Delineation of the Zone of Influence of Pumping Wells using CAS and DGS

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Abstract

Groundwater pollution is a general concern in countries using pumping wells for water consumption. Also, the determination of protected zone around pumping wells is a practical concern. The delineation of this zone is frequently obtained using numerical models such as MODFLOW and or FEFLOW. In this contribution we derived the equation of the plane curve of the protected zone using a steady state solution of the groundwater flow equations and the theory of envelopes. Cassini ovals appear in some particular cases, providing new applications of these plane curves. A Dynamic Geometry System, such as GeoGebra and/or Desmos, is used to explore the protected zone. A Computer Algebra System may be used for the computations, in particular to characterise the delineation curves for the zones of influence for wells.

1 Introduction

Protection of pumping wells from pollution is one of the most important concerns in the modern management of water resources. In Israel, where almost seventy percent of the available water comes from groundwater, the protection of production wells against dissolved pollutants is of critical concern. In the past, well protection techniques have focused on the delineation of zones of influence of pumping wells using advanced modeling tools. The aim of this paper is to develop a new approach, based on analytical derivation of the sensitivity of the well locations to the zone of influence of a series of wells. Figure 1 shows a standard situation with different kinds of wells.

Inside the zone of influence of a pumping well a particle of polluted species will flow through the well. On the opposite, a particle of polluted species outside the zone of influence will not reach the pumping well. Also, in the case of recharging wells the zone of influence of these wells corresponds to the zone of depollution. For example, in the pump and treat strategy for depollution of groundwater aquifers, the zone of depollution corresponds to the zone of

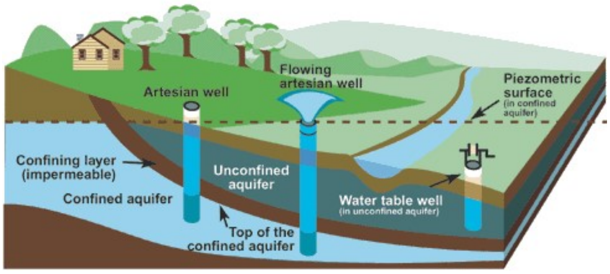


Figure 1: Acquifer and wells

influence of the recharge wells. When remediation design of a groundwater site is performed, the optimal locations of the pumping recharge wells and the pumping rates are certainly the significant parameters in the design. Therefore, a precise determination of the zone of influence with respect to the pumping rates and the distance between wells is needed for well design. A classical approach to delineation of pollution has been to use conceptual groundwater models such as MODFLOW or FEFLOW; see the USGS webpage devoted to Groundwater Modeling . These models are based on the analysis of geological, geophysical and geochemical data, without consideration of groundwater flow. Selection of an appropriate conceptual geohydrological model, based on hydrodynamics, requires the use of a calibrated three-dimensional flow and transport simulation for each conceptual scenario. This task, however, is time consuming and requires rather detailed knowledge of the model's input parameters. For this reason, a simplified and less computer-intensive model is useful in providing a good first assessment of the zone of influence of a polluted well [1].

This is precisely the situation that arises in the observation of source locations of salinization in many of Israel's deep aquifers. When many of the aquifer parameters are unknown or unavailable, the problem becomes one of identifying a conceptual model based upon the information that is available. Measurements, such as wellhead, well concentration, well pumping rate and the location of the wells must therefore be used to recover more knowledge of the contaminant transport characteristics.

We explore graphically the influence zones of the wells, using a Dynamic Geometry System, such as GeoGebra or Desmos. The curves which appear in this exploration are a parameterized family, containing Cassini ovals as a subfamily. Some envelopes and offsets of these curves have been studied by in [5]; in the present paper, these ovals are themselves the requested envelope, but with regards to another definition, as described in Section . The equations have to be transformed into polynomial equations. For this purpose, the algebraic computations are performed either with the CAS implemented into GeoGebra, or with another CAS. For this, we used Maple.

2 Flow equations

Tracing the movement of the elemental area over time, as determined by the flow field, then generates the particle paths. Because the concentration of the source, denoted by , in the Lagrangian interpretation of advective transport, is associated with a single particle and equals that of a particle emerging from the source, it must not change over time. It is therefore easy to see that the equation for advective transport in the form

$$\frac{DC}{DT} = \frac{q_s}{\theta}(C_s - C)$$

simplifies to become

$$\frac{DC}{DT} = 0.$$

Plainly stated, in purely advective transport the concentration associated with a fluid particle does not change with time as the particle moves along its path line. Therefore, in a series of wells, the pollution arrives to the well only when the particle flow through the well inside a zone of influence of the wells.

