On a Chasles construction of Cartesian ovals

Jiří Blažek
jirablazek@seznam.cz
Department of Mathematics
University of South Bohemia, Ceske Budejovice
South Bohemian Region
Czech Republic

Abstract

This article deals with Cartesian ovals and one of the many methods of their construction - the so-called Chasles construction. The paper first shows an alternative justification of why points constructed by the Chasles construction satisfy foci definition of Cartesian oval, and - as a part of this justification - shows why the oval has three foci and how to construct the third focus by Euclidean means (assuming the oval is defined by two foci). Finally, the special case in which one of the circles in the construction is replaced by a straight line is discussed. It is shown synthetically that in such a case the construction renders a conic, and moreover, it can be shown that the focal definition of a conic and the definition using a directrix line are equivalent (but proof is not included in the article).

1 Introduction

The foci definition of the Cartesian oval has the following form:

$$c_1|PF_1| + c_2|PF_2| = k \tag{1}$$

where F_1 and F_2 are given points (foci) and c_1, c_2 and k are given constants. All points P, satisfying the equation, belong to the oval.
There are many ways to construct the points of an oval geometrically. The most elementary one is to choose the distance PF_2 arbitrarily and calculate PF_1. If both distances are non-negative and satisfy the triangle inequality, the point P is constructed as the intersection of two circles. However, there are subtler and faster constructions, one of which is that of Chasles [1] (Michel Chasles, French mathematician, 1793-1880). His procedure follows (Figure 1):

Chasles construction

Let two circles $k_1(F_1, R_1)$ and $k_2(F_2, R_2)$ are given. Let us choose a fixed point X on the line F_1F_2. Let an arbitrary line through the point X intersects k_1 at P_1 and k_2 at P_2. Denote P the intersection of the lines P_1F_1 and P_2F_2. Then the point P belongs to the oval whose equation is:

$$\frac{|XF_1|}{R_1}|PF_1| - \frac{|XF_2|}{R_2}|PF_2| = |XF_1| - |XF_2| \tag{2}$$

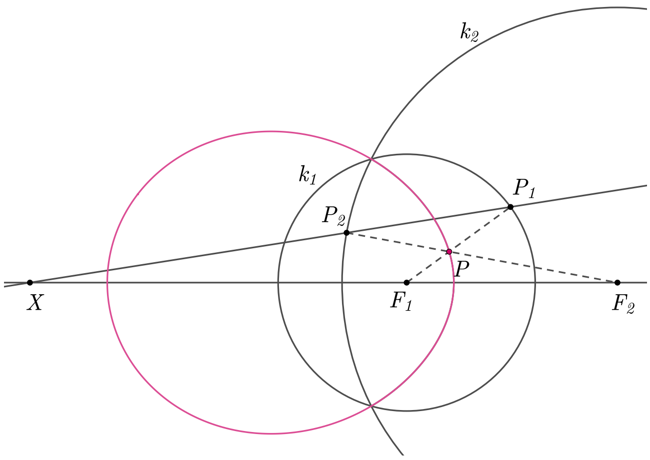

Figure 1: Chasles construction of oval

Since $|XF_1|/R_1$, $|XF_2|/R_2$ and $|XF_1|-|XF_2|$, are constants, this equation is equivalent to (1). This statement can be derived using Menelaus theorem (the line XP_1P_2 intersects the sides of the triangle F_1PF_2) or by spatial reasoning and subsequent projection onto a plane. These derivations are mentioned on the website [2], and we will not present them here.
Note: A line through X can intersect each circle at two points, so we can construct up to four points of P. It can be shown that two points belong to the oval with equation (2) and the remaining two points belong to the oval whose equation is similar to the equation (2), except that one of the mentioned three constants has the opposite sign. These two ovals are so called conjugate.
The article is divided into three parts. First, we give a new justification of why the points of P constructed using the Chasles construction belong to the oval. As a part of this justification, we show why a Cartesian oval has three foci and how to construct its third focus geometrically. In the second part, we show how to perform the Chasles construction of the oval, if two foci of the oval and its equation (1) are given. Finally, we consider the case where one of the circles is a straight line and show that in this case the construction depicts a conic. A by-product of this proof is a justification why the definition of a conic using a directrix line is equivalent to the definition using a foci definition.

