Ellipsoid is Tangent to its Locus under a Linear Transformation, Isometries and Sheared Maps

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Abstract

In ([9]) and ([10]), we describe an antipodal linear transformation L_E on an ellipsoid Σ, and see Σ is inside and tangent to $L_E(\Sigma)$. In this paper, we discuss how two geometry figures are congruent and are related by an isometry through this linear transformation L_E. We describe how a locus ellipsoid $L_E(\Sigma)$ can be written as a standard form of $XMX^t = 1$, where M is a symmetric and positive definite matrix, and how the rotated ellipsoid Σ_r stays being tangent to $XMX^t = 1$. Next, we explore that the minor and mean axis of $XMX^t = 1$ span a plane π which intersects the ellipsoid in the "smallest" possible ellipse. We rotate this plane by keeping the mean axis fixed, and tilting the minor axis towards the major axis. At some unique point one obtains a plane π' that intersects the ellipsoid in a round circle. We shall explore finding such sheared map T. This paper will benefit those students who have backgrounds in Linear Algebra and Multivariable Calculus. In particular, we need the eigendecomposition for the ellipsoid of $L_E(\Sigma)$.

1 Introduction

An isometry ([6]) is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space. In a two or three-dimensional Euclidean space, two geometric figures are congruent if they are related by an isometry. In this paper, we will discuss how the three-dimensional locus, two-dimensional space curve and a one-dimensional vector of a linear transformation will all stay as isometry after rotations.

The original locus problem in 2D was stated in ([7]), the corresponding 3D versions were discussed in ([8]) and ([9]). When the fixed point A is placed at an infinity, our locus problem from ([10]) becomes the following:

A Locus problem: *We are given a fixed point* $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ with $\rho \to \infty$ *and a generic point* C *on a surface* Σ. *We let the line* l *pass through* A *and* C

and intersect a well-defined D on Σ, we want to determine the locus surface generated by the point E, lying on CD and satisfying

$$\overrightarrow{ED} = s\overrightarrow{CD}, \tag{1}$$

where s is a real number parameter.

We briefly summarize the properties for the locus surface we have discussed in ([10]). The locus surface determined by E in (1) can be written as $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. In [10] we assume an ellipsoid Σ in $\mathbb{R}^3$ is given in either its standard form (2) or the parametric form in (3).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{2}$$

$$x(u,v) = a\cos u \sin v, y(u,v) = b \sin u \sin v, z(u,v) = c\cos v. \tag{3}$$

If $s \in \mathbb{R}^+ \backslash \{1/2\}$, we see the locus surface $\Delta_\infty(s, u_0, v_0)$ for an ellipsoid Σ is also an ellipsoid. Moreover, there exists a matrix $L_D = [l_{ij}]_{3\times 3}$ such that $L_D C = D_{\text{inf}}$. Consequently,

$$L_E = sI + (1-s)L_D \tag{4}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $L_E C = E_{\text{inf}}$, where $C \in \Sigma$, and therefore, the locus surface $\Delta_\infty(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L_E = [l^e_{ij}]_{3\times 3}$. We often use the notation $L_E(\Sigma) = \Delta_\infty(s, u_0, v_0)$. More importantly, the transformation L_E is such that Σ is in the interior of $\Delta_\infty(s, u_0, v_0)$ when $s > 1$, and Σ is tangent to $\Delta_\infty(s, u_0, v_0)$ at an elliptical curve, see [S1] for exploration. We refer to the Figure 1 that an ellipsoid Σ is shown in yellow, the locus $\Delta_\infty(s, u_0, v_0)$ is shown in blue for $s = 2$ and angles $u_0 = 1.0472$ and $v_0 = 0.7854$ are given for the fixed point A is given at an infinity.

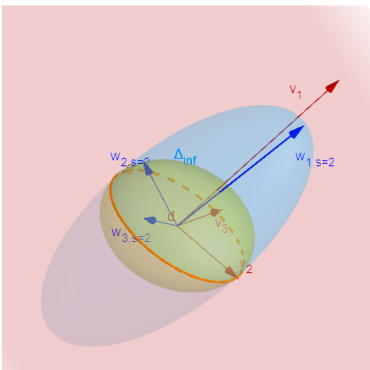

Figure 1. Ellipsoid and its locus

Three problems for explorations:
Assume that we fix the parameters $s > 1, u_0$ and v_0.

1. How can we transform the locus ellipsoid, $L_E(\Sigma) = \Delta_\infty$ into its standard implicit form, which is denoted by Δ_0?

2. Since the ellipsoid Σ in (2) or (3) is always in the interior of Δ_∞ when $s > 1$ and tangent to it. We want to find the followings:

   (a) The rotated Σ_r that is in the interior of Δ_0 and tangent to Δ_0.

   (b) The new intersecting curve between Σ_r and Δ_0.

3. We consider the locus ellipsoid $L_E(\Sigma)$ satisfying $XMX^t = 1$ with M being positive definite and symmetric matrix. The minor and mean axis of $XMX^t = 1$ span a plane π which intersects the ellipsoid in the "smallest" possible ellipse. We rotate this plane by keeping the mean axis fixed, and tilting the minor axis towards the major axis. At some unique point one obtains a plane π' that intersects the ellipsoid in a round circle. We want to find such shear map T, which shears the ellipsoid $XMX^t = 1$, by keeping this plane fixed, into another ellipsoid of rotation, E_1.

