Integer Partitions using Generating functions and their Applications in Graph Theory

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Abstract

Integer partitions is one of the main subjects in Mathematics and which is taught at various levels in colleges and Universities. The subject was introduced by Leibniz and Euler. The chief aim of this paper is to give the ideas about Euler generating function which is the main tool to learn various properties of partitions of integers along with bijective function. Bijection is used to find out various identities and it always helps to prove results. It is argued that Mathematica software is important to learn and teach the subject. We give some applications of integer partitions in graphs theory.

1 Basic concepts of Integer Partitions

A partition of a positive integer n is defined as a way of writing n as the sum of positive integers. We denote the number of partitions of n by P(n). An explicit formula for P(n) valid for all positive integers n was discovered by Rademacher in 1937 [1], but since it is a complicated infinite series and is not needed for the purposes of this paper. The theory of integer partitions is a rich source of identities, bijections, and interrelations at the confluence of number theory, combinatorics, algebra, analysis, and the physical sciences. Let $p = (p_1 + p_2 + p_3 + ...p_k)$ where p is any partition denote a generic partition, with integer parts $p_1 \geq p_2 \geq .... \geq p_k \geq 1$. It is very interesting to overview the concepts of Integer Partitions Function P(n). The integer Partition function $P(n)$ is essentially the quantity of all integer partitions of whole number n. The total number of integer partitions is finite and n is always itself a partition. A $\mathbf{PartitionFunction}$ $P(n)$ of integer n, $n \geq 1$, is a non increasing sequence $\{n_1, n_2, n_3, n_4, ...n_k\}$ such that $P(n) = \sum_{j=1}^{k} n_j = n$, where n_j is the part of the partition $P(n)$ and $P(-n) = 0$, $P(0) = 1$ by default and $P(1) = 1$.
There are two common diagrammatic methods to represent partitions: as Ferrer diagrams [2], named after Norman Macleod Ferrers, and as Young diagrams [3], named after the British mathematician Alfred Young. Both have several possible conventions. In the Figure 1, partitions of the integer 5 are given with the corresponding Ferrer diagrams.

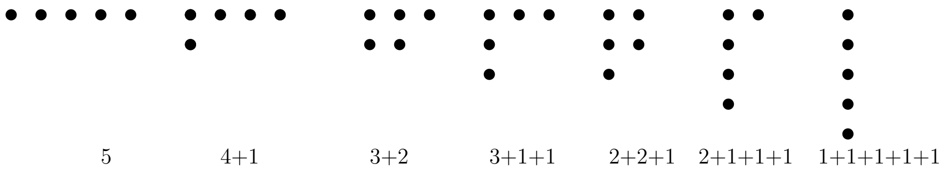


Figure 1: Ferrer diagram for integer 5

We can use *Mathematica* to find different sets about Integer Partitions.
`In[1]:IntegerPartitions[n]` gives a list of all possible ways to partition the integer n into smaller integers.
`In[1]:  IntegerPartitions[5]`
`Out[1]:{{5},{4,1},{3,2},{3,1,1},{2,2,1},{2,1,1,1}, {1,1,1,1,1}}`
`In[2]:IntegerPartitions[n,k]` gives partitions into at most k integers.
`In[2]:IntegerPartitions[8, 3]`
`Out[2]:{8},{7,1},{6,2},{6,1,1},{5,3},{5,2,1},{4,4},{4,3,1},{4,2,2},{3,3,2}}`
`In[3]:  IntegerPartitions[n,{k}]`gives partitions into exactly k integers.
`In[3]:IntegerPartitions[8,{3}]`
`Out[3]:  {{6,1,1},{5,2,1},{4,3,1},{4,2,2},{3,3,2}}`

