Geometry of planetary motion

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Abstract

This contribution is inspired by the lecture "The Motion of Planets around the Sun" given by Richard Feynman, American physicist and Nobel Laureate, in 1964, namely by steps of Feynman's geometrical proof of the law of ellipses, which was based on Isaac Newton's approach to the problem. The contribution focuses on selected passages of this proof, interprets them using the program of dynamic mathematics GeoGebra and offers their use in the form of activities that can be implemented at different levels of mathematics curriculum of lower and upper secondary school. The activities presented in the paper are suitable for the implementation of the STEM approach to mathematics education as they combine the topics of the mathematics and physics curriculum, in addition against the backdrop of the captivating story of discovering the essence of the functioning of the universe.

1 Introduction

This study presents several specific educational activities which, at various levels of school mathematics and with significant use of the dynamic mathematics software GeoGebra [3], are focused on the selected properties of the ellipse with emphasis on the role of this curve as the trajectory of the motion of planets around the Sun.

One of the ambitions of this work is to show how the use of GeoGebra, with its respective geometric, algebraic and numerical capabilities, will enable educators to bring the subject of ellipses and the physical laws of planetary motion closer to pupils at different levels of school education and to make meaningful use of their school knowledge of mathematics and physics, such as geometry of the triangle, properties of an axial symmetry, definition of conic sections, Kepler's laws of ellipses and of equal areas and Newton's laws of motion and of gravity. The presented materials are primarily intended for secondary school pupils and for students of the teaching of mathematics. The aim of their presence is to show how GeoGebra allows students not only to understand the meaning of the properties discussed, but also to practice their knowledge of the specific content of the school curriculum.

The activities we present are suitable for STEM (Science, Technology, Engineering, and Mathematics) education [19]. They appropriately combine the educational content of mathematics, geometry and physics, plus they reflect the history of the evolution of the knowledge of the universe and the principles on which its functioning is based. All materials created in GeoGebra are available online through the links featured in this text. Together they are available in the GeoGebra Book *Geometry of planetary motion* [7], based on the first version [8] presented at the Global GeoGebra Gathering in Linz in 2015, but which has not yet been mentioned in any publication.

The main inspiration for this study is the lecture *The Motion of Planets around the Sun* by American physicist, Nobel laureate, Richard Feynman, as recorded and further explained and illustrated in detail in the book *Feynman's Lost Lecture. The Motion of Planets Around the Sun* [4].

In 1964 Richard Feynman (1918 – 1988) gave a guest lecture titled *The Motion of Planets around the Sun* to the Caltech freshman class in order to introduce them to geometric proof of the elliptical motion of planets around the Sun. Inspired by Newton's geometric way of proving Kepler's laws of planetary motion, which he presented in his famous work *Philosophiæ Naturalis Principia Mathematica* [17] published in Latin in 1687 (For English translation, see e. g. [18]), Feynman created his own geometric proof using only elementary knowledge of plane geometry and the selected statements of Newton's laws of motion and gravity. As a result of the temporary disappearance of its records this lecture became known as *Feynman's Lost Lecture*. After the finding of it among other documents at the Physics Department of Caltech a number of years after the lecture was given it was professionally and literally edited by David and Judith Goodstein and published as an above mentioned book [4] in 1996.

For the sake of completeness, it should be noted that Richard Feynman addressed different approaches to the derivation of Kepler's laws of planetary motions applying Newton's laws of motion and gravity in his texts. In addition to the aforementioned purely geometrical method presented in his *Lost lecture*, which we will address further in this text, a method of numerical approximation, based on iterations recorded using a table, was published in the famous three volume book *Feynman's Lectures on Physics*, first issued in 1964, specifically in its first volume [5], Chapter 9, pages from 6 to 9 (alternatively the online [6], Chapter 9). The application of this method in GeoGebra and its use to represent the trajectory of both planets and other bodies, namely that of the peregrine falcon are presented in papers [9] and [10].

The standard method of proving the validity of Kepler's laws from Newton's laws of motion and gravity, presented in contemporary publications, employs an advanced calculus. Such proof of Kepler's Laws in a comprehensible and clear manner is presented, for example, in [20].

2 Related topics from school curriculum

As mentioned above, the activities presented in the paper cover a number of topics from the mathematics and physics curriculum. In this section we will mention the most important of them.