We start with the followings:

2 Background and Basic information

We recall that L_E has full rank of 3, and the eigenvalues for L_E are $\{2s - 1, 1, 1\}$. The corresponding eigenvectors are labeled as v_1, v_2 and v_3 respectively in (5),

$$\begin{aligned} v_1 &= [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t, \\ v_2 &= \left[\frac{-a^2}{c^2} \sec u_0 \cot v_0, 0, 1\right]^t, \\ v_3 &= \left[-\frac{a^2}{b^2} \tan u_0, 1, 0\right]^t. \end{aligned} \tag{5}$$

and they are linearly independent (except $v_0 = 0$ or π). The vectors v_1, v_2 and v_3 are shown in color blue in Figure 1. The orange curve $\gamma(t)$ (shown in Figure 1) is the intersecting curve between Σ and $\Delta_\infty(s, u_0, v_0)$, can be found using the techniques from ([9]):

1. We want to find the tangent plane T at a point P on the ellipsoid $C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$ such that T is passing through the fixed point $A = (x_0, y_0, z_0) = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$. If the ellipsoid is the level surface of $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. Then the gradient at a point of the ellipsoid is $\nabla F(x, y, z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$, then we see the tangent plane as follows:
$$T(x, y, z) = \nabla F(x, y, z) \cdot (x - x_0, y - y_0, z - z_0) = 0, \tag{6}$$
We thus solve $F(x, y, z) = 0$ and $T(x, y, z) = 0$ for the variables x, y, which we obtain two branches for x and y respectively.

2. Next we let $\rho \to \infty$ since the fixed point A is when $\rho \to \infty$, we denote the respective two branches by $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$.

3. We form two branches for $\gamma(t)$

$$\begin{aligned} \gamma^+(t) &= [x_1(t), y_1(t), z(t)], \text{ and} \\ \gamma^-(t) &= [x_2(t), y_2(t), z(t)], \end{aligned} \tag{7}$$

where $\gamma(t) = \gamma^+(t) \cup \gamma^-(t)$ and $z(t) = c\cos(t)$.

4. We include $\gamma(t)$ in [S2], and we remark that $\gamma(t)$ does not depend on the scaling factor s. The elliptical disk determined by $\gamma(t)$ is spanned by the vectors v_2 and v_3.

In this paper, unless otherwise specified, we assume that the parameters $s > 1, u_0$, and v_0 are given in advance, and we write the locus ellipsoid $L_E(\Sigma) = \Delta_\infty(s, u_0, v_0)$ simply as Δ_∞. We interchange the use of L_E as a linear transformation and its corresponding matrix $L_E = [l_{ij}^e]_{3\times 3}$ with no confusion. To keep the entirety of this subject, we briefly state how we can write Δ_∞ in its implicit form by applying the principle axes theorem as follows:

We recall from [10] that we applied L_E on the Σ in its parametric form, therefore, $L_E(\Sigma)$ will be expressed in its parametric form. We can transform $L_E(\Sigma)$ into its implicit form by making use of the conversion matrix for L_E as follows:

$$Q_\Delta = \left(\begin{pmatrix} [l_{ij}^e]_{3\times 3} & 0 \\ 0 & 0 \end{pmatrix}^{-t}\right)_{4\times 4} \begin{pmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & c^2a^2 & 0 & 0 \\ 0 & 0 & a^2b^2 & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{pmatrix} \begin{pmatrix} [l_{ij}^e]_{3\times 3} & 0 \\ 0 & 0 \end{pmatrix}^{-1}_{4\times 4}. \tag{8}$$

1. We find the eigenvalues and eigenvectors of matrix Q_Δ, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for eigenvalues, and $w_1, w_2, w_3,$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ for the eigenvectors.

2. If $X^* = [x, y, z, 1]$, then the implicit form of $L_E(\Sigma)$ or Δ_∞, can be expressed as

$$X^* Q_\Delta (X^*)^t = 0, \tag{9}$$

and Q_Δ is symmetric and can be written as

$$Q_\Delta = \begin{bmatrix} A & \frac{B}{2} & \frac{C}{2} & 0 \\ \frac{B}{2} & D & \frac{E}{2} & 0 \\ \frac{C}{2} & \frac{E}{2} & F & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{bmatrix}, \tag{10}$$

Subsequently, the implicit equation of Δ_∞ can be written as

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + J = 0, \tag{11}$$

where the coefficients A through J can be found in [S2] or [S3].

3. If we consider the submatrix $Q_{\Delta}^{'} = \left(\left([l_{ij}^{e}]_{3\times 3}\right)^{-t}\right)\begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}\left([l_{ij}^{e}]_{3\times 3}\right)^{-1}$ of the matrix Q_{Δ} (10), we remark that the matrix $[l_{ij}^{e}]_{3\times 3}$ is positive definite since all eigenvalues are positive for $s > 1$; therefore, the matrix $Q_{\Delta}^{'}$ is also positive definite, since it is a product of three positive definite matrices, and thus $Q_{\Delta}^{'}$ is a positive definite and symmetric matrix.

In the rest of this paper, we often consider the sub-matrix the following matrix M, which is derived from $Q_{\Delta}^{'}$

$$M = \frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}, \tag{12}$$

and use

$$XMX^t = 1 \tag{13}$$

to represent the implicit equation for which is associated with $L_E(\Sigma)$, where $X = [x, y, z]$. Equivalently, if we consider $L_E^{-1}(X)$, and let $G = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}$. Then the equation of

$$X\left(L_E^{-t}GL_E^{-1}\right)X^t = 1 \tag{14}$$

represents the implicit equation for Δ_{∞}.We note the implicit forms of (11), (13) and (14) are all identical. Furthermore, the eigenvalues of M are $\lambda_1, \lambda_2, \lambda_3$ respectively.

We remark that when considering the standard form of Δ_0, we may use the symmetric and positive definite matrix M ((12) and consider

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1, \tag{15}$$

see Theorem (3).

In this paper, we shall discuss how an isometry can be related to this linear projective transformation L_E.

