Exploring some elementary results of Ramanujan with modern tools

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Abstract

The Indian mathematician Srinivasa Ramanujan (1887–1920) was one of the most extraordinary mathematicians in history. Almost entirely self-taught, he produced results of such originality that nobody understood him until he gained the confidence of the English mathematician Godfrey Harold Hardy. As one of the most expert mathematicians of his time, even Hardy was forced to admit that some of Ramanujan's results "defeated me completely; I had never seen anything in the least like them before" [7]. Hardy and Ramanujan worked together for five years (during which time Ramanujan was admitted to both the Royal Society and to a Fellowship of Trinity College, Cambridge), after which Ramanujan, by then a very sick man, travelled home to India where he died aged only 32. His results have kept scores of mathematicians busy for the last century—indeed some have devoted their lives to his work—and his influence is if anything even greater now. Many of his results are deep and difficult, and quite beyond an article such as this. But there are also some lovely gems which are more easily grasped; and of varying difficulties to prove. In this paper we explore a few of his simpler results with the aid of modern technology.

1 Introduction

In 1913 the Cambridge mathematician Godfrey Harold Hardy received a letter from India; handwritten, and containing some mathematical results exhibiting quite extraordinary mathematical skills, and in particular, as Hardy said: "It was his insight into algebraical formulae, transformation of infinite series, and so forth, that was most amazing." Hardy and his collaborator John Edensor Littlewood compared Ramanujan to Euler and Jacobi, the two most

*With thanks to the reviewers who pointed out numerous typos and inconsistencies, which I have aimed to remove.

profoundly inventive “algorists”[1] in history. In his witty and entertaining (and often historically inaccurate) “Men of Mathematics” [2], the American mathematician Eric Temple Bell, in his chapter on Jacobi, said “In sheer manipulative ability in tangled algebra, Euler and Jacobi have had no rival, except possibly the Indian genius, Srinivasa Ramanujan, in our own century.” At one time Hardy drew up a small list giving various mathematicians a score out of 100 for their mathematical power; he gave himself a modest 25, his collaborator Littlewood 30, David Hilbert 80, and Ramanujan 100.

Ramanujan’s results were distinguished by their originality, his complete indifference to formal proof, and the curious gaps in his own mathematical education, which had basically stopped at high school, he being taken further only by G. S. Carr’s “Synopsis of Pure Mathematics” [5] which contained a list of some 6165 theorems with only the vaguest hints at proofs. The fame of this book is entirely due to the influence it had on Ramanujan’s mathematical development.

A modern and popular account of Ramanujan’s life has been given by Kanigel [8]. Such is Ramanujan’s allure that this book has been filmed.

In this article we explore the use of computer algebra systems to some of Ramanujan’s less abstruse results. This is a topic which has gained some interest, but is yet to reach its potential. One example [3] is co-authored by Bruce Berndt, a Ramanujan scholar who understands his work as few other current mathematicians do. However, the article—which mentions the use of a computer algebra system to verify some results—lacks any code examples.

2 Continued fractions and an anecdote

Continued fractions are objects made up of fractions of fractions, such as:

$$2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cdots}}}} \quad \text{or} \quad 3+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cdots}}}}$$

A continued fraction that terminates is a rational number; if not, then irrational. In general a continued fraction has the form

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\cfrac{b_4}{a_4+\cdots}}}}$$

where all of the a_k, b_k values are integers. If $b_k = 1$ for all k the fraction is a *simple* continued fraction; otherwise a *general* continued fraction. Even now continued fractions are curious,

[1]We use the term “algorist” to refer to a mathematician with a great facility for manipualting algebraic expressions; as a distinction from “algebraist” who is an expert in modern algebra.

recondite objects about which much is unknown. To save space a continued fraction is normally written as

$$a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\cdots \text{ or } a_0 + \frac{b_1 |}{| a_1} + \frac{b_2 |}{| a_2} + \frac{b_3 |}{| a_3} + \cdots \text{ or } a_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{b_k}{a_k}.$$

(the second of these is known as *Pringsheim notation*; the third is *Gauss's notation*, where the K stands for the German word for continued fraction: Kettenbruch. If the continued fraction is simple, it is most commonly written as

$$[a_0; a_1, a_2, a_2, \cdots]$$

For any continued fraction, the sums of the first terms after termination are its *convergents*:

$$a_0, \quad a_0 + \frac{b_1}{a_1}, \quad a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2}, \quad a_0 + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3}, \quad \ldots$$

If the k-th convergent is p_k/q_k then it is easy to show by induction that

$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_k p_k + b_k p_{k-1}}{a_k q_k + b_k q_{k-1}}$$

and it is convenient to set $p_{-1} = 1$, $q_{-1} = 0$.