1.1 Generating Function

There are many ways to formulate different properties in number theory, enumerative combinatorics and in discrete mathematics but one of the ways is generating function which helps to give a precise formula to various terms. Generating functions were first introduced in 1730 by Abraham de Moivre to solve the problem of common linear repetition. In order to generalize a formal power sequence to several asymmetric numbers, information about an infinite multidimensional array of numbers can be encoded. Generating functions work better when a discrete sequence is known [7]. Any problem which has a sequence in the result can be converted into a generating function. For this task, let us say an infinite sequence $\{e_0, e_1, e_2, e_3, ..., e_r, ..\}$ can be expressed as a function $f(x) = \sum_{i=0}^{\infty} e_i x^i$ or $f(x) = e_0 + e_1x + e_2x^2 + ... + e_rx^r + ...$, where e_r is number of ways to get $r - objects$. Any finite sequence can be used in the same manners as $\{e_0, e_1, e_2, e_3, ..., e_r, 0, 0, 0\}$ is a finite sequence but can be expressed as infinite sequence. Every finite or infinite sequence can be expressed as a generating function which gives a polynomial. Some sequences with their generating functions are given in Table 1. Binomial theorem plays a huge role in the solving of any generating function which

Table 1: Some sequences with respective generating functions

Sequence	Series
$\{1, 1, 1, 1, 1, ..., 1, ...\}$	$1 + x + x^2 + x^3 + ...$
$\{0, 1, 2, 3, ..., n\}$	$1 + x + 2x^2 + 3x^3 + ... + nx^n$
$\{0, 2, 4, 6, ...\}$	$1 + 2x + 4x^2 + 6x^3 + ...$
$\{1, 3, 5, 7, ...\}$	$x + 3x^2 + 5x^3 + 7x^4 ...$
$\{\binom{\beta}{0}, \binom{\beta}{1}, \binom{\beta}{2}, \binom{\beta}{3}, ..., \binom{\beta}{n}\}$	$\binom{\beta}{0} + \binom{\beta}{1}x + \binom{\beta}{2}x^2 + \binom{\beta}{3}x^3 + ... + \binom{\beta}{n}x^n$

is stated as

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

which is the expansion of the function $(x+y)^n$ and $\binom{n}{i}$'s are the coefficients of x^i's in it.
The generalized form of this theorem is stated as

$$(1 \pm x)^\alpha = 1 \pm \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 \pm \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + ... + (-1)^r \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-r+1)}{r!} x^r ...$$

where and $y = 1$. Using this theorem we can have many identities, some of them are given in Table 2. The integer partitions function $P(n)$ can be expressed using generating function.In this paper, there is a frequent use of generating function for understanding some complicated results.

Table 2: Some generating functions and there expansions

Generating Function	Expansion
$(1-x)^{-1}$	$1 + x + x^2 + x^3 + ...$
$(1-x)^{-2}$	$1 + 2x + 3x^2 + 4x^3 + ...$
$(1-x)^{-n}$	$1 + \binom{1+n-1}{1} x + \binom{2+n-1}{2} x^2 + \binom{3+n-1}{3} x^3 + ... + \binom{r+n-1}{r} x^r + ...$
$(1-x^m)^n$	$\binom{n}{0} - \binom{n}{1} x^m + \binom{n}{2} x^{2m} - \binom{n}{3} x^{3m} + ... + (-1)^r \binom{n}{r} x^{rm} + ...$

2 Generating Functions and $P(n)$

Generating functions plays very important role in calculating of different identities of integer partitions function $P(n)$. There is a very efficient and popular generating function to find sequence for integer partition function that is given in [10]. As we have discussed above the partitions of any integer can be partitioned into to the positive integers less or equal to that integer. Each partition has different parts of different sizes i.e $1's$, $2's$, and $3's$ so on. For instant, $5 = 3 + 2$ is partition of integer 5 with two parts of sizes 3 and 2. For the concept of generating function, we will use sizes of k parts, that how many $1's$, $2's$, $3's$ and so on are appeared in k parts of the partition n. The generating function for the $P(n)$ will be generated using the polynomial of all the sizes available in the integer partitions of n, the polynomials for different sizes of parts are given in Table 3. Using the polynomials from Table 3, we can combine all the factors, as all the factors are distinct so we can multiply them to form a function. Such that