2.1 Locus is an isometry for spheres

Let the fixed point A be given for parameters u_0, v_0, and s, and L_E be applied on a sphere of $x^2+y^2+z^2 = r^2$, which is denoted by Σ_1^r. Since Σ_1^r is symmetric with respect to the origin, it is natural to expect the shape of $L_E(\Sigma_1^r)$ stays unchanged regardless of the projection angles u_0 and v_0, once the scaling factor s is fixed. Specifically, if we move the fixed points $A_1, A_2, ..., A_n$ sequentially:

$$A_1 \to A_2 \to ... \to A_n \tag{16}$$

with $A_n = A$. Then Δ_i, the locus surface of Σ with respect to A_i, for $i = 1, 2, ...n$, moves sequentially

$$\Delta_1 \to \Delta_2 \to ... \to \Delta_n, \tag{17}$$

and we would expect that $\Delta_n = \Delta$. In other words, Δ_i stays isometric under rotations if we apply L_E on spheres, once s is given. It is clear we have the following observation for spheres.

Theorem 1 *For given $s > 1$, let Σ_1^r be the sphere $x^2 + y^2 + z^2 = r^2$, $A_1 = (u_1, v_1)$ and $A_i = (u_i, v_i)$, where $i = 1, 2, ...n$. If we set $a = b = c = r$ and fix the parameter s, then for given angles u_0, and v_0, $L_E(\Sigma_1^r)$ are all isometric copies.*

Proof: If we set $a = b = c = r$ in M in ((12), the eigenvalues for M are respectively, $\lambda_1 = \frac{1}{(2s-1)^2 r^2}$, $\lambda_2 = \frac{1}{r^2}$, and $\lambda_3 = \frac{1}{r^2}$. The corresponding eigenvectors are v_1, v_2, and v_3 of L_E, when we substitute $a = b = c = r$ for v_2, and v_3 in (5). In other words, once we convert $L_E(\Sigma_1^r)$ into the standard form, they will be all identical as shown below:

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1. \tag{18}$$

Consequently, the following is obvious result to express $L_E(\Sigma_1^r)$ in its standard form:

Corollary 2 *If a linear transformation $L_E(s, u_0, v_0)$ is applied on a sphere Σ_1^r, with specified s, u_0 and v_0, then $L_E\left(s, 0, \frac{\pi}{2}\right) = \Delta_0$ is the copy of Δ_1 when it is written in its standard form.*

We shall next state one important concept of converting Δ_∞ to Δ_0, which students learn in a basic Linear Algebra class.

3 Transition matrix

We want to find a proper rotation so that it preserves the rigid transformation from (Σ, Δ_∞) to (Σ_r, Δ_0). The process of finding such proper rotation matrix is a standard exercise students learned from Linear Algebra. We describe two bases in $\mathbb{R}^3$, first let $\mathfrak{S} = \{e_1, e_2, e_3\}$ to be the standard basis for $\mathbb{R}^3$. We let v_1^*, v_2^* and v_3^* be the **unit** eigenvectors for the matrix M (12) associated with $L_E(\Sigma)$, since the matrix M (12) is positive definite and symmetric, the eigenvectors of M are linearly independent and orthogonal. The eigenvectors for M are demonstrated in Figure 1, which are labeled as $axis_1$, $axis_2$, and $axis_3$ respectively. Therefore, $\{v_1^*, v_2^*, v_3^*\}$ forms another orthonormal eigenvectors for M (12), and we let $\mathfrak{B} = \{v_1^*, v_2^*, v_3^*\}$ be the second set of basis for $\mathbb{R}^3$. Now we set the 3×3 transition matrix

$$P = [v_1^* : v_2^* : v_3^*]. \tag{19}$$

Then we see

$$P^{-1}MP = D, \tag{20}$$

where D is the diagonal matrix, consists of eigenvalues λ_1, λ_2 and λ_3 of the matrix M (12). We observe the following commutative diagram (26) associated with L_E that

$$\Delta_\infty = P(\Delta_0), \tag{21}$$

$$\Delta_0 = P^{-1}(\Delta_\infty). \tag{22}$$

Furthermore, we see

$$\Sigma_r = P^{-1}\left(\Sigma\right), \tag{23}$$

$\Sigma_r \subsetneq \Delta_0$ and Σ_r is tangent to Δ_0. If we view P as simply a rotation matrix, it also follows from diagram (26) that

$$\Delta_0 = P^{-1} L_E P\left(\Sigma_r\right), \tag{24}$$

$$\Delta_0 = P^{-1}\left(\Delta_\infty\right). \tag{25}$$

$$\begin{array}{ccccc} & \Sigma & \xrightarrow{L_E} & \Delta_\infty = L_E\left(\Sigma\right) & \\ P^{-1} & \downarrow & & \uparrow & P \\ & \Sigma_r & \xrightarrow{P^{-1}L_E P} & \Delta_0 & \end{array} \tag{26}$$

Theorem 3 *The ellipsoid Δ_0 is a copy in the standard form of the locus ellipsoid $\Delta_\infty = L_E(\Sigma)$. Explicitly, Δ_0 has the equation of*

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1, \tag{27}$$

where the diagonal entries of Λ^* are written in descending orders, and λ_1, λ_2, and λ_3 are the eigenvalues of the matrix M (12).

Proof

As noted from (12) that $M = \frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$, and $XMX^T = 1$ represents the implicit form of Δ_∞. It is clear that

$$\begin{aligned} XMX^t &= X\left(PDP^t\right)X^t \\ &= \left(XP\right)D\left(XP\right)^t \qquad (28) \\ &= 1. \qquad (29) \end{aligned}$$

We see that if $Y = XP$, then $YDY^t = 1$ is exactly (15), when Δ_∞ is written in its standard form.

The following observation answers the question in (2).

