Curved Patterns in the Graphs of PPTs

James M. Parks
parksjm@potsdam.edu
Dept. of Mathematics
SUNY POTSDAM
U.S.A.

Abstract: *In a graph of all Primitive Pythagorean Triples (PPTs), with legs up to length 10,000, conic section-like patterns can be observed. We show that they are indeed parabolic curves, which follow in a natural way from the mathematics of the subject matter.* This work lives at the intersection of Arithmetic, H.S. Algebra, and Analytical Geometry. It is easily accessible by students. Computer programs for Dynamic Geometry (*Sketchpad, GeoGebra*) and *Mathematica* (or *Maple*) were used to build the graphs.

1. Introduction

A *Pythagorean Triple* (PT) is a set of 3 positive integers (a,b,c), which satisfy the Pythagorean Equation $a^2+b^2 = c^2$. The numbers a, b, c are associated with the sides of a right triangle ΔABC [5].

For example, $(3,4,5)$ is a Pythagorean Triple, since, by the Pythagorean Theorem, the right triangle ΔABC with sides of length $3, 4,$ and 5, satisfies $3^2+4^2 = 5^2$.

Given a PT (a,b,c), if the integers a, b, c are relatively prime (no common factors other than 1, so $GDC\{a,b,c\} = 1$), then call the triple (a,b,c) a *Primitive Pythagorean Triple* (PPT).

For example $(3,4,5)$ is a PPT, but the PT $(6,8,10)$ is not a PPT, since all the terms are even.

We wish to analyzing a graph of PPTs, which is construct as follows. Given a PT (a,b,c), $a<b<c$, with associated right triangle ΔABC, there is a way to graph it which allows us to compare different PTs. Choose the ordered pair (a,b) and graph it in the 1st quadrant of the *xy-plane,* Figure 1.1.

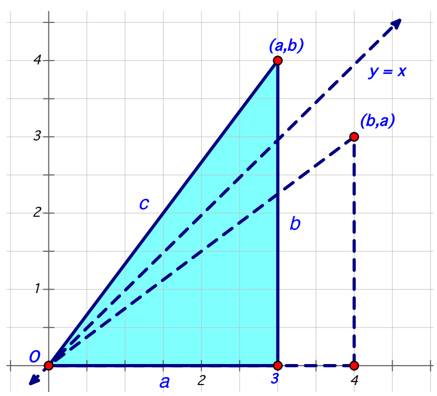


Figure 1.1 The Graph of a PPT

Then the right triangle ΔABC with sides a,b,c is congruent to the triangle formed by the origin O, the point $(a,0)$, and the point (a,b). Thus the point (a,b) corresponds to ΔABC.

Another congruent copy of ΔABC is provided by the graph of the point *(b,a)*. The point *(b,a)* is the reflection of *(a,b)* about the line $y = x$.

We assumed that $a<b<c$, when referring to the PT *(a,b,c)*, but occasionally it will happen, by way of an argument, that we have the PT *(b,a,c)*, with $a<b<c$, as happened in the triangle figures just above. The line $y = x$ divides the first quadrant into two regions, one which contains the points *(a,b)*, $a<b$, and one which contains the points *(b,a)*, $a<b$. Since the points *(a,b)* and *(b,a)* are symmetric about the line $y = x$, we call *(b,a)* the *reflected point* of *(a,b)*.