2.1 Generation of Flow Field

We begin first with the generation of the flow field from which particle tracking will be computed. Flow fields can be generated either by exact analytical solutions or through the use of numerical approximations. In the case of two-dimensional flow in a homogeneous single aquifer of idealized geometry, the solution of steady state flow may be derived analytically using the potential flow theory as described in [12, 17]. Although analytical techniques have a greater degree of accuracy, they lack the flexibility to model irregular boundaries and complicated boundary conditions. For irregular boundaries, realistic aquifer geometries and aquifers with heterogeneous properties, numerical methods are generally used to solve the flow and transport equations. This document will focus on the combination of analytical and numerical techniques, or semi-analytical solutions, which are sometimes referred to in the literature as Analytical Element Methods (AEM).

2.2 Numerical solutions

The two most common numerical techniques for solving groundwater flow equations are the finite difference (including integrated finite difference of constant volume methods) and finite element methods [11]. These methods use a series of nodes or elements to solve the governing equations of groundwater flow and transport. The most significant restriction in numerical methods is the need to discretize the domain into a three-dimensional network of nodes or elements. This type of limitation can be lessened by solving regional Dupuit-Forchheimer flow through superposition of analytic elements, as proven in [16, 9].

In the case of finite differences, the error introduced is that identified with the Taylor series approximation and will be affected by the size of the discretization and the behavior of the function being approximated. While finite elements allow slightly more flexibility when dealing with boundaries and have a similar degree of numerical accuracy, they require more computational effort and therefor more computer time.

2.3 Analytic solutions

In analytical modeling the flow field is generated by utilizing formal mathematical closed-form solutions to generate the appropriate flow field. Analytical solutions are restricted to simple geometries, but provide an extremely accurate way to study the behavior of groundwater flows under hypothetical conditions. The analytic element method, as developed in [17] for groundwater flow, uses the method of images and the principle of superposition to produce flows associated with various aquifer features.

2.4 Steady flow

In the analytical element method, simulation of regional flow is accomplished by using superposition to combine the equations for the fluid potential for each feature, or element. Features such as injection wells and pumping wells are simulated by using the equation describing the fluid potential in an infinite domain for a straight-line source [17] and the method of images. Derivation of the solution of a straight-line source with a given strength is obtained from the integration of a point source along a prescribed length. Aquifer heads and flow velocities may then be obtained by combining the equations of potentials for all of the prescribed elements.

In order to then transform the domain from an infinite region to one that adheres to the specified boundary conditions, the method of images is used [17]. The method of images consists of locating image conditions such that the combination of an aquifer feature and its image produces the desired equipotential. Aquifer heads and flow velocities are, consequently, obtained by combining the equations for the potentials and the images for all of the prescribed elements. For features located on the boundary, such as areas of infiltration, special analytical solutions are used. For example, the modeling of an area of infiltration can be simulated analytically using the potential for a circle derived by Haitjema [8] or through the use of the potential for a rectangle derived by Steward [15].

Closed form solutions for circular and rectangular surfaces exist in the form of specified constant discharges with varying heads. When closed form solutions are not available, one must apply the boundary element method [11]. The boundary element method combines a discretized numerical solution on the boundary, with an analytical solution inside the domain, to model more complicated boundary conditions when combined with general geometries.

3 Definition of the zone of influence

3.1 General setting

Before the pumping starts, the hydraulic head is at a level let say . After the pumping starts around each pumping well, the *drawdown* due to the pumping from a groundwater level may be expressed as in [17]:

$$\left|Drawndown\right| = |\phi(x, y) - \phi_w|$$

where $|Drawdown|$ is the hydraulic head at a point $M(x, y)$. The zone of influence of a given well may be defined as the zone of very small perturbation from the initial groundwater level. Therefore, the zone of influence or and/ or the envelop of the disturbed zone may be expressed as:

$$\begin{cases} \left|Drawdown\right| = |\phi(x, y) - \phi_w| \ < \ \varepsilon \\ \qquad \varepsilon \ll 1 \end{cases}$$

Analytical formula for the computation of the formula is difficult and we propose to use a Computer Algebra System (CAS) to derive this zone.

