

The loci of virtual points constructed with elementary models of planetary orbits

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Abstract

The orbits of the planets around the Sun are ellipses, close to circles. A simple model is given by concentric circles centered at the Sun. The radius of the orbit and the linear velocity of the planet may be found in numerical form in the literature. The motion of a planet on its orbit is then described by a trigonometric parametrization. Using such parametrizations for two planets, various mobile "virtual points" such as their midpoint or their center of gravity, are defined and their trajectories are studied. A good opportunity to experience different kinds of animations with software, depending on the affordances of the package. By that way, classical plane curves appear in a dynamical way. It appears quickly that the level of accuracy of the involved numerical data influences strongly the shape of the constructed curves. Despite the fact that loci and envelopes are often related, and automated commands exist for both in the same packages, their respective appearances may be different, the one very rich and the other much apparently less. The plane curves which appear as loci of some of the virtual points have been known for a long time, but separately. We have here an opportunity to see them as members of a larger family. A byproduct of the activities is an incitement to study classical plane curves and their properties, in a dynamical framework and in relation with real world. Despite the physical non-accuracy of the models, this is a typical STEAM approach.

1 Introduction

A few months ago, the news were filled with announcements about three space probes launched in July 2020 towards Mars and which arrived to Mars surface (the American rover called Perseverance, by NASA) or were inserted in a martian orbit (a Chinese probe, and Hope, a martian orbiter from the United Arab Emirates) in February 2021. The various national space agencies had a wonderful opportunity to attract public attention to space science and technology. Numerous websites have been developed and are freely accessible to a general audience. There are good reasons for mathematics educators to be part of this atmosphere, showing applications of mathematics to real world. Maybe not a daily concern, but the same world which appears in news.

According to the 1st Kepler's law ([11], p. 127), the orbit of a planet around the Sun is an ellipse, with the Sun at one of the foci. As the foci are very close, and in order to make the example as simple as possible, we consider an approximation of the orbits as coplanar concentric circles, centered at the Sun. It will be enough to know radius (i.e. distance to the Sun, actually a mean distance) and velocity. Table 1 gives some data, not the velocity. Kepler's 2nd law is illustrated by Figure 1, taken from [11] p. 129: the areas of the shaded sectors, covered by the radius in equal times (i.e. it takes equal times to travel distances AB , CD and EF), are equal. In our very simple models, we will consider motion with constant angular velocity on circular

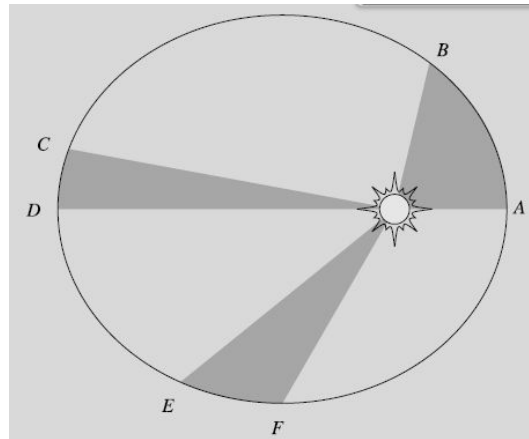


Figure 1: Kepler's 2nd law of planetary motion

orbits. We compute the velocities according to the year length of the planet, with Earth year equal to 1. Note that Table 1 displays only the eight objects, officially called planets by the international astronomical organization. According to 3rd Kepler's law, the orbital velocity is a function of the distance to the Sun.

Planet	Distance to the Sun (km)	Period (1=terrestrial year)
Mercury	57.91 10^6 km	0.2408
Venus	108.2 10^6 km	0.6152
Earth	149.6 10^6 km	1
Mars	227.9 10^6 km	1.8808
Jupiter	778.5 10^6 km	11.862
Saturn	1.434 10^9 km	29.457
Uranus	2.871 10^9 km	84.018
Neptune	4.495 10^9 km	164.78

Table 1: Some orbital data

Because of the huge differences between the distances and the hardware constraints (we mean mostly the size of the screen and the number of available pixels), we will consider examples with Earth and Mars only. The same activities can be done with the pair Venus-Earth, they will produce the same family of curves. Nte that in order to make the first examples easy, we use approximations less precise than in Table 1.

For the simulations and the animations, we use GeoGebra, a freely downloadable software, whose main characteristic is devoted to Dynamic Geometry. For some uses, including automated determination of loci and envelopes, it can be supplemented with the package GeoGebra-Discovery, also freely downloadable. A general analysis of the automated methods for loci and envelopes is given in [13]. We will also use the Computer Algebra System Maple for its specific animated affordances.

The goal of this paper is to present mathematical situations with a STEAM¹ approach where plane curves, either algebraic or not, are presented and some of their properties explored using technology. We see this kind of study as an opportunity to explore also classical families of plane curves. Nothing beyond that.

