Augmented intelligence with GeoGebra and Maple involvement

M. Pilar Vélez, Tomás Recio
pvelez@nebrija.es, trecio@nebrija.es
Escuela Politécnica Superior
Universidad Antonio de Nebrija
Madrid, Spain.

Abstract: *We address Clough's conjecture, restricted to the case of an equilateral triangle, using GeoGebra Discovery. On the one hand, we obtain a necessary and sufficient condition for the Clough conjecture to hold true. On the other hand, this example is used here to illustrate both some technical issues (namely, real vs. complex automated reasoning approaches in GeoGebra Discovery) and what we have called "augmented intelligence", by means of machine-human interaction, impacting in mathematics education as well as in today's mathematical research work.*

1. Introduction

The digital revolution we have experienced with the massive arrival of pocket calculators and personal computers since the last quarter of the 20th century continues to have a significant influence on the way that Mathematics is learned and developed. Until then, mathematics has essentially developed through writing and, as a result of this, it seems, in some sense, that it continues to favour discourse. At the same time, several mathematicians who have been able to integrate technological tools into their research have begun to use them in their researching and teaching, renewing the traditional relationship of writing to do and to learn mathematics. The result is a kind of unavoidable tension between traditional mathematics and technological mathematics with poorly defined outlines, but which will obviously thrive.

Recently published, the book "Mathematics Education in the Age of Artificial Intelligence" [15] addresses human-machine interaction in both directions. It contains ideas, questions, and inspiration related to the creation of artificial intelligence (AI) milieus to learn and to do mathematics, to the AI-supported learning of mathematics and to the coordination of "usual" paper/pencil techniques that invite the challenge of artificial intelligence contributing to human learning of mathematics.

In the spirit of this book, in this article we reflect on the inevitable human-machine symbiosis in today mathematical work and illustrate it with an example from our own experience (Clough's conjecture). The emergence of computers, and more recently everything related to artificial intelligence, has shaped a new way of doing and learning mathematics. The role of mathematics in this scenario is twofold. On the one hand, mathematics is the hidden support of artificial intelligence and, therefore, of the digitization of everything; on the other hand, personal computers provide digital tools that perform incredible calculations (including most of the tasks required in the current mathematics curriculum) or facilitate the drawing of dynamic graphs that help visualize mathematical objects, and even perform mathematical demonstrations. This symbiosis should result in what has been called "augmented intelligence". According to the Gartner Glossary[1]:

[1] https://www.gartner.com/en/information-technology/glossary/augmented-intelligence

> "Augmented intelligence is a design pattern for a human-centered partnership model of people and artificial intelligence (AI) working together to enhance cognitive performance, including learning, decision-making, and new experiences."

In section 2 we make a not exhaustive, but illustrative, summary of different digital tools for mathematics that can be considered AI tools because of their performance or their development. In view of all these tools with different purposes, we could assume that mathematical work will be reduced to using a certain software that solves all problems or makes demonstrations. Nothing could be further from the truth today (who knows in the future!).

Section 3 shows how digital tools, far from substituting human mathematical work, enrich it by providing new ideas and new points of view. We show how, through our own experience with automatic reasoning tools in GeoGebra and the checking of examples, we managed to learn geometry, detect problems and improve the tool itself. In this case we use as example the Clough's conjecture.

Section 4 contains the theoretical conclusions in the algebraic geometry framework of automated reasoning in geometry that have led us to analyze the Clough's conjecture example, demonstrating the powerful of human-machine interaction in mathematical work.

2. Machine thinking: Digital tools for Mathematics

The design of digital tools for doing mathematics requires a deep reflection, taking into account the opportunities they offer, as well as the way in which learning and research are modified in the last fifty years. These tools, which in many cases we can consider Artificial Intelligence (AI) offer both great benefits and enormous challenges for mathematicians. Since the work of Turing (1950), the capabilities of digital machines to perform symbolic processing have been progressing, with the goal of making computers capable of performing tasks that are generally considered "intelligent" [11].

Symbolic AI techniques were dominant in research from the 1950s to the 1990s. Their basis are algorithms that operate on high-level symbolic representations of entities in a mathematical domain. Computer algebra systems (CAS) can be considered under this perspective, as they are symbol manipulators that operate on representations of algebraic entities, following rules, in some cases driven by heuristics. Since their introduction in the 1970s-1980s, CAS have been a facilitating media for the work of mathematicians, but it has also influenced mathematical practices, including teaching and learning (see for instance, the computer-mediated thinking proposal [3] of Corless).