3.2 Conceptual Model of the Aquifer

The model developed for identifying salinization sources comprises the following steps and simplifying assumptions:

(i) The aquifer is modeled as one homogeneous unit of constant thickness with parallel vertical planar boundaries (box like). Constant head or given flow rates on the boundaries are selected.

(ii) The wells are represented by singularity lines of known strength while the pollutant source is represented as an area of given water flux (which may be small). To account for boundary conditions, appropriate images are added.

(iii) A mean, effective, value of the hydraulic conductivity for the formation is determined by calibrating computed heads against measured heads at a few points, while also taking into account that the hydraulic conductivity can be expressed as the ratio between the specific discharge potential and the pressure head.

(iv) The velocity field is determined analytically by differentiation of the potential and division of the result by the effective porosity. A simple algorithm leads to the velocity values at each selected point.

The aquifer to be modeled is considered fully-saturated, confined and incompressible. It is presumed to be at steady state with a homogeneous hydraulic conductivity. The aquifer is of constant thickness and contains two impervious boundaries formed by planar surfaces on the top and the bottom.

3.3 Specific cases

3.3.1 Two wells cases

Derivation of the flow field around a well by analytical methods originates with the solution of the potential in an infinite domain for a straight-line source [17, 12]. Wells are treated as lines with a given strength derived from the integration of a point source along a prescribed length. Consider a two dimensional infinite confined aquifer of transmissivity T and two pumping wells separated by a distance x_0.

1. Same rates:The first well is located at the origin of the coordinates system and both wells are pumping with the same rate Q. The drawdown due to the pumping from a groundwater level ϕ_w is given by the following equation [17]:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T}\ \ln\frac{R}{r} + \frac{Q}{2\pi T}\ \ln\frac{R}{r_1},$$

where R is the radius of influence of both wells, $r = \sqrt{x^2 + y^2}$ and $r_1 = \sqrt{(x - x_0)^2 + y^2}$. One may express the drawdown as:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \ln \frac{R^2}{rr_1}.$$

The zone of influence of the wells corresponds to the domain where $\phi(r) - \phi_w \neq 0$. Therefore, the boundary of the zone of influence is given by the following equations:

$$\begin{cases} \phi(r) - \phi_w = \frac{Q}{2\pi T} \ln \frac{R^2}{rr_1} = 0 \\ R^2 = rr_1 = \sqrt{(x^2 + y^2)((x - x_0)^2 + y^2)} \end{cases} \tag{1}$$

The last equation defined a family of surfaces with a parameter :

$$F(x, y, x_0) = \sqrt{(x^2 + y^2)((x - x_0)^2 + y^2)} - R^2. \tag{2}$$

2. Different rates: The drawdown due to the pumping from a groundwater level ϕ_w is given by the equation [17]).

$$\phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R}{r} + \frac{Q_2}{2\pi T} \ln \frac{R}{r_1},$$

One may express the drawdown as:

$$\phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R^{q+1}}{rr_1^q}, \quad \text{where } q = \frac{Q_2}{Q_1}.$$

Therefore, the boundary of the zone of influence is given by the following equations:

$$\begin{cases} \phi(r) - \phi_w = \frac{Q_1}{2\pi T} \ln \frac{R^{q+1}}{rr_1^q} = 0 \\ R^{q+1} = rr_1^q = (x^2 + y^2)^(1/2) \; ((x - x_0)^2 + y^2)^{q/2}. \end{cases} \tag{3}$$

The last equation defined a family of surfaces with two parameters :

$$F(x, y, x_0, q) = R^{q+1} - (x^2 + y^2)^(1/2) \; ((x - x_0)^2 + y^2)^{q/2}.$$

3.4 A line of wells

Consider a two-dimensional infinite confined aquifer of transmissivity T and a line of n pumping wells separated by a distance X_0. The first well is located at the origin of the coordinates system and both wells are pumping with the same rate Q. The drawdown due to the pumping from a groundwater level ϕ_w is given by the following equation [17]:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \sum_{i=0}^{n-1} \ln \frac{R}{r_i},$$

where R is the common radius of influence of all the wells. We have:

$$r_i = \sqrt{(x - ix_0)^2 + y^2}, \; i = 1...n$$

One may express the drawdown as follows:

$$\phi(r) - \phi_w = \frac{Q}{2\pi T} \sum_{i=0}^{n-1} \ln \frac{R^n}{\prod_{i=0}^{n-1} r_1}$$