Table 3: Polynomial corresponding to sizes of parts

Sizes	Polynomial
Size 1	$x^0 + x^1 + x^2 + x^3 + ...$
Size 2	$(x^2)^0 + (x^2)^1 + (x^2)^2 + (x^2)^3 + ...$
Size 3	$(x^3)^0 + (x^3)^1 + (x^3)^2 + (x^3)^3 + ...$
...	...
Size k	$(x^k)^0 + (x^k)^1 + (x^k)^2 + (x^k)^3 + ...$

$$f(x) = (x^0+x^1+x^2+x^3+...)(x^0+x^2+x^4+x^6+...)(x^0+x^3+x^6+x^9+...)...(x^0+x^k+x^{2k}+x^{3k}+...)...$$

Using instantly Table2, we will get

$$f(x) = \frac{1}{1-x} . \frac{1}{1-x^2} . \frac{1}{1-x^3} \cdots \frac{1}{1-x^k} \cdots$$

As we are working on integer partitions function $P(n)$, let us assume $f(x) = \sum_{i=0}^{\infty} P(i)x^i$, where $P(i)$ is total number of partitions of integer i. So that the generating function for integer partition function is defined as

$$\sum_{i=0}^{\infty} P(i)x^i = \prod_{i=1}^{\infty} \frac{1}{1-x^i} \tag{1}$$

In above the coefficients of x^i indicate the value of P(i).
We can use $Mathematica$ to see the expansion of this generating function as it will be hard to expand manually.

In[1]:Series[Product[1/(1-x^k),{k,1,10}],{x,0,10}]
Out[1]:1+x+2x^2+3x^3+5x^4+7x^5+11x^6+15x^7+22x^8+30x^9+42x^{10}+O[x]^{11}

In which the coefficients of x^i indicates the number of partitions of integer i i.e $P(3) = 3, P(6) = 11, P(10) = 42$ and $P(19) = 490$.

3 Some Identities of $P(n)$

There are so many interesting identities available in literature[6]. Euler was the one of the amazing mathematicians who did different experiments on polynomials. His one of the interesting theorems is on pentagonal numbers, known as ”Euler's Pentagonal number theorem”. Pentagonal numbers are the total polygon numbers which are generating recursively from a pentagon, see Figure 2. The

Figure 2: Recursive Pentagonal shapes starting from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

sequence we get from recursive pentagonal shapes is $1, 5, 12, 22, 35, 51, 70....$ This sequence is known as ”Pentagonal numbers $\S(r)$”. This sequence can be calculated using the formula

$$\S(r) = \frac{r(3r-1)}{2} \tag{2}$$

The generating function for this sequence is

$$\frac{x(2x+1)}{(1-x)^3} = x + 5x^2 + 12x^3 + 22x^4 + ... \tag{3}$$

We can use software Mathematica [4] to find this sequence, the code is given as

```
In[2]:PolygonalNumber[5, Range[0, 15]]
```

Out[2]:{0,1,5,12,22,35,51,70,92,117,145,176,210,247,287}
There is another beautiful sequence named "triangular numbers" [11] which consists of the number of dots used in the formation of a triangle. the structure is given in the Figure 3. The sequence obtained from the shapes is $1, 3, 6, 10, 15, 21, 28, ...$, which can be calculated using formula $t(r) = \frac{r(r+1)}{2}$. The

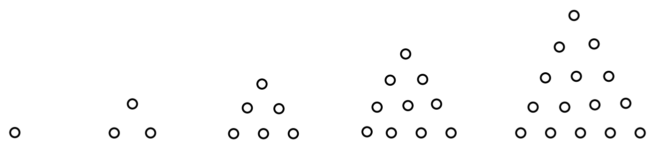


Figure 3: Recursive triangular shapes starting from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

"Euler's Pentagonal number theorem" is stated as

Theorem 3.1 *[6] Let* $E = P(n|even\ number\ of\ distinct\ parts)$ *and*

$$O = P(n|odd\ number\ of\ distinct\ parts)$$

then

$$|E| - |O| = \begin{cases} (-1)^r & n = \frac{r(3r\pm1)}{2} \\ 0 & otherwise \end{cases} \tag{4}$$

This theorem can be explained using triangular number, pentagonal numbers and ferrer diagrams. If we rearrange the pentagonal numbers given in Figure 2, we can write them in dotes only, as shown in Figure 4. The pattern in the rearranged pentagonal shapes is related to the triangular shapes, so the