Dynamic Geometry (DG) systems, such as GeoGebra[2], have become very popular for teaching purposes. Initially based on direct manipulation and analytic computations of numerical objects, these environments evolve to integrate a computer algebra system, as is the case of Giac in GeoGebra, thanks to which these objects can be managed as symbolic entities. This allows an automatic reasoning capability in geometry. In section 3 we will discuss the case of GeoGebra automated proving tools. See [12] for a history of Automated Deduction in Geometry from the earliest developments of automated theorem provers to the current application systems combining dynamic

[2] https://www.geogebra.org/

geometry and automated deduction to create mathematical environments pursuing mathematical rigor.

At the present, different digital tools for mathematics based on different AI techniques have emerged [16]. For instance, "camera calculators", such as Photomath[3], based in image recognition technology, to point a phone camera at any equation and instantly obtain the solution, including the detailed steps of reasoning. As well as intelligent tutorial systems, such as QED-Tutrix [5], based on data mining, are able to follow and guide students during the completion of a task, and suggest next step. In [16], a taxonomy of AI techniques that are used in digital tools for mathematics education is proposed and exemplified.

A step even further are systems known as interactive proof assistants [2], for example Lean[4]. The Lean theorem prover support not only proving but mathematical reasoning in general:

> "The Lean Theorem Prover aims to bridge the gap between interactive and automated theorem proving, by situating automated tools and methods in a framework that supports user interaction and the construction of fully specified axiomatic proofs. The goal is to support both mathematical reasoning and reasoning about complex systems, and to verify claims in both domains" ([1], pp. 1).

In a proof assistant, a set of mathematical concepts are introduced, then the program generates a library of computer code upon which other researchers can build and use to define higher-level mathematical objects. Thus, proof assistants force the user to state the logic of their arguments rigorously and to complete simpler steps that human mathematicians had skipped [2].

A reflection about the need for new approaches to teaching proof can be found in [6]. The irruption of proof assistants or automatic proving tools has not been reflected, in general, in the teaching of mathematics, even at undergraduate level, therefore "as a result, there is no solid evidence for the degree to which proof assistants in the undergraduate class-room might help students construct and understand valid proofs" ([6], pp. 1).

Although automated proving tools and proof assistants could surprise how far they can go, they still need continuous feeding by humans. On the other hand, it seems they cannot decide whether a mathematical statement is interesting or relevant, only whether it is correct or point out consequences of known facts that the mathematicians had not noticed.

Despite all the digital systems for mathematics, of which only a few are mentioned here for illustrative purposes, it seems that machines are far from replacing humans in mathematical work.

3. Interacting with intelligent machines: the case of GeoGebra Discovery

GeoGebra dynamic geometry software includes, since 2016, several tools and commands (Relation, LocusEquation, Prove, ProveDetails and Envelope) and commands (Relation, LocusEquation, Prove,

[3] https://photomath.app/
[4] https://leanprover.github.io/

ProveDetails and Envelope) to automatically prove and discover theorems in plane geometry constructions. This based in the CAS system Giac. An updated experimental version of GeoGebra, called GeoGebra Discovery [10], extends the automated reasoning capabilities of GeoGebra towards achieving a kind of mechanical geometric program that does not require human intervention. GeoGebra Discovery can be downloaded from https://github.com/kovzol/geogebra/releases.

These automated reasoning tools for elementary geometry are based on the algebraic geometry algorithmic approach of [13]. Roughly speaking, starting from a geometric configuration drawn in the graphical window, it translates the hypotheses H and the theses T into sets of polynomial equations, and proves the inclusion of the set of solutions of the hypotheses $V(H)$ in the set of solutions of the thesis $V(T)$ over an algebraically closed field, such as the complex field.

But this straightforward formulation is hardly useful in practice, because $V(H)$ often includes unexpected cases in which the thesis is not satisfied, for example, related to degenerate cases such as a segment defined by two points, when these two points coincide, etc. Thus, a more sophisticated proposal was included in [13] which involved algorithmically decrypting and discriminating (in a sense) these degenerate instances, identifying a distinguished set of geometrically significant free variables for $V(H)$.

Thus, proving the $V(H) \subset V(T)$ is reformulated by algorithms concerning the elimination of the ideals $(H, T * t - 1)$ and (H, T) over the chosen set of free variables, giving rise to the concepts of "generally true", "generally false" and "true on parts, false on parts" when the result of the corresponding elimination is different from or equal to zero [8].

Through an example, the Clough's conjecture, we would like to show the power of human-machine collaboration, which we have called augmented intelligence, both for learning and for researching.

