# THE POSSIBILITIES OF GEOGEBRA IN RESEARCH IN BRAZIL

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**Abstract:** *This paper aims to present the contribution of GeoGebra in research carried out in Brazil. Dynamic geometry systems (DGS) have evolved significantly and the contribution to research and education is noticeable in the papers presented here. The easy access and dynamism offered by GeoGebra make it possible to reduce the gaps between what is taught at school and the context of the students, in addition to contributing greatly to the training of teachers. Between 2009 and 2021, researchers identified 140 studies aimed at teacher training throughout Brazil and, in another study, on papers published in a specific journal on GeoGebra, it appears that all of them focus on didactic issues, concerning about the analysis of the effects on the teaching and learning process of students and analysis of the types of thinking/reasoning developed in the proposals presented. In the research, we identify strategies that enable teacher training that reflects positively on schools and indicates ways to improve teaching practice. Hence the importance of some theoretical contributions presented and considered in the studies, which allow validation of the results obtained. It appears that using GeoGebra is an alternative in the search for ways to build situations that help overcome teaching and learning difficulties, not only in Mathematics, but also in other contexts of science. The available and constantly evolving tools, as in the experimental proposal of GeoGebra Discovery, make it go beyond the context of mathematics to other sciences, revealing multidisciplinary aspects.*

## 1. Introduction

The evolution of information technology in society, from the end of the 1970s, created new possibilities of work in the school context by allowing the manipulation of objects on the computer screen, enabling new discoveries.

With the evolution of technologies, the so-called dynamic geometry systems (DGS) emerged, which in Brazil began with the use of software *CabriGéomètre*, created in France around 1985 by Jean-Marie Laborde [1]. Forerunner of the idea of dynamics in a computer resource, the *CabriGéomètre* was adopted in our institution in the postgraduate program in Mathematics Education, following the strong influence of French didactics in research in education, and with its use in some schools through teacher training projects.

Other DGS emerged around this time, such as *The Geometer's Sketchpad*, an interactive geometry software created by Eugene Klotz and Doris Schattschneider from 1986 to 1991 with the participation of Nicholas Jackiw, as the original designer, programmer of the software and inventor of their approach registered as "Geometry Dynamics". In late 1990s [2], *Cinderela* was introduced as a DGS, including the possibility of working with non-Euclidean geometries and with internal calculations on the set of complex numbers.

In 2001, GeoGebra created by researcher Markus Hohenwarter [3] is presented as a DGS, with a significant difference given its free access, and involving the collaboration of a community of users in all parts of the world that remains until the present day.

With the evolution of DGS, from 2D to 3D and the possibilities of computer algebra systems (CAS), new resources are constantly incorporated, making it a powerful tool for study and research, surpassing the context of mathematics to other sciences, revealing multidisciplinary aspects.

With the emergence of DGS, math educators who already presented observations of students working on the computer in their research, now had interactive learning environments, in which students prepare adaptations to solve problems through the dynamism they are allowed, aimed at learning, and which can be analyzed from this point of view. Such adaptations in problem solving attempts can be a source of new knowledge.

## 2. Research with GeoGebra in Brazil

The use of digital technologies in Brazil is a concern among mathematics teachers and managers, mainly due to the gaps between what is taught at school and the context of the students. Considering that the teaching of Mathematics must involve the use of digital technologies, in Brazil, the Brazilian National Common Core Curriculum (BNCC) [4] indicates its use in teacher training and presents digital culture as one of the general competences, as it is an important part of the Brazilian Basic Education curriculum.

In this context, master's and doctoral research, using GeoGebra, is developed in order to identify strategies that enable teacher training that reflects positively on schools and indicate ways to improve teaching practice.