The anecdote to which the section title refers is that Prasanta Chandra Mahalonobis (who would have a distinguished career as a statistician, and who would found the Indian Statistical Institute) visited Ramanujan in his Cambridge rooms while Ramanujan was cooking dinner, and proposed a question from "The Strand" magazine: on a street with houses numbered 1, 2, 3, 4 and so on, there is a house for which the sum of all numbers to its left is equal to the sum of all numbers to its right. If the street has between 50 and 500 houses, what is the house number, and the number of houses in the street? This can be solved by trial and error, as Mahalonobis did. Without pausing in his stirring, Ramanujan said "Please write down the answer" and dictated a continued fraction which embodied all the answers to all such problems. He explained: "On immediately hearing the question I knew the answer was a continued fraction." The particular fraction, he went on "just came to my head". Indeed Ramanujan's mastery of continued fractions has never been equalled; Hardy said that his "mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world."

Any quadratic surd can be expressed as a periodic simple continued fraction. (Conversely, any periodic continued fraction represents a quadratic surd.) For example, suppose $x = \sqrt{6}$. Then $x^2 = 6$ and so $x^2 - 4 = 2$. Factorizing the left and dividing, we get

$$x - 2 = \frac{2}{2 + x}$$

so that

$$x = 2 + \frac{2}{2 + x}.$$

Substituting the right hand expression into itself we have

$$x = 2 + \cfrac{2}{2 + 2 + \cfrac{2}{2 + x}} = 2 + \cfrac{2}{4 + \cfrac{2}{2 + x}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + x}}.$$

Repeating this process on this new fraction produces:

$$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + 2 + \cfrac{1}{2 + \cfrac{1}{2 + x}}}}$$

and we see that we can now write, using the simplified notation given earlier:

$$\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, \ldots] = [2; \overline{2, 4}]$$

For The Strand problem, if the house number is k in a street of n houses, we are asked to solve

$$\frac{k(k-1)}{2} = \frac{n(n+1)}{2} - \frac{k(k+1)}{2}$$

using the standard formula for the sum of integers. This can be expanded as

$$k^2 - k = n^2 + n - (k^2 + k) \quad \rightarrow \quad 2k^2 = n^2 + n.$$

Completing the square on the right produces

$$2k^2 + \frac{1}{4} = \left(n + \frac{1}{2}\right)^2$$

or as

$$8k^2 + 1 = (2n + 1)^2.$$

Replacing $2n + 1$ with m we have

$$m^2 - 8k^2 = 1$$

which is classic equation known as *Pell's equation*. Now the continued fraction for $\sqrt{8}$ is

$$[2; \overline{1, 4}]$$

and the first few convergents are

$$\frac{2}{1}, \frac{3}{1}, \frac{14}{5}, \frac{17}{6}, \frac{82}{29}, \frac{99}{35}, \frac{478}{169}, \frac{577}{204}, \frac{2786}{985}, \frac{3363}{1189}, \frac{16238}{5741}, \frac{19601}{6930}$$

For the values p_k/q_k in this list, it turns out that

$$p_k^2 - 8q_k^2 = \begin{cases} -4 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd} \end{cases}$$

So we take the odd fractions, and for each numerator p_k compute $(p_k - 1)/2$. This leads to the numerator, denominator pairs:

$$(1, 1),\ (8, 6),\ (49, 35),\ (288, 204),\ (1681, 1189),\ (9800, 6930)$$

which are the beginning of an infinite sequence of answers (n, k) to the original problem. The particular answer for the question given in magazine, with $50 \leq n \leq 500$, must be $(288, 204)$.

A slightly different approach is given by Butcher [4]. The theory behind continued fractions and Pell's equation is given excellent treatment by Davenport [6].