2.1 Area of a triangle

The area of a triangle is equal to one half its base times its altitude. Therefore, two triangles with a common base and the same altitude have the same area, regardless of the difference in their shapes, see Fig. 1.

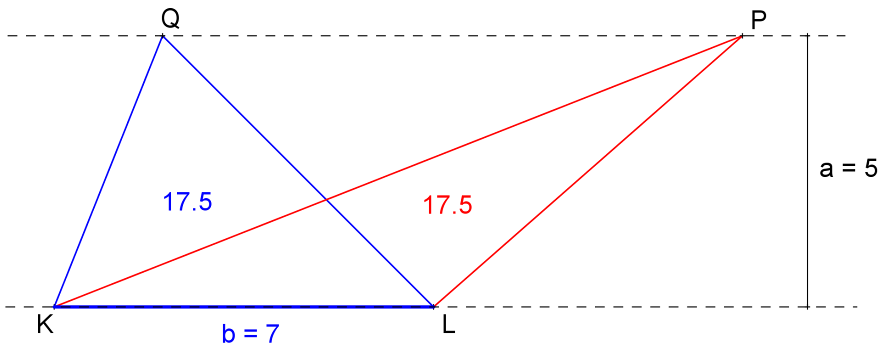

Figure 1: Triangles with a common base (b) and the same altitude (a) have the same area

2.2 Definition of an ellipse

An ellipse can be defined in various ways, see [2]. Here we use the definition of an ellipse as a locus of points, common in upper secondary school mathematics: *Given two points F and F', called foci, and a distance $2a$ greater than the distance $|FF'|$, the ellipse is a locus of points P such that the sum of the distances $|PF|$ and $|PF'|$ is constant and equal to $2a$.*

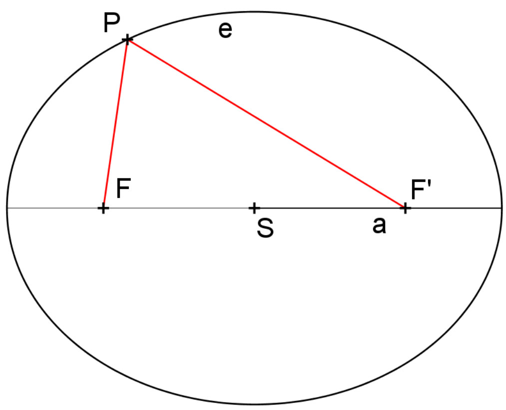

Figure 2: $e = \{P \in E_2;\ |PF| + |PF'| = 2a\}$

Without using the equation, this definition of an ellipse is already being introduced to lower secondary school pupils, often even to primary school pupils, in the form of the so-called *"gardener's construction of an ellipse"* (i. e. elliptical bed): *At two points, drive the pins into the ground, tie a rope longer than their distance to them, fix a pole into it and then, while keeping the rope taut, you carve an ellipse on the ground with this pole.*

GeoGebra has a command `LocusEquation` that allows the given definition to be used to construct an ellipse, and will also determine its equation. Its syntax corresponding to this purpose, specifically for the configuration shown in Figure 3, is as `LocusEquation(f + g == 8, P)`.

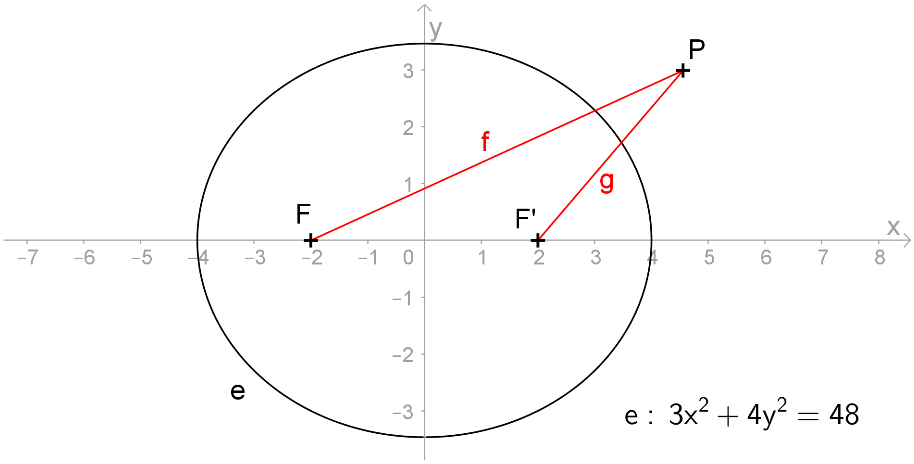

Figure 3: An ellipse determined from the definition using the `LocusEquation` command

2.3 Heron's shortest distance problem

Given two points A and B on one side of a straight line, to find the point P on the line so that $|AP| + |PB|$ is as small as possible [1]. It is a typical task, which illustrates the use and properties of axial symmetry for lower secondary school pupils, and in addition, the use of the triangle inequality in proving the correctness of its solution.