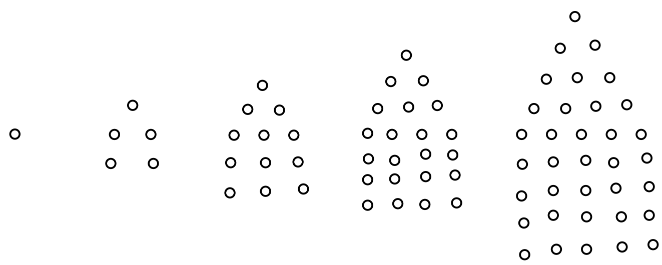


Figure 4: Rearranging Pentagonal Shapes from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively

formula can be modified as

$$\S(r) = \frac{r(r+1)}{2} + r(r-1) = \frac{r(3r-1)}{2} \tag{5}$$

Now again arrange the pentagonal numbers from Figure 4 in form of ferrer diagrams by converting them into straight rows, shown in Figure 5. the partitions corresponding to the ferrer diagrams are all distinct. Now we need to establish bijection between the even number of distinct parts and odd number of distinct parts for the formation of Euler's Pentagonal theorem. Let us take different ferrer diagrams of distinct partitions, shown in Figure 6, and try to construct another corresponding ferrer diagram by removing the smallest part and place all the dots of that part at the right side of the parts

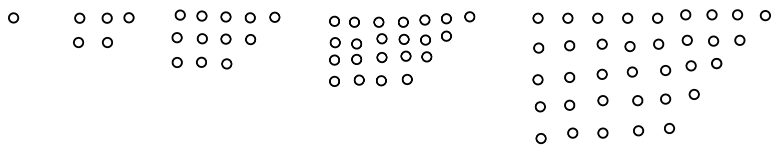


Figure 5: Rearranging Pentagonal Shapes from $r = 1$ then $r = 2, r = 3, r = 4$ and $r = 5$ respectively into Ferrer diagrams, the partitions corresponding to the ferrer diagrams are: $1, 3 + 2, 5 + 4 + 3, 7 + 6 + 5 + 4, 9 + 8 + 7 + 6 + 5$ respectively

from top to bottom. In the procedure, we see first two obtained diagrams are of again distinct parts, but the third diagram won't. Every ferrer diagram can be converted into other ferrer diagram of the same integer and there are two type of situations we yield. First will be in the form of first two examples in which parts of the obtained ferrer diagrams are less than or equal to the original diagram and the second will be in the form of third example in which the parts in the obtained diagram is greater than the original one. In the opposite direction, we will take a point from some large segments and create

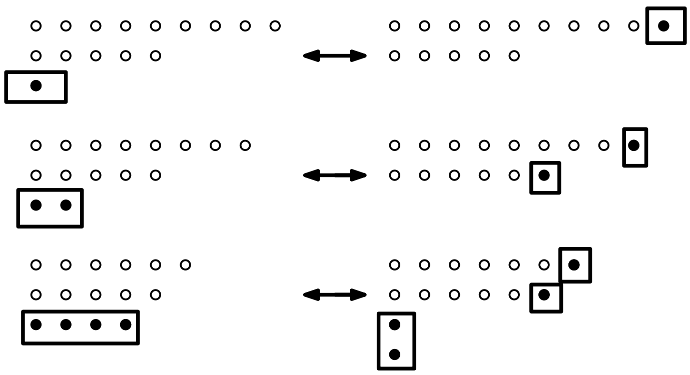


Figure 6: Three examples with transformations

a new small row. A well-defined number of points to move will be the number of rows separated by a single point, starting with the largest row. In other words, we will find the most accurate diagonal of the Ferrer graph. Figure 7. Now we need to decide which transformation is used and when? for

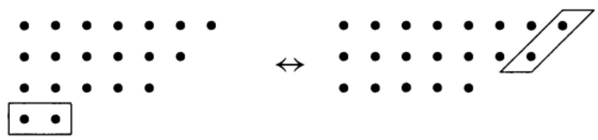


Figure 7: Inverse transformations

this, we check if the rightmost diagonal is shorter than the last row? If it is, then move it otherwise move the latter. The Ferrer graphs in the first case are the pentagons of size $r(3r - 1)/2$ dots that we considered in the beginning of this section. here we say $e(n)$ is added to make the theorem complete for this case, so we have

$$n = \frac{r(3r - 1)}{2} \tag{6}$$

However, there is a case where the above modification does not create a valid Ferrer graph, i.e., when the smallest row actually intersects the right diagonal in the lower right corner of the graph, and the row is the same length or point longer than the diagonal, Figure 8. In the second case, the Pentagon's