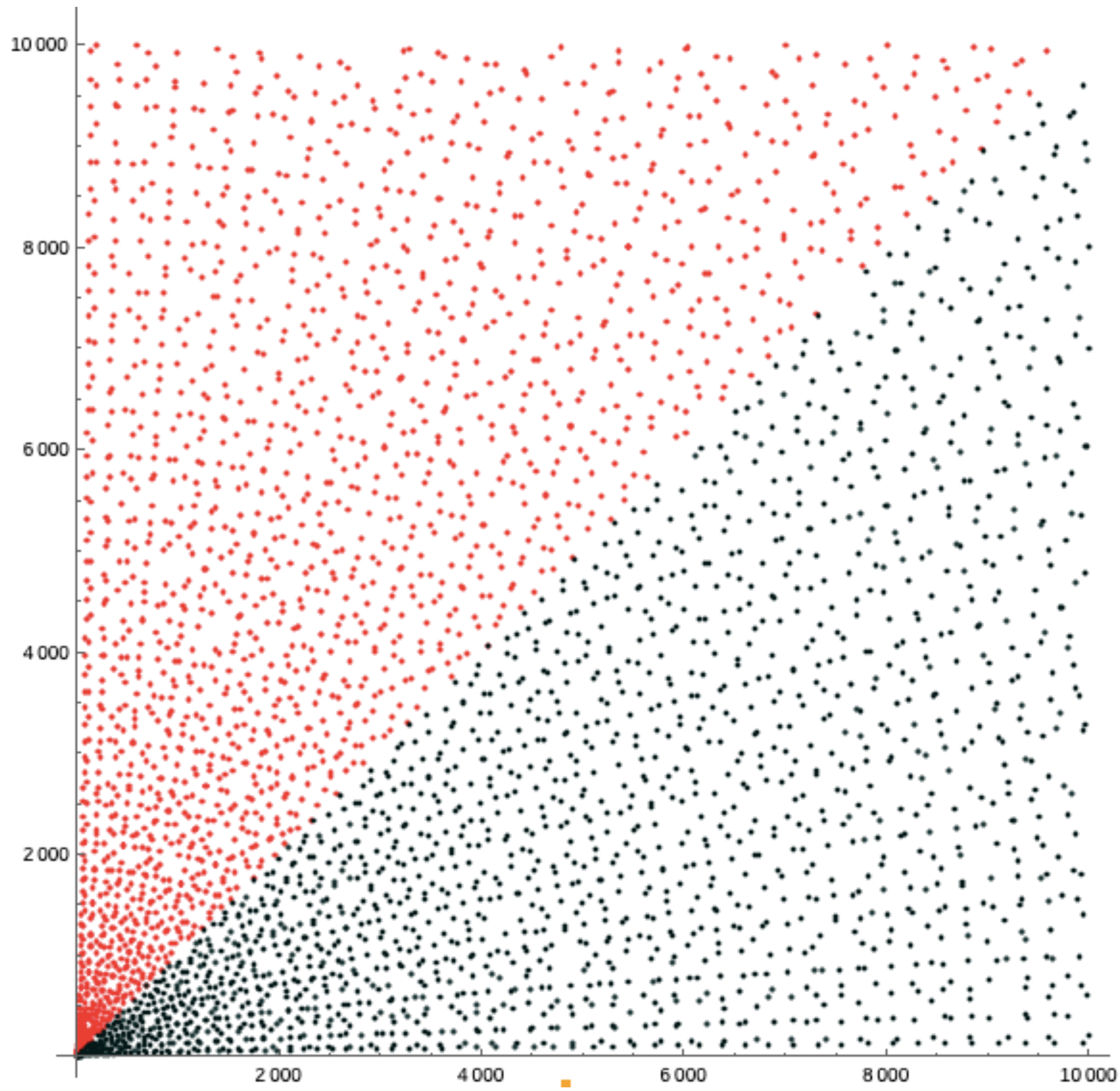


Figure 1.2 The Main Graph of all PPTs with $0<a<b<10{,}000$

Observe that if *(a,b,c)* is a PPT, then one of *a, b* must be odd, and the other must be even, while *c* is always odd. Also, it cannot happen that $a = b$, or that *a* or $b = 1$ [5].

If we list all PPTs with $0<a<b<10{,}000$, and graph them and the reflected points on *Mathematica* [9], using the graphing method above, the amazing graph shown in Figure 1.2 is the result [6]. An identical graph was constructed by R. Knott [3]. Both graphs were constructed using *Mathematica*, but using different codes.

The red region is the set of points (a,b), where $a < b$, and the black region is the set of reflected points (b,a). Thus the two regions are symmetric about the line $y = x$. We call Figure 1.2 the Main Graph.

2. Analysis of the Main Graph

In order to analyze this graph, consider the following set of the first *18* PPTs ordered by the size of the short leg *a.*

3, 4, 5	***5, 12, 13***	***7, 24, 25***	*8, 15, 17*	***9, 40, 41***	***11, 60, 61***
12, 35, 37	***13, 84, 85***	***15, 112, 113***	*16, 63, 65*	***17, 144, 145***	***19,180, 181***
20, 21, 29	*20, 99,101*	***21, 220, 221***	***23, 264, 265***	*24, 143, 145*	***25, 312, 313***

Those PPTs which are in **bold** print are meant to attract your attention. These particular *12* PPTs all have terms "*a*" which are odd numbers, and they also have the form $(a, b, b+1)$, that is b and c are consecutive numbers, $c = b+1$.

