Approaching Cesàro's inequality through GeoGebra Discovery

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Abstract

We illustrate the possibility of developing, using the automated reasoning tools implemented in the dynamic geometry program GeoGebra Discovery, a certain parallelism with Michael de Villiers reflection on the 'discovery function' of proof, as described in his 2012 paper concerning the formulation by one student of a certain geometric conjecture (Clough's conjecture).

1 Motivation and goals

Michael de Villiers is a well know emeritus professor from the University of KwaZulu-Natal and an honorary professor at the University of Stellenbosch, both in South Africa, with worldwide leading research contributions along the past 30 years on Dynamic Geometry and on the role of proof and reasoning in mathematics education. Visit, for example, his web page `http://dynamicmathematicslearning.com/homepage4.html` for a relation of his publications with links to some of them.

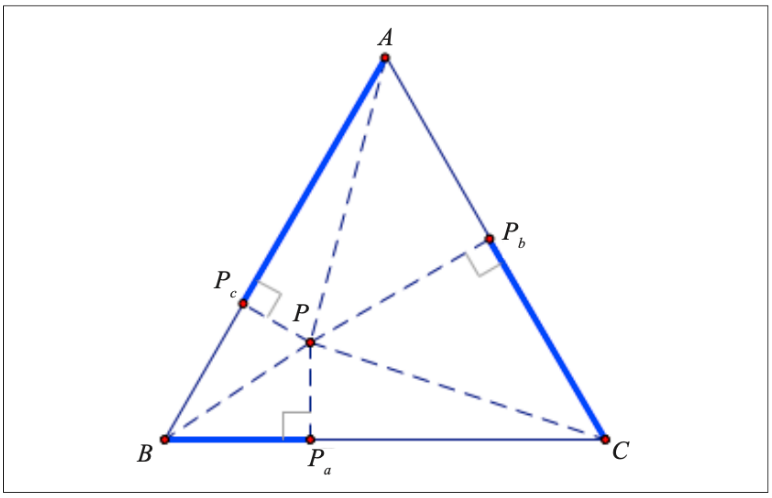
In 2004 he conducted workshops at two different conferences (AMESA[1], South Africa, and NTCM[2], USA), dealing, in particular, with what he labeled as "Clough's Conjecture"[6]. Later on, based on a presentation by himself at the 12th International Congress on Mathematics Education (ICME, July 2012, Seoul, Korea), de Villiers published an article [7], mentioning his contributions to the above Conferences and describing its content as follows ([7], p.3; Figure 1 below is reproduced from there):

> The main purpose of this article is to contribute further to the theoretical aspects of the role of proof by providing a heuristic description of some of my personal experiences of the explanatory and discovery functions of proof with a geometric conjecture made by a Grade 11 student.
>
> [...]

[1] 10th Association for Mathematics Education of South Africa (AMESA) Conference, 30 June-4 July 2004, Potchefstroom, South Africa, `http://www.amesa.org.za/AMESA2004/`

[2] National Council of Teachers of Mathematics (NCTM) Annual Meeting, April 2004, Philadelphia, USA

During 2003, a Grade 11 student from a high school in Cape Town was exploring Vivianis theorem using dynamic geometry. The theorem says that the sum of distances of a point to the sides of an equilateral triangle is constant (i.e. in Figure [below] $PP_a + PP_b + PP_c$ is constant, irrespective of the position of point P inside triangle ABC). The students further exploration led him to measure the distances AP_c, BP_a and CP_b, and then add them. To his surprise, he noticed that $AP_c + BP_a + CP_b$ also remained constant no matter how much he dragged P inside the triangle. However, he could not prove it.



His teacher eventually wrote to me to ask whether I could perhaps produce a simple geometric proof, as he himself could only prove it algebraically by means of coordinate geometry. Below is the geometric proof I first produced, followed by further proofs, explorations and different generalisations of what has become known as Clough's conjecture ([6]).

[...]

The underlying heuristic reasoning is carefully described in order to provide an exemplar for designing learning trajectories to engage students with these functions of proof.

In summary: in [7] de Villiers describes a Grade 11 student (Clough) experiment with Dynamic Geometry, initially aiming to prove Viviani's theorem, but at some point deriving towards the formulation of a different statement, conjectured by the student. And de Villiers develops the different steps (proving, discovering, conjecturing, generalizing, proving again ...) involved in this experience, as a useful example for the analysis and design of learning paths concerning the role of proof in mathematics.