Theorem 4 *In reference to Theorem (3), consider the ellipsoid $\Sigma_r = P^{-1}\Sigma$. Then,*

a. For $s \in \mathbb{R}^+ \setminus \{1\}$, the ellipsoid Σ_r and the ellipsoid in standard form Δ_0 intersect themselves tangentially at an elliptical curve, say γ_r .

b. If

$$\underline{\Delta_0} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \frac{\tilde{x}^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} \leq 1 \right\}$$

the solid ellipsoid which boundary is Δ_0. Then we have $\Sigma_r \subsetneq \underline{\Delta_0}$ when $s > 1$.

Proof

We recall the matrix P is formed by the orthonormal eigenvectors for M (12), P is orthogonal with $PP^t = P^tP = I_3$, a 3×3 identity matrix. We see

$$PP^t = P^tP = I_3 \tag{30}$$

$$P^t = P^{-1}, P = \left(P^t\right)^{-1} \tag{31}$$

Since the linear transformation defined by the orthogonal matrix P is injective, so is P^{-1}, we see

$$\Sigma_r \cap \Delta_0 = P^{-1}(\Sigma) \cap P^{-1}(\Delta_\infty) = P^{-1}(\Sigma \cap \Delta_\infty) = P^{-1}(\gamma), \tag{32}$$

where γ is the elliptical curve where Σ and Δ_∞ intersect themselves tangentially.

Now, let us denote by $\underline{\Delta_\infty}$ the solid ellipsoid which boundary is Δ_∞. Again, since P^{-1} is injective, we see

$$\Sigma \subsetneq \underline{\Delta_\infty} \Rightarrow \Sigma_r = P^{-1}(\Sigma) \subsetneq P^{-1}\left(\underline{\Delta_\infty}\right) = \underline{\Delta_0}. \tag{33}$$

4 A sheared ellipsoid with circle cross section

In [4], it posts the question of 'Find the radius of the largest circle on the ellipsoid $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with semi-axes $a > b > c$.' The existence of such circle can be found in [4]. In this section, we will explore *problem 3* by constructively finding such cross section circle C for the ellipsoid locus $L_E(\Sigma)$. Furthermore, we will find the shear map T so that the cross section containing two semi-axes of tilted ellipsoid of $L_E(\Sigma)$ is a circle C, which is also the intersection of the tilted ellipsoid of $L_E(\Sigma)$ and $L_E(\Sigma)$. In general, since an ellipsoid can be thought as an image of a linear transformation on the unit sphere. One may explore the TRD decomposition for a matrix of an ellipsoid using the idea from [2]. Consequently, we can transform an ellipsoid E back to the unit sphere using three steps, a shear map T, a rotation map R and a dilation D, which is discussed in details in [5].

4.1 Finding a sheared map for an ellipsoid when $a > b > c$

Before we consider the general case of *problem 3* (3), we consider the ellipsoid Σ of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, We refer to the following Figure 2. We let $OA_1 = a, OA_2 = b$ and $OA_3 = c$ be major, mean and minor axes for the ellipsoid Σ respectively. We note that the plane π=OA_2A_3 in dark red of Figure 2 contains the median and minor axes of Σ. Our objective is to rotate Σ into the ellipsoid Σ', which contains an the circle C lying on the plane $\pi' = OA_2A_3'$ containing two equaled semi-axes (see the circle in orange lying on the plane $\pi' = OA_2A_3'$ in Figure 2).

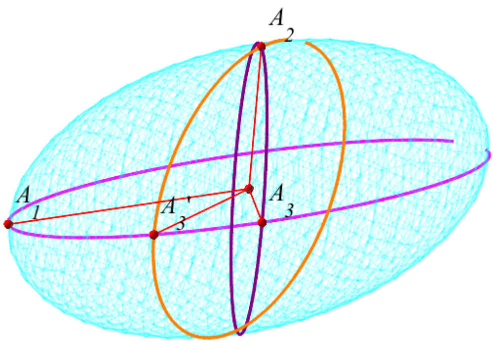

Figure 2. Tilting Minor axis

In other words, we need to rotate the minor $z-$axis or OA_3, toward the major $x-$axis or OA_1, until we obtain an the circle C lying on the plane containing two equaled axes (see the circle in orange lying on the plane OA_2A_3' in Figure 2). Therefore, the median $y-$axis is fixed, and hence we apply a rotation matrix around y-axis, say

$$R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & 0 & \pm 1 \end{pmatrix}. \tag{34}$$

We note that $R_y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ Ax \pm z \end{pmatrix}$. Without loss of generality, we consider $R_y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ Ax + z \end{pmatrix}$, and the ellipsoid $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ becomes $\Sigma' : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(Ax+z)^2}{c^2} = 1$. These two equations of Σ and Σ' are reduced to $(Ax+z)^2 - z^2 = 0$ or $(Ax + z + z) = 0$, and the plane equation

$$z = -\frac{Ax}{2} \tag{35}$$

is the intersecting plane equation where the circle lies.

1. Consider the cross section with $y = 0$, the original ellipsoid Σ becomes

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \tag{36}$$

and we substitute (35) into the equation yields,

$$x^2 = \frac{1}{\frac{1}{a^2} + \frac{A^2}{4c^2}} = \frac{1}{\frac{4c^2 + a^2A^2}{4a^2c^2}} = \frac{4a^2c^2}{4c^2 + a^2A^2}.$$

2. Use Eq. (36) again, we obtain

$$z^2 = c^2 \left(1 - \frac{4a^2c^2}{a^2(4c^2 + a^2A^2)}\right) \tag{37}$$

3. Since we are rotating the minor (z) toward the major axis (x), and if $P(x,y)$ denotes the intersection point for the ellipses on the plane of $y=0$, we want the distance P to $O=(0,0,0)$ to be equal for both ellipses. Therefore, it should be

$$\begin{aligned} x^2+z^2 &= b^2, \text{ or} \qquad (38)\\ \left(\frac{4a^2c^2}{4c^2+a^2A^2}\right)+c^2\left(1-\frac{4a^2c^2}{a^2\left(4c^2+a^2A^2\right)}\right) &= b^2 \qquad (39)\end{aligned}$$