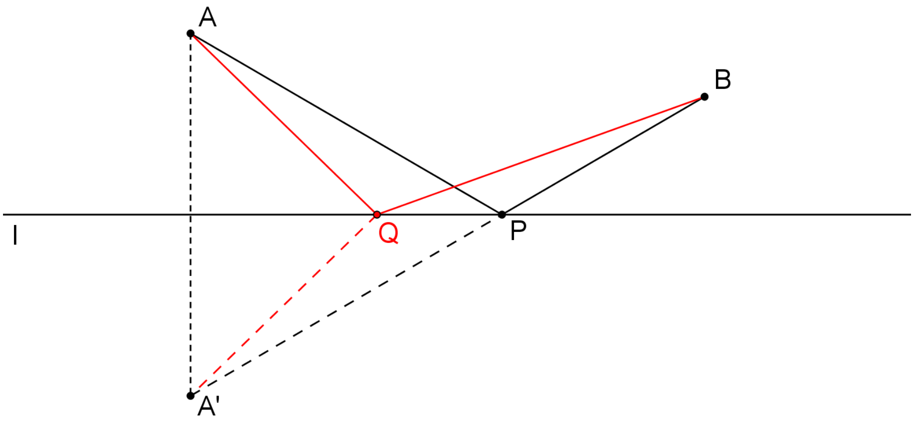

Figure 4: Heron's shortest distance problem

Using GeoGebra we can take advantage of its dragging function. It allows us to illustratively prove the solution of Heron's problem by contradiction, applying *the triangle inequality theorem*. See Fig. 4, where obviously $|AP| + |PB| = |A'P| + |PB| = |A'B|$; $|AQ| + |QB| = |A'Q| + |QB|$, while according to the triangle inequality is $|A'Q| + |QB| > |A'B|$. Consequently $|AQ| + |QB| > |AP| + |PB|$.

2.4 Focal properties of an ellipse

A typical feature of the ellipse explored with pupils already at lower secondary school is its reflective property: *All the light rays starting at one focus will be focused to a point at the other focus.* This property is equivalent to the fact that the tangent of the ellipse bisects the outer angle of the focal radii of its point of tangency.

Proof of it by using the triangle inequality belongs to the high school mathematics curriculum, solved by applying the same principle as in the case of solving the Heron's problem. See Fig. 5. If the line t is the angular bisector of the focal radii of P, in particular that one which

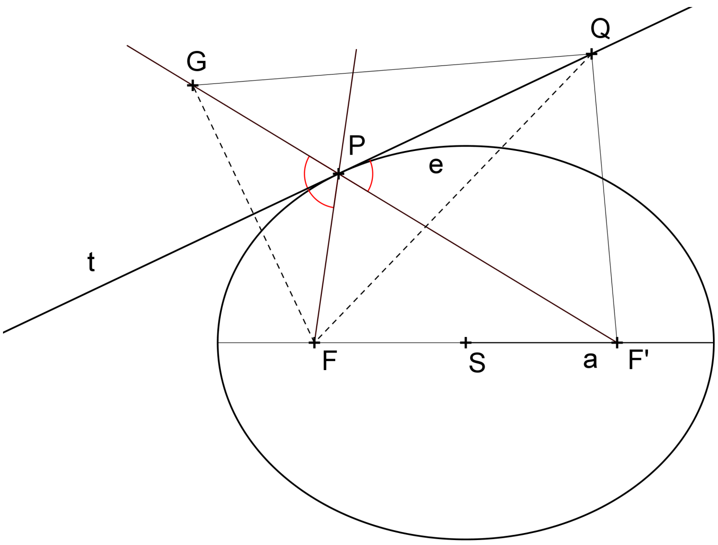

Figure 5: Tangent line t bisects the outer angle of the focal radii of its point of tangency P

does not intersect the segment FF', then the reflection G of the focus point F in t lies on the ray $F'P$. Consequently $|PG| = |PF|$, hence

$$|F'G| = |F'P| + |PG| = |F'P| + |PF| = 2a. \tag{1}$$

It is therefore sufficient to prove that each point Q of the line t different from P lies outside the ellipse, i.e. $|F'Q| + |QF| > 2a$. However, the latter inequality always pays, thanks to the triangle inequality applied to the triangle $F'GQ$. The line t is therefore the tangent of the ellipse e.