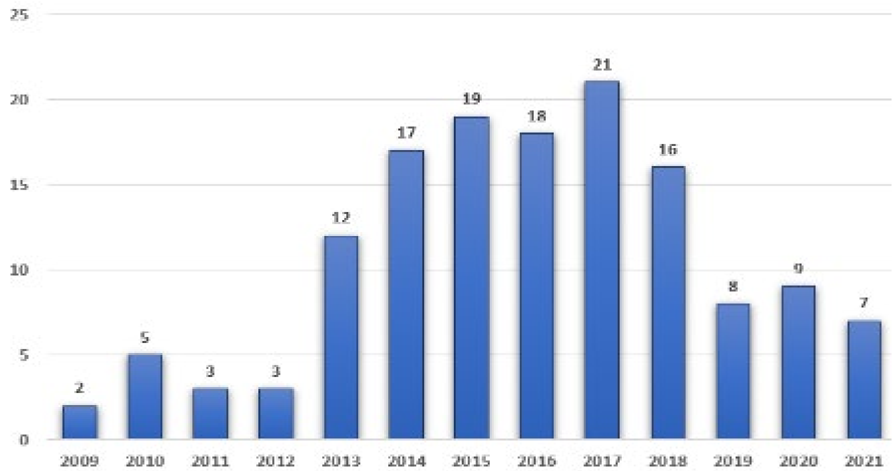

**Figure 1.** Year of Defense of Theses and Dissertations.
**Source:** Research data from Rodrigues and Azevedo, 2021.

According to [5], 140 research studies can be found between 2009 and 2021. Of this amount, 124 are master's and 16 doctoral studies, involving the theme of teacher training using GeoGebra and distributed as presented in Figure 1.

The authors consider that, in 2013, there was a significant increase in the number of defenses due to the professional master's program institutionalized in Brazil, whose research is directed towards teaching practice. The number of research studies in 2020 may reflect the Covid19 pandemic experienced by Brazil. Such pandemic persists.

[5] present a map of Brazil with the distribution of research by Brazilian State, which shows the knowledge of the possibilities of using GeoGebra in all regions and involving basic school mathematics content.

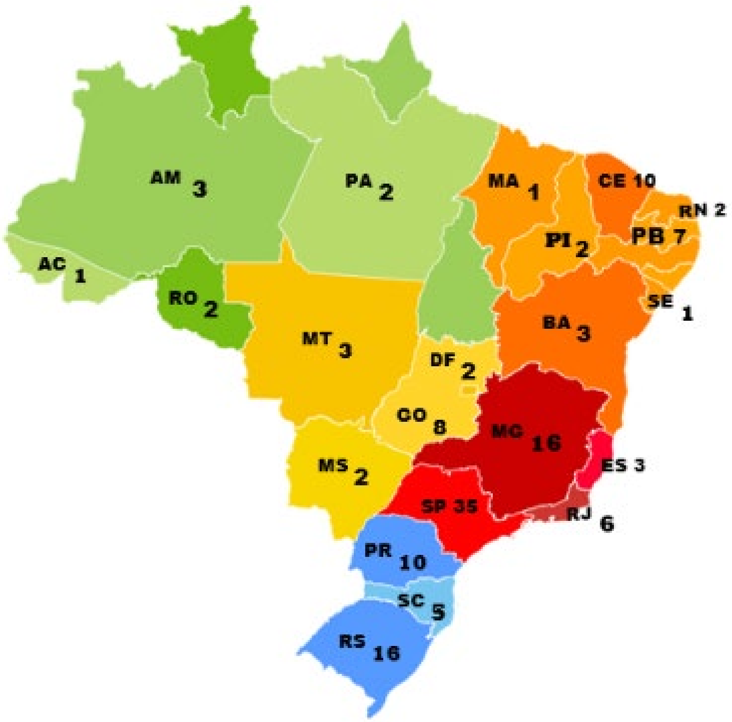


**Figure 2**. Distribution of Theses and Dissertations in the States of Brazil.
**Source:** Research data from Rodrigues and Azevedo, 2021.

Research on GeoGebra in the context of continuing education for Mathematics teachers, identified by Rodrigues and Azevedo, was developed in different contexts and presented several features that could be explored in the educational environment.

In [6], the authors present a paper that maps the papers published in the *Revista do Instituto GeoGebra Internacional de São Paulo*, between 2013 and 2018, focusing on the content of the published studies, in order to identify indicators of scientific production, in a way to check the profile of studies on the subject.

The analyzed papers focus on the description and explanation of teaching processes using GeoGebra, with detail and evidence of contributions to student learning. However, only four of the papers explicitly fit as research papers, presenting perfectly defined theoretical and methodological frameworks. Despite this situation, all papers address didactic issues, focusing on tasks and sequences of activities applied in the teaching context, concerned with analyzing the effects on the teaching and learning process of students and analyzing the types of thought/reasoning developed in the proposals presented.

In most of the papers analyzed, the mathematical activity development is presented through explorations, supported by visualization and, in about half of them, they are concerned with the consequences that this activity triggers in the explanation/demonstration and in the ability to conjecture. Another issue addressed in about a third of the papers is mathematical modeling activities.