It is our goal here to follow, in some sense, the same storyline, but replacing Clough's protagonist role by the performance of our dearest "personal geometry assistant", the program *GeoGebra Discovery*[3], whose main characteristics and features have been already described in

[3]See `https://github.com/kovzol/geogebra-discovery`, and `http://autgeo.online/geogebra-discovery/`. *GeoGebra Discovery* is available in two options: GeoGebra Classic 5, for Win-

[2], [16] or in the Asian Technology Conference in Mathematics (ATCM) 2020 invited lecture, entitled “GeoGebra Reasoning Tools for Humans and for Automatons” [11]. See the next section for a short collection of basic examples illustrating the different tools and possibilities of *GeoGebra Discovery*.

On the other hand, as a more challenging instance of the performance of this program, Figure 1 shows the answer of *GeoGebra Discovery* to Clough’s conjecture, namely, when asked for the relation between

- the sum of segments $l = EC$, $m = FB$, and $n = GA$, where E, F, G are the feet of the perpendiculars to the sides of the equilateral triangle from an arbitrary point D,
- and $3p/2$, where p is the length of side AB, i.e. $3p/2$ is half the perimeter $3p$ of the equilateral triangle ABC.

Here *GeoGebra Discovery* declares that such relation is “true on parts, false on parts”, an answer that we will explain and comment in the last part of this paper and that illustrates some of the issues we would like to address here.

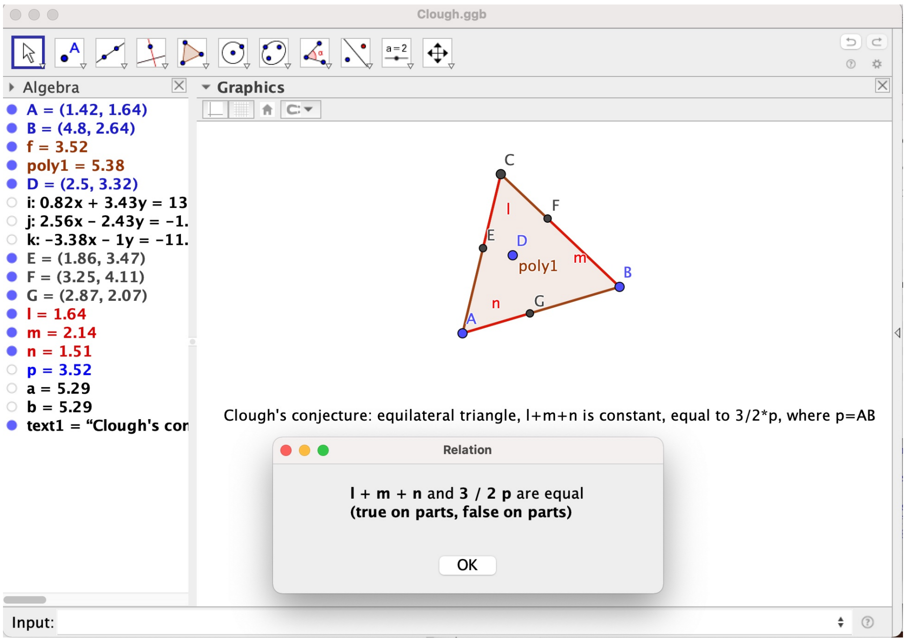


Figure 1: Checking Clough’s conjecture yields ”’true on parts”

Of course, we are aware that the intended parallelism between de Villiers’ mentioned paper [7] and our work here, has several limitations. Leaving aside the obvious differences of expertise from the authors of both articles, it is clear that de Villiers’ aims to exemplify the educational relevance of the heuristics involved in the production of traditional proofs. This interaction: proof/heuristics, i.e. what he calls “…the ’looking back’ discovery function of proof…” [7], p. 7, could

dows, Mac and Linux systems; and GeoGebra Classic 6, made for starting it in a browser, mainly ready for use on tablets and smartphones

> ...at least acquaint students with the idea that a deductive argument can provide additional insight and some form of novel discovery ... Problem posing and generalisation through the utilisation of the 'discovery' function of proof is as important and creative as problem-solving itself, and ways of encouraging this kind of thinking in students need to be further explored.

And then he emphasizes that this 'discovery' function of proof is, in his own words, something to remark over the more traditional functions, such as 'verification':

> ...Instead of defining proof in terms of its verification function (or any other function for that matter), it is suggested that proof should rather be defined simply as a deductive or logical argument that shows how a particular result can be derived from other proven or assumed results; nothing more, nothing less. It is not here suggested that fidelity to the verification function of proof is sacrificed at all, but that it should not be elevated to a defining characteristic of proof. Moreover, the verification function ought to be supplemented with other important functions of proof using genuine mathematical activities as described above.