3.1. The inspiring Clough's conjecture

The expert in mathematics education and geometry, Prof. M. De Villiers introduces and discusses in 2004 the so-called Clough's conjecture for equilateral triangles. In October 2021 the young Spanish student Alvaro Gamboa proposes in the section Problem Corner of The Electronic Journal of Mathematics and Technology the following generalization (Fig. 1) to any regular polygon.

The problem received two solutions, the first one with pencil and paper by Alvaro Gamboa[5] himself using trigonometry, and the alternative one by Kovács, Recio and Vélez[6] using GeoGebra Discovery automated reasoning tools see [10].

[5] https://php.radford.edu/~ejmt/ProblemCornerDocs/eJMT_ProblemCorner_Solutions_to_Oct2021.pdf
[6] https://php.radford.edu/~ejmt/ProblemCornerDocs/eJMT_Alternative_Solutions_to_Oct2021.pdf

PROBLEM 1

Let $X_1, X_2, \ldots, X_n$ be the vertices of a regular n-gon P and let P be any point interior to P. We denote by P_{ij} the projection of P onto $r(X_i, X_j)$. By abuse, we denote $X_{n+1} := X_1$, and so $P_{n,1} := P_{n,n+1}$. Prove that the sum

$$\sum_{i=1}^{n} X_i P_{i,i+1}$$

is constant, that is, it does not depend on the point P (see Figure 1 for an example).

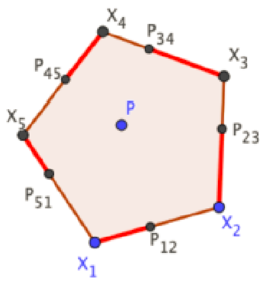

Figure 1. Clough's conjecture generalized to regular polygons.
https://php.radford.edu/~ejmt/ProblemCornerDocs/eJMT_ProblemCorner_Problems_Oct2021.pdf

Following the alternative solution [9], to address Clough's conjecture using GeoGebra Discovery, for our purposes here, we will restrict ourselves to the case of an equilateral triangle (which is the . In Figure 2 we use the *ProveDetails* command of GeoGebra to ask about the equality between the sum of the three segments l, m, n, and the semi-perimeter 3/2 *f, where f is the side of the equilateral triangle.