More than half of the analyzed papers present the positive effect of experiences on students' motivation to learn and some present evidence of increased student confidence.

In view of the content analysis as a contribution, it was found that using GeoGebra may be considered as an alternative in the search for ways to build situations beyond the geometric content and help in overcoming the difficulties of teaching and learning, not only those related to mathematics, but also in other contexts of science.

## 3. Theoretical contributions that support research with GeoGebra

Oriented towards education and considering its pedagogical potential, DGS, particularly GeoGebra, are widely used in teaching and student practices in research, especially in the context of Mathematics Education, considering the evolutionary and dynamic aspect of mathematics.

Today, in Brazil and Latin America, GeoGebra is widely used both in teaching practice and in research, especially in teacher education. Academic productions on this theme have frequently occurred in master's and doctoral courses.

The main feature of GeoGebra lies in combining geometry, algebra and calculus in a single environment, in addition to allowing working with dynamic spreadsheets and using tools for data analysis, among other possibilities. In research conducted in Mathematics Education, in which GeoGebra is present, adequate theoretical and methodological contributions that can validate the results obtained are required.

The GeoGebra software is not just another technological resource, but one that helps develop mathematical concepts. By itself, the *software does not do mathematics*. In addition to this aspect, the resources presented by dynamic software allow the institutionalization of the knowledge of mathematical objects when they are explored by the different records visually presented on the computer screen, as considered by [7].

The fact that the teaching and learning processes are very complex means that the problems that mathematics teachers face in their professional activity give rise to many questions, from different categories related to many aspects: the mathematical content, the students' learning, the social context, the use of certain materials and technologies, student motivation, among others. The complexity of mathematical objects together with the complexity of their teaching and learning process are two of the reasons for the existence of a plurality of theories that support research in Mathematics Education, particularly those involving DGS.

Mathematics works with abstract objects. That is, mathematical objects are not directly accessible to perception, and require the use of a representation to be understood. In this case, the representations through symbols, signs, codes, tables, graphs, algorithms, and drawings are quite significant, as they allow the communication between the subjects and the cognitive activities of thought, allowing different representation records of the same mathematical object. [8].

According to the Theory of Registers of Semiotic Representation [7], in order to understand Mathematics, one needs to study the functioning of cognitive systems that provide the development of reasoning, analysis and visualization capabilities, also considering which are the necessary cognitive systems and if they are specific to the mathematics activity.

For the functioning of the cognitive activity required by Mathematics, which is differentiated from other domains of knowledge, semiotic representation is an essential condition for the evolution of mathematical thinking for two reasons: first, the possibilities of mathematical treatment depend on the representation system used; second, the existence of a great variety of semiotic representation used in Mathematics, such as geometric figures, algebraic writings, graphic representations and natural language. [7].

With the presence of Dynamic Geometry software, especially GeoGebra, the aspects considered by Duval can be explored in the different windows that are displayed and allow that a mathematical object be constructed. This is possible in GeoGebra, as all windows (2D and 3D graphics, algebraic, CAS, spreadsheet) can be seen at the same time, allowing the construction of an environment for possible conversions, by students, of different semiotic registers that can help them in the appropriation of mathematical objects. In addition, dynamic movements are key elements in the simultaneous variations of records for the construction and association of mathematical meanings between the concepts proposed in the didactic interventions.

According to [7], there are two types of transformations of semiotic representation registers: conversion and treatment, which represent the different signs used in Mathematics, such as figures, graphs, symbolic writings, natural language, and numerical register.

A conversion is a transformation of a representation, which changes from one register to another. For example, when using a Cartesian graph to represent a system of equations, a conversion from the graphic register to the algebraic register takes place. In our proposal, we will use the commands offered by GeoGebra in a symbolic writing specific to it, inserted in the respective Input and that allow the construction of mathematical objects. Then, symbolic writing register is converted into the graphic and algebraic registers.

The treatment is an operation performed within the same representation register, for example, when multiplying a system equation by a real number different from zero to scale it, this algebraic register is treated. In our case, the treatment occurs when different symbolic writings are used in the Input, resulting in the same Mathematical object.

When using dynamic geometry software, a treatment is performed on the graphic register when moving the figure or, a treatment is applied in the natural language, when the statement of an activity is rewritten in another form.

Another aspect to consider in research refers to the fact that verbs *explain, prove* and *demonstrate* are sometimes considered synonymous. This may become an obstacle for investigations around the topic and, thus, it is necessary to establish a distinction between them [9].