On the contrary, as described in the ATCM 2020 proceedings [11], *GeoGebra Discovery* facilitates

> ...the exploration, by humans, of geometric tasks by using GeoGebra as a kind of "symbolic geometry calculator": the user poses a concrete geometric task and GeoGebra provides a mathematically sound answer.

That is, as a "symbolic geometry calculator", *GeoGebra Discovery* main feature is, precisely, the verification of geometric statements, without bringing any human readable argument for their truth or falsity. Roughly speaking, as we will show in the next section, it provides just a "yes/no" answer to a certain query posed by the user. Therefore, in some sense, the use of this technological tool can not enhance the 'discovery function' of proof, since there is no proof at all!

Thus, bearing in mind this drastic affirmation, why do we regard in this paper the possibility to follow with *GeoGebra Discovery* a parallel path to the one established in de Villiers' cited work?

Answer: in three different ways. First of all, trying to imitate de Villiers' discourse on the opportunities brought by the 'discovery function' of proof, but now concerning the 'extended discovery opportunity' of geometric properties that comes when having an 'oracle' at hand, just as having a numerical calculator at our disposal can help us finding out numerical properties. This has already been sufficiently argued in some recent works of ours, such as [13], [10], [18], [12], [16], thus we will not address in more detail this issue here.

Second, replicating de Villiers' route towards proving and generalizing Clough's conjecture, now through the analysis of the problems and difficulties shown by the algorithms involved in the *GeoGebra Discovery* commands when dealing with different statements. We will exhibit how these elements conform a sort of 'discovery function' of automated proof, describing a helpful learning path to discover algebro-geometric properties of the involved figures for more advanced mathematics students, teachers, researchers....

Third, as a way to describe, again, a learning path, this time not for the human user, but for the "personal geometry assistant", i.e. for the researchers and programmers involved in its debugging and improvement.

2 GeoGebra Discovery: a short digest

We have described *GeoGebra Discovery* as a kind of 'oracle'. Indeed, the most obvious command in this context (see [13] for a sort of tutorial) is the one that answers to the quest to *Prove*... a certain relation between two geometric objects in a figure. For example[4] consider a parallelogram $ABCD$ and let E, F, G, I be the midpoints of the sides. Then we would like to verify if the line $j = EF$ and the line $k = IG$ are parallel. The answer is displayed in Figure 2.

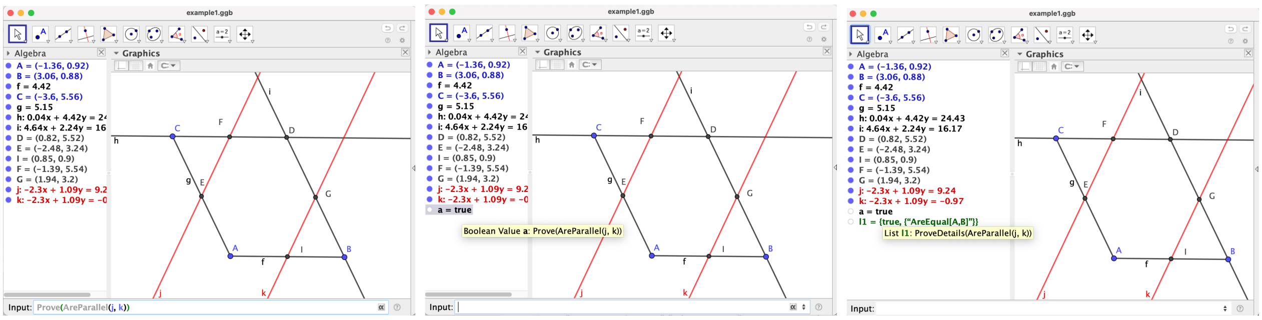


Figure 2: Left, input: Prove(AreParallell(j,k)). Center, output: True. Right, details: true in the case of non-degeneracy (i.e. with non-coincident initial points, $A \neq B$).

To the left of the Figure, in the input line, the command *Prove(AreParallell(j,k))*. The output is presented in the central image, with the concise *true* reply. To the right, after introducing the user the demand for details through the *ProveDetails(AreParallell(j,k))* order, *GeoGebra Discovery* points out that it is required –for the parallelism of j and k– that the construction does not colapse, i.e. that points A and B, that are the starting points for constructing the parallelogram (together with point C, displaying some segments $f = AB, g = AC$ and then some lines h, i parallel to f, g (respectively) passing through C and B, etc.), are different.