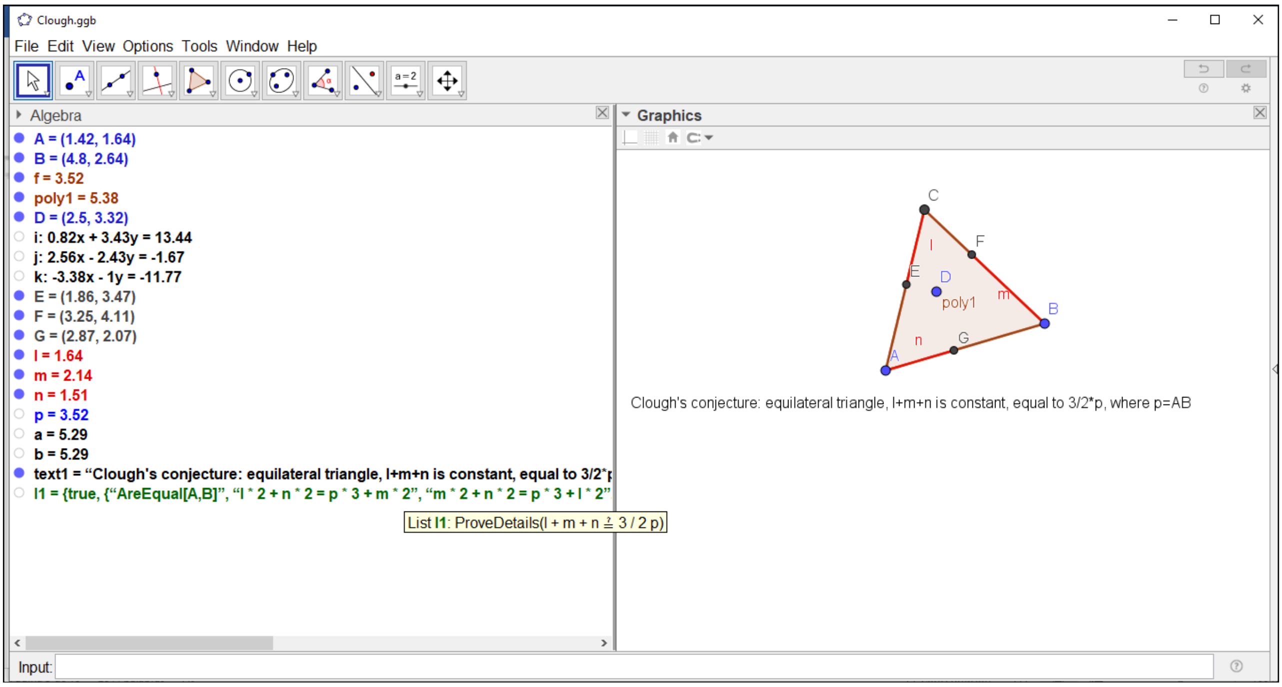

Figure 2. Asking to GeoGebra Discovery *ProveDetails(l+m+n=3/2*p)*

We were surprised by the answer, because GeoGebra Discovery says that the statement is true except when some of the following seven equalities happens:

*l1 = {true, {"AreEqual[A,B]", "f * 3 + n * 2 = l * 2 + m * 2", "l * 2 + f * 3 + n * 2 = m * 2", "l * 2 + n * 2 = m * 2 + f * 3", "m * 2 + f * 3 + n * 2 = l * 2", "m * 2 + n * 2 = l * 2 + f * 3", "n * 2 = l * 2 + m * 2 + f * 3"}}*

The first one is A = B, that is the case of degenerate triangles. The remaining ones, expressed more closely related to the statement, are as follows:

$$3f - 2(l + m - n) = 0, 3f - 2(-l + m - n) = 0, 3f - 2(l - m + n) = 0,$$
$$3f - 2(l - m - n) = 0, 3f - 2(-l + m + n) = 0, 3f - 2(-l - m + n) = 0.$$

We can see that the six equalities correspond to all possible sign choices for the addition of l, m, n except $l + m + n$ and $-l - m - n$. Of course, the case $l + m + n$ is precisely the one we are checking for its validity. And the case $-l - m - n$ is not geometrically meaningful, since f, l, m, n are segment lengths (thus positive) and $-l - m - n$ is negative.

Our intuition let us to conjecture that these equalities had to do with the position of the point D in the different regions in the real plane determined by the 3 straight lines that contain the sides of the equilateral triangle, and precisely when D is in the interior of the triangle, $3f - 2(l + m + n) = 0$ holds.

Using the *Relation* command, another simpler way to ask for truth with GeoGebra Discovery, the answer to the equality between $2 * (l + m + n)$ and the perimeter we found, that "numerically" (that, is, for the concrete triangle configuration draw in the GeoGebra window) the equality is true for D inside the triangle and fails for D outside the triangle (see Fig. 3)

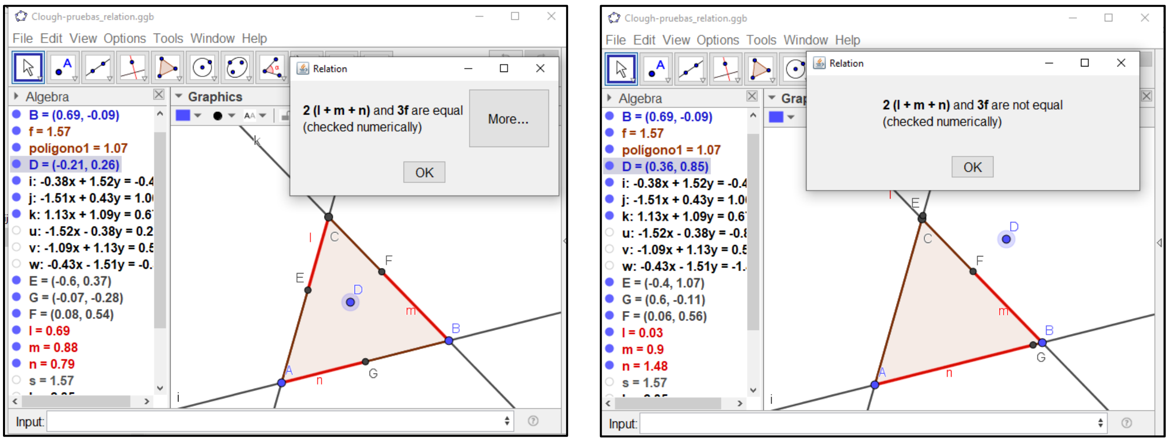

Figure 3. GeoGebra" numerical" answer when asking $\boldsymbol{Relation}(\mathbf{2} * (\boldsymbol{l} + \boldsymbol{m} + \boldsymbol{n}), \mathbf{3} * \boldsymbol{f})$, where $\boldsymbol{f}$ is the length of segment $\boldsymbol{AB}$, for $\boldsymbol{D}$ inside (left) and outside (right) the triangle.