The explanation lies at the level of the person who produces it. It is expressed in the person's discourse and intends to make the truth of a proposition already acquired by that person intelligible. The explanation is not necessarily reduced to a deductive chain, and can be discussed, rejected or accepted. The passage from explanation to proof is a process in which the discourse that assures the validity of a proposition changes its role, becoming accepted by a certain community at a given moment. This role is not definitive and may evolve together with advances in the knowledge on which it is based [9].

On demonstration, Balacheff notes that "the dominant type of proof in mathematics has a particular form. It is a sequence of statements that are organized according to a well-defined set of rules" [9. p. 13].

In the context of demonstration ideas, according to Balacheff, mathematics teachers perceive that students do not understand their needs, especially when it is a visually observable fact, and do not recognize the role of the demonstration [10].

The challenge of explaining the veracity of a specific result can arouse curiosity for the preparation of a demonstration when the empirical verification is not enough to motivate this search [10] and the DGS allow, from inductive verifications, the preparation of conjectures.

Using technological resources to introduce problems involving demonstration can contribute to a disruption between pragmatic proofs and intellectual proofs, i.e., those prepared by observation and those based on mathematical rigor [11]. In this aspect, GeoGebra is a technological didactic resource for the teaching of Mathematics, in which activities and situations can be created to allow students to explore and prepare conjectures, and learn through discovery or experimentation [12].

According to [10], the traditional function of demonstration is to verify the validity of mathematical statements, while convincing someone of a result is not necessary. While this is the main function, Villiers presents other ones:

Verification (concerning the truth of the statement); Explanation (providing explanations as to whether it is true); Systematization (the organization of the various results in a deductive system of axioms, main concepts, and theorems); Discovery (discovery or invention of new results); Communication (the transmission of mathematical knowledge); Intellectual challenge (the personal fulfillment/gratification resulting from building a demonstration). [10, p. 32].

Mathematicians consider demonstration to be an intellectual challenge, as something that provides gratification and personal fulfillment, which is one of the functions established by [13].

Another theory comes from [14] and [15] when they consider the process of didactic transposition that develops in different stages: it starts with scientific knowledge (mathematical knowledge), moves to pedagogical texts (knowing how to teach) and ends with the knowledge of pedagogical practice (knowledge taught).

A content of knowledge, which is intended to the knowledge to be taught, undergoes a number of changes in order to adapt more efficiently its place among the objects of education. This 'work' that takes place with the knowledge to be taught is called didactic transposition. [14].

Some rules stated, according to [15], support activities that may have the following purposes: Rule 1 – Modernize school knowledge; Rule 2 – Update the knowledge to be taught; Rule 3 – Articulate "old" knowledge with "new" knowledge; Rule 4 – Transform knowledge into exercises and problems; Rule 5 – Make a concept more understandable.

[15] explain the rules arguing that modernization consists of including, in the training programs of future professionals, new theories, models and scientific and technological interpretations that emerge from scientific development over time. Updating the knowledge to be taught means opening space for the introduction of new relevant knowledge in the present, to the detriment of that knowledge that is no longer relevant. Articulating "old" knowledge with "new" knowledge means introducing "new" objects of knowledge, articulating them with the "old" ones; the new presents itself by better clarifying the old content, and the old one by giving validity to the new one. Transforming knowledge into exercises and problems has the sense of selecting that wise knowledge, whose format allows a wider range of exercises, instead of those less "operationalizable" contents. Making a concept more understandable arises from the need to make elaborated knowledge, often with a significant degree of complexity, undergo a transformation so that its learning is facilitated in the school context.

## 4. Research conducted and in progress with GeoGebra

In this item, we will present some studies included in master's, doctoral and teacher training research.

### 4.1. Research 1

Following the ideas of [15], one of the research studies conducted was dedicated to the analysis of some theorems from Book I of Regiomontanus [16] on the construction of triangles, having some given conditions been satisfied, and from which the research question emerged: "which functions of demonstration reveal themselves in the geometric situations of the theorems of Regiomontanus on triangles when explored in GeoGebra?"

It was verified that GeoGebra allows the exploration of the different functions of demonstration and allows a form of representation of mathematical results through the visualization of new cases or in the creation of other possibilities to be explored.

Other theorems of Regiomontanus [16] were dynamically studied using GeoGebra [17], which suggests an interesting didactic strategy to be explored in teaching practice.

