all today cannot be achieved without considering technological factors that "can facilitate universal access to education, reduce learning gaps, support teacher development, improve the quality and relevance of learning, strengthen inclusion and improve the administration and governance of education" [38].

ICTs are tools that can innovate the way mathematics is taught and can facilitate student learning. Combining mathematics and technology is undoubtedly advantageous for the dynamics of teaching, especially when new knowledge is built through technological tools. The teacher's challenge is to assert correct software applications for the learning of mathematics to materialise in a significant way. The combination of active methodologies with technological tools allows many experiences, when well-planned and organized, can allow greater student involvement in classes and activities. It is essential that the teacher has full knowledge of the resources and methodologies to be used, since this factor is decisive for achieving a good result.

As Lang refer that,

> "a good teacher is not only a specialist in his discipline, but also an actor, sensitive to the public's reactions. He also explained that he was very happy to have this new experience: talk and do mathematics with people who are not students and show them what mathematics is by 'doing mathematics' with them." [17, p. 2]

Mathematics and Technology: Does it work? It is difficult to find a fair balance in the use of technologies (because there are abuses), in implementation (because sometimes it is better not to implement) and in reaching potential (because ICTs are not a panacea), this balance, this 'right dose' is a challenge we have to face it.

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