2 Chasles construction and third focus of oval

We start from the construction described in the previous section. Our aim is to justify the following
Theorem 2.1 The points P constructed by Chasles' construction satisfy the equation (2).
Proof. Construct inverse points X_1, X_2 of the point X with respect to the circles k_1 and k_2, respectively. Then construct the intersection E of the lines X_1P_1 and X_2P_2 and finally construct the intersection F_3 of the lines EP and F_1F_2 (Figure 2).

Due to the definition of inversion,

$$\frac{|XF_1|}{|P_1F_1|} = \frac{|P_1F_1|}{|X_1F_1|},$$

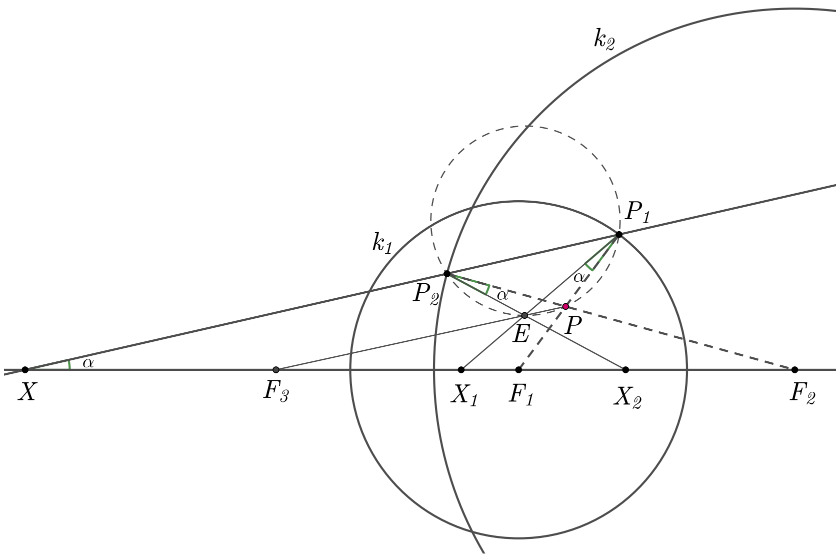

Figure 2: Chasles construction and the third focus

the triangles $X_1F_1P_1$ and P_1F_1X are similar. Therefore

$$\angle F_1P_1X_1 = \angle F_1XP_1 = \alpha$$

In the same way we derive the similarity of the triangles F_2P_2X and $F_2X_2P_2$ and the equality

$$\angle F_2P_2X_2 = \angle F_2XP_2 = \alpha$$

From the above equations follows that the quadrilateral P_2P_1PE is cyclic, therefore $\angle XP_2X_2 = \angle XP_2E = \angle P_1PE$. Hence, the triangles EPP_1 and X_2P_2X are similar. Since the triangles $F_2P_2X_2$ and F_2XP_2 are also similar, it holds:

$$\frac{|EP|}{|PP_1|} = \frac{|P_2X_2|}{|P_2X|} = \frac{|P_2F_2|}{|XF_2|} \tag{3}$$

In the same way, we arrive at equality

$$\frac{|EP|}{|PP_2|} = \frac{|P_1X_1|}{|P_1X|} = \frac{|P_1F_1|}{|XF_1|} \tag{4}$$

Therefore,

$$\frac{|PP_1|}{|PP_2|} = \frac{|F_1P_1| - |F_1P|}{|F_2P_2| - |F_2P|} = \frac{\dfrac{|P_1F_1|}{|XF_1|}}{\dfrac{|P_2F_2|}{|XF_2|}} = \frac{|P_1F_1|}{|XF_1|} \cdot \frac{|XF_2|}{|P_2F_2|}, \tag{5}$$

which, after easy rearrangement, gives the equation

$$\frac{|XF_1|}{|P_1F_1|} \cdot |PF_1| - \frac{|XF_2|}{|P_2F_2|} \cdot |PF_2| = |XF_1| - |XF_2| \tag{6}$$

identical to the equation (2).

Theorem 2.2 Point F_3 is the third focus of the oval.