As an advanced problem to solve, related to above mentioned properties, an upper secondary school math textbook on analytical geometry [14] presents the following task: *Show that all reflections of one focus point of an ellipse through all its tangent lines form a circle centered in the other focus point. Determine the radius of this circle.* The circle in question is known as a *circular directrix related to the focus of an ellipse*, its radius is $2a$, see Fig 6. Its existence is justified by (1).

2.5 Kepler's laws of planetary motion

Kepler's three laws [11], which describe the motion of planets around the Sun, belong to the usual secondary school curriculum. *Johannes Kepler* (1571−1630), German mathematician

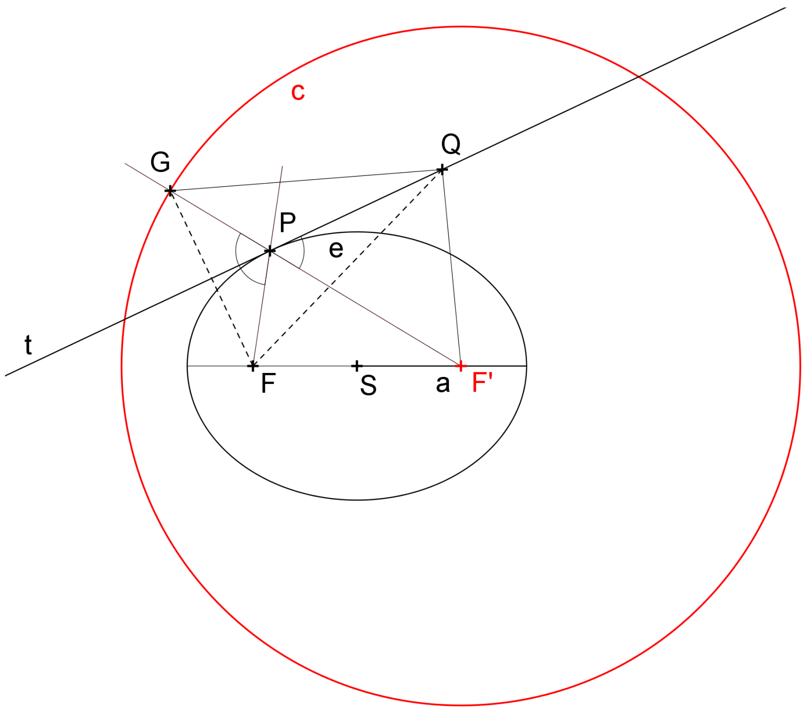

Figure 6: Circular directrix related to the focus F of the ellipse e

and astronomer who spent a significant part of his life in Graz and Linz in Austria, and in Praha in Bohemia, published the three laws over a period of time in two books. His findings were based on observations of the Danish astronomer *Tycho Brahe* (1546−1601). First, in 1609 in *Astronomia nova* [12], Kepler published two statements that are known as his 1$^{\text{st}}$ and 2$^{\text{nd}}$ laws of planetary motion, according to their focus as *the law of ellipses*, and *the law of equal areas*, respectively, then, in 1619 in *Harmonices mundi* [13], he added the 3$^{\text{rd}}$ law, *the law of harmonies*:

The law of ellipses: Orbits of all the planets are ellipses with the Sun at one focus.

The law of equal areas: A line segment from the Sun to a planet sweeps out equal areas in equal time.

The law of harmonies: The orbital period of a planet is proportional to the three-halves power of the size of the semi-major axis of its orbit.

2.6 Newton's laws of motion and gravity

Isaac Newton (1643−1727), English physicists and mathematician, published three laws of motion [15] and the law of gravity [16] in 1687 in his work *Philosophicae Naturalis Principia Mathematica* [17]. Although Newton in this work also developed a mathematical method that enabled him to derive Kepler's laws as consequences of his (Newton's) laws, as we know, instead of this he provided geometric proof of Kepler's laws in the *Principia*. We are particularly interested in Newton's 1$^{\text{st}}$ and 2$^{\text{nd}}$ laws of motion, *the law of inertia* and *the law of force and* *acceleration*, respectively, and in his law of gravity, often descriptively referred to as *the inverse square of the distance law of gravity*:

The law of inertia: An object keeps its state of motion, i.e. a rest or a motion at a constant speed in the same direction, unless an external force is impressed on it.