Figure 8: Inverse transformations

rectangular section contains an additional column, giving a total size of $r(3r+1)/2$ dots.We say $e(n)$ is added to make the theorem complete for this case, so we have

$$n = \frac{r(3r+1)}{2} \tag{7}$$

We have described a change that divides each partition n into an odd number of different parts and even number of distinct parts except these pentagonal partitions. Therefore using equations 6 and 7, we have

$$|E| - |O| = \begin{cases} (-1)^r & n = \frac{r(3r\pm1)}{2} \\ 0 & otherwise \end{cases}$$

where $e(n)$ is 0 unless $n = \frac{r(3r\pm1)}{2}$ for some integer r, in which case it shall be 1 if the number of parts is even and -1 if odd. This proves Euler's pentagonal number theorem.

4 Application of $P(n)$ in Graph theory

In the following, we give few results for the structural properties of graphs G_n , where G_n is defined in [9] using integer partitions , see Figure 9(a), graph G_3 is given for $n = 3$. Here we give some theorems.

Theorem 4.1 *For the graph G_n, the total number of vertices $N(G_n)$ is given as*

$$N(G_n) = \sum_{i=1}^{n} P(i), n \in \mathbb{Z}^+ \tag{8}$$

where $P(i)$ is the number of partitions of integer i.

Proof: We use mathematical induction to prove the theorem. It is trivial to see that $N(G_1) = P(1) = 1$ Suppose that it is true for $n = m$. We have to show that the statement is also true for $n = m + 1$.As

$$N(G_m) = \sum_{i=1}^{m} P(i), m \in \mathbb{Z}^+ \tag{9}$$

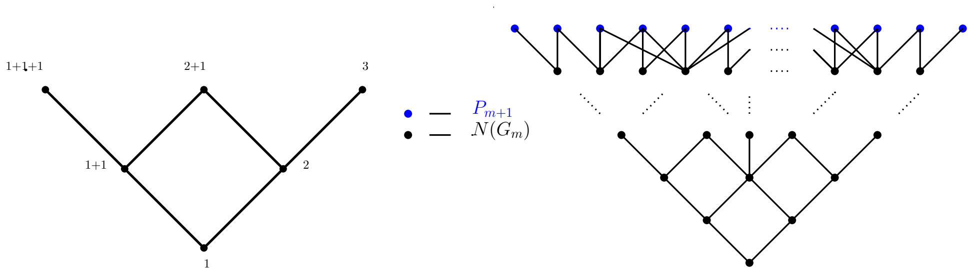
$$N(G_m) + P(m+1) = P(1) + P(2) + P(3) + ... + P(m) + P(m+1). \tag{10}$$

By using 1 in (10), we get

$$\prod_{p=1}^{1}\frac{1}{1-x^p}+\prod_{p=1}^{2}\frac{1}{1-x^p}+\ldots+\prod_{p=1}^{m}\frac{1}{1-x^p}+\prod_{p=1}^{m+1}\frac{1}{1-x^p} \tag{11}$$

$$=\sum_{i=1}^{m+1}\prod_{p=1}^{i}\frac{1}{1-x^p}=\sum_{i=1}^{m+1}P(i) \tag{12}$$

Now, according to the graph in Figure 9(b)



(a) G_3 (b) G_{m+1}

Figure 9: Integer based graphs

$$\Longrightarrow N(G_{m+1})=\sum_{i=1}^{m+1}P(i)$$

This completes the proof.
Here is another result in the form of theorem.

Theorem 4.2 *Let $m(G_n)$ is the total number of edges of the graph G_n, then*

$$m(G_n)=(n-1)P(0)+\sum_{i=1}^{n-1}(n-i)P(i) \tag{13}$$

where $n\geq 1$ and $P(0)=1$ by default.