The boundary of the zone of influence is given by the equation:

$$\begin{cases} \phi(r) - \phi_w == 0 \\ R^n = \prod_{i=0}^{n-1} r_1 \sqrt{(x - ix_0)^2 + y^2}. \end{cases} \tag{4}$$

The last equation defined a family of surfaces with a parameter x_0:

$$F(x, y, x_0) = \prod_{i=0}^{n-1} r_1 \sqrt{(x - ix_0)^2 + y^2} - R^n. \tag{5}$$

3.4.1 Computation of the zone of influence

With this simplified approach, the optimal design for a remediation using pump and treat system will correspond to the design that assures a total recover of the zone of influence. Mathematically speaking, it corresponds to an envelope, in the sense of Definition 4 in next Section, of the parameterized families of surfaces (either with one or two parameters) given by the functions in Equations (2) or (5).

4 Different definitions of envelopes of 1-parameter families of plane curves

Envelopes of 1-parameter families of plane curves have been studied for a long time, but there exist 4 different definitions of this kind of objects. Kock [10] gives 3 different definitions of an envelope of a 1-parameter family of plane curves:

Let $\{C_k\}$ be a family of real plane curves dependent on a real parameter k.

Definition 1 (Synthetic) *The envelope $\mathcal{E}_1$ is the union of the characteristic points M_k, where the characteristic point M_k is the limit point of intersections $C_k \cap C_{k+h}$ as $h \to 0$. In other words, the envelope $\mathcal{E}_1$ is the set of limit points of intersections of nearby curves C_k.*

Definition 2 (Impredicative) *The envelope $\mathcal{E}_2$ is a curve such that at each of its points, it is tangent to a unique curve from the given family. The locus of points where $\mathcal{E}_2$ touches C_k is called the characteristic point M_k.*

Definition 3 (Analytic) *Suppose that the family of curves is given by an equation $F(x, y, k) = 0$ (where k is a real parameter and F is differentiable with respect to k); then an envelope $\mathcal{E}_3$ is determined by the solution of the system of equations:*

$$\begin{cases} F(x, y, k) = 0 \\ \frac{\partial F}{\partial k}(x, y, k) = 0 \end{cases}$$

i.e., the envelope is the projection onto the $(x, y)-$plane of the points in the 3- dimensional $(x, y, k)-$space, belonging to the surface with equation $F(x, y, k) = 0$.

Simple examples are given in [7]. With other notations, Bruce and Giblin ([3], Chap. 5), show that $\mathcal{E}_1 \subset \mathcal{E}_3$ and $\mathcal{E}_2 \subset \mathcal{E}_3$, and give several examples. They add a 4^{th} definition, different from the previous three.

Definition 4 *The envelope $\mathcal{E}_4$ is the boundary of the region filled by the curves C_k.*

Among the above definitions, the only one which is easily computable is Definition3. This is the only definition given by Berger [2](sections 9.6.7 and 14.6.1). Examples for Definition 4 have been studied in [4, 5]; in this 1^{st} paper in reference, the question was related to the determination of a safety zone around a mobile device. This is the meaning of an envelope that interests us in what follows to determine zones of influence of wells.

5 Some case studies

5.1 Two wells

For two wells, one at the origin $(0, 0)$ and another one at the position $(a, 0)$. The drawdown of the head for steady state solution of the flow equation for a confined aquifer is given by:

$$\begin{cases} h(x,y) = \phi(x,y) = h_0 \\ \phi(x,y) = \frac{Q_1}{2\pi Kb} \ln \frac{r_0}{\sqrt{x^2+y^2}} + \frac{Q_2}{2\pi Kb} \ln \frac{r_0}{\sqrt{(x-a)^2+y^2}} \end{cases}$$

where h_0 denotes the initial head before pumping, K the hydraulic conductivity, b the depth of the aquifer, Q_1 the pumping rate of the 1^{st} well and Q_2 the pumping rate of the 2^{nd} well. A simple way of building the zone of protection is to compute the zone determined by the equation $\phi(x, y) = 0$. This equation is equivalent to the following:

$$\ln \left(\frac{r_0}{\sqrt{x^2 + y^2}} \right) \left(\frac{r_0}{\sqrt{(x-a)^2 + y^2}} \right)^q = 0,$$

where $q = \frac{Q_2}{Q_1}$.