4. Consequently, we get

$$A=\frac{2\sqrt{\left(b^2-c^2\right)\left(a^2-b^2\right)}c}{\left(b^2-c^2\right)a}. \qquad (40)$$

Therefore, the plane $z=-\frac{Ax}{2}=-\left(\frac{\sqrt{(b^2-c^2)(a^2-b^2)}c}{(b^2-c^2)a}\right)x$ will intersect both Σ and Σ' at a circle. If we denote the intersecting circle by $[x(t),y(t),z(t)]$, then we have

$$\begin{aligned} x(t) &= t \qquad (41)\\ y(t) &= \pm\frac{\sqrt{\left(b^2-c^2\right)\left(a^2b^2-a^2c^2-a^2t^2+t^2c^2\right)}b}{\left(b^2-c^2\right)a}\\ z(t) &= -\frac{At}{2},\end{aligned}$$

where $t\in[0,2\pi]$. In other words, the intersecting curve is the union of $\gamma_1(t)\cup\gamma_2(t)$, with $\gamma_1(t)=r_1(t)\cup r_2(t)$ and $\gamma_2(t)=r_3(t)\cup r_4(t)$, where $t\in[0,2\pi]$, and

$$\begin{aligned} r_1(t) &= \left(t,\frac{\sqrt{\left(b^2-c^2\right)\left(a^2b^2-a^2c^2-a^2t^2+t^2c^2\right)}b}{\left(b^2-c^2\right)a},-\frac{At}{2}\right), \qquad (42)\\ r_2(t) &= \left(t,-\frac{\sqrt{\left(b^2-c^2\right)\left(a^2b^2-a^2c^2-a^2t^2+t^2c^2\right)}b}{\left(b^2-c^2\right)a},-\frac{At}{2}\right), \qquad (43)\\ r_3(t) &= -r_1(t), \qquad (44)\\ r_4(t) &= -r_2(t). \qquad (45)\end{aligned}$$

Next, we want to find the furthermost point on Σ to the plane of $z=-\frac{A}{2}x$. We do this by applying Lagrange Multipliers, which is the place we need to switch to numerical computations using a CAS ([3]).

1. We let $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ and $f(x,y,z)=z+\frac{Ax}{2}=0$.

2. We let $L(x_1,y_1,z_1,x_2,z_2,k_1,k_2)=(x_1-x_2)^2+y_1^2+(z_1-z_2)^2+k_1g(x_1,y_1,z_1)+k_2\left(z_2+\frac{A}{2}x_2\right)$, and set $\nabla L=0$ to solve x_1,y_1,z_1,x_2,z_2,k_1, and k_2.

3. We select the nonzero solutions of k_1 and k_2, and make

$$v_L=(x_1,y_1,z_1), \qquad (46)$$

which is the desired furthermost point on Σ. For simplicity, we use the vector $\overrightarrow{v_L}=\overrightarrow{Ov_L}$.

4. The unit normal vector for the $z = -\frac{A}{2}x$ is

$$\overrightarrow{n} = \frac{\left(\frac{A}{2}, 0, 1\right)}{\left\|\left(\frac{A}{2}, 0, 1\right)\right\|}. \tag{47}$$

5. The projection vector of $\overrightarrow{v_L}$ along $\overrightarrow{n}$ is

$$\overrightarrow{v_P} = \left(\|\overrightarrow{v_L}\| \cos\theta\right) \overrightarrow{n}, \tag{48}$$

where θ is the angle between $\overrightarrow{v_L}$ and $\overrightarrow{n}$, i.e. $\theta = \cos^{-1}\left(\frac{\overrightarrow{v_L} \cdot \overrightarrow{n}}{\|\overrightarrow{v_L}\|}\right)$. Finally, we see that $\overrightarrow{v_m} = (0, b, 0)$ and $\overrightarrow{v_\perp} = \overrightarrow{v_P} \times \overrightarrow{v_m}$ spans the circle with the radius being equal to $\|\overrightarrow{v_m}\| = \|\overrightarrow{v_\perp}\| = b$, and the the direction and the length of the semi-major axis for the sheared ellipsoid Σ' is $\overrightarrow{v_P}$ and $\|\overrightarrow{v_P}\|$ respectively.

6. The matrix T for the sheared map of the standard form of the ellipsoid should map the matrix $V = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_L}]$, which contains three vectors from the ellipsoid Σ, to a new corresponding matrix $W = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_P}]$ on the sheared ellipsoid Σ'. In other words, we need $TV = W$ and solve for T as follows:

$$T = WV^{-1}. \tag{49}$$

4.2 Sheared map for a locus written in a general form

Now we consider the case when Σ is written in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and $L_E(\Sigma)$ is written in a general form of $XM_{\Delta_\infty}X^t = 1$, with $M_{\Delta_\infty} = \frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$ and $X = [x, y, z]$. We describe how we can find the the sheared map for the ellipsoid of $XM_{\Delta_\infty}X^t = 1$.

1. Since M_{Δ_∞} is symmetric and positive-definite, it is diagonalizable, and we find the transition matrix P for M_{Δ_∞} such that

$$P^{-1}M_{\Delta_\infty}P = D_{M_{\Delta_\infty}}, \tag{50}$$

where $D_{M_{\Delta_\infty}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ is the diagonal matrix, which consists of the eigenvalues of M_{Δ_∞}, $\lambda_1 < \lambda_2 < \lambda_3$.