Now, a key feature of *GeoGebra Discovery* is the *Relation* command, that allows the user to ask the program to formulate possible relations holding between two elements of a figure. That is, while the *Prove* command requires the user to "guess" a statement that will be confirmed or denied by the program, the *Relation* automatically tests, numerically, a collection of potential statements involving two selected elements and outputs, in a first step, some relations that seem (apparently) to hold true. Then, after clicking on the *More...* button, *GeoGebra Discovery* confirms or denies the rigorous truth of the automatically suggested statement. See Figure 3.

Next, an ample generalization of the *Relation* command is the *Discover* command, that does not even require the user to point out two possible elements, but just one, for example, point E in Figure 4. Then, *Discover(E)* launches a collection of *Relation* tests between E and other elements in the geometric construction, yielding, and visually highlighting, a list of statements that are true involving point E. See Figure 4.

Finally, let us mention (and exemplify with an elementary, yet surprising, result) the recent extension of these *GeoGebra Discovery* tools to deal with statements involving inequalities [17].

[4]Following de Villiers' inspiring paper, we will on purpose restrict in general to simple examples, of school level, although *GeoGebra Discovery* is able to deal with quite complicated ones, arising for instance, in mathematical contests, university entrance or professional selection exams, see [14], [15], [16].

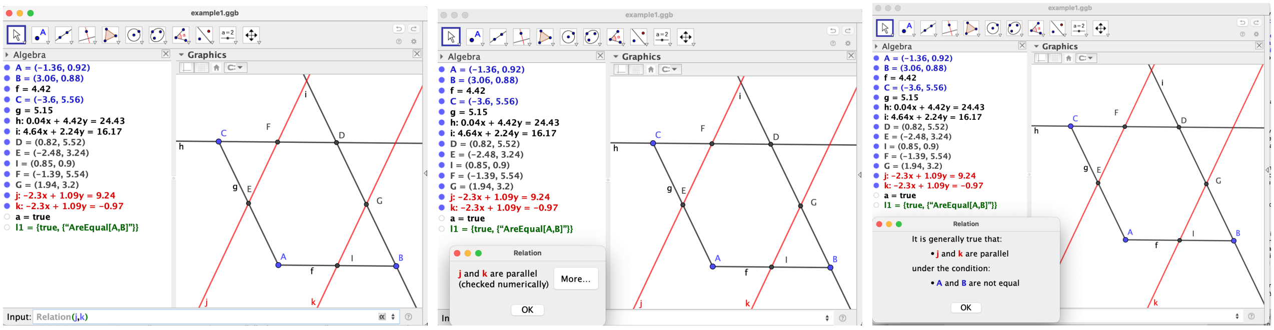


Figure 3: Left, input question: is there any Relation(j,k)? Center, initial output: numerical parallelism of the two lines. Right, symbolic verification: parallelism of j, k is a mathematically sound statement.

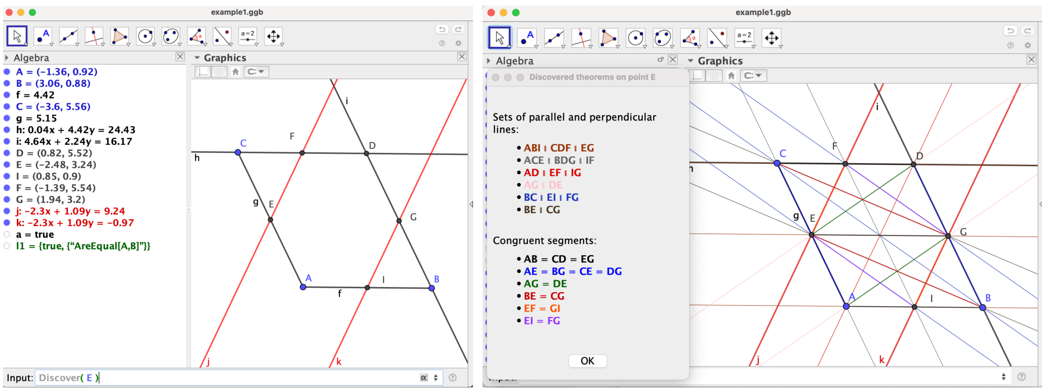


Figure 4: Left: input, Discover(E). Right: output, a collection of true statements on parallel or perpendicular lines or congruent segments, involving E.