The law of force and acceleration: The change in motion is proportional to the motive force impressed; it is made in the direction of the straight line in which that force is impressed.

The inverse square of the distance law of gravity: The force of gravity diminishes as R^{-2}, where R is the distance from a planet to the Sun.

3 Proof of the 2nd Kepler's law

The geometric proof of the second Kepler's law, *the law of equal areas*, presented by Richard Feynman corresponded to the proof which was published by Isaac Newton in his *Philosophiæ Naturalis Principia Mathematica* (usually referred to as the *Principia*), 1697 [17]. Feynman used an almost identical illustration to the original one by Newton, see [17], page 32.

This geometric proof is based on the elementary properties of a triangle, particularly the determination of its area, combined with the Newton's laws of inertia and gravity. Its partial steps can be interpreted as a pair of consecutive exercises, the solution of which is the application of relevant topics from the school curriculum of mathematics and physics as mentioned in the previous section 2.1.

Exercise 1 *Given triangles SAB, SAC, SBC and SBD (D is movable along p), see Fig. 7, so that B is the midpoint of the segment AC and p is parallel to SB. Determine the relations of areas of all the given triangles. Justify your claim.*

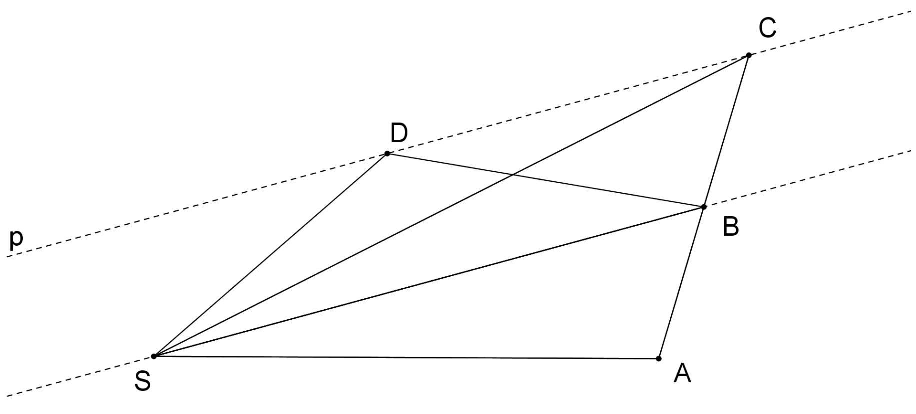

Figure 7: What are the relations between the areas of the displayed triangles?

Exercise 2 *See Fig. 8. Points A, B, C, D and E are the successive positions of a planet in its orbit around the Sun (point S) at equal intervals of time, assuming gravitational action between the Sun and the planet only at the end of each interval. The change in position due to* *gravity is represented by vectors pointing from each of these points to point S. Then, due to the law of inertia $|AB| = |Bc|$, $|BC| = |Cd|$ and $|CD| = |De|$ (Explain why!). In accordance with statement of the 2nd Kepler's law (the law of equal areas) it follows that the areas of triangles SAB, SBC, SCD and SDE are equal. Prove it! You can do it gradually, using triangles SBc, SCd and SDe. Be aware that cC, dD and eE are parallel to BS, CS and DS, respectively. Use the knowledge from the solution of the previous exercise.*

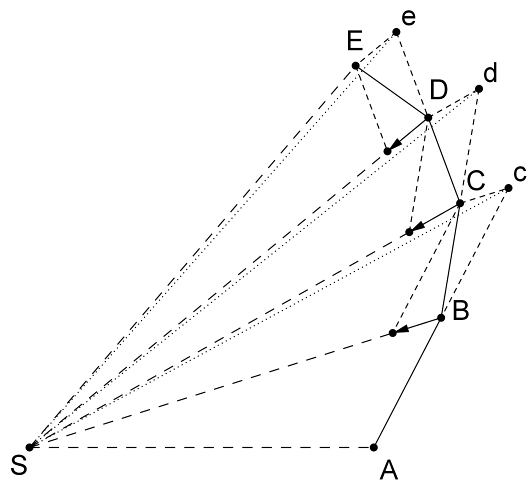

Figure 8: Newton's geometric proof of the law of equal areas

The principle of geometric proof of Kepler's second law is evident from several successive time intervals, as shown in Fig. 8, and as Newton also stated in his work [17]. With sufficient patience, however, we can draw, at least approximately, the whole ellipse of the trajectory of an imaginary planet, as shown in Fig. 9. The Kepler's *law of equal areas* thus appeared to be a consequence of the Newton's law of inertia and the fact that changes in motion of planets are caused by the gravitational force directed toward the Sun.