Remark that for D inside the triangle, where the equality holds, we can ask for a "symbolic" answer by clicking the "More…" button (Fig. 4), and the answer is "true on parts, false on parts" (see [8]), i.e. true just on some instances of the hypothesis configuration (namely, we have conjectured when the point D is placed inside the triangle).

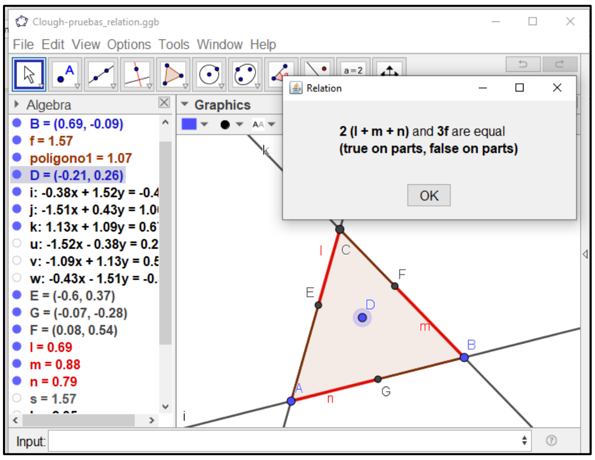

Figure 4. GeoGebra "symbolic" answer when clicking the "More…" button in the left situation of Fig. 3.

From the point of view of complex algebraic geometry, which is the context for GeoGebra Discovery implemented algorithms, we cannot be more precise, since to represent lengths by means of polynomials it is necessary to use polynomials of degree 2 and therefore their positive and negative roots are indistinguishable. See [9] for more details.

To distinguish a length of a segments as the positive root of degree 2 polynomial the option is to deal with real algebraic geometry algorithms. This actual approach is more accurate, but less efficient than the complex one.

At this point, several questions arise:

a) What is the relation between the list of degeneracy conditions of Fig. 2 and the primary components of the hypotheses ideal?
b) It seems that the real plane locus for D such that the semiperimeter is equal to $l + m + n$ is not an algebraic set, it is a region described by polynomial inequlities (that is, a semialgebraic set). Is there some hidden real information in the primary algebraic components of the hypotheses variety?

But our goal for this presentation is not to talk about the technical details of the computer algebra algorithms behind the automatic reasoning tools in GeoGebra, but to illustrate through this example what we have called "augmented intelligence", by means of machine-human interaction.

3.2. Researching: real vs complex

Our initial guess, about the relationship between the position of the point D in the different regions in the real plane determined by the 3 straight lines that contain the sides of the equilateral triangle and the equalities given by *ProveDetails* command of GeoGebra, turned out to be false. It suffices to do some checks in GeoGebra by dragging point D in different positions inside and outside the triangle (Fig. 5).

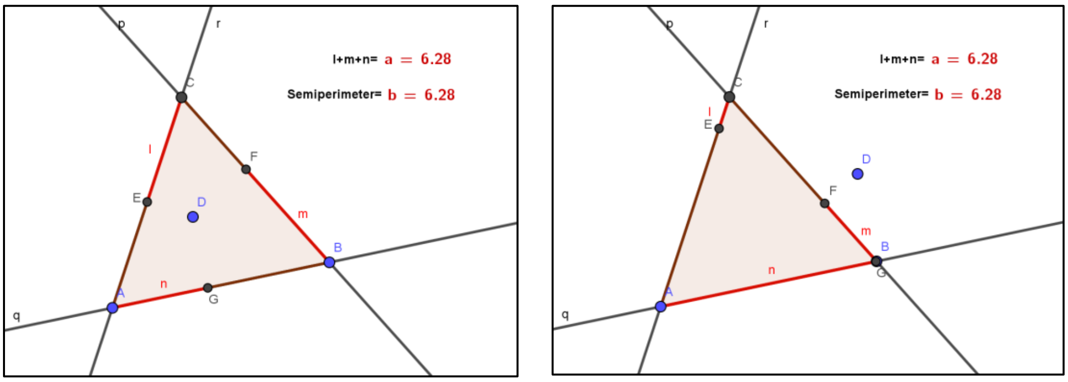

Figure 5. The figure on the left shows the equality between $\boldsymbol{l}+\boldsymbol{m}+\boldsymbol{n}$ and the semiperimeter when point $\boldsymbol{D}$ is dragged inside the triangle, while the figure on the right shows that the equality is still verified for certain positions of D outside the triangle but close to its sides.

Then, we could observe, by dragging again point D, that the bound to have the equality $3f-2(l+m+n)=0$ is reached when one of the points E, F or G arrives a vertex, and from this vertex out, the equality is not verified (Fig. 6).