Thus, in the next section we will describe an attempt to investigate the basic inequality number 1.4 (from E. Cesàro's [4], p. 140) included in the classic book by Bottema et al. [3], p. 12. Let us remark that the original proposal by Cesàro (see Figure 5) includes, as a footnote, the fact that for equilateral triangles the inequality is an equality, while Bottema's formulation of the same inequality adds that this only holds in this equilateral case.

How can *GeoGebra Discovery*'s very concise answers help developing 'learning paths' associated to some kind of 'discovery' function of proof in this context? We will attempt to exemplify our point of view on this issue in the next section.

3 Cesàro's inequality

Let us start by asking *GeoGebra Discovery* for the relation between the products $(a+b)\cdot(a+c)\cdot(b+c)$ and abc, where a, b, c are the lengths of the sides of a triangle. GeoGebra replies almost immediately (over an old MacBookPro 2.5 GHz) presenting the –perhaps– unexpected inequality $(a+b)\cdot(a+c)\cdot(b+c) \geq 8abc$. See Figure 6.

Next, we try to provide some reasons that justify GeoGebra's answer. Our first idea is to

Question 529.

Théorème. — *Dans tout triangle, le produit des rapports de chaque côté à la somme des deux autres, ne surpasse pas* $\frac{1}{8}$.

(E. Cesaro.)

Le théorème revient à prouver l'inégalité

$$\frac{abc}{(a+b)(b+c)(c+a)} < \frac{1}{8}. \quad \ldots \quad (1)$$

Or, la moyenne géométrique de deux nombres est plus petite que leur moyenne arithmétique; ainsi :

$$\sqrt{ab} < \frac{a+b}{2}, \quad \sqrt{bc} < \frac{b+c}{2}, \quad \sqrt{ac} < \frac{a+c}{2}.$$

On conclut, de ces inégalités,

$$abc < \frac{(a+b)(b+c)(c+a)}{8} \ (^*).$$

(E. Fauquembergue.)

Autres solutions par MM. Torrès, élève du Lycée de Bordeaux, Leinekugel, étudiant (Paris) et Cocheteux, élève de l'École des Mines (Liége).

(*) Il est bon d'observer que, si le triangle est équilatéral,

$$\frac{abc}{(a+b)(b+c)(c+a)} = \frac{1}{8}.$$

(E. C.)

Figure 5: Original formulation of E. Cesàro's inequality.

consider (and verify with GeoGebra) a much simpler case, the well known triangle's inequality: every side is smaller than the sum of the other two. Thus, $(a+b) \geq c$, and, likewise, $(b+c) \geq a$ and $(a+c) \geq b$. See Figure 7. But, using these inequalities, we would just obtain abc as a lower bound for $(a+b)\cdot(a+c)\cdot(b+c)$, not $8abc$. We wonder, then, if it could be true in general over any triangle that $(a+b) \geq 2c$, $(b+c) \geq 2a$ and $(a+c) \geq 2b$. After a moment's thought we discard this hypothesis, thinking, for instance, of the right triangle with sides $a = 3, b = 4, c = 5$, where $a + b \not\geq 2c$. We do not even need to recall such Pythagorean triple; we could just ask *GeoGebra Discovery* for the locus of, say, vertex C such that $a + b = 2c$. The answer 'seems' to be an ellipse, see Figure 7. Placing C inside the ellipse would yield $a + b \leq 2c$; and placing it outside would imply $a + b \geq 2c$.

But we would like to confirm this visual impression. Thus, we select some simple coordinates for $A = (0, 0), B = (1, 0)$, as the analysis of the investigated equality can be reduced, without loss of generality, to this particular case, by homothecy. Now we can handle easily the displayed output equation, which is, indeed, an ellipse with foci in A, B, center in the midpoint of AB, axis of size 2 and $\sqrt(3)$, respectively. And the ellipse includes the points $C = (1/2, \sqrt(3)/2)$ and $C = (1/2, -\sqrt(3)/2)$, corresponding to an equilateral triangle.