4 Proof of the 1st Kepler's law

The answer to the question of what causes the elliptical shape of a planet's orbit provides the proof of the Kepler's first law, *the law of ellipses*. This property arises from Kepler's 2nd and 3rd laws and from the fact that the gravitational force diminishes as R^{-2}, i. e. from *the inverse square of the distance law of gravity.*

4.1 The property of reflection of an ellipse

First of all Feynman was to prove that the property of reflection from one focus to the other of an ellipse is equivalent to the property that the sum $|F_1P| + |F_2P|$ is constant for any point P on the ellipse, using arguments analogous to those presented in section 2.4. Again, instead of proving it directly, we assign two consecutive exercises of the same nature.

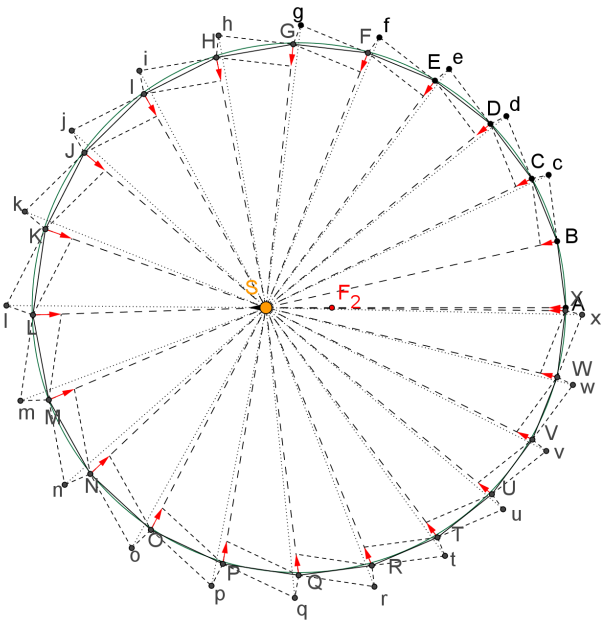

Figure 9: A line segment from the Sun to a planet sweeps out equal areas in equal time

Exercise 3 *See Fig. 10. Sort by size the following lengths:* $|F_1P| + |PF_2|$, $|F_1R| + |RF_2|$, $|F_2P| + |PG|$, $|F_2R| + |RG|$. *Then move the line* p *to align point* R *with point* P *of the ellipse and make a conjecture about the relationship between the line* p *and the ellipse in this configuration.*

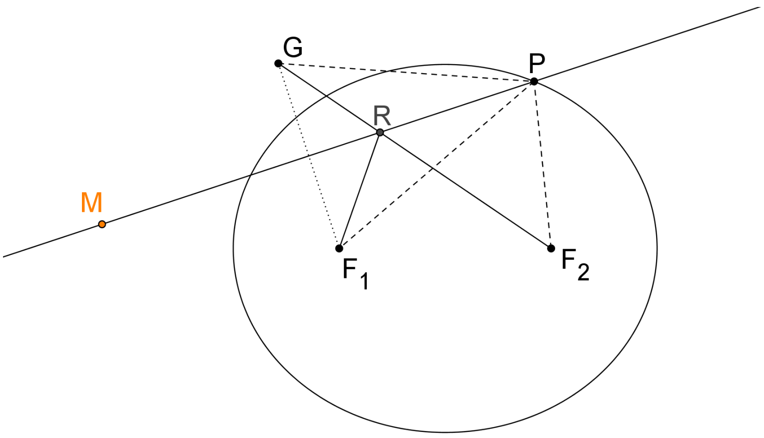

Figure 10: The property of reflection of an ellipse

Exercise 4 *An ellipse is given by its two foci and by the length of its major axis, see Fig. 11. Without drawing the ellipse, using the available tools construct its movable point and the tangent of the ellipse passing through this point.*