2 The midpoint Earth-Mars

Because of Mars ubiquity in the news during last year, we consider now the pair Earth-Mars and their midpoint. We choose the distance Sun-Earth as equal to 1, which is consistent with the general definition of the so-called *astronomical unit* ([11], p.24), the distance Sun-Mars is then almost equal to 1.5. We describe the respective orbital motions of the planets by the following parametrizations:

$$\begin{aligned} \text{Earth: } (x, y) &= (\cos t, \sin t) \\ \text{Mars: } (x, y) &= \left(r \cos \frac{t}{h}, r \sin \frac{t}{h} \right) \end{aligned}$$

where $1/h$ encodes the angular velocity of a planet on its orbit and r denotes the ratio between the radii of the orbits. For Mars, we will take $h = 1.881$ or a coarser approximation. The t -interval on which to consider these parametrizations has to be defined later, either experimentally after some trials with the software, or by computation of the number of needed orbits, let's say for the Earth, in order for the pair of planets to return to the initial position. For integer values of the parameters, a LCM of two integers has to be computed; otherwise, the computation is harder and an experimental approach will be often preferred. This general setting allows to construct a general applet, useful for different pairs Earth-other planet, and will be translated by GeoGebra into a slider bar.

Of course, when proposing to students such an activity, an educator should make clear that this midpoint has no physical meaning. Figure 3 illustrates this. In July 2020, the NASA launched a rover called *Perseverance* which landed on Mars in February 2021. The trajectory was not a segment of line, and the trajectory in space was roughly the union of arcs of ellipses. Figure 3(b) shows the midpoint on the trajectory; it is easy to see that this point has no relation with a midpoint of a segment Earth-Mars. The simplified model that we will use in the activities contributes to this understanding.

¹STEAM = acronym for Science, Technology, Engineering, Arts and Mathematics.

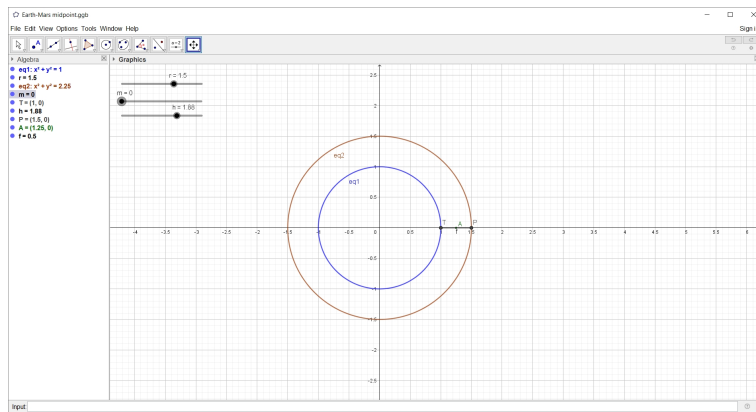
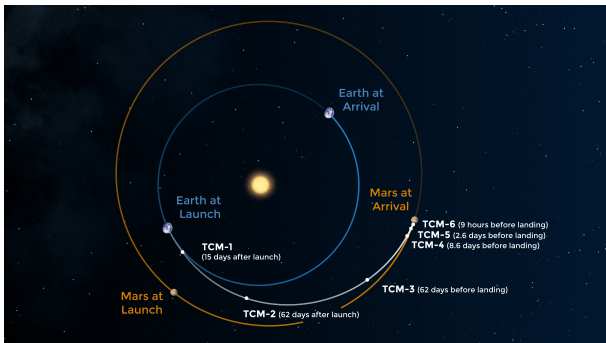
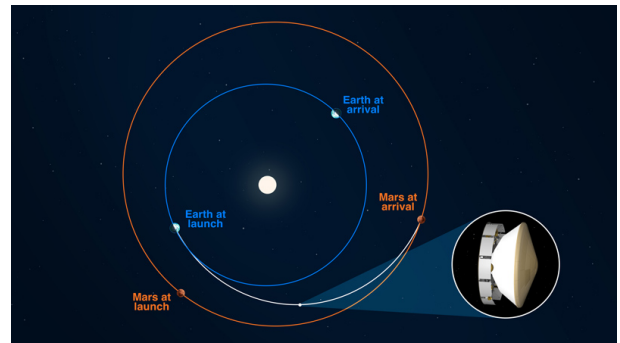


Figure 2: A snapshot of the general applet for the midpoint of two planets



(a) The trajectory - an arc of ellipse



(b) At midpoint of the space trip

Figure 3: Perseverance's trajectory from Earth to Mars (Credit: NASA/JPL-Caltech)

2.1 First example: $r = 1.5$, $h = 1.9$

The midpoint of the pair Earth-Mars is described by the parametrization

$$\begin{cases} x = \frac{1}{2} \left(\cos t + 1.5 \cos \frac{t}{1.9} \right) \\ y = \frac{1}{2} \left(\sin t + 1.5 \sin \frac{t}{1.9} \right) \end{cases} \quad (1)$$