Now, in the next Figure 9 we show how we have extended this computation and displayed the locus of C for $2c = a + b$, the red ellipse; the locus of C for $2a - b - c$, the blue quartic;

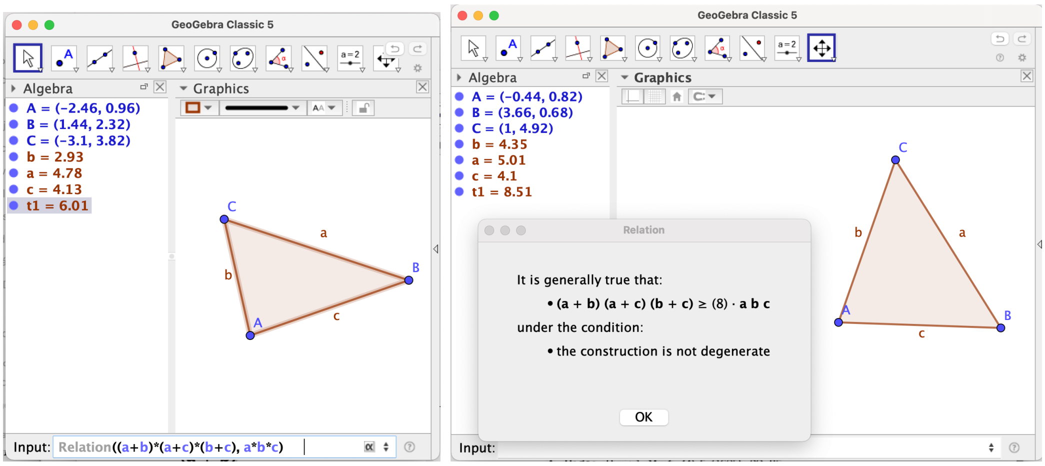


Figure 6: Left: input, asking for the relation between $(a+b)\cdot(a+c)\cdot(b+c)$ and $a\cdot b\cdot c$. Right: output, the inequality $(a+b)\cdot(a+c)\cdot(b+c) \geq 8abc$.

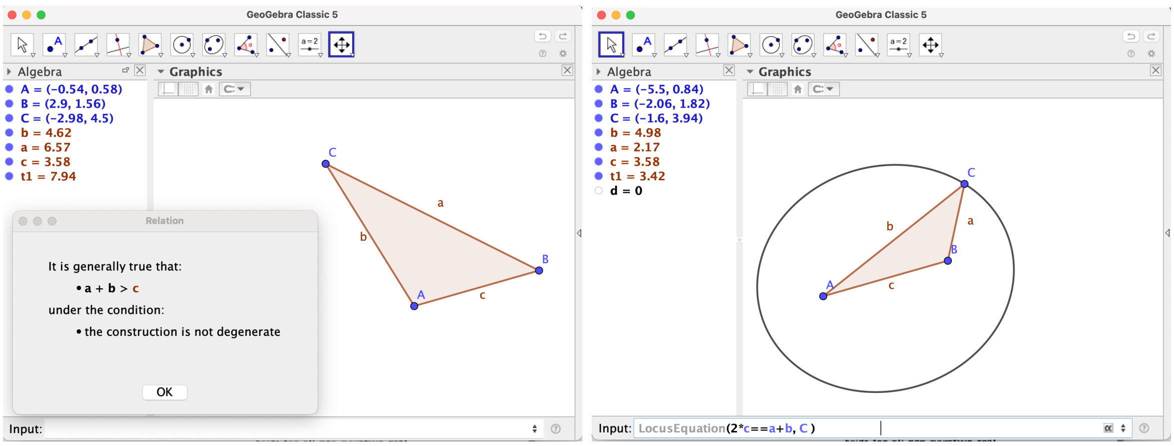


Figure 7: Left: A side of a triangle is smaller than the sum of the other two. Right: Locus of C for $a+b=2c$.

and the locus of C for $2b-a-c$, the black quartic. We have labeled as d, e, f the values of $2c-a-b$, $\quad 2b-a-c$, $\quad 2a-b-c$, respectively. Notice that placing C inside the red ellipse makes d positive (and negative outside of the ellipse), while placing this point inside the blue or black quartics makes both e, f negative (positive, otherwise).

But finding in a geometrically precise way the intersection of the three curves seems challenging for GeoGebra (there is no command for finding the intersection of three curves). Yet, in this case the intersection is easy to describe: indeed, the conjunction of $\{a+b-2*c=0, \quad a+c-2*b=0, \quad b+c-2*a=0\}$ yields as solution $a=b=c$, which means that C should be the third vertex of one of the two equilateral triangles with vertices at $A=(0,0), B=(1,0)$. Notice that, conversely, for any equilateral triangle it is true that $\{a+b-2*c=0, \quad a+c-2*b=0, \quad b+c-2*a=0\}$, since $a=b=c$. In particular this means that over such triangles $(a+b)\cdot(a+c)\cdot(b+c)=8abc$, as remarked by Cesàro.