Finally the equation

$$[x - a)^2 + y^2]^{q/2}[x^2 + y^2)^{1/2}] = r_0^{q+1} \tag{6}$$

defines the curves of delineation of the zone of influence of the two wells. It depends on three parameters:

- the ratio between the pumping rates: $q = r = \frac{Q_2}{Q_1}$;
- The distance between the two wells: a=L;
- The radius of the well: $r_0 = p$.

It is important to understand the engineering aspects of these three parameters.

- $q = r = \frac{Q_2}{Q_1}$ described the effect of the pumping rate on the zone of influence;
- $a = L$ analyses the effect of the distance between the wells on the zone of influence;

- $r_0 = p$ analyses the effect of the well design of the singular well on the zone of influence. Depending on the internal radius of the well the the well has a " potential radius of influence" r_0.

In the following figures, obtained with Desmos,for $Q_1 = Q_2$ we present three different types of graphs. These graphs show three cases of "zone of influence".

a. Two separate zones of influence; see Figure 2.

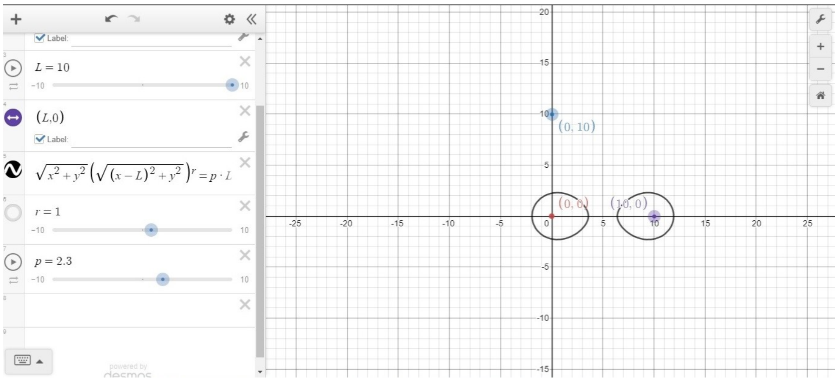


Figure 2: Equal pumping; $r = 1, L = 10, p = 2.3$

b. A narrow zone of influence, as illustrated un Figure 3.

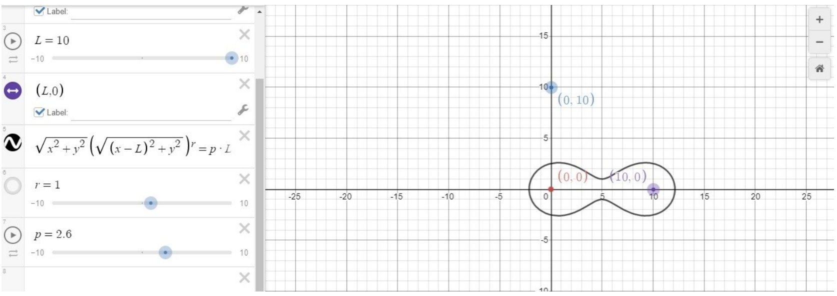


Figure 3: Equal pumping; $r = 1, L = 10, p = 2.6$

c. A large zone of influence, as shown in Figure 4.

The equations appearing in the figures have been written in a simplified form, suitable for the specific cases, namely:

$$\sqrt{x^2 + y^2} \cdot \sqrt{(x - L)^2 + y^2}^{\,r} = pL. \tag{7}$$

Consider the particular case for which $r = 1$. Squaring both sides of Equation (7), we obtained a quadratic equation

$$(x^2 + y^2) \cdot ((x - L)^2 + y^2)) = p^2 L^2. \tag{8}$$

The delineation curve is now a bicircular quadratic of a specific kind[1]

[1] Recall that a complete catalog of quadratic curves exists; as soon as a plane algebraic curve is of degree 4, it is easy to determine which kind of curve it is.