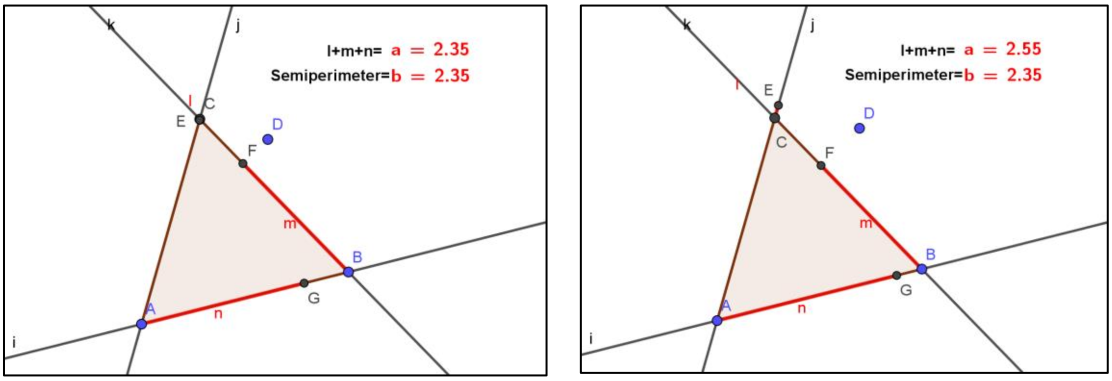

Figure 6. The equality between $\boldsymbol{l}+\boldsymbol{m}+\boldsymbol{n}$ and the semiperimeter fails when point E leaves the side AC to the right of vertex C due to the effect of the dragging of point D

This experimental fact led us down to achieve two different goals:

1) To extend ~~obtain a generalization of~~ the Clough's conjecture for equilateral triangles to positions of point D outside the triangle.
2) To extend our theoretical research field to the study inspired by points a) and b) above.

3.2.1 *Extended Clough's conjecture for equilateral triangles*

After dragging point D around the equilateral triangle ABC, we can conjecture that l+m+n is equal to the semiperimeter if and only if the point D is inside the triangle $\mathcal{T}$ described by the following lines: the perpendicular to AB in A, the perpendicular BC in B and the perpendicular to CA in C (see Fig. 7). Remark that this new triangle contains triangle ABC.

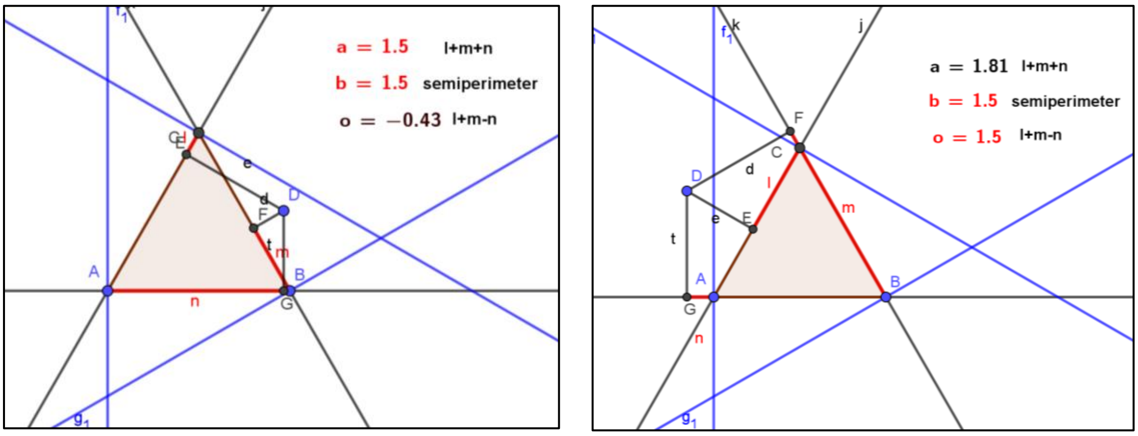

Figure 7. Some positions of point D inside and outside triangle $\boldsymbol{\mathcal{T}}$. The left side show that $\boldsymbol{l+m+n}$ is equal to the semiperimeter for $\boldsymbol{D}$ inside $\boldsymbol{\mathcal{T}}$ but outside $\boldsymbol{ABC}$. While the right side shows $\boldsymbol{D}$ outside $\boldsymbol{\mathcal{T}}$ in a region where the semiperimeter coincides with $\boldsymbol{l+m-n}$.