Python with its SymPy package [9] can be used to manage the continued fractions for the street problem:

```
> import sympy as sy
> cf8 = sy.continued_fraction_iterator(sy.sqrt(8))
> cc8 = sy.continued_fraction_convergents(cf8)
> for i in range(8):
   ...: print(next(cc8))
   ...:

2
3
14/5
17/6
82/29
99/35
478/169
577/204
2786/985
3363/1189
16238/5741
19601/6930
```

In fact it is possible to create a continued fraction which produces only the solutions to the house number problem. In the list of fractions given previously, the odd numbers fractions are the ones we want. And by the formula for convergents, we have

$$\begin{aligned}\frac{p_{2k+1}}{q_{2k+1}} &= \frac{p_{2k} + p_{2k-1}}{q_{2k} + q_{2k-1}}\\ \frac{p_{2k}}{q_{2k}} &= \frac{4p_{2k-1} + p_{2k-2}}{4q_{2k-1} + q_{2k-2}}\end{aligned}$$

We now create a recurrence using just odd terms:

$$\begin{aligned}p_{2k+1} &= p_{2k} + p_{2k-1}\\ &= 4p_{2k-1} + p_{2k-2} + p_{2k-1}\\ &= 5p_{2k-1} + p_{2k-2}\\ &= 5p_{2k-1} + (p_{2k-1} - p_{2k-3})\\ &= 6p_{2k-1} - p_{2k-3}.\end{aligned}$$

For the sake of simplicity, we write $g_k = p_{2k+1}$ and $h_k = q_{2k+1}$ so we have the recurrences

$$\begin{aligned}g_k &= 6g_{k-1} - g_{k-2}\\ h_k &= 6h_{k-1} - h_{k-2}.\end{aligned}$$

Treating these as convergents, we have

$$\frac{g_k}{h_k} = \frac{6g_{k-1} - g_{k-2}}{6h_{k-1} - h_{k-2}}$$

Including the first term, these are the convergents of the continued fraction

$$3 + \frac{-1}{6+}\ \frac{-1}{6+}\ \frac{-1}{6+} \cdots$$

This fraction is not quite simple, as the numerators are equal to -1. But we can multiply through by -1 on alternative numerators (and their denominators) to obtain the simple continued fraction

$$3 + \frac{1}{-6+}\ \frac{1}{6+}\ \frac{1}{-6+} \cdots$$

To evaluate this fraction, note that we can rewrite it as

$$3 + \frac{1}{-6+}\ \frac{1}{3+3+}\ \frac{1}{-6+} \cdots$$

and so if x is the value of this continued fraction, we have, in more familiar notation:

$$x = 3 + \cfrac{1}{-6 + \cfrac{1}{3 + x}}$$

This fraction can be simplified into a quadratic equation one of whose roots is $\sqrt{8}$.

Which of the two continued fractions for $\sqrt{8}$ did Ramanujan dictate to Mahalonobis? No one knows...

Although simple continued fractions are (relatively) straightforward, general continued fractions are quite complicated objects. In the spirit of Ramanujan's profound mastery of continued fractions, we show how to create one that represents π.

We start with a result of Euler, and which can be proved by induction; if

$$x = u_0 + u_0u_1 + u_0u_1u_2 + \cdots + u_0u_1u_2 \ldots u_n$$

then

$$x = \frac{u_0}{1-}\ \frac{u_1}{(1+u_1)-}\ \frac{u_2}{(1+u_2)-} \cdots \frac{u_n}{1+u_n}$$

and this can be extended to an infinite continued fraction representing an infinite sum, supposing that both converge. (An infinite continued fraction converges if its convergents p_n/q_n converge.)

With the well-known sum

$$\begin{aligned}\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= u_0 + u_0u_1 + u_0u_1u_2 + u_0u_1u_2u_3 + \cdots\end{aligned}$$

we have

$$u_0 = 1, \quad u_k = -\frac{2k-1}{2k+1}$$

and so $1 + u_k = 2/(2k+1)$. This gives rise to the continued fraction

$$\cfrac{1}{1 - \cfrac{-1/3}{2/3 - \cfrac{-3/5}{2/5 - \cfrac{-5/7}{2/7 - \cdots}}}} = \cfrac{1}{1 + \cfrac{1/3}{2/3 + \cfrac{3/5}{2/5 + \cfrac{5/7}{2/7 + \cdots}}}}$$

Now we can clear the inner fractions by multiplying the numerator and denominator of the first inner fraction by 3, the second by 5, the third by 7, and so on:

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{3^2/5}{2/5 + \cfrac{5/7}{2/7 + \cdots}}}} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2/7}{2/7 + \cdots}}}} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\cdots}}}}}$$

This particular fraction was first discovered by Lord Brouncker from an analysis of Wallis' infinite product for π.

