Learning Congruence Through Ornaments And Tiling

Šárka Voráčová
voracova@fd.cvut.cz
Department of Applied Mathematics
Czech Technical University in Prague, Faculty of Transportation
Czech Republic

Abstract: *In order to investigate movement in the plane and congruent transformations, instead of traditional logic development, we can conveniently use intuitive involvement in the geometric ideas of ornaments with the dynamic geometry software GeoGebra and 3D software Tinkercad. In this paper, we will explore selected examples of ornaments and segmented geometric patterns. We bring up some ideas about what a teacher of mathematic can find in geometric patterns and how he can fit them into his teaching curriculum. In this paper, we present a few proposals of methodologies for teaching symmetry in the plane with digital technologies, manipulative activities, and controlled exploration. We also present the results of the research of this method.*

1. Introduction

In math, finding patterns is extremely important. Patterns make tasks simpler because problems are easier to solve when they share patterns. Once we recognize a pattern, we can then use the same problem-solving solution wherever that pattern exists.

In coding – like in math - patterns are made from ideas. Mathematicians and computer programmers use patterns to express themselves and to make their work more efficient. For example, they might use loops to allow for the repetition of a sequence of code multiple times.

The highly symmetric ornaments from real-life provide interesting examples of geometric construction for use in the mathematics classroom. Symmetry is the basic concept of harmony in all cultures, found in churches, mosques, and minarets throughout the world. Understanding the mathematics behind the symmetry and construction of patterns is the key to then using these intricate patterns as motivational tools in the classroom.

Geometric methods needed to construct the tiling have the potential of introducing students to traditional techniques of straightedge and compass constructions as well as modern geometric design with computer software like GeoGebra. By comparing the traditional with the modern, each student may be challenged to use their individual creativity to design their own pattern.

2. Symmetry as a tool for improving picture perception and spatial ability

Spatial ability is one of the most widely studied domains of cognitive ability. Broadly defined, the spatial ability comprises the processes involved in perceiving, memorizing, and manipulating mental representations of visual scenes, including two-dimensional and three-dimensional objects and the relationships between them (see [8] and [10]).

Developing the spatial imagination of schoolchildren is an important task in the educational process, especially with regard to the practical application of acquired competencies in real life [8]. Some studies state that there are certain time periods that are particularly favorable for the development of spatial imagination. When these periods are missed, one loses the opportunity to develop his/her abilities to the level given by genetic predispositions (see [5], [10], and [11]). There are two periods in childhood, which are connected with the rapid and intensive evolution of spatial abilities. The first of them is the period between 5th and 6th years of life and then, the second period occurs between 10–15 years, which is the second half of compulsory school education. While the first, pre-school period is the time of spontaneous manipulations with materials and tools, in the second, school plays the important role. This is the period of structured and knowledgeable growth, the period of evolution of the ability of synthesis and abstract deduction. It's the optimal period in which to develop student's spatial abilities and to stimulate and train their spatial imagination [3].

Investigation of ornaments and tessellations can help the students understand geometry terms like basic shapes, sides, vertices, and interior angles of a polygon. Tessellations introduce pupils to slightly advanced concepts like irregular shapes, their dimensions, surface areas of irregular polygons, and complementary shapes. Tessellations for kids are a way to make math easy. Because of repetition, patterns are usually easy to identify, but making a pattern – or even describing how it is made – requires a much deeper level of thinking.

3. Using IT technology in geometrical education

The use of digital technologies in solving and expressing problems allows us observe, explore, and formulate conjectures about possible suppositions, and verify those. While it is difficult to say which skills or concepts develop what aspect of computational thinking, it is not hard to see that these can lead to developing some of the useful habits. For instance, abstracting information and describing them as variables in a computer program helps develop one's mind to think of factors in the real world as mathematical variables.

Repetition in geometrical patterns is an opportunity for practicing for-loop control flow statement as well as block structure of the code. We'll show how the geometric ideas of ornaments could be explored in the classroom with the dynamic geometry software *GeoGebra* and 3D software *Tinkercad*. Also, with the aid of modern tools students may gain further appreciation to what was accomplished by artists and geometers of bygone centuries who possessed the simplest of tools.