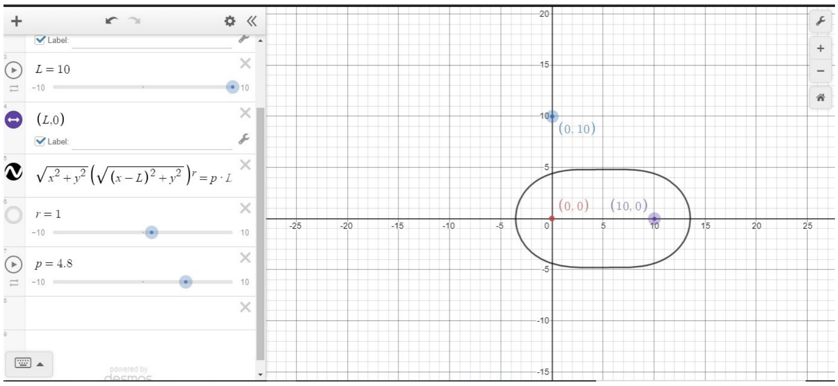


Figure 4: Equal pumping; $r = 1, L = 10, p = 4.8$

Proposition 5 *For $r = 1$, the delineation curves are Cassini ovals.*

Proof. Denote $F_1(L/2, 0)$ and let $r = 1$. Now look at the equation $\sqrt{x^2+y^2}\sqrt{(x-L^2+y^2} = pL$ which appears in the algebraic window of Figures 2, 3 and 4. It describes the geometric locus of points $M(x, y)$ such that $OM \cdot F_1M = pL$, i.e. the given curve is a Cassini oval with foci O and F1.

We can see that also by an algebraic computation. Apply the change of coordinates $(x, y) = (X + L/2, Y)$. Then we have:

$$\sqrt{x^2+y^2} \cdot \sqrt{(x-L)^2+y^2} = pL$$

i.e.

$$\sqrt{\left(x+\frac{L}{2}\right)^2 + y^2} \cdot \sqrt{\left(x-\frac{L}{2}\right)^2 + y^2} = pL.$$

Squaring both sides and expanding them, we have:

$$(X^2+Y^2)^2 - \frac{1}{2}L^2X^2 + \frac{1}{2}L^2Y^2 + \frac{1}{16}L^4 - p^2L^2 = 0,$$

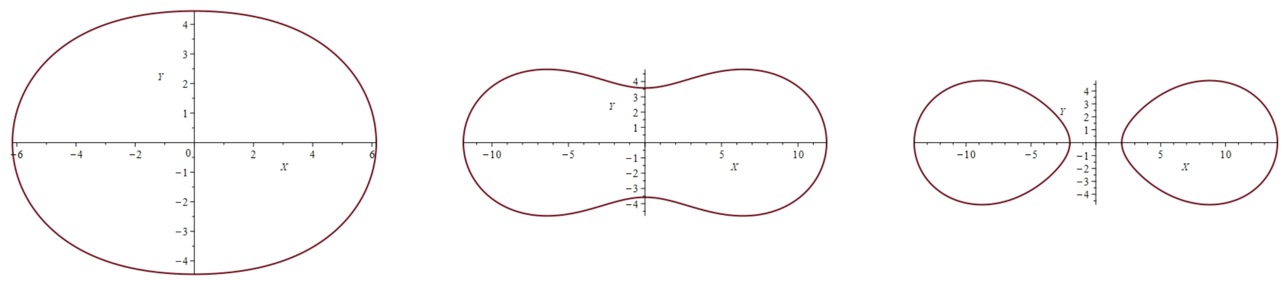
which is easily identified as the equation of a Cassini oval; see [5] and the references there. ■

Recall that, even if the plot shows two components, the polynomial is irreducible and the curve is irreducible. This is easy to check with the **factor** command of any CAS. The more the wells are distant, the more the curve shows points of inflexion, until it has two components, shaped as loops. This is illustrated in Figure 5.

Here are a few rows of Maple code for Figure 5.

```
restart:with(plots):setoptions(scaling = constrained):setoptions(thickness = 2);
F := (X^2 + Y^2)^2 - 1/2*L^2*X^2 + 1/2*L^2*Y^2 + 1/16*L^4 - p^2*L^2 = 0;
p := 4.8;
for k from 10 by 2 to 16 do
implicitplot(subs(L = k, F), X = -15 .. 15, Y = -15 .. 15);
end do;
```

Remark 6 *Cassini ovals are defined by equations whose general form is $(x^2+y^2)^2 + ax^2 + by^2 + c = 0$, where a, b, c are real numbers. They may have one or two components, which are*



(a) $L = 6$ (b) $L = 16$ (c) $L = 20$

Figure 5: The influence of the distance between wells

not distinguished by factorization of the polynomial. Another description of Cassini ovals is as the intersection of a torus with a plane parallel to the torus axis. The general setting in the literature is with a regular torus. Equations as above describe sometimes the intersection of a self-intersecting torus with a plane parallel to the axis; in such a case, the intersection may have two components, one inside the other. Details are explained in see [6], where Cassini ovals are called by their other name: spiric curves. The physical meaning of the question under study here is enough to understand why such a situation does not occur here and we don't have a component in the interior of the other one.