Proof: Let $m(P(i))$ is the number of edges that can produce by $P(i)$, where $P(i)$ is partition of integer i, $1\leq i\leq n$. As the number of 1's in all partition for the integer n is exactly same as $m(P(n-1))$. Let $\psi(n)$ denotes the number of 1's in all partition of n; $n\geq 2$, then

$$\psi(n)=\sum_{z=0}^{n-1}P(z)=m(P(n-1))$$

Here $m(P(n-1))$ denotes the number of edges that are added in G_{n-1} graph to produce G_n.

$$\Longrightarrow \sum_{k=2}^{n}m(P(k-1))=\sum_{k=2}^{n}\sum_{z=0}^{k-1}P(z)$$

$$\implies m(G_n) = \sum_{k=2}^{n} m(P(k-1)) = \sum_{k=2}^{n}\sum_{z=0}^{k-1} P(z)$$

So to prove given result we have to show that

$$\sum_{k=2}^{n}\sum_{z=0}^{k-1} P(z) = (n-1)P(0) + \sum_{i=1}^{n-1}(n-i)P(i)$$

We use mathematical induction to prove this result. for n = 2,

$$\sum_{k=2}^{2}\sum_{z=0}^{k-1} P(z) = (2-1)P(0) + \sum_{i=1}^{2-1}(2-i)P(i)$$

$$\sum_{z=0}^{1} P(z) = P(0) + (2-1)P(1)$$

$$P(0) + P(1) = P(0) + P(1)$$

Now for induction step, let us assume that the result is true for $n = r$.

$$\implies \sum_{k=2}^{r}\sum_{z=0}^{k-1} P(z) = (r-1)P(0) + \sum_{i=1}^{r-1}(r-i)P(i)$$

We have to prove that it is also true for $n = r+1$.

$$\sum_{k=2}^{r+1}\sum_{z=0}^{k-1} P(z) = (r+1-1)P(0) + \sum_{i=1}^{r+1-1}(r+1-i)P(i)$$

$$\sum_{k=2}^{r}\sum_{z=0}^{k-1} P(z) + \sum_{z=0}^{r+1-1} P(z) = (r-1)P(0) + P(0) + \sum_{i=1}^{r}(r-i)P(i) + \sum_{i=1}^{r} P(i)$$

$$\implies \sum_{k=2}^{r}\sum_{z=0}^{k-1} P(z) = (r-1)P(0) + \sum_{i=1}^{r-1}(r-i)P(i)$$

. This implies that

$$m(G_n) = (n-1)P(0) + \sum_{i=1}^{n-1}(n-i)P(i)$$

We give the relationship between the number of vertices and the number of edges.

Theorem 4.3 *Let $N(G_n)$ and $m(G_n)$ be the number of vertices and edges respectively then*

$$m(G_n) = m(G_{n-1}) + N(G_{n-1}) + P(0) \tag{14}$$

Proof: Using (8) & (13), $m(G_{n-1}) + N(G_{n-1}) + P(0) = (n-2)P(0) + \sum_{i=1}^{n-2}(n-1-i)P(i) + \sum_{i=1}^{n-1} P(i) + P(0)$

$$\implies (n-2+1)P(0) + \sum_{i=1}^{n-2}(n-i)P(i) - \sum_{i=1}^{n-2} P(i) + \sum_{i=1}^{n-1} P(i)$$

$$\Longrightarrow (n-1)P(0) + \sum_{i=1}^{n-1}(n-i)P(i) - (n-(n-1))P(n-1) - \sum_{i=1}^{n-1} P(i) + P(n-1) + \sum_{i=1}^{n-1} P(i)$$

$$\Longrightarrow (n-1)P(0) + \sum_{i=1}^{n-1}(n-i)P(i) - P(n-1) + P(n-1) \Longrightarrow m(G_n)$$

This completes the proof.

5 Conclusions

We have briefly reviewed the basic concept of integer partitions. The idea of generating function is the basic tool to generate partitions of any finite integer. Another important topic which is covered in the paper is the use of bijective techniques which helps to prove results in getting new results and particularly identities for partitions of integers. At the end of this paper, we give a few new results for the applications of the partitions and particularly P(n) which is used to find out the structural properties of a family of graphs.

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