Proof. To prove this statement, it is necessary to justify two facts:
1) The point F_3 is fixed, i.e. it does not depend on the position of the line XP_2P_1. 2) Points P of the oval with foci F_1F_2 also lie on the (identical) ovals with foci F_3F_1 and F_3F_2

Due to the equality $\angle X_1EX_2 = \angle P_2EP_1 = \angle P_2PP_1$ and $\angle X_2X_1E = \angle F_1X_1P_1 = \angle F_1P_1X = \angle PP_1P_2$ the triangles X_2EX_1 and P_1PP_2 are similar. Since the ratio of of the sides $|PP_1|/|PP_2|$ is constant according to equation (5), the ratio of the sides $|EX_2|/|EX_1|$ is also constant. Let us denote this constant e:

$$\frac{|EX_2|}{|EX_1|} = e$$

Furthermore, the triangles F_3EX_1 and F_3X_2E are also similar since they share the angle at the vertex F_3 and the following equality holds:

$$\angle X_1EF_3 = \angle P_1EP = \angle P_1P_2P = \angle P_1P_2F_2 = \alpha + \angle P_2F_2X_2 = \angle P_2X_2F_3$$

Hence:

$$\frac{|F_3X_2|}{|F_3E|} = e \quad \text{and} \quad \frac{|F_3E|}{|F_3X_1|} = e \tag{7}$$

Multiplying these identities one gets

$$|F_3X_2| = e^2 \cdot |F_3X_1|$$

But

$$|X_2X_1| = |F_3X_2| - |F_3X_1| = |F_3X_1| \cdot (e^2 - 1)$$

Since the length of $|X_2X_1|$ is constant, the length of $|F_3X_1|$ is constant too and the point F_3 is fixed. (More precisely, it is the centre of the Apollonius circle for the triangle X_1EX_2.) This completes the first part of the proof.

In the second part, it suffices to consider the foci of F_3F_1. In the case of the foci of F_3F_2 the procedure is analogous. Let's consider an equation

$$\frac{|XF_2|}{|P_2F_2|} \cdot |F_3P| + |F_1P| = k \tag{8}$$

As $|XF_2|/|P_2F_2| = c$ is a constant, it is sufficient to show that k is also a constant. Putting the identity $|F_3P| = |F_3E| + |EP|$ into the equation (8) and using the relation (3), in the form $|XF_2|/|P_2F_2| \cdot |EP| = |PP_1|$, one gets

$$\frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + \frac{|XF_2|}{|P_2F_2|} \cdot |EP| + |F_1P| = \frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + |PP_1| + |F_1P| = \frac{|XF_2|}{|P_2F_2|} \cdot |F_3E| + |F_1P_1|$$

The right-hand side of the equation is constant if and only if $|F_3E|$ is a constant. Equation (7) implies

$$|F_3E| = e \cdot |F_3X_1|$$

Since we have proved that $|F_3X_1|$ is constant, $|F_3E|$ is also constant. The proof is complete.

3 Chasles construction of an oval given by focal equation

In this section we solve the following
Problem 3.1
Let an oval is given by two foci F_3 and F_2 and by the equation

$$c_3 \cdot |PF_3| + c_2 \cdot |PF_2| = k$$

where c_3, c_2 and k are constants. How to perform the Chasles construction of the oval?

Before we show the solution to this problem, we will return to the classical construction mentioned in the introduction. We will show that it is not unique, namely, that there are an infinite number of constructions giving the same oval. We choose one of them suitable for solving our problem.
Let the circles k_1, k_2 and the point X on the line of their centres be given. These objects, as we already know, determine the oval by the construction. Next, let us choose another point $Y \neq X$ on this line. How to modify the circles k_1 and k_2 to l_1 and l_2, respectively, so that l_1, l_2 and Y determine the same oval? (Figure 3)