The influence of the parameters p and L can be explored separately, using the following rows of Maple code:

- For the influence of L (note that here the value of F is fixed, but this can be easily changed, even introduced in a **for** loop):

```
plots[animate](implicitplot, [subs(p = 4.8, F) = 0, X = -20 .. 20,
                               Y = -10 .. 10], L = 0 .. 20);
```

- For the influence of p (with a similar remark as above, this time regarding the value of L):

```
plots[animate](implicitplot, [subs(L = 3, F) = 0, X = -20 .. 20,
                              Y = -10 .. 10], p = 0 .. 10);
```

For other values of the parameter r, other shapes are obtained, and have to be studied separately. Figure 6 shows an example with $r = 2$. If $r = 2$ the obtained delineation curve is as sextic[2]. For such curves, no complete catalog exist, but an algorithm is available[3], to determine the topology of a given sextic. The website Mathcurve presents rational sextics and a few non rational ones. A list of 64 cases is given. The pear-shaped curve which we obtained here does

[2]An algebraic curve of degree 6.

[3]Developed in 2021 at Max Planck Institute for Mathematics in the Sciences, and implemented in Mathematica

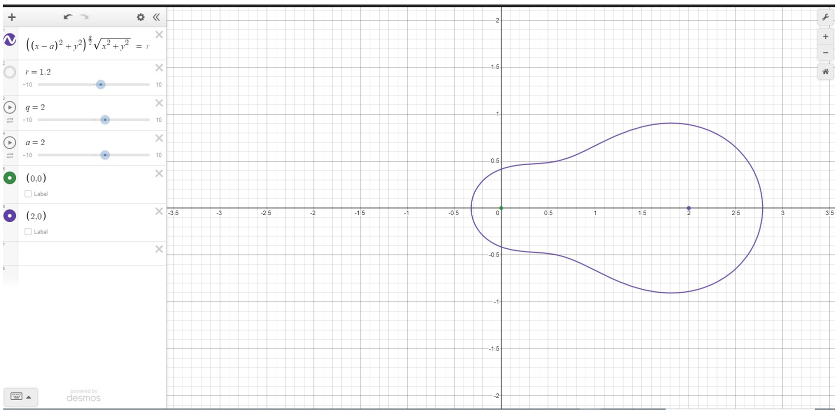


Figure 6: Inequal pumping; $r = 2$

not appear in the list, and checking whether this curve is rational or not is beyond the scope of the present work. See also [13] (the classification is performed for sextics over $\mathbb{C}$, which is not exactly our concern here) and the references there.

Figures 7 and 9 corresponds to $r = 3$ and $r = 4$ respectively. Here too, it is possible to derive from the data a polynomial equation for the delineation curve. Of course degrees are higher, making the identification with a classical curve harder, if possible. A general exploration of these