It is given that these particular triples all satisfy the Pythagorean equation $a^2+b^2 = c^2$, and the terms are obviously relatively prime, since consecutive numbers cannot have any common divisors. They also satisfy the equivalent equation:

$$b = (a^2 -1)/2 \tag{2.1}$$

This means that a^2 (and thus a) must be an odd integer, so that b is then an even integer.

So if an arbitrary odd positive integer "*a*" is given, then the triple $(a,b,b+1)$ is a PPT, whenever b satisfies equation (2.1). Thus these PPTs all satisfy the form $(a, (a^2-1)/2, (a^2+1)/2)$, for "*a*" an odd positive integer. This determines a one-to-one correspondence between the odd positive integers "*a*" and those PPTs which have the form $(a, b, b+1)$.

Note that if any two integers are relatively prime, then listing a third number with them makes a relatively prime list of three numbers.

The above formulas for $(a,b,b+1)$ above are not a new result, they are well known. According to Proclus (410-485 AD), these PPTs were known to the Pythagoreans (570-495 BCE), and perhaps before [1]. However, this does not seem to prevent the result from being 'rediscovered' occasionally.

Using *Sketchpad* [7], graphs of these PPTs in the *xy-plane* are obtained by graphing the set of points $(a, (a^2 -1)/2)$, and the reflected points $((a^2 -1)/2, a)$, for "*a*" an odd positive integer.

For example, the odd integer *3* determines the points $(3,4)$ and $(4,3)$, and the odd integers $5, 7, 9,\ldots, 25$, determine the points corresponding to the PPTs given in bold in the list above.

A graph of these points and their reflected points in the *xy-plane* in the range $0 < a,b < 120$ is given in Figure 2.1.

It suggests parabolic shaped curves about the positive x and y *axes*. The red points have the form $(a, (a^2 -1)/2)$, so they satisfy the parabola equation $y = (x^2 -1)/2$, and the reflected points have the form $((a^2 -1)/2, a)$, so they satisfy the reflected parabola equation $x = (y^2 -1)/2$.

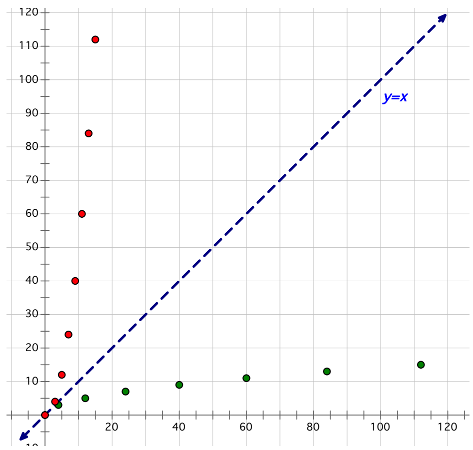
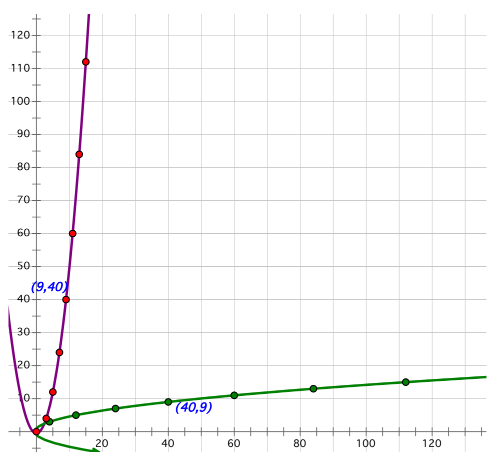


Figure 2.1 The Graph of PPTs for $d = 1$

The first parabola opens about the positive *y-axis,* with focus at O, and vertex $(0, -1/2)$. We denote this set of PPTs by $d = 1$. The second parabola opens about the positive *x-axis*, has focus at O, and vertex $(-1/2, 0)$, Figure 2.2. We will give a notation for it below.