Tinkercad codeblocks

Tinkercad[1] is free software that can handle the majority of situations that beginners would see in both modeling and 3D printing environments. An illustrative user interface with a limited toolbox makes *Tinkercad* more intuitive than *SketchUp*[2] and easier to learn which is an excellent asset in an educational setting. Challenges in individual or group development to design a structure that can be printed in 3D play a noticeable role in the motivation of the students. Research [2] founded, that there was a positive relationship between pupil's perception of *Tinkercad* and pupil's computational thinking skills.

Tinkercad Codeblocks coding graphical language is similar to *Scratch*[3]. Solid shapes can be stretched, reshaped, copied and pasted, grouped, and varying shapes can be connected to design new shape blocks. Blocks for basic generator shapes of an ornament give the readable structure of the code and push pupils to procedural thinking[4]. Moreover, we can have library objects as an

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[1] https://www.tinkercad.com
[2] https://www.sketchup.com/
[3] https://scratch.mit.edu/
[4] Tenedorio-Carty, K. (2019). Code-Generate patterns in Tinkercad, Hour of Code 2019, https://blog.tinkercad.com/hour-of-code-2019, available on Tinkercad for Education,

output of programs. Thus, if a student or teacher created a complex geometric object, anyone can reuse it in many other designs.

This is what we often do while designing a complex pattern – draw a part of a construction, copy it on a separate paper and add to it some elements, then copy it again and add something else. If one of the steps is wrong, we can always go back to the previous drawing and start again.

GeoGebra

Dynamic software GeoGebra[5] is a freeware software designed for primary education. Thanks to a long history of sharing materials on common space you are free to reuse or edit any project from the other teachers, students, or GeoGebra enthusiasts. Together with [3] and [7], we can conclude that GeoGebra is basic tool for developing digital literacy and computation thinking in mathematical middle school education.

Benefits of using dynamic geometry tools in the classroom:

- Saving a lot of time with precise drawing.
- Deductions and creation of perceptions. Active cognition develops memory, understanding the context, constructive thinking, and motivation of the pupils.
- Explaining and examining all possible relative positions of given elements.

The geometrical patterns could be designed by using GeoGebra command *sequence*[6] with arbitrary geometrical objects, such as polygon, spline, or conic. Command *sequence* returns list, that could be the new object for designing frieze, rosette, or wallpaper pattern. The code in *Algebra View* or in *GGBScript* is similar to the block structure in *Tinkercad* explained in the previous paragraph. As an example, we use the frieze on Figure 1. This symmetrical geometrical pattern is created by using the triangle *ABC* (name *Triangle1*) and vector *u* for translation.

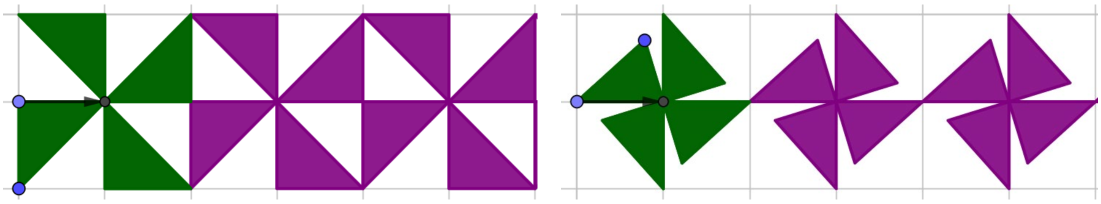

Figure 1 Two variants of frieze created from triangle by using the code
Ornament = sequence(Rotate,(Triangle1,90°,B),i,1,4)
*Frieze1 = sequence(Translate(Ornament, 2*j*u), j, 1,3)*

Moving with vertices of base triangle *ABC* helps to understand the geometry of the pattern. We could easily inspect the special position of the triangle (generator) with more global symmetries of the frieze.

[5] https://www.geogebra.org/
[6] https://wiki.geogebra.org/en/Sequence_Command

4. Geometrical patterns and symmetry groups

A mathematician while designing geometric patterns may use precise geometric constructions executed with a compass and ruler. He may also investigate shapes included in geometric patterns, proportions, and relations between particular elements of a pattern, symmetries global and local. In this case, we deal with classical constructive geometry and the tessellation theory [9].

Analysis of a wallpaper pattern – or even describing the wallpaper symmetrical groups – requires knowledge of abstract algebra, experience, and perfect visual perception. The best introduction is to start with a simple ornament and its symmetry. We can take a star with 8 vertices, a square, a long kite, a pentagon, or a photo of symmetrical object. For instance, rosettes in Figure 2 have rotation symmetry as well as mirror symmetry. Evidence of these invariants is necessary for further work.