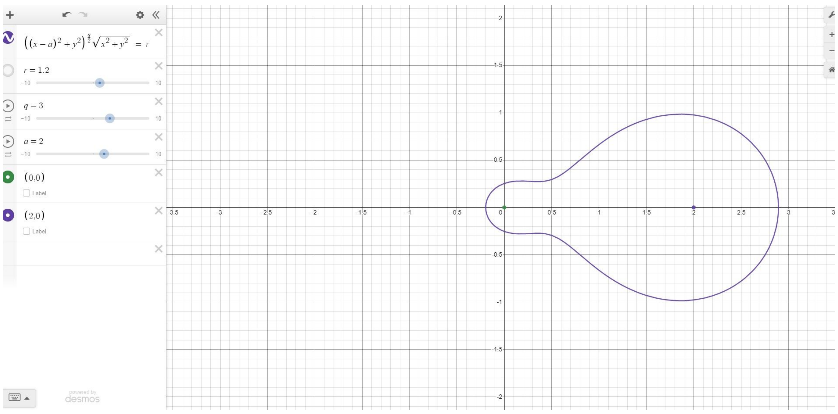


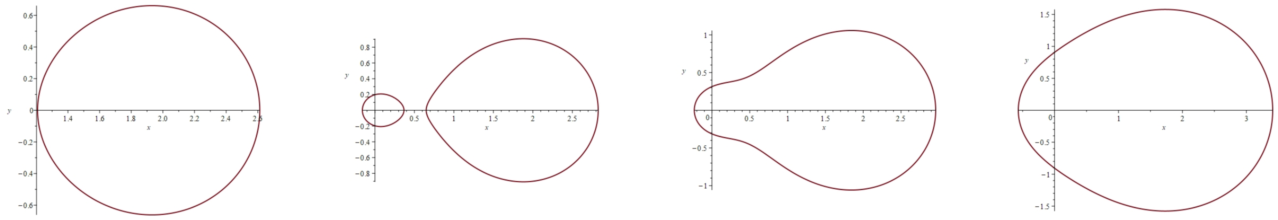
Figure 7: Inequal pumping; $r = 3$,L=10,p=4.8

sextics with similar code as above shows that for pL close to 0, the curves are convex, then have two disjoint components, then have one component with points of inflexion (shaped somehow like cougurds), and again tend to convex shapes. See Figure 8. This surprising behaviour from a single component to a single component and in-between cases with two components is not intuitive. It reinforces the need to make a computerized exploration for different values.

Similar exploration can be performed for any value of the parameter r.

6 Conclusions and directions for future work

Starting from the end, we wish to emphasize the importance of finding Cassini ovals. Jean-Dominique Cassini (8 June 1625 – 14 September 1712), an Italian and French naturalized



(a) $pL = 0.6$ (b) $pL = 1.6$ (c) $pL = 2.6$ (d) $pL = 9.6$

Figure 8: The influence of the distance between wells - sextics

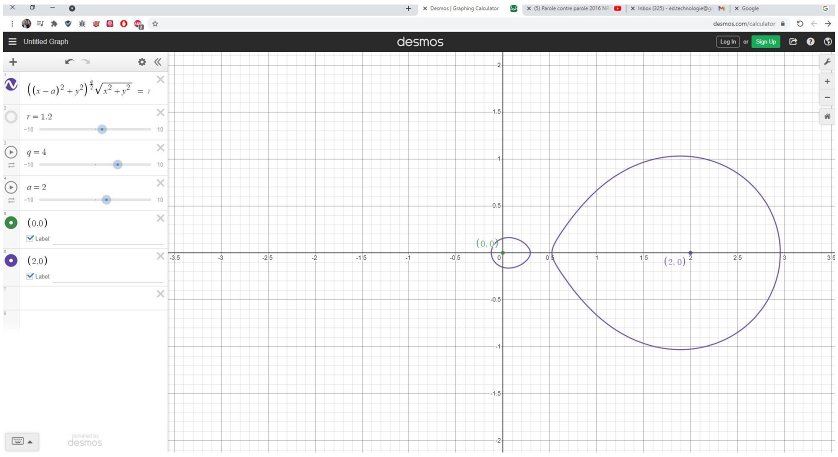


Figure 9: Inequal pumping; $r = 4, L = 10, p = 4.8$

mathematician and astronomer conjectured that the planetary orbits around the Sun were the ovals which will be later called after his name. After Kepler proved that these orbits were actually ellipses (1^{st} Kepler law), Cassini ovals seemed to have lost of their importance and some mathematicians considered them as a nice topic in mathematics and not more. Actually, Cassini ovals appear in electrostatics and to describe some magnetic fields. A mathcurve page is devoted to numerous properties of these ovals. We describe here another application of Cassini ovals and of some of their generalizations. For a reader non familiar with Cassini ovals, a GeoGebra applet is available to check the various possible shapes, according to the choice of the foci and of the parameter..

Regarding the contents of the work above, we wish to make the following (not so) final remarks:

1. One can compile easily using a CAS the zone of influence of a *series* of wells.
2. We present the different solutions for the 2 wells problem with constant pumping wells. This can be generalized to any number of wells in a line. Subsequent work will address the issue of non aligned wells.
3. Parameters such as distances between wells are also considered. Here we did not analyse this effect available for at least three wells.

4. More work is needed for the analysis of the well design of each singular well on the zone of influence. We focused on the distance between wells. The other parameters encode the volume of pumping per time unit, whose influence deserves a separate amelioration of our model.

5. More work is needed for understanding the influence of different pumping rates and well design. In particular, we discovered through a CAS assisted dynamical exploration that for a fixed rate and different values of the distance, very different shapes for the influence zone are obtained. This can have very important consequences in the field, for actual wells connected to the aquifer.

With our approach, it will be possible to analyze the optimal location of the wells.

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