But we can do all of this, at least experimentally, with a bit of computing. We first note, that by multiplying each "inner fraction" by r_0, r_1, r_2 in turn, we have

$$\cfrac{b_0}{a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}} = \cfrac{r_0 b_0}{r_0 a_0 + \cfrac{r_0 r_1 b_1}{r_1 a_1 + \cfrac{r_1 r_2 b_2}{r_2 a_2 + \cfrac{r_2 r_3 b_3}{r_3 a_3 + \cdots}}}}$$

Consider

$$e^{-1} = 1 + 1(-1) + 1(-1)\left(-\frac{1}{2}\right) + 1(-1)\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) 1(-1)\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\cdots$$

then by Euler's continued fraction construction above, and ignoring the first two terms (which cancel out), we have $u_0 = 1/2$ and $u_k = -1/(k+2)$ for all other k. Now, in Python:

```
R = lambda x,y:sy.Rational(x,y)
u = [R(1,2)]+[-R(1,k+3) for k in range(9)]
b = [u[0]]+[-x for x in u[1:]]
a = [1]+[1+x for x in u[1:]]
```

Here the b and a values are the numerators and denominators of Euler's continued fraction.

Now we can choose some multipliers r_k and multiply as above:

```
r = list(range(2,12))
ar = [x*y for x,y in zip(a,r)]
br = [x*y*z for x,y,z in zip(b,r,[1]+r[:-1])]

br
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]

ar
[2, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

This has produced the continued fraction:

$$e^{-1} = \cfrac{1}{2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cdots}}}} \Rightarrow e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cdots}}}$$

Since the a_k and b_k values are the same, we can create the convergents by

$$\frac{p_k}{q_k} = \frac{k(p_{k-1} + p_{k-2})}{k(q_{k-1} + q_{k-2})}$$

with $[p_{-1}, p_0] = [1, 2]$ and $[q_{-1}, q_0] = [0, 1]$. Now to compute the convergents:

```
p = [1,2]
q = [0,1]
for k in range(2,10):
    p += [k*(p[-1]+p[-2])]
    q += [k*(q[-1]+q[-2])]
p
[1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800]

q
[0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961]
```

Notice that the q values are the number of derangements D_n of n values: the number of permutations where no object is in its original place. Finally:

```
[float(x/y) for x,y in zip(p[1:],q[1:])]
[2.0,
 3.0,
 2.6666666666666665,
 2.727272727272727,
 2.7169811320754715,
 2.7184466019417477,
 2.7182633317602645,
 2.71828369389345,
 2.718281657666403 7,
```

```
  2.7182818427778273,
  2.7182818273518743]
```

What might have Ramanujan managed with a computer algebra system at his disposal?

3 Another anecdote; another equation

In a story which has become famous, Hardy, on visiting Ramanujan in hospital, commented that he'd come in taxi number 1729, which seemed to him to be a dull number, and he hoped it wasn't a bad omen. No indeed, responded Ramanujan, 1729 is an extraordinarily interesting number, being the smallest number expressible as a sum of two cubes in two different ways: $1729 = 9^3 + 10^3 = 1^3 + 12^3$.

On the face of it, this would seem evidence of Littlewood's comment that for Ramanujan, "every positive integer was one of his personal friends". But in fact Ramanujan had been exploring such topics for a long time, to the extent that he had developed a generating function to provide solutions to *Euler's diophantine equation* $a^3 + b^3 = c^3 + d^3$. These are described by George Andrews and Bruce Berndt in "Ramanujan's Lost Notebook, Part IV", pp 199–205 (Section 8.5) [1]

Ramanujan's result is that if

$$f_1(x) = \frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \alpha_0 + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \cdots$$

$$f_2(x) = \frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3} = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots = \beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \frac{\beta_3}{x^3} + \cdots$$

$$f_3(x) = \frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots = \gamma_0 + \frac{\gamma_1}{x} + \frac{\gamma_2}{x^2} + \frac{\gamma_3}{x^3} + \cdots$$

then for every value of n we have:

$$a_n^3 + b_n^3 = c_n^3 + (-1)^3$$

and

$$\alpha_n^3 + \beta_n^3 = \gamma_n^3 - (-1)^3$$

thus providing infinite sequences of solutions to Euler's equations. The values $\alpha_1, \beta_1, \gamma_1$ are $9, -12, -10$ giving rise to

$$(9)^3 + (-12)^3 = (-10)^3 - (-1)^3$$

which can be rewritten as $9^3 + 10^3 = 12^2 + 1^3$.

Andrews and Berndt comment that:

> This is another of those many results of Ramanujan for which one wonders, "How did he ever think of this?"

We shall show how these formulas can be verified by with the help of a computer algebra system; in our case Python and its SymPy package.