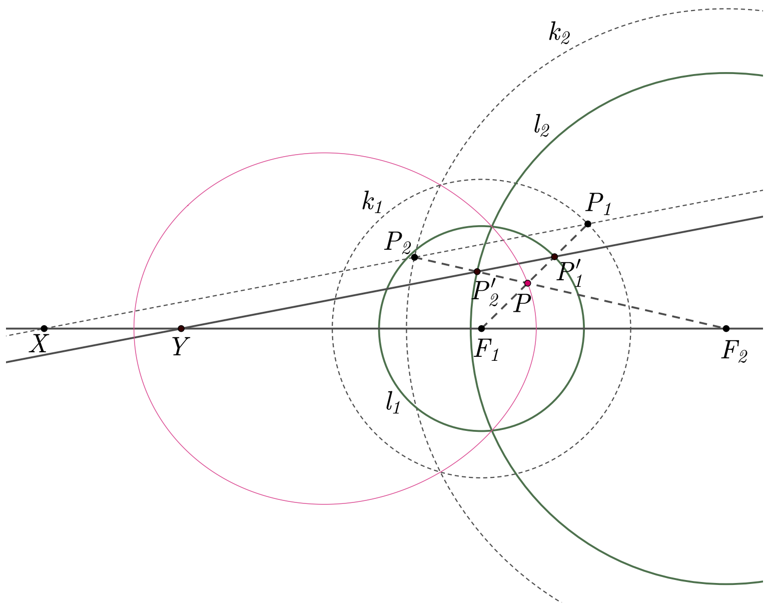

Figure 3: Chasles constructions of the same oval

Let the line XP_1P_2 determine the point P of the oval. Let us draw a parallel to this line through the point Y, which intersects the lines P_2F_2 and P_1F_1 at the points P_2' and P_1', respectively. Then the intersection of the lines $P_2'F_2$ and $P_1'F_1$ coincides with the point P. It remains to justify that the points P_2' and P_1' move on the circles with centres F_2 and F_1, respectively. From the similarity of the triangles XP_2F_2 and $YP_2'F_2$ follows:

$$|P_2'F_2| = |P_2F_2| \cdot \frac{|F_2Y|}{|F_2X|}$$

As the right-hand side of the equation is a constant, the point P_2' moves on the circle l_2 centred at F_2. Similarly, we prove that the point P_1' moves on the circle l_1 centred at F_1.
Also, the focal point F_3, whose construction is described in the previous section, remains the same since (Figure 2 and Figure 3)

$$\angle P_1PF_3 = \angle XP_2X_2 = \angle XP_2F_2 - \alpha = \angle YP_2'F_2 - \alpha = \angle YP_2'Y_2 = \angle P_1'PF_3$$

where Y_2 is the inverse of the point Y with respect to the circle l_2.
Now it is possible to choose a point Y in such a way that its inverse image Y_1 with respect the circle l_1 is identical to the inverse image Y_2 of Y with respect to the circle l_2. Such a point always exists unless the point X - and Y - is not centre of similitude of the circles. In such case the solution is a conic and the third focus does not exist.

The proofs of these two propositions will be only sketched. Firstly, we show the existence of the point Y. Since Y_1 is inverse of Y in l_1, it holds:

$$R_{l_1}^2 = |F_1Y| \cdot |F_1Y_1|$$

where R_{l_1} is radii of the circle. Dividing the equation by $|F_1Y|^2$ we get

$$\left(\frac{R_{l_1}}{|F_1Y|}\right)^2 = \frac{|F_1Y_1|}{|F_1Y|}$$

But the left hand side of the equation is constant for all points Y. Hence, the point Y_1 is the image of Y in homothethy h_1, with centre at F_1 and coefficient $(R_{l_1}/|F_1Y|)^2$. Similarly, we arrive at the conclusion that the point Y_2 is the image of Y in a homothethy h_2. Hence

$$h_1 \circ h_2^{(-1)}(Y_2) = Y_1$$

Now, composition of two homotheties $h_1 \circ h_2^{(-1)}$ is a third homothethy h_3 if and only if the product of the coefficients of homotheties is not equal 1, in which case it is a translation. It is clear that the centre Z of the h_3 fulfils

$$h_3(Z) = Z$$

Therefore we can select Y in a way that $Z = Y_1 = Y_2$.
The second case (translation) occurs only if the point Y is the centre of similitude of the circles. In such case, the lines P_1F_1 and P_2F_2 intersect for the points P_1 and P_2 being antihomologous (for details see website [3]). They intersect in a centre of a circle tangent to the circles l_1 and l_2. It is possible to show that the intersections belong to hyperbola with foci F_1, F_2, if the tangent circle touches both circles externally or internally, and to ellipse, if the tangent circle touches one circle externally and second internally.