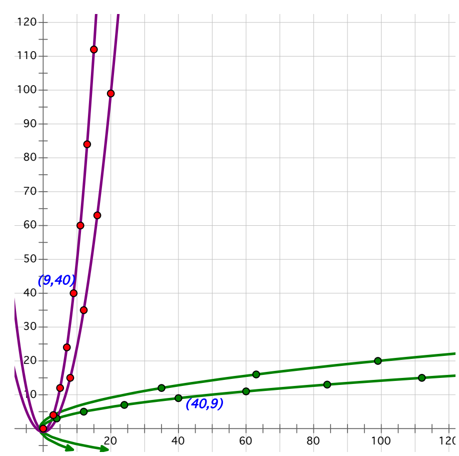


Figure 2.2 Graph of Parabolas for $d = 1$ **Figure 2.3** Graph of Parabolas for $d = 1, 2$

Look again at the list of the first *18* PPTs above, and notice that *5* of the PPTs have the form $(a,b,b+2)$. Denote the set of all such PPTs by $d = 2$, since the 'difference' $c - b = 2$. The set of PPTs which satisfy $d = 2$ also satisfy the equivalent equation:

$$b = (a^2 -4)/4 \tag{2.2}$$

This means that a^2, and thus a, must be even, and b must be odd. The numbers a, b, c are relatively prime, since the odd terms $b, c,$ are 2 units apart, so they cannot have any common divisors. This form is not new, Proclus attributed this result to Plato (429-347 BCE) [1].

From the list of $d = 2$ PPTs above, note that "a" is even and $a = 0 mod 4$. But if "a" is even, and not a multiple of 4, then $a = 2 mod 4$, so the triple is a PT, but not a PPT, hence it is not on the list. The "*mod*" notation is short-hand from modular arithmetic. If the integer $z > 1$, and $x, y,$ are integers, then the equation $x = y mod z$, means $x - y = zk$, for some integer k, or equivalently, $x = zk + y$, for some integer k.

Thus, if $a = 0 mod 4$, then $a = 4k$, for $k > 0$, and $(a, b, b+2) = (4k, 4k^2 -1, 4k^2 +1)$, a PPT. But if $a = 2 mod 4$, then $a = 4k +2$, and $b = 4k^2 +4k$, so $b+2 = 4k^2 +4k +2$. This PT is not a PPT, as all terms are even. So every other even number "a" determines a PPT. The curves which are for these PPTs with $d=2$, satisfy the parabola equations $y = (x^2 -4)/4$, and $x = (y^2 -4)/4$. They have vertex points at $(0,-1)$ and $(-1,0)$, resp., and both have focus at the origin O, Figure 2.3.

In general, the parabolas for arbitrary values of $d>0$ are given by the equations:

$$y = (x^2 -d^2)/2d \tag{2.3}$$
$$x = (y^2 -d^2)/2d \tag{2.4}$$

where "a" and "d" are either both odd or both even positive integers, and $d|a$. However we find that a lot of the values of "d" determine PTs, but not PPTs, see below.

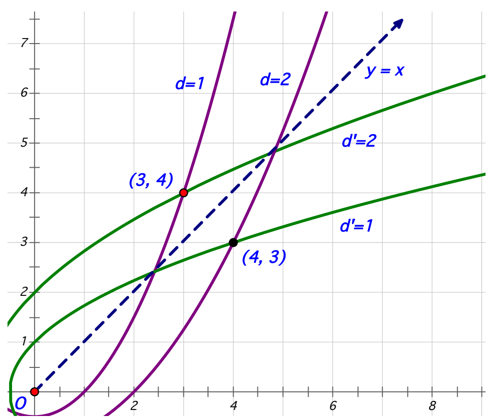


Figure 2.4 Graph of Parabolas for $d = 1$, $d' = 2$, and $d' = 1$, $d = 2$

There is another d-value associated with the PPT (a,b,c), which is determined by the form (b,a,c), namely $d' = c - a$. The point (a,b) (and (b,a)) will occur at the intersection of parabolas for d and d'. For example, the graph above, Figure 2.4, shows the point $(3,4)$ at the intersection of the parabolas labeled $d=1$ and $d'=2$, and the point $(4,3)$ at the intersection of the parabolas labeled $d'=1$, and $d=2$.

Any integer point (s,t) which is on a parabola $y = (x^2 -d^2)/2d$, equation (2.3), for some d will satisfy $t = (s^2 -d^2)/2d$, and (s,t,u) is a PT for that d, with $u = (x^2 +d^2)/2d$. Each integer point on the graph of the PPTs above is on a parabola for some value of d.