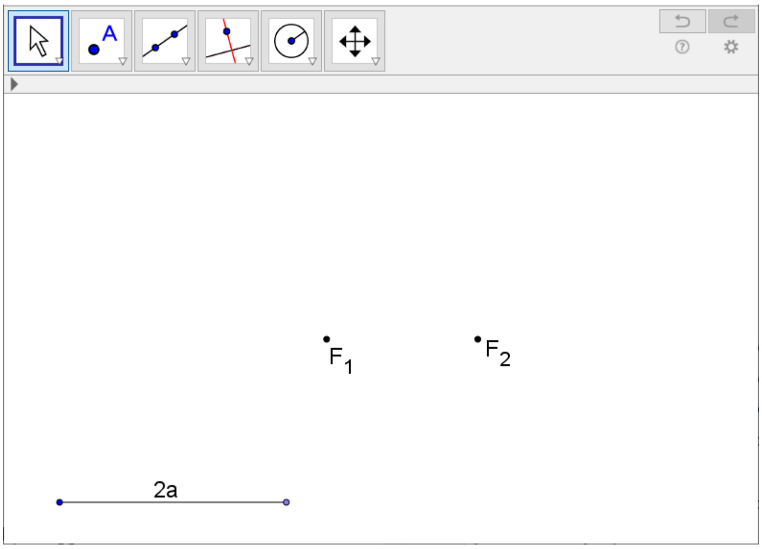

Figure 11: Using the available tools, construct an ellipse given F_1, F_2 and a (GeoGebra with the selected tool menu)

4.2 Feynman's proof of the law of ellipses

The ongoing steps of Feynman's geometric proof of Kepler's first law are quite complex and require a detailed description. For that, we refer the reader directly to [4]. Here we will only briefly describe these steps. All dynamic applets are available in [7].

First, Feynman proved that equal angles correspond to equal velocity changes. He illustrated this property by two diagrams: an orbit diagram and a velocity diagram. From the construction and from the relevant physical laws it follows that the velocity diagram is always the shape of the regular polygon, see Fig. 12.

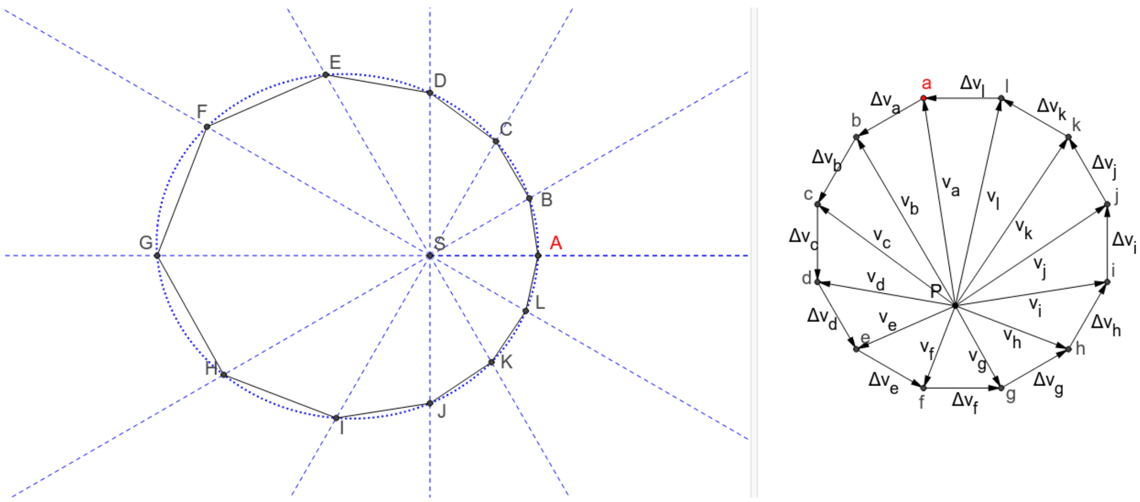

Figure 12: An orbit diagram and a velocity diagram

Then, he found the geometrical correspondence of these two diagrams based on the equality of the angle swept by a planet with the relevant central angle in the velocity diagram, see Fig. 13.