Let's return to the problem. The intersection E (Figure 2 with substitution of point labels Y, Y_1 and Y_2 for the labels X, X_1 and X_2) is identical to this common image, and since it lies on the line F_1F_2, it must be the third focus F_3.

Let us denote this point Y by X again, this time knowing that $F_3 = X_1 = X_2$. The updated Figure 2 then looks as follows (Figure 4):

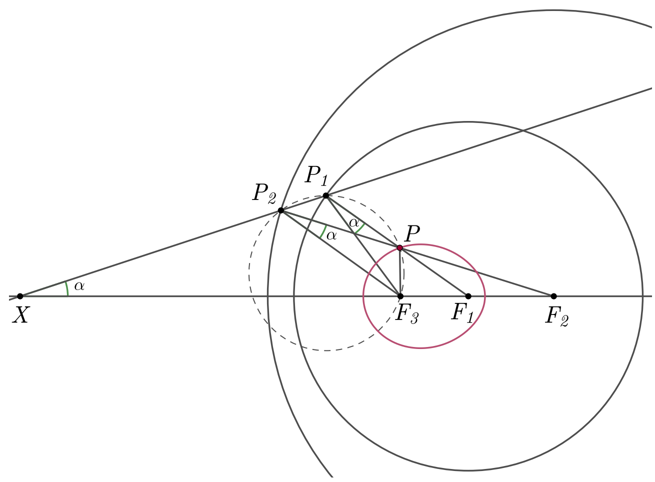

Figure 4: Chasles constructions with $F_3 = X_1 = X_2$

Denoting the radii of the circles $|P_1F_1| = R_1$ and $|P_2F_2| = R_2$, the new equations (3), (4), (6) and (8) are

$$\frac{|F_3P|}{|PP_1|} = \frac{|P_2F_3|}{|P_2X|} = \frac{R_2}{|XF_2|}$$

$$\frac{|F_3P|}{|PP_2|} = \frac{|P_1F_3|}{|P_1X|} = \frac{R_1}{|XF_1|} \tag{9}$$

$$\frac{|XF_1|}{R_1} \cdot |PF_1| - \frac{|XF_2|}{R_2} \cdot |PF_2| = |XF_1| - |XF_2| \tag{10}$$

$$\frac{|XF_2|}{R_2} \cdot |F_3P| + |F_1P| = R_1 \tag{11}$$

$$\frac{|XF_1|}{R_1} \cdot |F_3P| + |F_2P| = R_2 \tag{12}$$

Equations (10), (11) and (12) express an oval with foci F_1F_2, F_3F_1 and F_3F_2.

Solution of problem 3.1

In order to solve the problem 3.1, we will modify the oval equation to the form (12). Then we will proceed as follows:
1) Construct the circle $k_2(F_2, R_2)$.
2) Construct the inverse image X of the point F_3 with respect to circle k_2.
3) Construct the Apollonius circle k_1 with the ratio distances of its points P_1 to the points F_3 and X equal to (9), $\frac{|P_1F_3|}{|P_1X|} = \frac{R_1}{|XF_1|}$. This ratio is given by equation (12). Denote the centre of this circle by F_1.
4) The circles k_1, k_2 and the point X determine the oval given by equation (12).
Negative constant in equation (12) does not change the above procedure - we work with absolute values of these constants. It may seem that absolute values of the two constants determine a total of four different equations (12) but the Chasles construction determines only two of them. It can be shown, however, that two of these four ovals are empty sets. Particularly, the oval of the equation (12) with $|XF_1|/R_1 > 0$ and $R_2 < 0$ is excluded immediately. One of the remaining three equations is always excluded due to the improper focal distance F_2F_3.

4 Chasles construction for one of the circles replaced by a line

Now we prove that replacing one circle in the Chasles construction by a line, the set of points P is a conic. For completeness we describe the construction (Fig. 5) in detail.