Note that the value $d = 3$ does not determine any PPTs, for all such PTs have a common value of 3 in their coordinates. The same thing happens for the d values $4, 5, 6, 7, 10, 11,$ and $12.$ The next PPTs, after $d = 1$ or $2,$ occur when $d = 8$ or $9,$ and these values have mixed results similar to those we found for $d = 2.$

3. Results and Conclusions

The list of d-values which determine PPTs begins as follows: *1, 2, 8, 9, 18, 25, 32, 49, 50, 72, 81, 98, …,* [2]. We call these numbers the *allowable values* of $d.$ This list is in fact the OEIS sequence *A096033*. The graph of some of the representative points and parabolic curves for some of these d and d' values is shown in Figure 3.1.

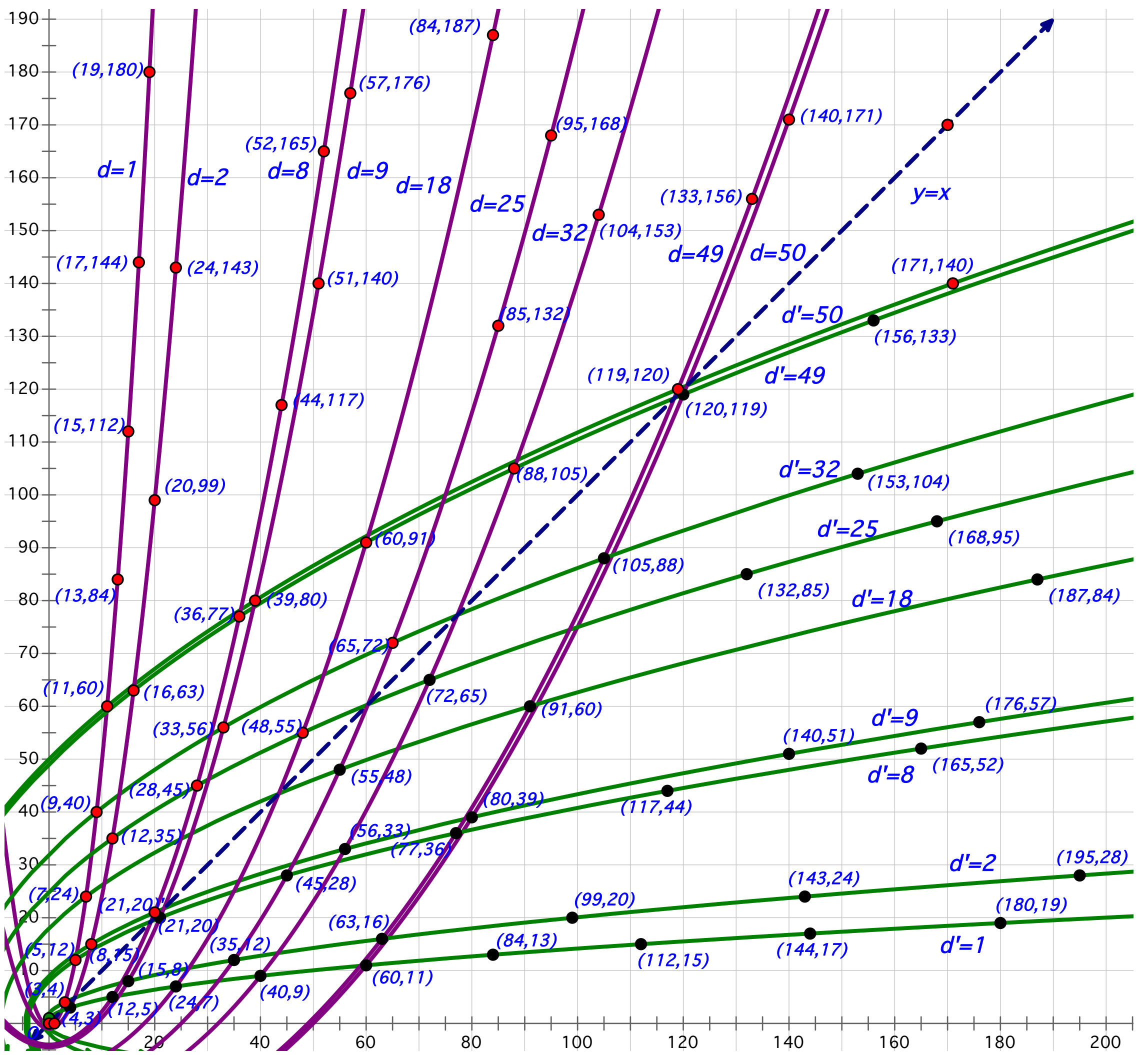


Figure 3.1 Representative Points and Parabolic Curves for Beginning d and d' Values

Comparing this graph with the Main Graph it becomes clear where some of the curved patterns originate. This is only a very small part of the lower left corner of the Main Graph, Figure 1.2, but the general picture is starting to become apparent.