3.2.2 *Inspiring real vs complex behavior of automated reasoning*

The verification of Clough's conjecture using GeoGebra Discovery (Fig. 4) shows the need (and the involved mathematical, algorithmic and user-interface difficulties) to improve proof assistants to output some answer that could be clearly understood by most users. Here let us just succinctly state that 'true on parts, false on parts' refers to the fact that the algebraic translation of the construction involves different components (but, for a standard user, there is only one, the one that is graphically and intuitively perceived), because the idea of 'length' of a side is, in the complex geometry algorithmic background for GeoGebra Discovery, some square root that can take positive or negative values. And, of course, the involved conjecture is true for the component where these roots are positive, and fails for the others. The option to associate signs to the lengths of segments involves real algebraic geometry and it is on-going work.

Otherwise, as we remark above, we should have to work on the realm of computational real algebraic geometry (more accurate, but less efficient than the complex one), which allows us to introduce polynomial inequalities and thus to distinguish the real roots of a polynomial. But, is it posible to find some bridge between real and complex approches?

Let us therefore analyze the behavior of our example from the algebraic geometry point of view by using the potential of Maple[7] to deal with real and complex polynomial ideals and with semialgebraic sets. In particular, we will deal with the hypothesis ideal which is an algebraic model of the geometric construction where the coordinates of the points are the variables of the polynomials.

For simplicity in computations, we take the equilateral triangle ABC as $A(00), B(10), C(\frac{1}{2}, \frac{\sqrt{3}}{2})$ with sides of lengh 1; and take a free point $D(a,b)$. Then, G must be (a,0) and take E(u,v), F(r,s). Now instead of working with $\sqrt{3}$, we use the equivalente formulation $k = \sqrt{3}$ including $k^2 - 3$, considering the following hypotheses ideal:

[7] https://www.maplesoft.com/

$$H:=< -\frac{k}{2}u+\frac{1}{2}v, \frac{1}{2}(u-a)+\frac{k}{2}(v-b), \frac{k}{2}\,(r-1)+\frac{1}{2}\,s-, \frac{1}{2}(r-a)\ +\ \frac{k}{2}(s-b),$$
$$n^2-a^2, l^2-\left(\left(u-\frac{1}{2}\right)^2+\left(v-\frac{k}{2}\right)^2\right), m^2-((r-1)^2+s^2), k^2-3 >$$

The thesis ideal is $T\ =< l+m+n-3/2 >$. Note that H is a two dimensional ideal with *{a,b}* as free variables.

Recall that GeoGebra has revealed a "true on parts, false on parts" situation (Fig. 4), that means that our statement is true on some components of the hypothesis ideal and false on others [8]. Therefore, our interest is to know in detail the components of the hypotheses ideal and to check the validity of out conjecture in each of them.

We use the *PolynomialIdeals* package of Maple to compute a primary decomposition of *H* (running the *PrimaryDecomposition(H)* command) and we get 8 primary components (Fig. 8). Remark that 8 is the number of additions of *l*, *m* and *n* with different signs. It is easy to check that all components have dimension 2 (*HilbertDimension(H)*) with *{a,b}* as free variables (*EliminationIdeal(H,{a,b})*).

$$\begin{aligned}
&\langle -a+n, -2r+m+2, -2a+2l+4r-1, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle -a+n, -2r+m+2, -4r+1+2l+2a, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle -a+n, 2r+m-2, -2a+2l+4r-1, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle -a+n, 2r+m-2, -4r+1+2l+2a, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle n+a, -2r+m+2, -2a+2l+4r-1, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle n+a, -2r+m+2, -4r+1+2l+2a, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle n+a, 2r+m-2, -2a+2l+4r-1, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle\\
&\langle n+a, 2r+m-2, -4r+1+2l+2a, 4u+4r-2a-3, k^2-3, kr-k+s, -ak\\
&\quad +4kr+3b-3k, -2ak+4kr-3k+4v\rangle
\end{aligned}$$

Figure 8. The 8 primary components of H

Labelling componnets as $PP[1]\ \dots\ PP[8]$, we check, using elimination, the components where the statement $l+m+n=3/2$ is true (and not generally false) or not true (and generally false), it holds that it is true only in the fourth one. Indeed, it happens that each of the factors of type $\pm l\pm m\pm n-$

$3/2$ is only true on one corresponding component, but also belongs to the ideal defining the component, a stronger version of truth (see Fig. 9).