Special case of the Chasles construction

Let a circle $k(F_2, R_2)$ and a line l be given and let the line p passes through F_2 and is perpendicular to the line l. Choose a point X on the line p. Let an arbitrary line passing through X intersects the line l at P_1 and the circle k at P_2. Draw the perpendicular line p_1 to the line l at P_1 and denote the intersection of the lines P_2F_2 and p_1 as P.

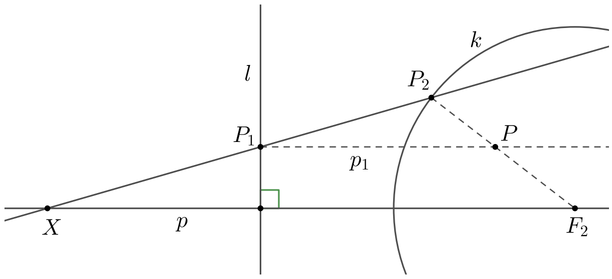

Figure 5: Special case of Chasles constructions

Theorem 4.1 The set of points P constructed in this way is a conic.
Theorem 4.2 The focus F_1 of this conic is constructed as follows (Figure 6):
1) Construct the axisymmetric image X_1 of the point X with respect to the line l.
2) Construct the inversion image X_2 of the point X with respect to the circle k.
3) Construct the intersection E of the lines P_1X_1 and P_2X_2.
4) Denote F_1 the intersection of the lines EP and p.

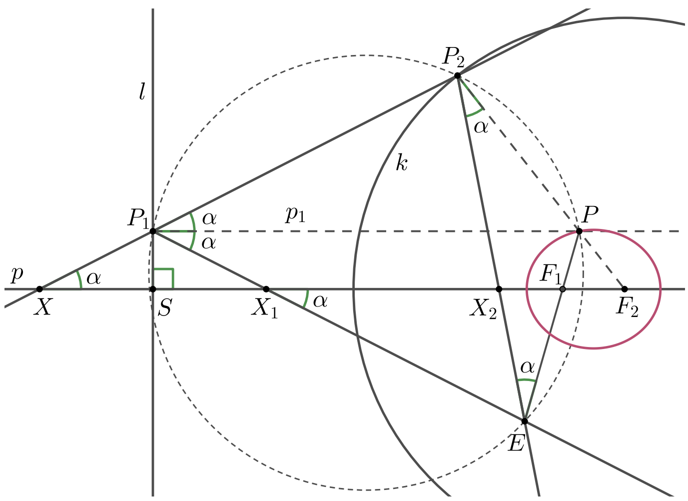

Figure 6: Construction of a focus of a conic

Proof of 4.1 and 4.2.

Let's start with a few obvious facts. It holds

$$\alpha = \angle F_2P_2X_2 = \angle F_2XP_2 = \angle PP_1P_2 = \angle PP_1E$$

Since $\angle PP_1E = \angle PP_2X_2 = \angle PP_2E$, the quadrilateral PP_2P_1E is cyclic and hence

$$\alpha = \angle PP_1P_2 = \angle PEP_2$$

Firstly, we will show that the point F_1 is fixed for any line XP_1P_2. As the triangles X_1X_2E and $P_2X_2F_2$ are similar, we have

$$|X_1X_2| \cdot |X_2F_2| = |X_2E| \cdot |X_2P_2|.$$

The left side of this equation is a constant, therefore the product on the right side is also constant.
From the similarity of the triangles X_2F_1E and X_2P_2X follows

$$|X_2E| \cdot |X_2P_2| = |X_2X| \cdot |X_2F_1|.$$

The product on the left side is constant and the length of the line segment $|X_2X|$ is fixed. Therefore, the length of the line segment $|X_2F_1|$ is also fixed and the point F_1 is common to all lines XP_1P_2.
Since $\angle PP_2E = \angle PEP_2$, the triangle P_2PE is isosceles.
Now we are at a crossroads: if the point X lies outside the circle k, the locus will be an ellipse (see below), if it lies inside the locus will be a hyperbola, finally if the point X lies at the intersection of p and k, the locus is parabolla. We only hint why this proposition is true in the conclusion of the article, but before it, lets focus on the case of ellipse (Figure 6).