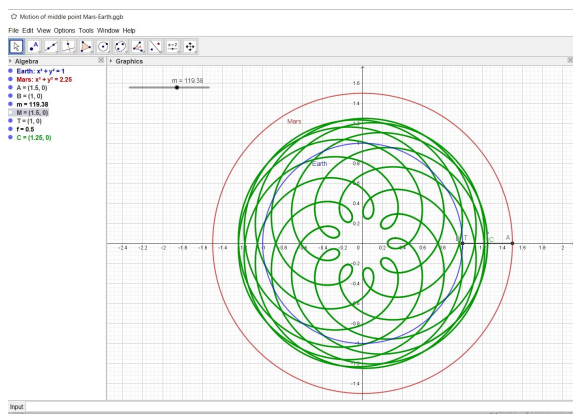
In Figure 4, the orbit of the Earth (point T) is represented in blue and the orbit of Mars (point M) in brown. The midpoint is denoted by A and is plotted in dark green. Note that in the applet, a slider for a parameter m instead of t is used, as t has a special meaning in GeoGebra. There are two ways to use the applet from which we have here a snapshot:

- Automated command: use the Locus command, either typed in the command line or using a specific button. The syntax is as follows:

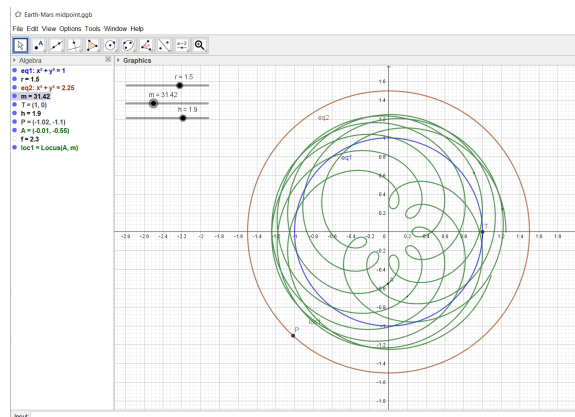
```
Locus(<Point creating the Locus line>,<slider>)
```

. Here we need **Locus(A,m)**.

- Dynamics: Use **Animation On** for the slider m . The curve is created dynamically.



(a) Animated construction



(b) Automated construction with **Locus** command

Figure 4: The locus of the midpoint Earth-Mars - first attempts with $h = 1.9$

In both cases, if a defining interval for the parameter is "too short", then the obtained curve is incomplete. Either experiments and successive trials or computations, depending on the computational skills of the user, lead to a suitable interval. The first option may be time consuming. Figure 4(a) has been obtained within 10 minutes with **Animation On** for the slider and **Trace On** for the point A . Figure 4(b) shows a incomplete attempt with the **Locus** command: the given interval for the parameter m is $[0, 30\pi]$ and is too small.

Remark 1 Figure 4 displays both the algebraic and the graphical windows of the GeoGebra session. The algebraic window is easily written, and the order of commands can be checked with the windows displaying **Object Properties**, or better, with the windows displaying the **Construction Protocol**, accessible from the main menu.

The same activity can be performed using the animate command of Maple. In the following code, the animations to simulate the moving of Earth and of Mars on their orbits are defined separately, then displayed together. The resulting plot is quite trivial, as it creates dynamically two concentric circles. This may be used as a preliminary control, before plotting the animation for the midpoint, which comes afterwards.

```
restart;
with(plots):with(plottools): setoptions = (thickness = 2):
earth := <cos(t), sin(t)>;
mars := <1.5*cos(t/1.9), 1.5*sin(t/1.9)>;
plotearth := plot([earth[1], earth[2], t = 0 .. 2*Pi], color = blue);
plotmars := plot([mars[1], mars[2], t = 0 .. 4*Pi], color = brown);
display(plotearth, plotmars);
animearth := plots[animate](plot, [[earth[1], earth[2], t = 0 .. A],
    color = navy], A = 0 .. 2*Pi);
animmars := plots[animate](plot, [[mars[1], mars[2], t = 0 .. A],
    color = red], A = 0 .. 4*Pi);
animmidpt := plots[animate](plot, [[1/2*(earth[1] + mars[1]), 1/2*(earth[2] + mars[2]),
    t = 0 .. A], color = green], A = 0 .. 28*Pi);
display(animearth, animmars, animmidpt);
```

At start, the plot is empty; it is made alive by a left click, then by clicking on the suitable button in the command row shown in Figure 5. The animation speed is controlled from the command row to. Eventually, to improve the visualization, the segment TM can be added to the animation, using the **line** command..