2. Next, we recall that $\frac{1}{\sqrt{\lambda_i}}, i = 1, 2, 3$, corresponds to **the length of each respective semi-axis**, and $M_\lambda = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} \end{pmatrix}$. We note that $XM_\lambda X^t = 1$ is the ellipsoid Δ_0. We recall from (49) that the matrix T for the ellipsoid, written in $XM_\lambda X^t = 1$, will map $V = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_L}]$ to $W = [\overrightarrow{v_m}, \overrightarrow{v_\perp}, \overrightarrow{v_P}]$. We obtain the sheared ellipsoid $E_{s,0} = T(\Delta_0)$ for Δ_0. In view of the nature of the construction of T and (39), the sheared ellipsoid will

have the cross section being a circle. Since the transition matrix P is simply a rotation matrix, the sheared ellipsoid of Δ_∞ is $P(T(\Delta_0))$, and $P(T(\Delta_0))$ shall have a cross section being a circle too. In other words, the shear map $T' : \Delta_\infty \to T'(\Delta_\infty)$ should satisfy the following commutative diagram and we see $T' = PTP^{-1}$:

$$\begin{array}{ccccc} & \Delta_\infty & \xrightarrow{T'} & T'(\Delta_\infty) & \\ P^{-1} & \downarrow & & \uparrow & P \\ & \Delta_0 & \xrightarrow{T} & T(\Delta_0) & \end{array} \tag{51}$$

5 Examples

Example 5 *We consider the linear transformation* $L_E : \Sigma \to \Delta_\infty$ *on an ellipsoid of* $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *with* $a = 5, b = 4, c = 3, s = 3, u_0 = \frac{\pi}{6}$, *and* $v_0 = \frac{\pi}{3}$, *we see that* $L_E = \begin{pmatrix} \frac{8755}{3571} & \frac{2700\sqrt{3}}{3571} & \frac{9600}{3571} \\ \frac{1728\sqrt{3}}{3571} & \frac{6271}{3571} & \frac{3200\sqrt{3}}{3571} \\ \frac{3456}{3571} & \frac{1800\sqrt{3}}{3571} & \frac{9971}{3571} \end{pmatrix} = \begin{pmatrix} 2.45169 & 1.30959 & 2.68832 \\ 0.83814 & 1.75609 & 1.5521 \\ 0.96778 & 0.87306 & 2.79222 \end{pmatrix}$. *Our tasks are (i) Expressing* Δ_∞ *in its standard form* Δ_0. *(ii) Find the rotated* Σ_r *that is inside and tangent to* Δ_0. *(iii) Find the intersecting curve between* Σ_r *and* Δ_0. *(iv) Find the vector that points to the major axis, and with the norm of the largest eigenvalues for* Δ_0.

We consider the positive definite and symmetric matrix:

$$M = \frac{1}{5^2 4^2 3^2} \begin{bmatrix} \frac{8376624}{89275} & -\frac{93312\sqrt{3}}{3571} & -\frac{331776}{3571} \\ -\frac{93312\sqrt{3}}{3571} & \frac{657675}{3571} & -\frac{172800\sqrt{3}}{3571} \\ -\frac{331776}{3571} & -\frac{172800\sqrt{3}}{3571} & \frac{814000}{3571} \end{bmatrix}. \tag{52}$$

The implicit form for Δ_∞ can be found through $XMX^t = 1$, where $X = [x, y, z]$. We remind readers that the parametric form for Δ_∞ can be obtained from $L_E(\Sigma)$.

1. The intersecting curve $\gamma(t)$ between Σ and Δ_∞ can be found by using the equation (7) with $a = 5, b = 4, c = 3, s = 3, u_0 = \frac{\pi}{6}$, and $v_0 = \frac{\pi}{3}$, which is expressed in two parts $\gamma(t) = \gamma^+(t) \cup \gamma^-(t)$, where

$$\gamma^\pm(t) = \left[\frac{-800\cos t}{219} \mp \frac{25\sqrt{3}\sqrt{1971 - 3571\cos^2 t}}{657}, \frac{-800\cos t}{657} \pm \frac{16\sqrt{1971 - 3571\cos^2 t}}{210}, 3\cos t \right], \tag{53}$$

   and $t \in [0, 2\pi]$. We depict $\gamma(t)$ in the red curve (see Figure 3). We depict the vector $\left(\sqrt{\frac{1}{\lambda_1}}\right) v_1^*$ in black in Figure 3.

2. The eigenvalues of M (found through the CAS ([3] after removing negligible complex portions), are respectively $\lambda_1 = 0.00245971750$, $\lambda_2 = 0.05324874268$ and $\lambda_3 = 0.08483262385$.

We found the transition matrix $P = [v_1^* : v_2^* : v_3^*]$, which consists of orthonormal eigenvectors of M below:

$$P = [v_1^* : v_2^* : v_3^*] \tag{54}$$

$$= \begin{pmatrix} 0.76082767296684587207 & -0.59247039192190548461 & -0.26480197647252925903 \\ 0.42914149371881194358 & 0.76542026211097316451 & -0.47955124931414372035 \\ 0.48680471486689723325 & 0.25121834536090880609 & 0.83660654583710139213 \end{pmatrix} \tag{55}$$

We see $P^{-1}MP = D$, where D is a diagonal matrix consisting of eigenvalues of matrix M, (52), λ_1, λ_2, and λ_3. In view of the diagram (26), the rotated ellipsoid $\Sigma_r = P^{-1}(\Sigma) = (s_{11}, s_{12}, s_{13})^t$ can be expressed in shown in parametric form (56), which can be shown that its implicit form is exactly shown in (57).