Obviously, this success also means that our initial conjecture about "...if it could be true in

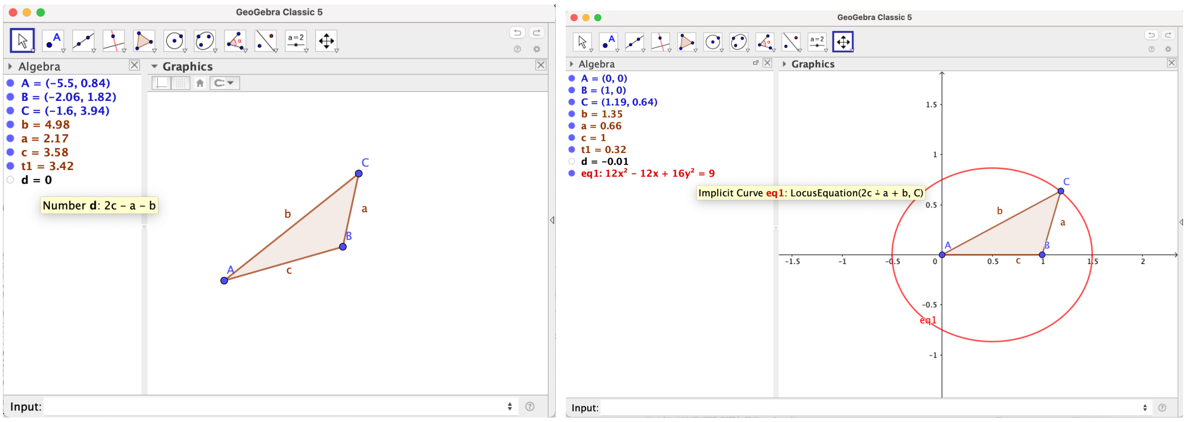


Figure 8: Left: $d = 2c - a - b$. Right: Equation of the locus of C for $a + b = 2c$, showing it is an ellipse.

general over any triangle that $(a+b) \geq 2c$, $(b+c) \geq 2a$ and $(a+c) \geq 2b$." is false. Thus, we have to look for a different approach towards proving Cesàro's inequality $(a+b)\cdot(a+c)\cdot(b+c) \geq 8abc$. Since it involves the sum and the product of every two sides of the triangle, we might try to find out if there is some simpler inequality holding between, say, $a + b$ and ab. The initial problem is that both expressions are not of the same degree and, thus, they can not be reduced to the case $A = (0, 0), B = (1, 0)$. Yet, we ask GeoGebra, through the *Relation* tool, for the relation between $a + b$ and ab, yielding that $(a + b)^2 \geq 4ab$, see Figure 10.

In order to have some explanation for this inequality (let us remark that GeoGebra's output is the result of some symbolic computation with real algebraic geometry tools, such as quantifier elimination, so it is already mathematically sound) we observe that the inequality is equivalent to $((a + b)/2)^2 \geq ab$. Thus, in Figure 10, right, following `https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means`, we have constructed a circle with diameter AB and radius $i = (a + b)/2$. We place now a point C on this circle in such a way that the foot of its height is the point where the segments a, b meet, yielding the triangle ABC and apply the altitude (or geometric mean) theorem $h^2 = ab$ (see `https://en.wikipedia.org/wiki/Geometric_mean_theorem`, and [8] for a proof with GeoGebra and a generalization), yielding that $i \geq h$, and thus $i^2 \geq h^2$, i.e. that $((a + b)/2)^2 \geq ab$.

Now, this inequality implies that $(a + b) \geq 2\sqrt{(ab)}$ and, likewise, $(a + c) \geq 2\sqrt{(ac)}$ and $(b+c) \geq 2\sqrt{(bc)}$. It is now clear that multiplying all these expressions we will arrive to Cesàro's inequality, Q.E.D.