### 4.2. Research 2

In another research, in the theoretical line of [14], the *Imagiciel* proposal was considered: a set of educational situations that were developed by researchers from the Centre de Recherche et d'Expérimentation pour l'Enseignement des Mathemátiques – [18] in France, in partnership with the French Ministry of Education and Culture. These situations address topics of numerical functions, probability and plane and spatial geometry, developed between the late 1980s and early 1990s using the *Imagiciel* software.

The sequence begins with the objective of working on the dependence and relationship between variables in a real interval and ends with working with functions defined by sentences in real intervals, considering situations that favor frame changes. Accordingly, there was an attempt to analyze the extent to which students were given an understanding of the concept of function and, in particular, real function defined by sentences in real intervals.

Imagine a square ABCD colored with two colors, as represented in the GeoGebra Visualization Window (Figure 3). For every point Z belonging to the segment AD, consider the square AWQZ such that W lies on the segment AB. Call x the measure of segment AZ, given in *cm* and s the area of the dark colored region of square AWQZ, given in $cm^2$. Using the slider control to move point Z, with the help of the table and Visualization Window 2, and with f(x) being the function that associates an s with each x, access the address and answer what is asked.

Open the link (https://www.geogebra.org/m/fMEsdZ63#material/PatEjPFp)
What is the domain of f?
What is the image of f?
What is the formation law of f? Describe how you arrived at each answer.

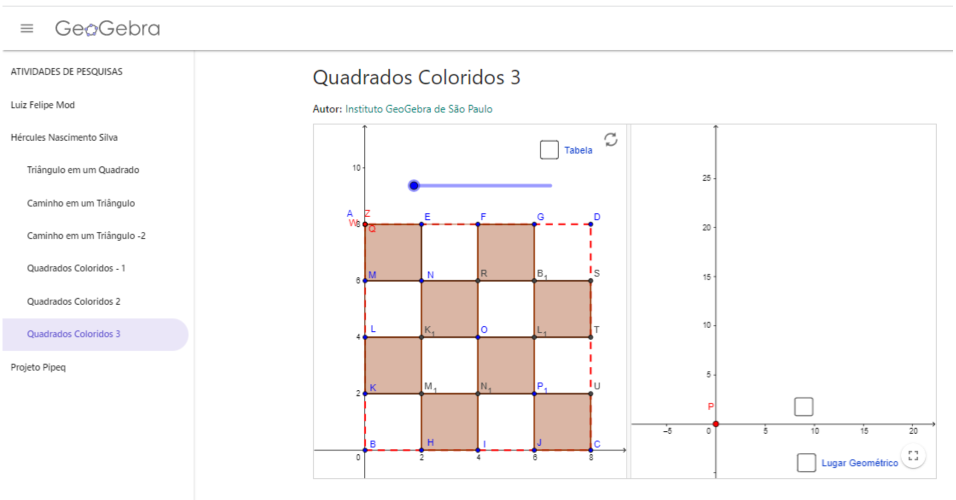


**Figure 3.** Colored squares available on the GeoGebra website.
**Source:** Author's construction in GeoGebra.

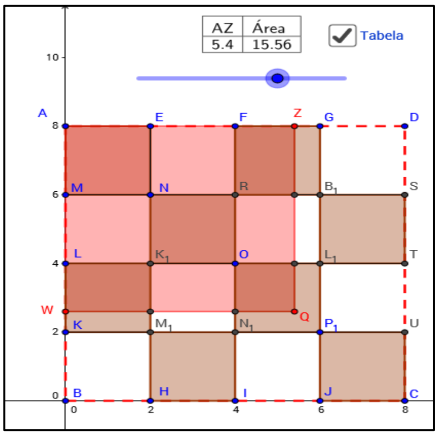

**Figure 4.** Colored squares.
**Source:** Construction in GeoGebra.

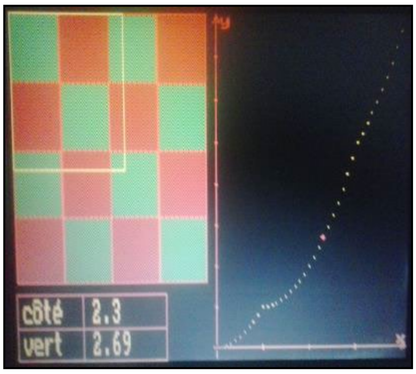

**Figure 5.** Construction in Imagiciel – Carrés colories.
**Source:** Imaciel's screenshot.

As proposed in the Imagiciel publication, the activities carried out favored the transition from the geometric to the tabular, from the tabular to the geometric and, finally, to the algebraic, a strategy rarely used in textbooks, as pointed out by [19], enabling the simultaneous work of different representations of the same function, which is not possible without the use of a computational environment.