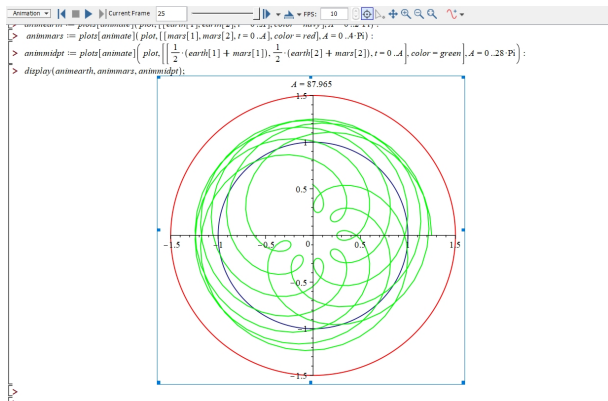


Figure 5: Snapshot of a Maple animation

2.2 Second example: $r = 1.5$, $h = 1.85$

We modify slightly the approximation of the parameter h encoding the period of Mars on its orbit, and take here $h = 1.85$ and $h = 1.86$. Snapshots of the resulting geometric loci of the midpoint of the two planets are displayed in Figure 6. The small modifications of the value of

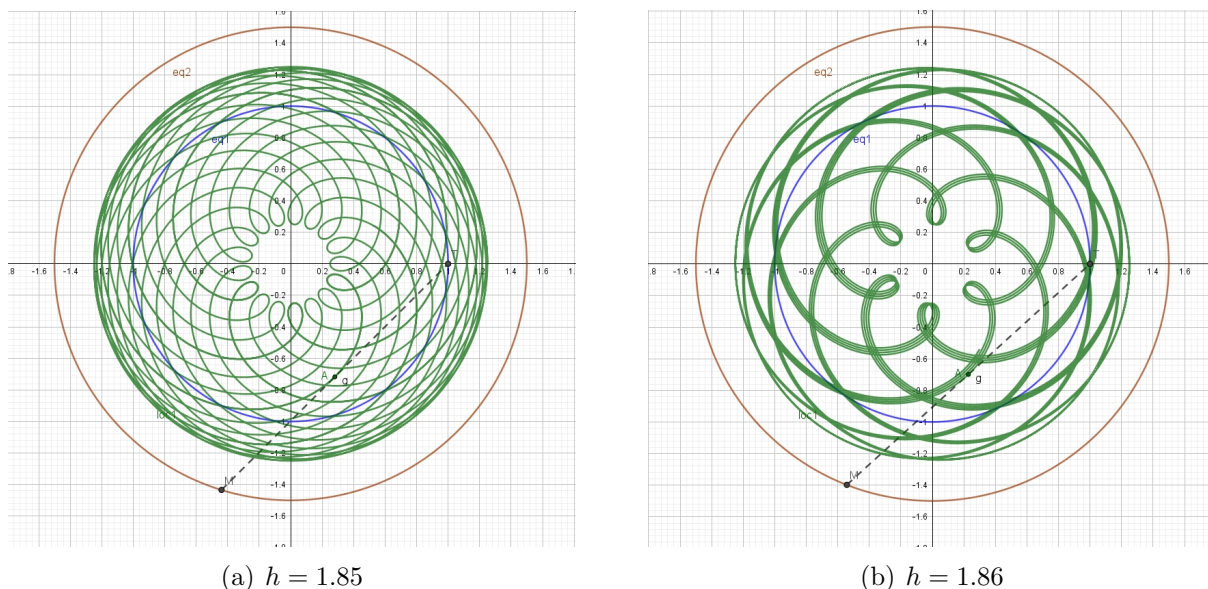


Figure 6: The locus of the midpoint Earth-Mars: the influence of the approximations

the parameter h result in very different geometric loci. The existence of rotational and axial symmetries has been preserved, but their nature has changed.

2.3 Generalisations.

A GeoGebra applet is available to explore the influence of the different parameters and of changing the rounding for numerical values. A screen snapshot is displayed in Figure 7. Actually, the ratio of the radii of the orbits is much closer to 1.881. The resulting geometric locus is still more complicated than Figure 6(a). Note that rounding has to be changed to 3 digits in order to use efficiently the slider and fix $h = 1.881$; to change the increment in the **Object Properties** windows is necessary, but not enough. Pay attention to the 3 sliders on the screen. Their range

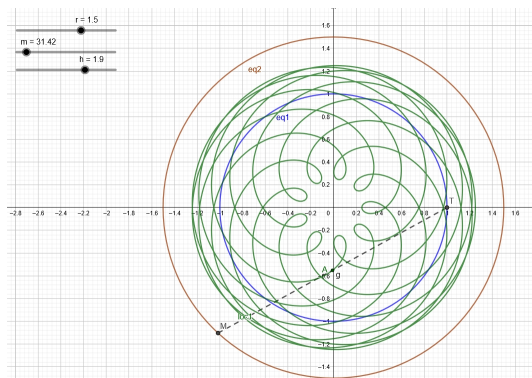


Figure 7: An applet to generalize the previous study

can be modified, most of the possible values cannot be considered as good approximations of the celestial parameters, but enable a broader exploration of the family of curves.

Further generalization is provided as follows:

- The ratio r of the radii of the orbits can be changed. For example for Venus, we have $h = 0.6152$ and $r = 0.723$ (see Table 1).
- The masses of the two planets may be taken into account, if interested in the center of gravity of the pair of planets.

A complete GeoGebra applet is available for such an exploration. Once again, nice plane periodic curves are created. Note that this applet allows any value for each parameter, which is not true in the real world, first because we use a very simplified model with circular orbits, and further according to Kepler's laws, which establish a relation between the distance to the Sun and the velocity on the orbit. Once again with the simplified model, Figure 8 shows the locus of the center of gravity of the pair Earth-Mars, with a rounding of 4 digits.