These results are stated in the following Proposition on parabolas about positive *x,y-axes* with allowable d values.

Proposition 3.1 If (a,b,c) is a PPT with $a<b<c$, and values d, d', such that $d<d'$, then (a,b) is a point at the intersection of the parabolas with equations $y = (x^2-d^2)/2d$, and $x = (y^2-d'^2)/2d'$, and (b,a) is a point at the intersection of the reflection of those parabolas about $y = x$, which have equations $x = (y^2-d^2)/2d$, and $y = (x^2-d'^2)/2d'$.

For the Main Graph, Figure 1.2, notice that there also appear to be parabolic curve patterns which open about the negative *x,y-axes*. These curves are mentioned in [3]. The equations for these curves are formed in a different manner from those above.

First consider the parabolic curves which open about the negative *y-axis*.

An example is the parabola with equation:

$$y = -x^2/(2\cdot 3^2) + 3^2/2 \tag{2.5}$$

shown here in orange, Figure 3.2. This parabola has vertex point $(0,3^2/2)$, and focus at O. It also contains the point $(3,4)$.

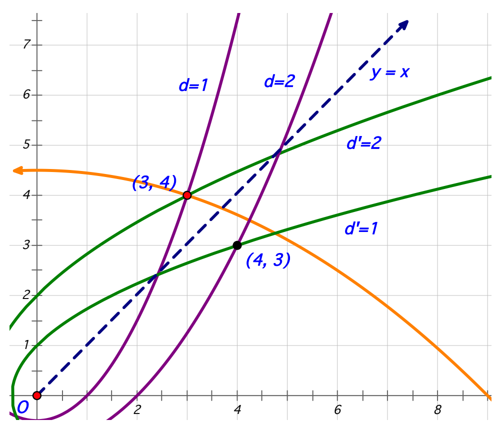


Figure 3.2 The Graph of Equation (2.5) in Orange

When studying the graphs in Figure 3.1 and Figure 3.2, notice that certain sets of points seem to be in a parabolic shaped pattern which opens downward. For example, the *4* points *(20,99), (60,91), (140,51),* and *(180,19)* appear to form a parabolic curve which opens about the negative *y-axis*. Note that the 1st coordinates of these *3* points are all multiples of *10.* The LH point *(20,99)* is from the triple *(20,99,101),* and is on the parabola labeled $d = 2$. The equation for this parabola is:

$$y = -x^2/(2\cdot 10)^2+10^2 \tag{2.6}$$

the vertex point is *(0,10²),* and the focus point is *(0,-10²).*

For another example, consider the *3* points *(28,45), (56,33),* and *(84,13).* They also appear to be on a parabola which opens about the negative *y-axis*. The 1st coordinates here are all multiples of *28.* The LH point *(28,45)* is from the triple *(28,45,53),* and is on the parabola labeled $d = 8$. The equation for this parabola is:

$$y = -x^2/(2\cdot 7)^2 + 7^2 \tag{2.7}$$

the vertex point is $(0, 7^2)$, and the focus point is $(0, -7^2)$.

These examples provide a pattern for the equations of the parabolas in the Main Graph which open about the negative *y-axis*. Given a series of points with *even* 1st coordinates all multiples of, say $2\cdot n^2$, the general equation of the these parabolas is:

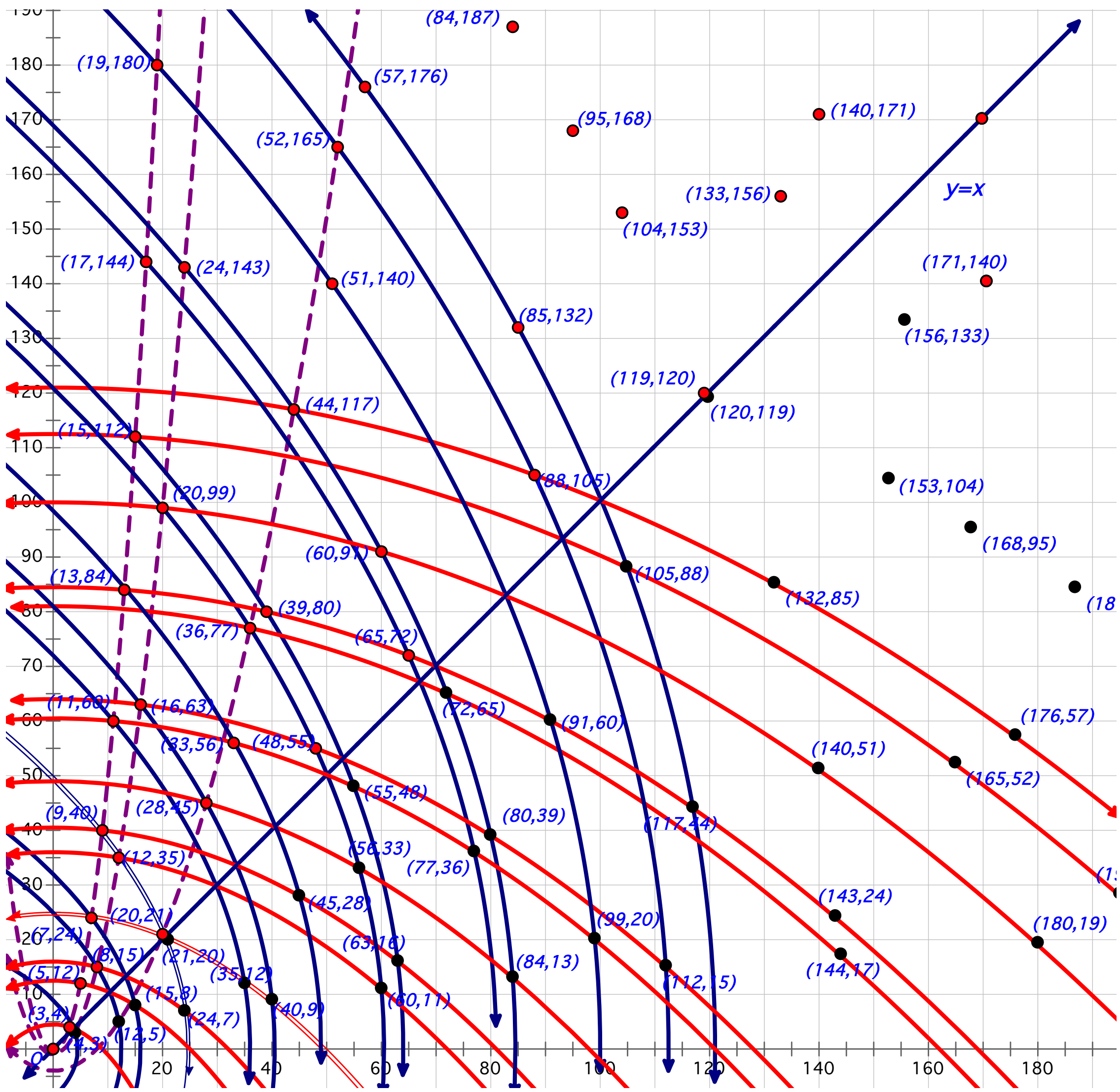


Figure 3.3 Examples of Graphs which Open about the Negative *y-axis*

$$y = -x^2/(2 \cdot n)^2 + n^2 \tag{2.8}$$

where $n = a/m$, for *"a"* the 1st coordinate in the LH point in the series, and $m = 2$, if this point is on the $d = 2$ parabola, and $m = 4$, if it's on the $d = 8$ parabola. Thus every point from a PPT on the $d = 2$ and $d = 8$ parabolas (the middle and RH red dashed lines in the graph below) determines a parabola in this set. These curves are shown in orange in Figure 3.3.

If "*a"* is *odd*, as in the example with the point $(3,4)$ above, then the equation is derived as in the following example. The points $(13,84)$, $(39,80)$, $(65,72)$, $(91,60)$, $(117,44)$, and $(143,24)$, appear to be on a parabolic curve which opens about the negative *y-axis.* The 1st coordinates of these points are all multiples of 13, the LH point $(13,84)$ is on the parabola labeled $d = 1$, and it's from the triple $(13,84,85)$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 13^2) + 13^2/2 \tag{2.9}$$

It has vertex point is $(0,13^2/2)$, and the focus point is $(0,-13^2/2)$.