$$R_2 = |F_2P_2| = |PF_2| + |PE| = |F_2P| + |F_1P| + |F_1E|$$

Otherwise written:

$$R_2 - |F_1E| = |F_2P| + |F_1P| \tag{13}$$

This equation represents a focus definition of an ellipse if and only if $|F_1E|$ is constant. The similarity of the triangles F_1EX_2 and F_1X_1E implies

$$|F_1X_1| \cdot |F_1X_2| = |F_1E|^2.$$

The product on the left-hand side is constant and the proof is completed.
As in the case of the classical Chasel construction, the "special one" is not unique. Let S is the intersection of the lines l and p. Choose a point $Y \neq X$ on the line p, construct a circle k_2 with centre F_2 and radius $R_2' = R_2 \cdot |YF_2|/|F_2X|$ and a line l_2 parallel to the line l such that the distance of intersection S_2 of lines p and l_2 to the point S is equal, $\overline{SS_2} = \overline{XY}$ (equality of oriented distances). Then the line l_2, the circle k_2 and the point Y determine the same conic.

As in the previous case, the point Y can be chosen in such a way that the points Y_1 (axisymmetric point with respect to the line l_2) and Y_2 (the inverse image of the point Y with respect to the circle k_2) are identical, i.e. $Y_1 = Y_2 = F_1$. (To be exact: such a point always exists, unless the point Y does not lie on the circle k_2, in which case the focus F_1 does not exist and the locus is a parabola.) By this transformation the relative position of the point Y and the circle k_2 is preserved. It means that the points Y lie either inside of the circles k_2 in all of the configurations or outside in all of them. So we have the following configuration: points Y, F_1, line l_2 (symmetrical axis of the points) and the circle k_2 with the property, that inverse image of Y is F_1 and vice versa. In this configuration the circle k_2 and the line l_2 are nonintersecting and it is possible to show: If the point Y lies outside of the circle, the locus is an ellipse If it lies inside, the locus is a hyperbola. We only sketch the proof. Let's begin the proof with the former case, Figure 7:

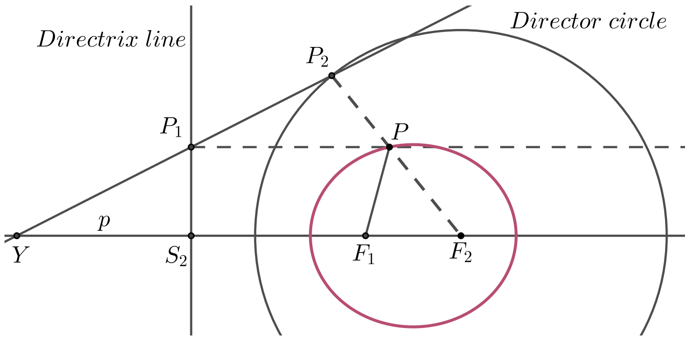

Figure 7: Directrix line and director circle in Chasles construction

We know, that $|EP| = |PP_2|$ and (in this configuration) $E = F_1$, so $|F_1P| = |PP_2|$. From the Figure 7 it is obvious that P lies always within the segment F_2P_2. Hence

$$Constant = |F_2P_2| = |F_2P| + |PP_2| = |F_2P| + |F_1P|,$$

which is focal definition of ellipse.
The case of a hyperbola is analogous with the exception that the labels of the points Y and F_1 in the Figure 7 are interchanged (since the point Y lies inside of the circle). Again $|EP| = |PP_2|$, $E = F_1$ and $|F_1P| = |PP_2|$, but in this case, the point P lies outside of the segment F_2P_2. Hence:

$$Constant = |F_2P_2| = |F_2P| - |PP_2| = |F_2P| - |F_1P|,$$

which is focal definition of the hyperbola. The exact proof why the point P lies in/out – side of the segment in respective cases is left to the reader. It is not hard to show that in this configuration the line l_2 is the directrix line of the conic.

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[2] Ferréol R., 2017, ENCYCLOPÉDIE DES FORMES MATHÉMATIQUES REMARQUABLES, https://mathcurve.com/courbes2d.gb/descartes/descartes.shtml

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