Remark 2 *Different values of the parameters yield different closed curves with rotational symmetries. This is clear already from the trigonometric parametrization given for the curves and the midpoint. When an algebraic curve is given, an important question is whether it has a rational parametrization or not [15]. It is well known that rational parametrizations are available for a circle. Here above, it was more efficient to use trigonometric parametrizations, not only to have a more "linear" behaviour of the animations, an issue mentioned and studied in [4, 7], but in order to reflect the motion of each planet on its orbit, with constant velocity. This will be useful in Section 3.*

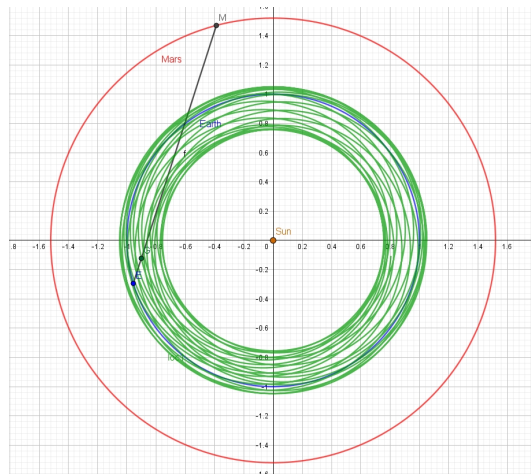


Figure 8: The trajectory of the center of gravity Earth-Mars

3 Identification of classical curves

The parametric presentations that are in use in the previous sections have a too specific form not to be already known among the plane periodic curves. A quick websearch reveals that they are special cases of the family of *epitrochoids* (from the Greek "trochos"=wheel). These curves are defined by the following equations (see [18], p.233-234, and the MacTutor website:

$$\begin{cases} x = (a + b) \cos t + c \cos \left(\frac{a}{b} + 1\right) t \\ y = (a + b) \sin t + c \sin \left(\frac{a}{b} + 1\right) t \end{cases} \quad (2)$$

where a, b, c are real parameters. With these notations, let C_1 be a circle of radius a and C_2 a circle of radius b rolling outside C_1 . A point P at a distance c from the center of C_2 rolls together with C_2 . Its locus is an *epitrochoid*.

With the same simple model of coplanar concentric circular orbits as above, another description of epitrochoids is as the apparent trajectory of a planet when seen from another planet. Derivation of equations in this setting is beyond the scope of this paper. The obtained animations explain why, in certain circumstances, a planet seems to travel in reverse direction (what is called *retrograde motion*). Figure 9 shows an illustration by Kepler of Mars motion on its orbit, viewed from Earth (see for example [10] p. 86).

For special values of the parameters, well-known plane curves are obtained. For example, if $a = b$, i.e. C_1 and C_2 have the same radius, the corresponding curve is a Pascal Limaçon, for other values it is a nephroid, etc.. The illustrations in Figure 10 have been obtained with the following Maple code with $p = 1$:

```
x := 1/2*(p*cos (t) +k*cos(t/k));
y := 1/2*(p*sin(t) + k*sin(t/k));
for k from 2 to 6 do
  plot([x, y, t = 0 .. 20*Pi]);
end do
```

With $p = 3$, Figure 11 is obtained:

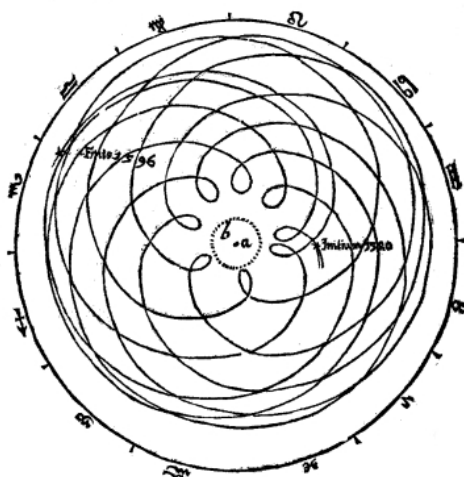


Figure 9: Kepler's drawing of Mars orbit viewed from Earth

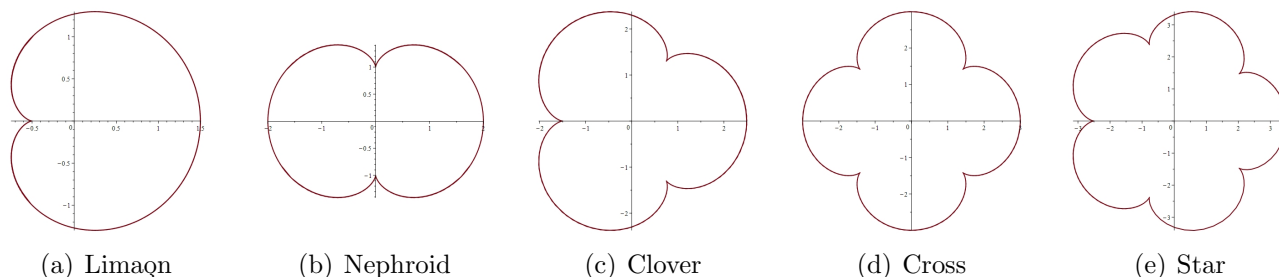


Figure 10: Special epitrochoids

3.1 Envelopes

A detail appearing in Figure 6 has not been mentioned until now: the segment TM is there, dotted. It may help to visualize the coordinated motion of the 3 points T, M, A . Another feature may be worth of an exploration: does the family of segments TM have an envelope?