For another example, consider the points $(11,60)$, $(33,56)$, $(55,48)$, $(77,36)$, and $(99,20)$. They also appear to be on a parabola which opens about the negative *y-axis.* All of the 1st coordinates of the points are multiples of 11, the LH point $(11,60)$ is from the triple $(11,60,61)$, and is on the parabola labeled $d = 1$. The equation for this parabola is:

$$y = -x^2/(2 \cdot 11^2) + 11^2/2 \tag{2.10}$$

the vertex point is $(0,11^2/2)$, and the focus point is $(0,-11^2/2)$.

The general form for the equations of the parabolas which open about the negative *y-axis,* when "*a"* is odd, is:

$$y = -x^2/(2 \cdot a^2) + a^2/2 \tag{2.11}$$

where "*a"* is the value of the *x*-coordinate of the first LH point (a,b) on the parabola labeled $d = 1$, in the series of points being considered (the LH red dashed line) Figure 3.3.

Thus every point (a,b), from a PPT point (a,b,c), on the $d = 1$ labeled parabola determines a parabola in this set. These curves are shown in orange in the graph, Figure 3.3.

When these 2 sets of parabolas, for "*a"* even or "*a"* odd, are reflected about the line $y = x$, we have the corresponding parabolas for the *d'*-values. These parabolas open about the negative *x-axis,* and are shown in dark blue in the graph above. Not all of the curves which exist in this range are shown here.

These results are stated in the following Proposition on parabolas about negative *x,y-axes.*

Proposition 3.2 Let $(a_1,b_1), \dots ,(a_k,b_k)$, be a finite sequence of points in the Main Graph such that the first coordinates $a_1, \dots ,a_k$ form an increasing sequence, and a_1 divides all of the other a_i, $i = 2, \dots , k$. If a_1 is odd, then (a_1,b_1) is on the parabola labeled $d = 1$, and the equation of the parabola which contains the points $(a_1,b_1), \dots ,(a_k,b_k)$ is then $y = -x^2/(2 \cdot a_1^2) + a_1^2/2$. If a_1 is even, then (a_1,b_1) is on the parabola labeled $d = 2$ *or* 8. The equation of the parabola which contains the points $(a_1,b_1), \dots ,(a_k,b_k)$, is then $y = -x^2/(2 \cdot (a_1/m))^2 + (a_1/m)^2$, where $m = 2$, when $d = 2$, and

$m = 4$, when $d = 8$. The reflection of these parabolas about the line $y = x$ determines parabolas which open about the negative x-axis. In the case for $d = 1$, the equation of the reflected parabola is $x = -y^2/(2 \cdot a_1^2) + a_1^2/2$, and in the cases for $d = 2$ or 8, the equation of the reflected parabola is $x = -y^2/(2 \cdot (a_1/m)^2) + (a_1/m)^2$, where $m = 2$, when $d = 2$, and $m = 4$, when $d = 8$.

That every parabola shown which opens about the negative *y-axis* has a point *(a,b)* from a PPT *(a,b,c)* on the parabolas labeled $d = 1, 2,$ or 8, follows from the observation that these parabolas are the closest ones to the *y-axis* for even or odd values of *d*.

References

[1] Heath, T. (1956). *Euclid, The Thirteen Books of the Elements, Vol. 1: Books 1-2, 2nd ed.* NY: Dover Publ.

[2] Joyce, D. (2010). *Primitive Pythagorean Triples.* Worcester, MA: Clark U. *https://mathcs.clarku.edu/~djoyce/numbers/pyth.pdf*

[3] Knott, R. (2021). *Pythagorean Right Triangles, Section 4.4.1.* UK: U. of Surrey. *http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Pythag/pythag.html#section3.4*

[4] Parks, J. (2021). *On the Curved Patterns Seen in Graphs of PPTs.* Preprint, arXiv, 2104.09449.

[5] ______ (2021). *Computing Pythagorean Triples.* arXiv 2107.06891.

[6] ______ (2021). *PPT Graph (Copy)(Copy).nb.* Mathematica Notebook Archive. *https://www.notebookarchive.org/ppt-graph-nb--2021-06-6y23baq/*

[7] KCP Technologies (2012). *Geometer's Sketchpad.* Emeryville, CA.

[8] International GeoGebra Institute (2016). *GeoGebra.* geogebra.org.

[9] Wolfram Foundation (2021). *Mathematica*. wolfram.org.

[10] Waterloo Maple (Maplesoft) (2020). *Maple.* maplesoft.com.