Rosettes

Pattern with a central point, without any translations or glide reflections but, may have rotations about the central point or reflections in lines through the central point. Thus, rosettes fall into two major categories. These ornaments that contain only rotations form a cyclic group C_n and those that contain rotations and reflections are from dihedral group D_n. Index n denotes the number of symmetries.

Rosettes are typical in gothic and Islamic architecture. Figure 2 shows hexagram in Spanish Synagogue in Prague with D_6 symmetry and D_{12} rosette from the neo-gothic Catholic church in Prague. Is worth mentioning here that in our times, the gothic tracery is no longer developed while Islamic patterns are still used, and new patterns are still created [9].

Figure 2 left: hexagonal rosette in Spanish Synagogue, Prague (D_6 symmetry); right: dodecagonal rosette on Church of St. Ludmila, Prague's Vinohrady, 12 identical petals (D_{12} symmetry)

Frieze

The pattern on a belt that is repetitive in one direction has symmetries from Frieze group. Such patterns occur frequently in architecture and decorative art. The study of frieze patterns reveals that they can be classified into seven types according to their symmetries. The notation F_1^j give us reference to type of mirroring line (horizontal, vertical, glide reflection) and presence of reflection in point. Another notation used the IUCr symbol for more complex wallpaper groups. Frieze on Figure 1 has only 2-fold reflection ad translation, which means F_2 symmetry (UICr symbol *p*2).

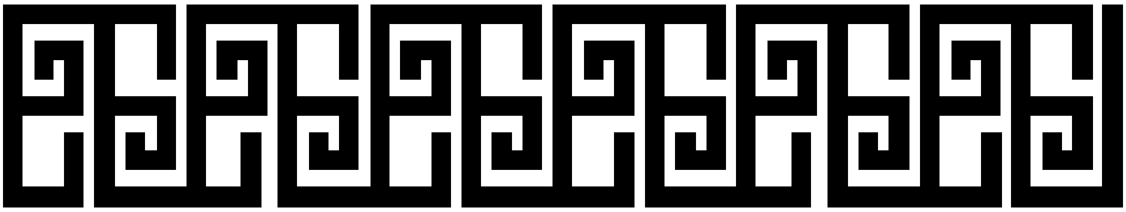

Figure 3 Frieze group F_1^3- absence of reflection in point, glide reflection (*p11g*)

Wallpaper pattern

A wallpaper pattern contains translational symmetries in two independent directions. These patterns are classified by the smallest angle of rotation that appears in the pattern. By a property known as the crystallographic restriction, the only angles that may appear in rotations for these patterns are 60°, 90°, 120°, 180°, or the pattern may have no rotational symmetry. If each of these patterns is considered to continue repeating in all directions then their symmetry groups are infinite. It can be shown that there are exactly seventeen symmetry groups for these types of patterns and they are referred to as the crystallographic or wallpaper groups [4].

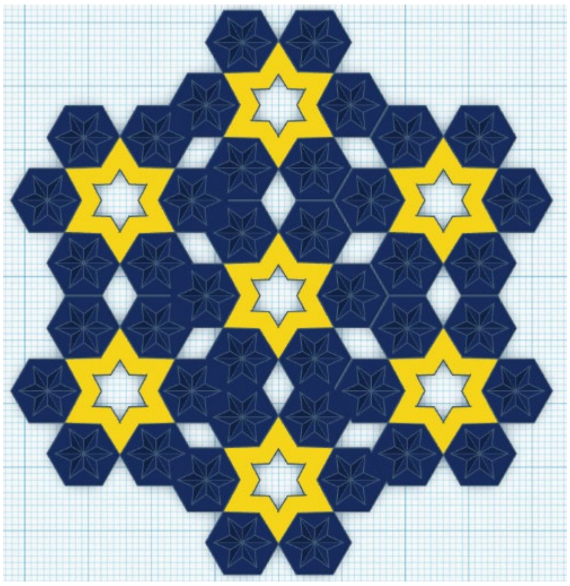

Figure 4 Fragment of the hexagonal tiling, pattern with symmetry group *p6m*, designed in *Tinkercad*

Hermann-Mauguin (also called IUCr) notation of the wallpaper group is given by appearance of the symmetry. It begins with either ***p*** or ***c***, for a *primitive cell* or a *face-centered cell*. Digit *n* indicates the highest order of rotational symmetry: 1-fold (none), 2-fold (reflection in point) ... The next two symbols indicate symmetries relative to the main translation axis of the pattern. The symbols are either *m*, *g*, or 1, for mirror, glide reflection, or none. Many groups include other symmetries implied by the given ones[7].