A first exploration may be performed using the DGS. Figure 12 shows two trials with different values of r and h .

With the current settings, i.e. when everything depends on the parametric presentation of the points T and M , we cannot use GeoGebra's command **Envelope**. It has been developed on another basis. Four different definitions of envelopes are given and their similarities and differences analyzed in [3]. Here we consider one of their definitions, the only one considered by Berger [2]3](sections 9.6.7 and 14.6.1). Some recent developments and examples can be found in [6].

Definition 3 Let C_m be a family of plane curves given by an equation of the form $F(x, y, m) = 0$, where x, y are real variables and m is a real parameter. An envelope of this family, if it exists, is the solution set of the system of equations:

$$\begin{cases} F(x, y, m) = 0 \\ \frac{\partial F}{\partial m} F(x, y, m) = 0 \end{cases} \quad (3)$$

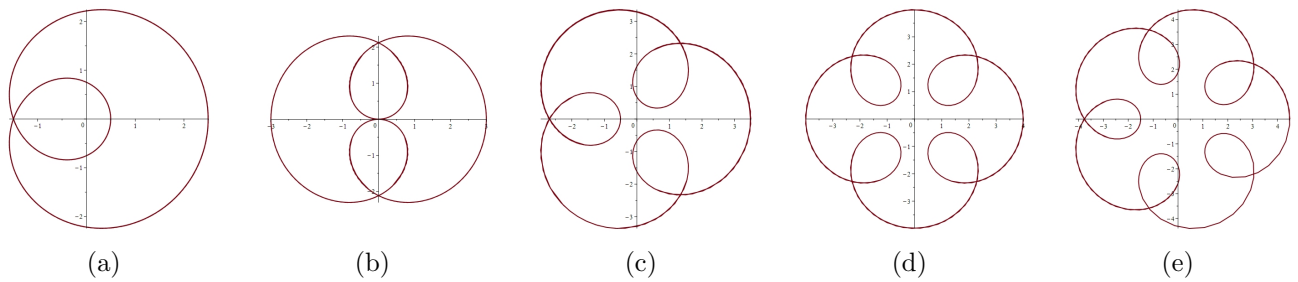


Figure 11: Other epitrochoids

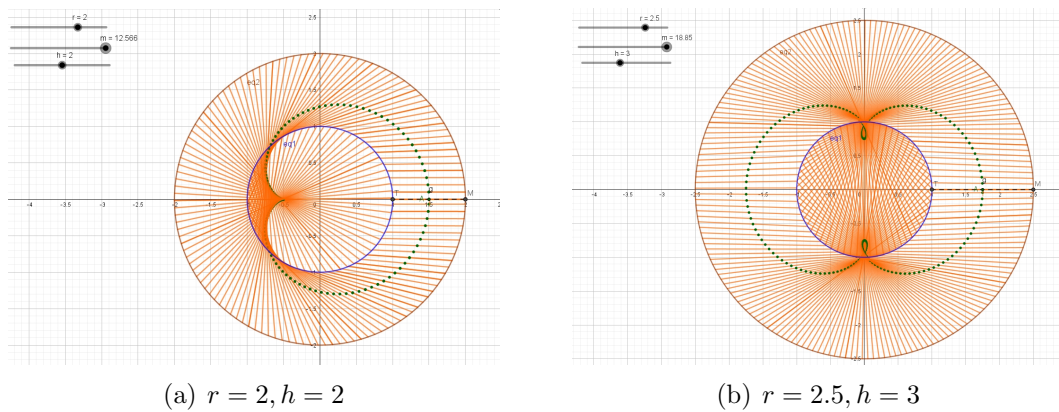


Figure 12: Experiments to determine envelopes graphically

Actually we will look for an envelope of the family of lines TM . It is easily shown that the family has no fixed point, so an envelope should exist. First we compute an equation for the lines; the role of the parameter m in Definition 3 will be played by the parameter t in the parametric presentation of the moving planets. Working with Maple, here is the source code of a suitable session:

```
restart;
with(plots);with(LinearAlgebra);
T := <cos(t), sin(t)>; # Earth
M := <r*cos(t/h), r*sin(t/h)>; #Mars
mat := Matrix(2, 2, [[x - T[1], M[1] - T[1]], [y - T[2], M[2] - T[2]]]);
line := Determinant(mat); # implicit equation of the line TM
dline := diff(line, t);
solve({dline = 0, line = 0}, {x, y}); #the solutions determine the desired envelope
env:=simplify(%);
```