In the sequence applied in this research, an approach was possible in which the concept of function was not singularized to a set of rules and algorithms insofar as the changes of frame, allowed by the applets, made it possible for students to mobilize old knowledge to build new knowledge, and that new knowledge could be used for the construction of other new knowledge.

### 4.3. Research 3

In another research project, called *GeoGebra as a Strategy for Remote Teaching: Creating Activities with Automatic Feedback*, a study was developed with Mathematics teachers, to use GeoGebra in their teaching practice in order to create of automatic evaluation processes allowed by the software.

The project was developed with a group of teachers from Brazil, Portugal, Cape Verde, and Angola through the ZOOM platform. The team of researchers joined this group, making their knowledge available to teachers who volunteer to integrate it, seeking common answers to the challenges set out in the project.

In the research project, it was suggested the use of GeoGebra in the creation of resources by teachers for the teaching of mathematics in two aspects: adapting or creating materials, adjusted to the interests, needs and problems faced by teachers in schools, and investigating the use of these materials in school contexts and their effect on improving the results of students.

The justification for the use of GeoGebra is that it is a software that proves useful for the teaching of mathematics and other sciences, and establishes connections between mathematical and other scientific fields, ensuring principles of equity in the teaching and learning of mathematics. [3]

To illustrate the processes inherent in an automatic feedback task with GeoGebra, we present an example of one built for the first years, conceived by a teacher of the 1st cycle of Basic Education in Portugal.

The task proposes a problem of adding two natural numbers, a and b, between 1 and 5. Numbers a and b are automatically generated whenever the user clicks the "New Challenge" button, changing the dynamic text of the problem, in case the correct or wrong answer is given visual feedback (Figure 6).

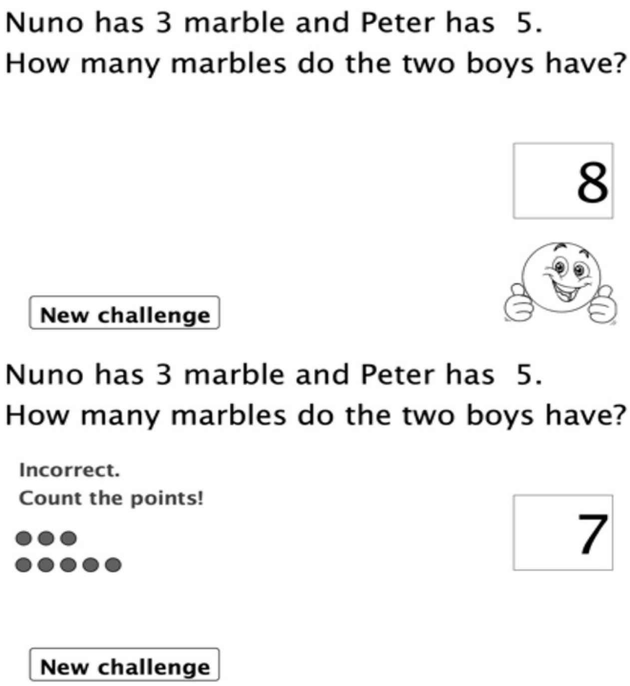


**Figure 6**. Task created in GeoGebra, with correct and wrong answers

In the case of an incorrect answer, immediate help is to be immediately provided in more detail to the user. This strategy has a critical role for less confident students so that they do not give up and can be helped in their response. In the example presented here, visual feedback is appropriate, because the didactic strategy is the identification of two sets with cardinals *a* and *b*. Even after this visual feedback is given, if the user gives a wrong answer, visual feedback appears that prompts you to count the points that appear in the visual feedback. See Figure 1.

### 4.4. Research 4

In a doctoral study, an experimental version, GeoGebra Discovery [20], has been used with the objective of researching the functionalities of its resources in the process of investigating the properties of plane geometry by students of the 8th year of Basic Education, covering the levels of understanding and development of the geometric thinking of [21].

GeoGebra Discovery is a DGS with automatic reasoning tools through commands such as Relation, Prove, Prove Details, Discover, Compare and LocusEquation. After performing a geometric construction and accessing some of the commands above, one mathematically obtains rigorous information about whether a geometric statement is true or false, which is requested through the construction objects.