```python
import sympy as sy
sy.init_printing(num_columns=120)

x,a,b,c = sy.var('x,a,b,c')
n = sy.Symbol('n', positive=True, integer=True)
```

Start by entering the three rational functions.

```python
g = 1-82*x-82*x**2+x**3
f1 = (1+53*x+9*x**2)/g
f2 = (2-26*x-12*x**2)/g
f3 = (2+8*x-10*x**2)/g

display(f1)
display(f2)
display(f3)
```

$$\frac{9x^2+53x+1}{x^3-82x^2-82x+1}$$

$$\frac{-12x^2-26x+2}{x^3-82x^2-82x+1}$$

$$\frac{-10x^2+8x+2}{x^3-82x^2-82x+1}$$

```python
fp1 = f1.apart(x,full=True).doit()
fp2 = f2.apart(x,full=True).doit()
fp3 = f3.apart(x,full=True).doit()
fs1 = [z.simplify() for z in fp1.args]
fs2 = [z.simplify() for z in fp2.args]
fs3 = [z.simplify() for z in fp3.args]

display(fs1)
display(fs2)
display(fs3)
```

$$\left[-\frac{43}{85x+85}, \frac{8(101+11\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{8(101-11\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

$$\left[-\frac{16}{85x+85}, \frac{28(-37-4\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{28(-37+4\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

$$\left[-\frac{16}{85x+85}, \frac{6(-139-15\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{6(-139+15\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

Note that the denominators of each fraction is the same (as we'd expect).

Now we use the fact that

$$\frac{a}{bx+c}$$

has the infinite series expansion

$$\frac{a}{c}\left(1-\frac{b}{c}x+\left(\frac{b}{c}\right)^2x^2-\left(\frac{b}{c}\right)^3x^3+\cdots\right)$$

This means that the coefficient of x^n is

$$\frac{a}{c}\left(-\frac{b}{c}\right)^n$$

Because of the denominators, the values of b and c are always the same. We start by considering `fs1`, which consists of the partial fraction sums of `f1`.

```
a1_s = [sy.numer(z) for z in fs1]
b1_s = [sy.denom(z).coeff(x) for z in fs1]
c1_s = [sy.denom(z).coeff(x,0) for z in fs1]

ac1_s = [sy.simplify(s/t) for s,t in zip(a1_s,c1_s)]
bc1_s = [sy.simplify(s/t) for s,t in zip(b1_s,c1_s)]

display(ac1_s)
display(bc1_s)
```

$$\left[-\frac{43}{85}, \frac{64}{85}-\frac{8\sqrt{85}}{85}, \frac{64}{85}+\frac{8\sqrt{85}}{85}\right]$$

$$\left[1, -\frac{83}{2}+\frac{9\sqrt{85}}{2}, -\frac{83}{2}-\frac{9\sqrt{85}}{2}\right]$$

Now we can determine the coefficient of x^n in the power series expansion of $f_1(x)$:

```
a_n = sum(s*(-t)**n for s,t in zip(ac1_s,bc1_s))
display(a_n)
```

$$-\frac{43(-1)^n}{85}+\left(\frac{64}{85}-\frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{64}{85}+\frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

And repeat all of the above for $f_2(x)$ and its partial fractions `fs2`.

```
a2_s = [sy.numer(z) for z in fs2]
b2_s = [sy.denom(z).coeff(x) for z in fs2]
c2_s = [sy.denom(z).coeff(x,0) for z in fs2]
```

```
ac2_s = [sy.simplify(s/t) for s,t in zip(a2_s,c2_s)]
bc2_s = [sy.simplify(s/t) for s,t in zip(b2_s,c2_s)]

b_n = sum(s*(-t)**n for s,t in zip(ac2_s,bc2_s))
display(b_n)
```

$$-\frac{16(-1)^n}{85}+\left(\frac{77}{85}-\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{77}{85}+\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

Continuing for $f_3(x)$:

```
a3_s = [sy.numer(z) for z in fs3]
b3_s = [sy.denom(z).coeff(x) for z in fs3]
c3_s = [sy.denom(z).coeff(x,0) for z in fs3]

ac3_s = [sy.simplify(s/t) for s,t in zip(a3_s,c3_s)]
bc3_s = [sy.simplify(s/t) for s,t in zip(b3_s,c3_s)]

c_n = sum(s*(-t)**n for s,t in zip(ac3_s,bc3_s))
display(c_n)
```

$$-\frac{16(-1)^n}{85}+\left(\frac{93}{85}-\frac{9\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{93}{85}+\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

In order to see their similarities and differences, we now show them together:

```
display(a_n)
display(b_n)
display(c_n)
```

$$-\frac{43(-1)^n}{85}+\left(\frac{64}{85}-\frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{64}{85}+\frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

$$-\frac{16(-1)^n}{85}+\left(\frac{77}{85}-\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{77}{85}+\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

$$-\frac{16(-1)^n}{85}+\left(\frac{93}{85}-\frac{9\sqrt{85}}{85}\right)\left(\frac{83}{2}-\frac{9\sqrt{85}}{2}\right)^n+\left(\frac{93}{85}+\frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2}+\frac{9\sqrt{85}}{2}\right)^n$$

The first few coefficients can be checked against a Taylor series expansion:

```
s1 = f1.series(n=6)
display([s1.coeff(x,k) for k in range(6)])
a_s = [a_n.subs(n,k).simplify() for k in range(6)]
display(a_s)
```

$[1, \quad 135, \quad 11161, \quad 926271, \quad 76869289, \quad 6379224759]$

$[1, \quad 135, \quad 11161, \quad 926271, \quad 76869289, \quad 6379224759]$

```
s2 = f2.series(n=6)
display([s2.coeff(x,k) for k in range(6)])
b_s = [b_n.subs(n,k).simplify() for k in range(6)]
display(b_s)
```

$[2, \quad 138, \quad 11468, \quad 951690, \quad 78978818, \quad 6554290188]$

$[2, \quad 138, \quad 11468, \quad 951690, \quad 78978818, \quad 6554290188]$

```
s3 = f3.series(n=6)
display([s3.coeff(x,k) for k in range(6)])
c_s = [c_n.subs(n,k).simplify() for k in range(6)]
display(c_s)
```

$[2, \quad 172, \quad 14258, \quad 1183258, \quad 98196140, \quad 8149006378]$

$[2, \quad 172, \quad 14258, \quad 1183258, \quad 98196140, \quad 8149006378]$

Now, if everything has behaved properly, we should now have

$$a^3 + b^3 = c^3 + (-1)^3$$

and we can check the first few values:

```
[s**3+t**3-u**3 for s,t,u in zip(a_s,b_s,c_s)]
```

$[1, \quad -1, \quad 1, \quad -1, \quad 1, \quad -1]$

And now for the general result:

```
sy.powsimp(sy.expand(a_n**3 + b_n**3 - c_n**3),
    combine='all',
    force=True).factor()
```

$(-1)^n$

Which is exactly what we wanted.

Now for the other expansions, in negative powers of x; in other words based on the the functions $f_k(1/x)$. We'll rename these functions: $g_k(x) = f_k(1/x)$. After that it's pretty much a carbon copy of the preceding computations.

```
g1 = f1.subs(x,1/x).simplify()
g2 = f2.subs(x,1/x).simplify()
g3 = f3.subs(x,1/x).simplify()

display(g1)
display(g2)
display(g3)
```

$$\frac{x(x^2+53x+9)}{x^3-82x^2-82x+1}$$

$$\frac{2x(x^2-13x-6)}{x^3-82x^2-82x+1}$$

$$\frac{2x(x^2+4x-5)}{x^3-82x^2-82x+1}$$

```
gp1 = g1.apart(x,full=True).doit()
gp2 = g2.apart(x,full=True).doit()
gp3 = g3.apart(x,full=True).doit()
gs1 = [z.simplify() for z in gp1.args]
gs2 = [z.simplify() for z in gp2.args]
gs3 = [z.simplify() for z in gp3.args]

display(gs1)
display(gs2)
display(gs3)
```

$$\left[1, \frac{43}{85x+85}, \frac{8(1429+155\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{8(1429-155\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

$$\left[2, -\frac{16}{85x+85}, \frac{14(839+91\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{14(8397-91\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

$$\left[2, \frac{16}{85x+85}, \frac{12(1217+132\sqrt{85})}{85(2x-83-9\sqrt{85})}, \frac{12(1217-132\sqrt{85})}{85(2x-83+9\sqrt{85})}\right]$$

For ease of writing in Python, we'll use `d`, `e` and `f` instead of α, β and γ.