An implicit equation for the line TM is

$$x \left(r \sin \frac{t}{h} - \sin t \right) + y \left(\cos t - r \cos \frac{t}{h} \right) + r \left(\cos t \cos \frac{t}{h} + \sin t \sin \frac{t}{h} \right) = 0$$

and, after simplification:

$$x \left(r \sin \frac{t}{h} - \sin t \right) + y \left(\cos t - r \cos \frac{t}{h} \right) + r \sin \left(\frac{h+1}{h} t \right) = 0. \quad (4)$$

We denote by $F(x, y, t)$ the polynomial in left hand side. Solving now Equations (3) we obtain the output of last command row above::

$$\begin{cases} x = \frac{(h \cos(\frac{t}{h}) - \cos(t))(-\sin(\frac{t}{h}) \sin(t) + \cos(t) \cos(\frac{t}{h}) - 1)}{\cos(t)(h+r) \cos(\frac{t}{h}) + \sin(t)(hr+1) \sin(\frac{t}{h}) - h - r} \\ y = \frac{(\cos(t) \cos(\frac{t}{h}) hr - \cos(t)^2 + h + 1) \sin(\frac{t}{h}) + \sin(t) (\cos(\frac{t}{h})^2 hr - \cos(t) \cos(\frac{t}{h}) - r(h+1))}{\sin(t)(hr+1) \sin(\frac{t}{h}) + (\cos(t) \cos(\frac{t}{h}) - 1)(h+r)} \end{cases} \quad (5)$$

The first output displayed by the CAS is quite complicated, but **simplify** commands are helpful. Nevertheless, the final display with general parameters is still heavy and trying to identify the curve using catalogs of curves and websearch is unilluminating, even hopeless. Moreover, in order to try and implicitize the parametric presentation, methods such as substituting rational expressions for sine and cosine may be considered, but even for the CAS it can be a heavy task. Substituting specific values for the parameters, the plot command yields a plot of the curve, and the result is somehow surprising (an example is given in Figure 13). Graphical superposition of the curves obtained with an animation as in Figure 12 and 13 showed differences. We have here still an open question.

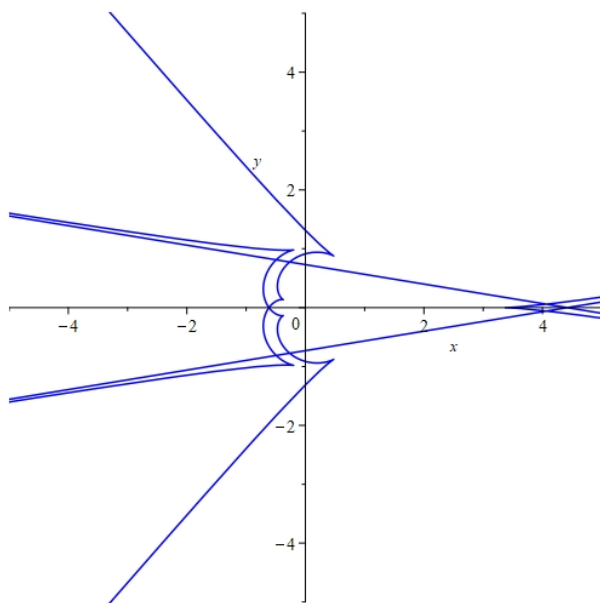


Figure 13: The envelope of lines TM for $r = 1.5$ and $h = 1.9$

And, after all, the literature, either printed or electronic, devoted to the epitrochoids, show only integer values for the parameters and do not display any implicit equation, which can mean that these curves are not algebraic. We invite the reader to check.

4 Discussion

This is not the first time that we analyze the different contributions of a CAS and of a DGS for exploring the same question. Generally, the roles are complementary, and with the help of both kinds of software together, letting them having a dialog, a precise insight into the problem is obtained [6, 7].

Here a first exploration has been done with the DGS, using the following features:

- The numerous options for the animation (interval, speed, oscillating, etc.) may be changed in the **Object Properties** window, which has to be opened. Sometimes, numerous trials are needed.
- The animated construction and the plot of the locus are driven by different commands.
- Zoom-in and zoom-out are easy to perform with the mouse. This is important to discover properties of the curves in neighborhoods of singular points. As an example, see the study of the relations between the cusps of a given curve and of its offsets [6]. A strong zooming with the DGS provided conviction, and an algebraic proof was not needed. Here with non integer values of the parameters, zooming is strongly needed to distinguish arcs through multiple points, as can be seen in Figure 8.
- Modifications (size, proportions, colors, etc.) may be performed quickly with the mouse, for example with a right-click on the object.

Exploration with the CAS is slightly different.

- After a command line to define a parametric curve, an **animate** command has to be entered. Its output is not immediately visible.
- A left-click on the graphical window is necessary, and it switches automatically to the row of graphical buttons.
- Here too, the relevant values for the parameters (number of frames, speed, etc.), in order to obtain a significant graphical output have to be looked for experimentally, using the buttons.
- Other modifications of the output may require changes in the written commands.