One may notice that pattern in Figure 4 is the periodical repeating of one hexagonal ornament. Thus, to construct this pattern we need to construct the content of one tile and then use multiple copies of it. This way we can easily plan how a given space can be covered with this pattern. These polygons form a tessellation of the plane. Each of these polygons has the same symmetry lines as the pattern on it. This tessellation forms the hidden geometry of the pattern. As shown in [9] the same tessellation can be used to create a few different patterns and often the same geometric pattern can be created using two different tessellations.

A geometric pattern may have global as well as local symmetries. We recommend starting with local symmetries of a tile and then investigate special points on the pattern where multiple symmetry lines cross or points that can be treated as invariant points for rotations.

5. Education of geometrical pattern

Some basics in geometry can be best understood with the help of tessellations. For kids, it's simpler to grasp things when ideas are supported by pictures and models as examples. For example, it's after they try to tile a surface with random shapes that they shall understand the considerations for irregular shapes to achieve a uniform tessellation [12].

Investigation of tessellation and geometric pattern has an educational benefit in a geometric competence in every level of mathematical education:

- Geometric constructions – dividing angles and segments into equal parts, constructing perpendicular and parallel lines, constructing regular polygons and shields.
- Polygons and tessellations – tessellations with regular or symmetric polygons and shields.
- Symmetries – global and local symmetries of patterns and tessellations.
- Transformations – translations, rotations, mirror reflections, dilations, and iterations.
- Connection with technology – all examples discussed here can be developed using traditional drawing techniques with compasses and ruler, or they can be created with geometry software – *Tinkercad*, *Geometer's Sketchpad* or *GeoGebra*.
- Opportunities for students' projects – individual or collective.

The tasks are posed to the class along with a potential sequence of problems through which the class may be directed in solving the problem:

- How should construction of the tiling begin? Is there a particular tile that may produce the overall tiling? Could you describe a strategy for constructing a pattern similar to it?
- What are the symmetries of the tile?
- What is the minimal tile that is needed to be able to reproduce the entire pattern?
- How does the introduction of color affect the symmetry of the pattern?
- Does your tiling contain the same symmetries as the original tiling? Can you find a different tiling that contains the same symmetries as your tiling?

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[7] We recommend visual tools for symmetrical wallpaper patterns: https://eschersket.ch/ or https://math.hws.edu/eck/js/symmetry/wallpaper.html (cit. 2021)

6. Experiment in the classroom

The ideas for teaching symmetry were tested in two classrooms at the elementary school. Pupils of the 4th and the 5th grade were involved (the 5th grade is considered as a transitional grade between primary and secondary education). The activities took place during several classes and were focused mainly on programming in *Tinkercad Codeblocks* and *GGBScript*. Education was focused on constructivist principles. The tasks of activities were achieved in the dimension of cognitive process, understanding, application, analysis, and evaluation. During research, we used the methodology of qualitative research.

Activity 1: Designing with Algorithm in Tinkercad

The simple *Codeblocks* scripts will enable your students to create a rosette that includes text or scrawl that they can easily modify to personalize their design[8]. Encourage students to modify and remix the *Codeblocks*.