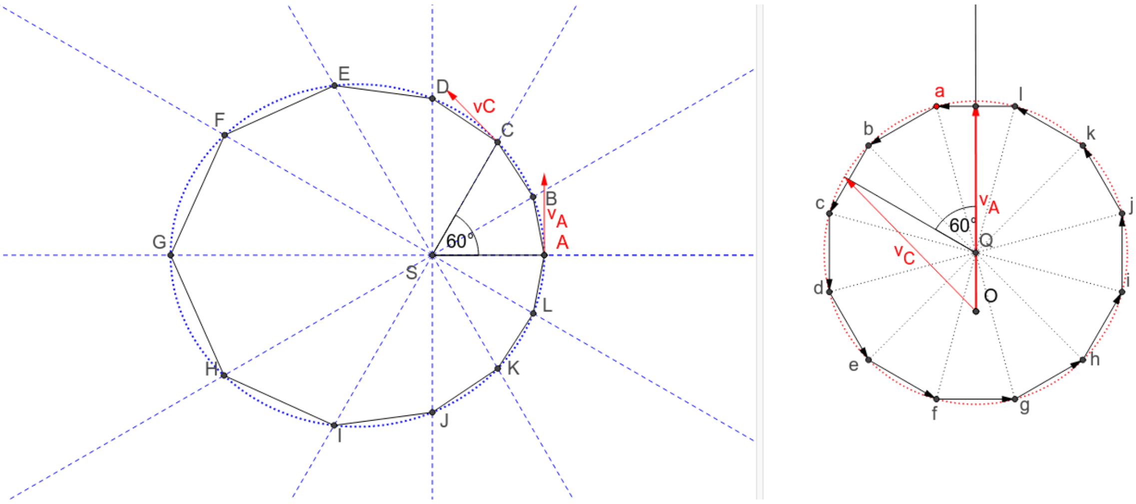

Figure 13: Geometrical correspondence of two diagrams

Rotating the velocity diagram and using the properties of ellipse that we mentioned above (section 2.4 or exercise 4) he finally proved that the shape of a planet's path is an ellipse, see Fig. 14. Of course, its size does not fit, it is only a proof of the shape of the trajectory.

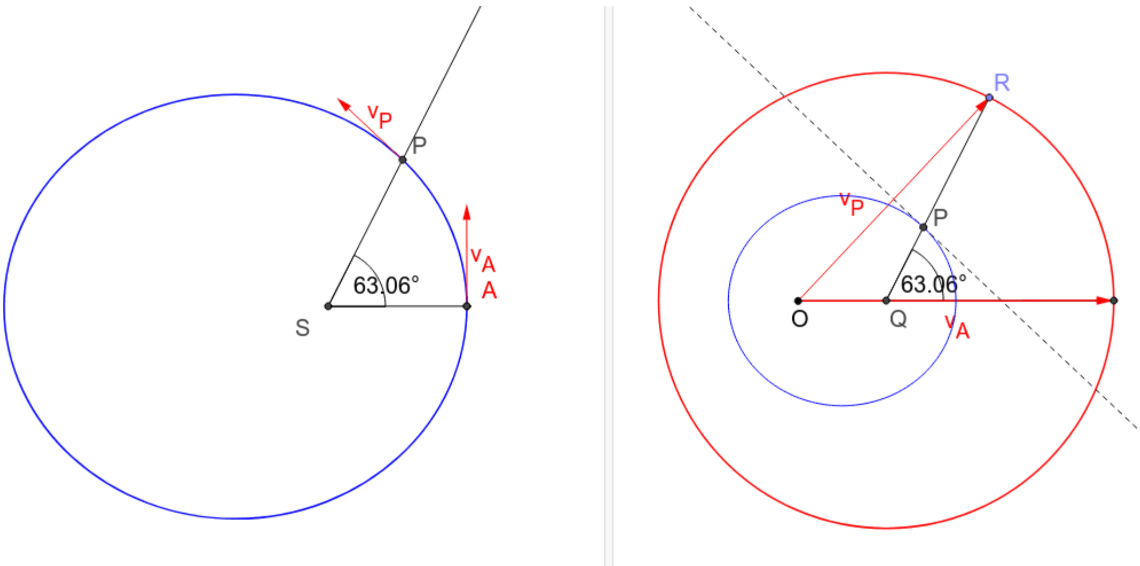

Figure 14: The shape of a planet's path is an ellipse

5 Conclusions

The aim of the paper was to show the classical secondary school geometric problems solved in the context of a physics, moreover in connection with the stories of the discovering of the physical nature of the universe and of the life of Richard Feynman, Nobel Laureate. We will be happy if the reader tries out the presented exercises in her teaching practice and provides us with feedback.

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