The animation speed is quite different with the two software packages². A complete plot for the midpoint of Earth-Mars with $h = 1.881$ (see Figure ??) has been obtained with Maple within a few seconds; with GeoGebra it took much longer. Interesting conjectures and results can be obtained with a single software only, but joint work (we should say collaboration between packages) provides a richer insight. We do not always have a strong dialog between the two kinds of software, as in [7], but the collaboration man-and-machine in this triangular setting is efficient and enriches the study.

The starting point of the study is STEAM oriented, namely using a scientific model from an item in the news. Students may have prior interest in the domain, without having a strong knowledge. The present topic offers an opportunity to collaborate between educators, between man and machine, of course between students. The study output is multiple, and among the "rewards" we have:

- Acquisition of new mathematical knowledge: classical curves (epitrochoids), which are not part of the regular curriculum, have been discovered and studied. Epitrochoids are members of a larger family of curves, which involves roses, epicycloids, etc. Activities as

²We wish to mention that we worked with both packages on the same computer.

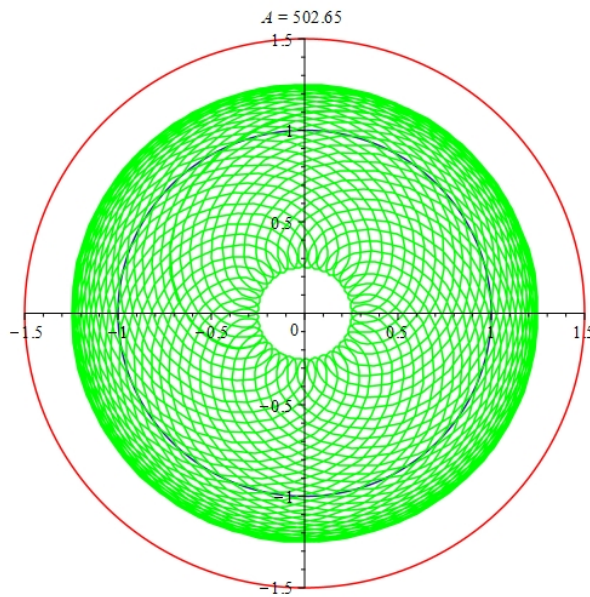


Figure 14: The locus of the midpoint Earth-Mars with $h = 1.881$

in this work may be a nice incitement to explore other situations and to broaden horizons. The literature describes generally the epitrochoids for integer values of the parameters, and our experimentations showed also more general settings.

- Emphasis on the importance of the data precision (in space, contrary to most classrooms, nothing is measured by integers) and of rounding. We considered non integer ratios of radii of orbits, and of orbital angular velocity, approximations and rounding became an important issue. We could discover that different precisions in the approximation yield very different output. This is probably a crucial product of this work: students do not always believe that mastering errors is important, and they believe that the answers provided by a numerical calculator are always accurate. Asking them which answer is true among the cases that we studied with different rounding should lead at least to some questioning.
- Emphasis on the differences between results obtained by a graphical method and results obtained by an algebraic method. There may be a difference between what you see and what actually exists. Zooming appeared as an important feature. More general studies are available in [12, 8, 19].
- Development on new technological skills, which are part of the new mathematical knowledge [1].
- Emphasis on multidisciplinary tasks, whence development of STEAM skills.

In the present work, the geometric locus of the midpoint of two planets (within a very simplified model of motion) has been explored using the dynamical features of the DGS. An automated method could be applied to find this locus for various values of the parameters. Then sliders allow to modify interactively the values of the parameters and to observe the

consequent changes in the obtained loci. Algebraic work was not strongly needed, but could add another layer to the study. We did not elaborate on this topic, as our discovery of the different loci for positive integer values of the parameters showed classical plane curves, namely the epitrochoids. Anyway, the different websites devoted to the curves that we discovered provide parametric presentations and polar equations, but no implicit equation. This is a good enough reason for the educator not to push in that direction, and purely algebraic methods have not been applied.

The fact that we started from a concrete question with planets had two goals: to attract students to explore mathematical situations linked to topics from their everyday life. We have here a good example of the importance of teaching mathematics in relation with the cultural background of the students. In [5, 9] various situations accessible at an early stage of the curriculum have been proposed. Here we propose a more advanced environment, where classical mathematics, sometimes already known in Antiquity, model very modern situations. Some websearch will enrich more the new developed knowledge. In 21st century, STEAM is the name of the game; around curves, models of planetary motion, usage of software, we propose a truly STEAM oriented activity.

WE devoted a brief section to envelopes. This, because envelopes are a classical topic with numerous applications, and they provides also new constructions of classical curves, as in [4, 6]. Moreover, the notions are related, and automated commands exist in the same technological environments for loci and envelopes. Nevertheless, it happens, and we have here an example, that one topic provides richer results than the other.

Finally, we wish to emphasize that among the components of the STEAM approach, we focused on STEM and did not address issues involving Arts. No need to say that the curves which have been discovered provide patterns for rosette-like design, generally more complicated than the traditional rosettes on European monuments. The reader's imagination will do the rest.

References

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