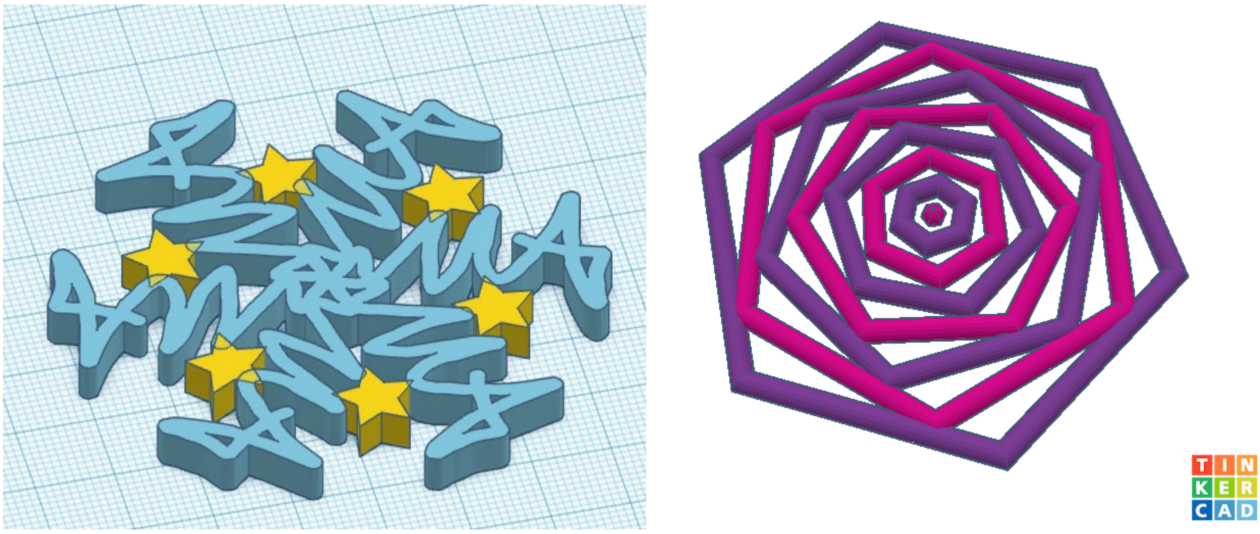

Figure 5 Rosettes with C_6 symmetry in *Tinkercad*, inspired by [*Tinkercad for Education*]

In this lesson pupils will:

- Discuss and debate an issue of human and computer interaction.
- Recognize, define, and solve computational problems in a rosette design in order to modify it.
- Explore the use of control structures to set parameters, allowing students to compute more design iterations quickly and efficiently.
- Communicate and collaborate clearly and express themselves using simple code.
- Make interesting, code-generated rosette or frieze designs that they can share through 3D printing.

[8] https://www.instructables.com/Designing-With-Algorithms-in-Tinkercad/

Activity 2: Drawing symmetric shapes in GeoGebra

GeoGebra book "Symetrie ornamentu" (Symmetry of Ornament)[9] demonstrates finite symmetric groups C_4, C_6, D_4 and D_6. During activity, students were supposed to draw in shapes in *GeoGebra* on a cartesian and isometric grid and experiment with position and shape of the basic objects. This activity significantly motivated the students as they had immediate feedback. Even weak students solved the problems with enthusiasm and search another symmetrical picture on the web independently on the teacher.

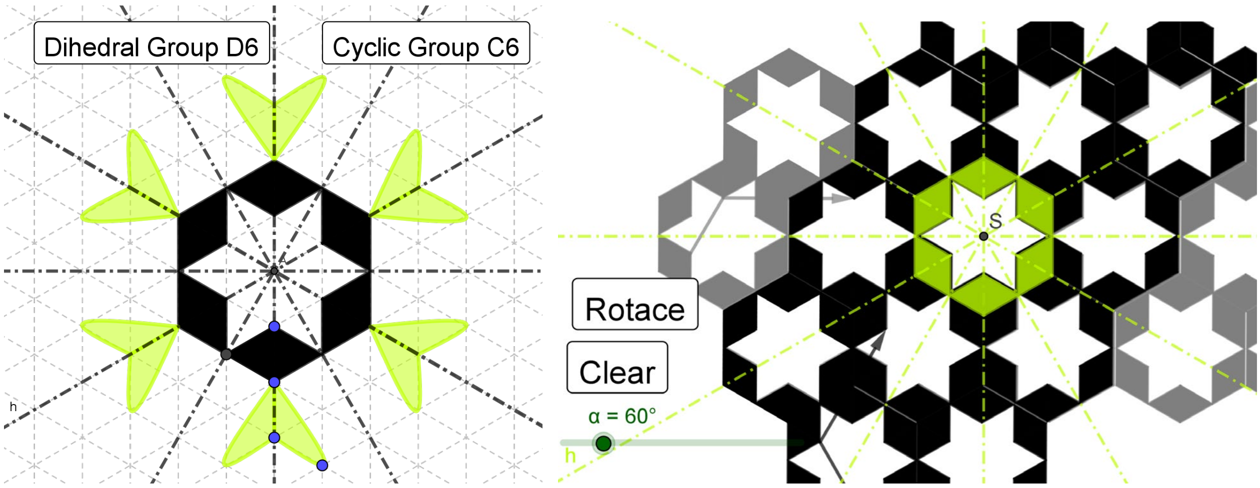

Figure 6 Ornament with 6-fold rotation and reflections in six lines (on the left). Regular hexagon (on the right), its symmetry and tilling with symmetry *p6m*.