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> -l-m+n-3/2 in PP[1];l-m+n-3/2 in PP[2]; -l+m+n-3/2 in PP[3];l+
m+n-3/2 in PP[4]; -l-m-n-3/2 in PP[5]; l-m-n-3/2 in PP[6]; -l+
m-n-3/2 in PP[7]; l+m-n-3/2 in PP[8];
                              true
                              true
                              true
                              true
                              true
                              true
                              true
                              true                              (3.
```

Figure 9. Verifying the stranger version of truth in the components of H.

In conclusion, we have approached the answer to question a) in Section 3.1.

We now ask ourselves about the geometrical sense of these components, that is, with respect to the position of $D(a,b)$ in the real plane. So we project each component over the a,b plane, assuming that k is positive (as the case when k is negative will be just symmetrical) and that l,m,n are positive, as they are distances. For example, let us consider the projection of the PP[4], where we know that statement $l+m+n-3/2$ is true. Using the $RegularChains$ package of Maple[8] we can compute this projection.

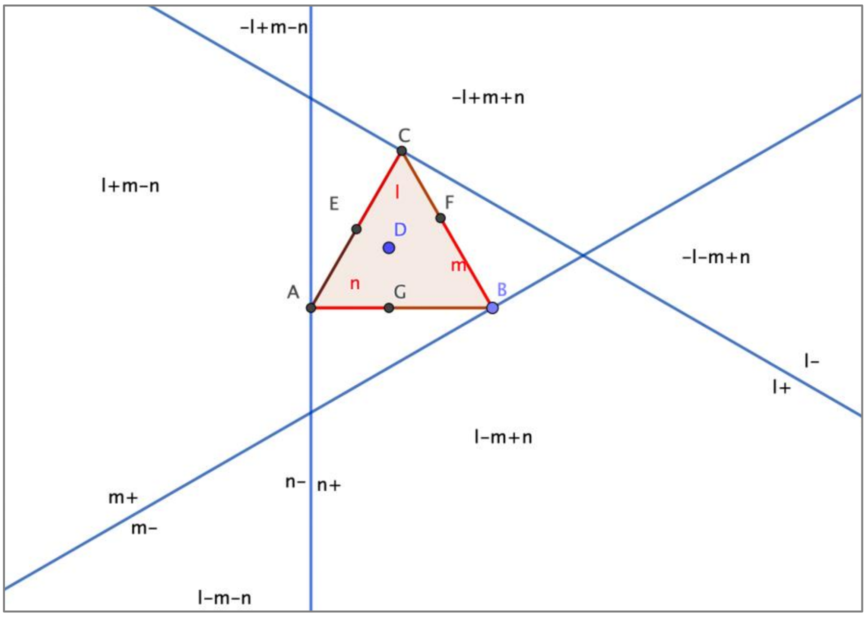

Figure 10. The projection of each of the 8 components of H give 7 different regions in real plane bounded by the tree lines in blue and in which only one of the possible equations $\boldsymbol{\pm l \pm m \pm n = 3/2}$ is true (as it is indicated in the figure). Remark that the component containing $\boldsymbol{-l-m-n=3/2}$ has not real points.

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[8] https://www.maplesoft.com/

The result is that the interior of a triangle limited by the perpendiculars to the sides of the given triangle ABC passing through one of the vertex (Fig. 10), confirming the conjecture made by dragging point D in GeoGebra.

Therefore, we have arrived to a proof of our conjecture and found a way to address question b) in Section 3.1: we can extract some real information in the complex primary algebraic components of the hypotheses variety.

4. Conclusions

After a surprising answer of automated reasoning tools in GeoGebra discovery, a direct observation with GeoGebra lead us to conjecture a necessary and sufficient condition for the Clough conjecture to hold true. This condition extends the locus for point D beyond the place given by De Villiers (the interior of triangle ABC) to a bigger triangle containing ABC. Furthermore, this example leads us to a better understanding of the relationship between the complex irreducible components and the real semialgebraic regions of the algebraic model (the tool used to implement automated reasoning in GeoGebra) of our construction.

Summarizing, by dissecting the anatomy of Clough's configuration, we arrive at a very coherent picture of the geometry behind this nice result. It has also given us light to broaden our theoretical field of study behind automatic reasoning algorithms in GeoGebra. Technical issues of real and complex algebraic geometry and algorithmic protocols are now work in progress. Preliminary results addressing these issues suggest the desirability of avoiding inequalities in the hypotheses and including them in the thesis (see [14]), as is the case for the algebraic formulation of Clough's conjecture.

As previously mentioned, it seems that machines are far from replacing humans in mathematical work. However, digital tools are a powerful resource for research in mathematics, not only as a collaborator for large computations, but also advance mathematics, when we use them by discovering and suggesting new paths, or when we develop them by the necessary mathematical algorithms that support them.

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