$$\begin{aligned} s_{11} &= 3.80413836483423 \cos(u) \sin(v) + 1.71656597487525 \sin(u) \sin(v) \\ &\quad + 1.46041414460069 \cos(v), \\ s_{12} &= -2.96235195960952 \cos(u) \sin(v) + 3.06168104844389 \sin(u) \sin(v) \\ &\quad + 0.753655036082727 \cos(v), \\ s_{13} &= -1.32400988236264 \cos(u) \sin(v) - 1.91820499725658 \sin(u) \sin(v) \\ &\quad + 2.50981963751130 \cos(v), \end{aligned} \tag{56}$$

$$219.5837x^2 + 115.8274xy + 175.1803xz + 207.6117y^2 + 48.1441yz + 341.80462 = 3600. \tag{57}$$

3. It can be shown that

$$\Delta_0 = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{(20.163)^2} + \frac{y^2}{(4.333\,6)^2} + \frac{z^2}{(3.4334)^2} = 1 \right\}. \tag{58}$$

The intersecting curve $\gamma_r(t) = P^{-1}\gamma(t)$ shown in yellow in Figure 4, and can be found as follows:

$$\gamma_r(t) = \begin{pmatrix} -1.4847 \cos(t) - 1.1052 \sin(t) \\ 4.2046 \cos(t) - 1.0179 \sin(t) \\ 0.7919 \cos(t) + 3.332 \sin(t) \end{pmatrix}. \tag{59}$$

4. We depict how the ellipsoid Σ_r (shown in green in Figure 4) is inside and tangent to Δ_0 (shown in light blue in Figure 4) and the vector $v = \left(\sqrt{\frac{1}{\lambda_1}}, 0, 0 \right)$ is shown in red (see

Figure 4).

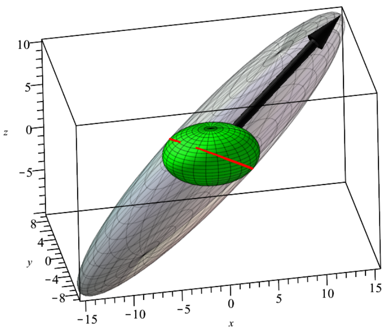

Figure 3. The graphs of Σ, $L_E(\Sigma)$ and its intersecting curve.

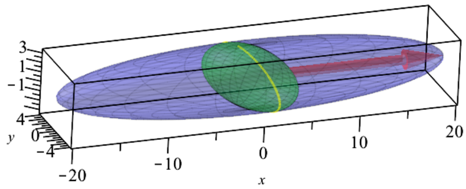

Figure 4. The graphs of Σ_r, Δ_0 and its intersecting curve

Example 6 *We consider the parameters of* $a = 5, b = 4, c = 1, u_0 = \frac{\pi}{3}, v_0 = \frac{\pi}{4}$, *and* $s = 3$ *for the linear transformation* L_E. *Then find the sheared map for the ellipsoid of* $XMX^t = 1$, *where* $M = \frac{1}{a^2b^2c^2}\begin{pmatrix} A & \frac{B}{2} & \frac{C}{2} \\ \frac{B}{2} & D & \frac{E}{2} \\ \frac{C}{2} & \frac{E}{2} & F \end{pmatrix}$. *(Complete computations can be found in [S3].)*

1. $M_{\Delta_\infty} = \begin{pmatrix} \frac{41891}{1056875} & -\frac{24\sqrt{3}}{42275} & -\frac{768}{42275} \\ -\frac{24\sqrt{3}}{42275} & \frac{1619}{27056} & -\frac{48\sqrt{3}}{1691} \\ -\frac{768}{42275} & -\frac{48\sqrt{3}}{1691} & \frac{155}{1691} \end{pmatrix}$,

2. $D_{M_{\Delta_\infty}} = \begin{pmatrix} 0.0175664583978446 & 0 & 0 \\ 0 & 0.0439013885141853 & 0 \\ 0 & 0 & 0.129669409149472 \end{pmatrix}$.

3. $P = \begin{pmatrix} 0.492115662759820 & -0.856262698599654 & -0.156959757159834 \\ 0.664928628967863 & 0.486100824357473 & -0.567076632295043 \\ 0.561864834700000 & 0.174700256800000] & 0.808571411300000 \end{pmatrix}$. We see

$$D_{M_{\Delta_\infty}} = P^{-1} M_{\Delta_\infty} P. \tag{60}$$

4. We proceed using the Lagrange method on the standard form, or $XM_\lambda X^t = 1$ below:

$$\frac{x^2}{\left(\sqrt{\frac{1}{\lambda_1}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{1}{\lambda_2}}\right)^2} + \frac{z^2}{\left(\sqrt{\frac{1}{\lambda_3}}\right)^2} = 1 \tag{61}$$

5. The plane equation that will intersect $XM_\lambda X^t = 1$ at a circle

$$\left(\frac{\sqrt{(b^2 - c^2)(a^2 - b^2)}c}{(b^2 - c^2)\,a}\right) x + z = 0, \text{ or} \tag{62}$$

$$0.5541194455x + z = 0. \tag{63}$$

6. The intersecting curve between $XM_\lambda X^t = 1$ and plane is $\gamma_1(t) \cup \gamma_2(t)$, with $\gamma_1(t) = r_1(t) \cup r_2(t)$ and $\gamma_2(t) = r_3(t) \cup r_4(t)$, where $t \in [0, 2\pi]$ and

$$\begin{aligned}
r_1(t) &= \left(t, 1.138916142 \cdot 10^{-9}\sqrt{-1.007646405 \cdot 10^{18}t^2 + 1.756055540 \cdot 10^{19}}, -0.5541194455t\right), \\
r_2(t) &= \left(t, -1.138916142 \cdot 10^{-9}\sqrt{-1.007646405 \cdot 10^{18}t^2 + 1.756055540 \cdot 10^{19}}, -0.5541194455t\right), \\
r_3(t) &= -r_1(t), \\
r_4(t) &= -r_2(t).
\end{aligned}$$