Activity 3: Construct your own pattern with a restricted family of transformation

In this activity, we observed a significant difference between classes, especially in pace and skill. None of the students were able to determine how to construct the given wallpaper pattern without the translation.

Survey research

Research data were collected using information form, formative test, and pupil perception questionnaire about *Tinkercad* and *GeoGebra* software. The results are not truly comparable, because there were different forms of learning (present/online) and also different instructional models (transmission / constructivist).

Not surprisingly, the survey shows that rosettes are more readable for all students: 75% find the symmetry in line, 43% of the pupils find the rotation symmetry. The wallpaper pattern *p6m* in Figure 6 were not fully recognized in the class without software.

In addition, when looking at students' perceptions of *Tinkercad*, it was determined that they were highly motivated for interest and appreciation and found *Tinkercad* to be generally useful and easy to use. Pupils appreciated the export to *.stl* file for 3D print, because "the end result is a physical object".

[9] https://www.geogebra.org/m/pzb5mfku, public GeoGebra book „Symetrie ornamentu" (in Czech)

7. Conclusion

The role of CAD and dynamic geometric software in the teaching of geometry is discussed for a long time. Could they contribute to the development of reasoning skills and spatial ability? Will the dissemination and accessibility of these tools mean that students will be no longer consider elementary geometry as the most relevant for their future career? It's evident, that tablets and computers can't substitute the manipulatives activity and compass and ruler constructions. On the other hand, there are a lot of problems, which could be educated more effectively with software by using different solving strategies. Some of the foreseeing changes could be similar to those that the emergence and dissemination of pocket calculators already implied for mathematics education, not only about how certain problems have to be solved, but, more generally, concerning what type of techniques and problems should be considered as the true objective of mathematics education.

By introducing history, culture, and art into discussions in the classroom, students have an opportunity to see mathematics as a multidimensional embroidery woven across time and space [9]. It is indeed a stimulating exercise for students to attempt to understand the beautiful symmetry of geometrical pattern and learn the geometry required to reconstruct them.

8. References

[1] Dana-Picard, T., Hershkovitz, S. (2019). "Geometrical features of a Jewish monument: study with a DGS", *Journal of Mathematics and the Arts*, 13:1-2, 60-71, 10.1080/17513472.2018.1552067

[2] EryilMaz, S., Deniz, G. (2021). "Effect of Tinkercad on Students' Computational Thinking Skills and Perceptions: A Case of Ankara Province", *Tojet*, Vol 20, pp. 25-38

[3] Gergelitsová, Š. (2007). "Computer Aided Development of Spatial Abilities" In WDS'07 Proceedings of Contributed Papers, Part I, 246–250. MATFYZPRESS

[4] Grünbaum, B., Shephard, C. (1987): *Tilings and Patterns*. New York: Freeman.

[5] Hejný, M. et al. (1989). *Teória vyučovania matematiky* 2. SPN, Bratislava.

[6] Owen, J. (1856). *The Grammar of Ornament*, Franklin Classics, edition 2018.

[7] Korenova, L. (2017). Symmetry in elementary education with the use of digital technologies and manipulations. *APLIMAT: 16th Conference on Applied Mathematics*. Bratislava

[8] Kuřina, F. (1991). "Představivost a vyučování matematice" In *Pokroky matematiky, fyziky a astronomie*, Vol. 36 (1991), No. 2, 117—122, http://dml.cz/dmlcz/139671

[9] Majewski, M. (2020). "Understanding Geometric Pattern and its Geometry (part 1)". *The Electronic Journal of Mathematics and Technology*. 14. 87–106.

[10] Molnár, J. (2009). *Rozvíjení prostorové představivosti (nejen) ve stereometrii*. Univerzita Palackého v Olomouci.

[11] Nagyová Lehocká, Z., Csáky, A., Žitný, R. (2021). Best Practices for Improving Spatial Imagination in Mathematics, In: Ad Alta: *Journal of Interdisciplinary Research*. Jun2021, p162-166. 5p.

[12] Tennant, R. (2004). Islamic Tilings of the Alhambra Palace: Teaching the Beauty of Mathematics In *Teachers, Learners and Curriculum Journal*, vol. 2, February. 2. 21-25.