```
d1_s = [sy.numer(z) for z in gs1]
e1_s = [sy.denom(z).coeff(x) for z in gs1]
f1_s = [sy.denom(z).coeff(x,0) for z in gs1]

df1_s = [sy.simplify(s/t) for s,t in zip(d1_s,f1_s)]
ef1_s = [sy.simplify(s/t) for s,t in zip(e1_s,f1_s)]

d_n = sum(s*(-t)**n for s,t in zip(df1_s,ef1_s))
```

```
d2_s = [sy.numer(z) for z in gs2]
e2_s = [sy.denom(z).coeff(x) for z in gs2]
f2_s = [sy.denom(z).coeff(x,0) for z in gs2]

df2_s = [sy.simplify(s/t) for s,t in zip(d2_s,f2_s)]
ef2_s = [sy.simplify(s/t) for s,t in zip(e2_s,f2_s)]

e_n = sum(s*(-t)**n for s,t in zip(df2_s,ef2_s))
```

```
d3_s = [sy.numer(z) for z in gs3]
e3_s = [sy.denom(z).coeff(x) for z in gs3]
f3_s = [sy.denom(z).coeff(x,0) for z in gs3]

df3_s = [sy.simplify(s/t) for s,t in zip(d3_s,f3_s)]
ef3_s = [sy.simplify(s/t) for s,t in zip(e3_s,f3_s)]

f_n = sum(s*(-t)**n for s,t in zip(df3_s,ef3_s))
```

```
display(d_n)
display(e_n)
display(f_n)
```

$$-\frac{43(-1)^n}{85} + \left(-\frac{64}{85} - \frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2} - \frac{9\sqrt{85}}{2}\right)^n + \left(-\frac{64}{85} + \frac{8\sqrt{85}}{85}\right)\left(\frac{83}{2} + \frac{9\sqrt{85}}{2}\right)^n$$

$$-\frac{16(-1)^n}{85} + \left(-\frac{77}{85} - \frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2} - \frac{9\sqrt{85}}{2}\right)^n + \left(-\frac{77}{85} + \frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2} + \frac{9\sqrt{85}}{2}\right)^n$$

$$-\frac{16(-1)^n}{85} + \left(-\frac{93}{85} - \frac{9\sqrt{85}}{85}\right)\left(\frac{83}{2} - \frac{9\sqrt{85}}{2}\right)^n + \left(-\frac{93}{85} + \frac{7\sqrt{85}}{85}\right)\left(\frac{83}{2} + \frac{9\sqrt{85}}{2}\right)^n$$

As before, a quick check:

```
ds = [d_n.subs(n,k).simplify() for k in range(6)]
es = [e_n.subs(n,k).simplify() for k in range(6)]
fs = [f_n.subs(n,k).simplify() for k in range(6)]
display(ds)
display(es)
display(fs)
```

$$[-1, 9, 791, 65601, 5444135, 451797561]$$

$$[-2, -12, -1010, -83802, -6954572, -577145658]$$

$$[-2, -10, -812, -67402, -5593538, -464196268]$$

```
[s**3+t**3-v**3 for s,t,v in zip(ds,es,fs)]
```

$$[-1, \quad 1, \quad -1, \quad 1, \quad -1, \quad 1]$$

And finally, confirming the general result:

```
sy.powsimp(sy.expand(d_n**3 + e_n**3 - f_n**3),
    combine='all',
    force=True).factor()
```

$-(-1)^n$

Again, exactly what was wanted.

The algebraic manipulations involved in these formulas only serves to heighten our amazement at Ramanujan's power, given that compared to the depth of most of his work, this was pretty much a "throw-away" result.

4 Where to now...

Ramanujan's works present one of the richest treasure troves of mathematics ever produced by a single person. His results range from the (relatively) straightforward to a depth almost unparalleled. And as Hardy observed in his comment on Ramanujan's first letter to him, the specific equations given clearly spoke of some more general results, known perhaps only to Ramanujan himself. Now just over a century past his death, his legacy remains as bright, exciting, and as enticing as ever. And the exploration of his results with computer systems has only begun.

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