Situations proposed in GeoGebra Discovery, from geometric constructions to the study of properties, can be verified with the available resources. They may also be an agent that will enable students to organize information, allowing them, through the commands used, to identify theoretical properties in the procedures of the constructions performed as an external sign of a theorem, and prepare conjectures that contribute to the development of the process of understanding geometric properties.

[22] argues that the progress of students' internal development is based on the construction and on different processes of conjecture, argumentation, proof and systematization of these tests, in which meaning and value are given. For the author, "the culture in the classroom is strongly determined by the use of mathematical discussion orchestrated by the teacher to change the spontaneous attitude of students towards theoretical validation". [22, p. 50].

The author adds that "the process of internalization of such signs determines the construction of the theoretical meaning of a construction problem and opens the theoretical perspective for geometric problems in general". [22, p. 49].

[23] adds that explanation and discovery should be used to present the proof as a meaningful activity for students, so that they should be introduced, from an early age, to the art of presenting problems by providing opportunities for conjecture, exploration, explanation and reformulation. For the author, geometry software with dynamic representation encourages these thoughts because they are powerful, not only to prove true conjectures, but also important for the construction of counterexamples and false conjectures.

Due to the pandemic experienced at the time of the investigation, the activities and interactions were carried out through Microsoft Teams platform, which allows sharing the desktop with everyone who is connected and allows participants to request control of the presenter's screen. It also allows recording the entire event with the permission of the attendees. After the screen sharing process, student control access request and GeoGebra Discovery ready to be used, the interactions began.

In a first experiment, the activities proposed to the students were related to the study of triangles. In the first activity, considering any triangle, the objective was to check which properties should have the measures of its sides referring to the triangular inequality, since not always three straight segments can form or make up the perimeter of a triangle.

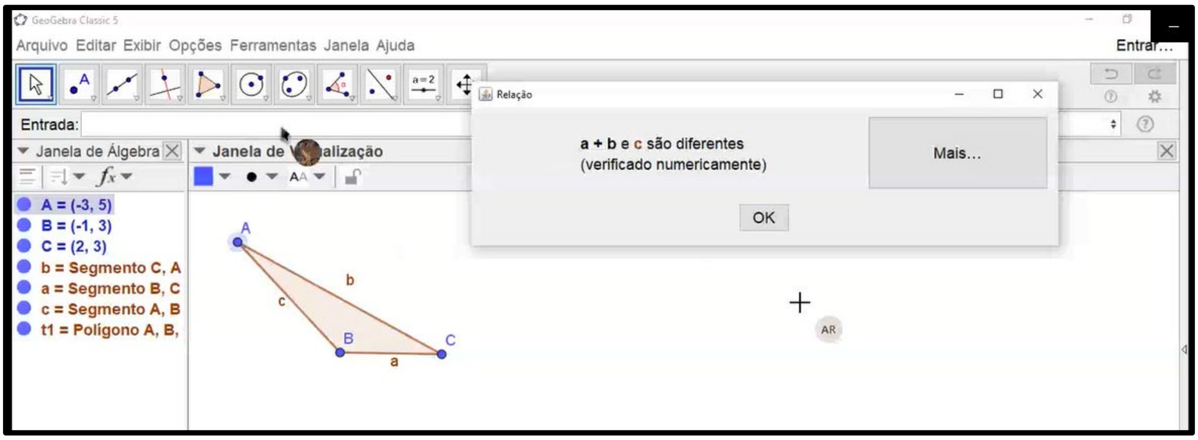


**Figure 7.** GeoGebra Discovery response window.

After the construction has been carried out and the relation command has been activated, GeoGebra Discovery returns a new window, bringing the information:(a + b and c are different (verified numerically), as we can see in (Figure 7).