7. The vector $v_m = (0, b, 0)^t = (0, 4.772664123, 0)^t$. After solving the Lagrange equation, we obtain

$$\begin{aligned}
\overrightarrow{v_L} &= (x_1, y_1, z_1)^t \qquad (64) \\
&= \begin{pmatrix} 6.284852576 \\ 0 \\ 1.536520588 \end{pmatrix}.
\end{aligned}$$

8. The unit normal vector for the plane is $\overrightarrow{n} = (0.484682749365491, 0, 0.874690020900000)^t$, and

$$\begin{aligned}
\overrightarrow{v_P} &= \left(\|\overrightarrow{v_L}\| \cos\theta\right) \overrightarrow{n} \qquad (65) \\
&= \begin{pmatrix} 2.12782456791426 \\ 0 \\ 3.84001064246040 \end{pmatrix},
\end{aligned}$$

where θ is the angle between $\overrightarrow{v_L}$ and $\overrightarrow{n}$. Furthermore, we have

$$\begin{aligned}
\overrightarrow{v_\perp} &= \overrightarrow{v_P} \times \overrightarrow{v_m} \qquad (66) \\
&= \begin{pmatrix} -4.17460168123777 \\ 0 \\ 2.31322796879084 \end{pmatrix}.
\end{aligned}$$

9. We depict the sheared ellipsoid (from M_λ) in blue, the intersecting curve $\gamma_1(t) \cup \gamma_2(t)$ (in red), and vectors, $\overrightarrow{v_P}$ (in black), $\overrightarrow{v_m}$ (in yellow), and $\overrightarrow{v_\perp}$ (in red) respectively in Figure 5 below:

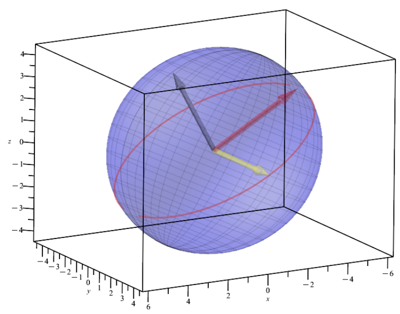

Figure 5. Sheared ellipsoid, intersecting curve and respective axes.

10. Now the matrix sheared matrix T for M_λ is

$$T = \begin{pmatrix} 0.541053294104186 & 0 & -0.828245082577262 \\ 0 & 1 & 0 \\ 0.254311294151032 & 0 & 1.45894670583444 \end{pmatrix},$$

11. We remark that if the ellipsoid $XM_\lambda X^t = 1$ is $\Delta_0 = \begin{pmatrix} \sqrt{\frac{1}{\lambda_1}} \cos(u) \sin(v \\ \sqrt{\frac{1}{\lambda_2}} \sin u \sin v \\ \sqrt{\frac{1}{\lambda 3}} \cos v \end{pmatrix}$. Then $T(\Delta_0)$ is the tilted ellipsoid of Δ_0 with a circle cross section. Furthermore, $P(T(\Delta_0))$ is the tilted ellipsoid of Δ_∞ with a circle cross section. If the intersection curve between Δ_0 and $T(\Delta_0)$ is $\gamma_1(t) \cup \gamma_2(t)$, then the intersecting curve between $P(T(\Delta_0))$ and Δ_∞ are $P\gamma_1(t) \cup P\gamma_2(t)$. We depict the pictures for Δ_∞ (in green), $P(T(\Delta_0))$ (in blue) and $P\gamma_1(t) \cup P\gamma_2(t)$ (in red) respectively in Figure 6.

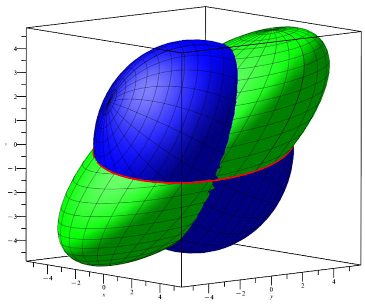

Figure 6. $P(T(\Delta_0))$, Δ_∞, and $P\gamma_1(t) \cup P\gamma_2(t)$

6 Conclusion

We hope that the linear transformation L_E described in this paper will link to interesting areas in projective geometry. Those three questions listed in (1), (2) and (3) provide good exploratory activities to those students who have learned multi-variable calculus and introductory linear algebra. Some natural and extended areas for readers to explore are the followings:

There are more areas need to be further investigated. For example, we recall that $L_E : \Sigma \longrightarrow \Delta_\infty$ is a bijection when the parameters a, b, c, s, u_0 and v_0 are given for L_E. Consequently, the implicit form of the locus ellipsoid of $L_E(\Sigma)$ is uniquely determined and can be represented by a positive and symmetric matrix M as seen in (12). Conversely, if Δ_∞ is given by such a matrix M, then it is hard to find a, b, c, s, u_0 and v_0 for such projective linear transformation L_E.

Since an ellipsoid can be thought as the image of the unit sphere under a linear transformation. After writing the image ellipsoid as a quadratic form of $XMX^t = 1$ for some matrix M with $X = [x, y, z]$, one may study how we can decompose the matrix M as $M = TRD$, where T is a sheared map, D is a dilation and R is a rotation, see ([5]).

We recall that the locus problem was originated from a college entrance exam (see [7]). With the help of technological tools, the problem was extended to more challenging forms as seen in ([8]), ([9]), and ([10]). Consequently, the explorations lead us to deeper areas in projective geometry, algebraic geometry and etc. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate more challenging areas in mathematics. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

7 Acknowledgements

Author would like to express sincere thanks to Antonio Morante and Vladimir Nodelman for making several constructive comments to the paper.

8 Supplementary Electronic Materials

[S1] A GeoGebra file to explore a given ellipsoid and its locus.

[S2] A Maple file for (5)-computing intersecting curve between the ellipsoid and its locus, and etc.

[S3] A Maple file for (6)-a tilted ellipsoid.

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