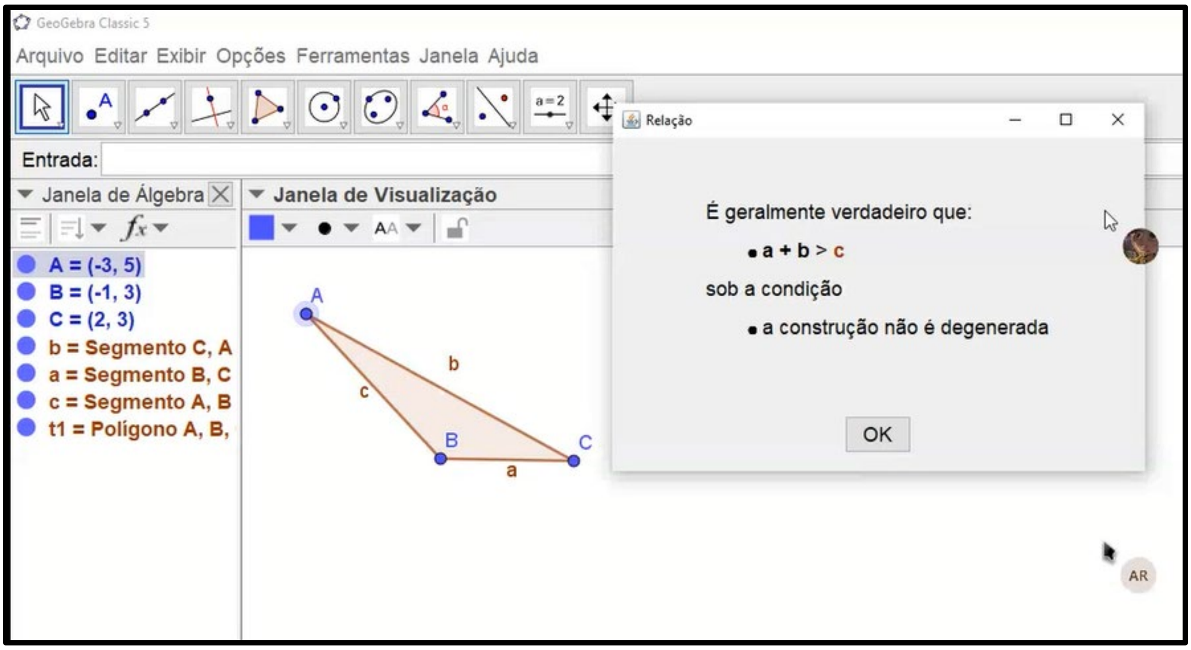


**Figure 8.** More... button response in GeoGebra Discovery.

The researcher guides the student to click on the More… button, (Figure 7), and to report the information that is provided on the screen. After clicking on the More… button, a new window opens with the following information: It is generally true that: a + b > c, under the condition that the construction is not degenerate), as seen in (Figure 8).

When asking the student to verbalize his understanding of the information provided by GeoGebra Discovery, he answers that “it showed the condition of existence of a triangle”. The student was guided to move the construction and encouraged to create conjectures for later verification with the GeoGebra Discovery commands.

In the proposed activity, according to the model of levels presented by [21], students reached Level 2 - informal deduction, as they are able to create an analysis of some properties, bringing them closer to certain geometric concepts and some features of the triangle.

It is understood that the possibility of interacting with the new features of GeoGebra Discovery allows the exploration, verification, and deduction of some properties of the plane geometry.

## 5. Concluding Remarks

Mathematics Education is always related to teaching and learning processes of Mathematics and such processes evolve, are innovated, and improve over time. This involves theories and methodologies that support them for this. Its transdisciplinary and innovative profile allows its improvement for the teaching and learning of Mathematics through the development of other areas of knowledge. In a globalized reality, education needs to renew itself with innovative strategies for a promising future for the youth involved.

In all the research studies described above, Mathematics Didactics in its field of investigation allowed us to identify, characterize and understand the phenomena and processes that conditioned the teaching and learning of Mathematics.

GeoGebra's possibilities for education in the world are evidenced in the information on its own website as follows:

> GeoGebra is a community of millions of users located in just about every country. It has become the leading provider of dynamic mathematics software, supporting science, technology, engineering and mathematics (STEM) education and innovations in teaching and learning worldwide. GeoGebra’s math engine powers hundreds of educational websites worldwide in different ways from simple demonstrations to full online assessment systems. (https://www.geogebra.org/about)
>
> In 2021, GeoGebra became part of the BYJU’S family with hundreds of millions of students on their learning platforms. GeoGebra's apps, classroom resources, GeoGebra Classroom & other features will continue to be available to the public for free. GeoGebra continues to operate as an independent unit within the BYJU’S group under the leadership of GeoGebra’s original founders and developers. (https://www.geogebra.org/about)

In a current and future vision of mathematics teaching and learning, in which knowledge and mathematical thinking are a mediator of positive experiences in science, engineering and technology, GeoGebra has been used in several projects in which researchers, teachers and students participate and will be the main instrument.

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