4 Cesàro's Equality locus

Finally, we wonder about when the equality $(a + b) \cdot (a + c) \cdot (b + c) = 8abc$ holds, beyond the already analyzed equilateral case. In Figure 11 we show how computing the locus of C verifying this equality yields an empty graph, associated to a complicated equation: $eq4 := \{252 * x^10 + 1264 * x^8 * y^2 + 2536 * x^6 * y^4 + 2544 * x^4 * y^6 + 1276 * x^2 * y^8 + 256 * y^10 - 1260 * x^9 - 5056 * x^7 * y^2 - 7608 * x^5 * y^4 - 5088 * x^3 * y^6 - 1276 * x * y^8 + 2151 * x^8 + 6596 * x^6 * y^2 + 6866 * x^4 * y^4 + 2548 * x^2 * y^6 + 127 * y^8 - 1044 * x^7 - 2092 * x^5 * y^2 - 1052 * x^3 * y^4 - 4 * x *$

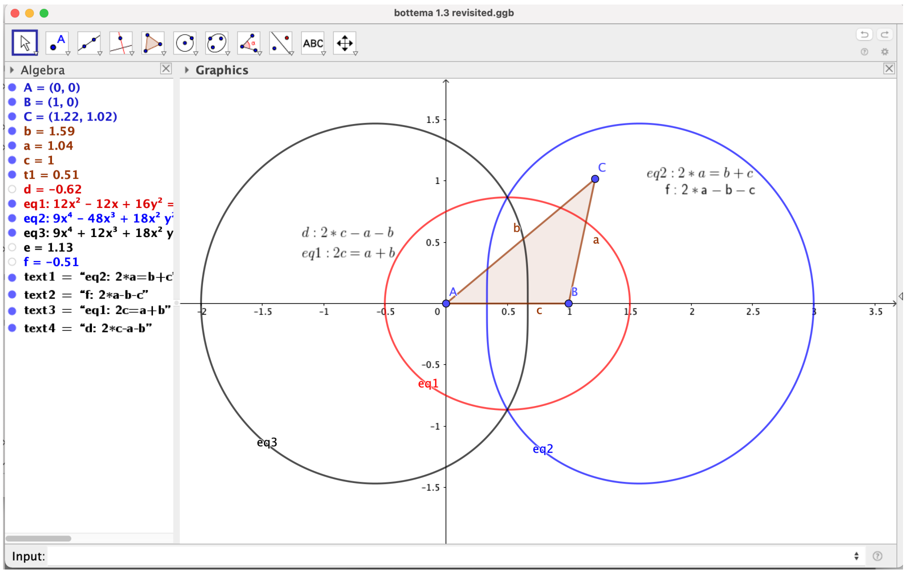


Figure 9: Intersection of the three curves corresponding to the loci of C for $2c - a - b = 0$, $2a - b - c = 0$, and $2b - a - c = 0$, respectively.

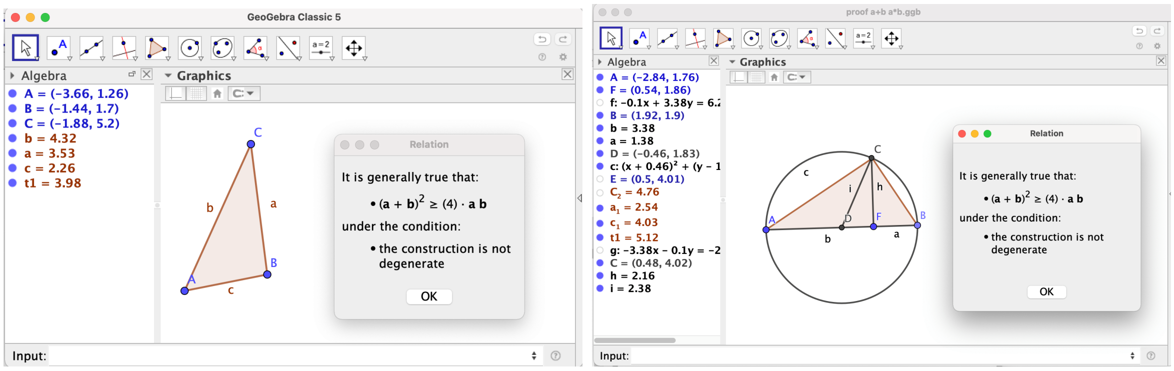


Figure 10: Left: $(a+b)^2 \geq 4ab$. Right: Explaining this inequality through the altitude theorem.

$y^6 - 198 * x^6 - 682 * x^4 * y^2 - 738 * x^2 * y^4 - 254 * y^6 - 1044 * x^5 - 1048 * x^3 * y^2 - 4 * x * y^4 + 2151 * x^4 + 2294 * x^2 * y^2 + 127 * y^4 - 1260 * x^3 - 1276 * x * y^2 + 252 * x^2 + 256 * y^2 = 0\}$

This "missing real points" problem (empty or not complete graphic output in GeoGebra, of the equation of a curve) happens in many other cases. For example, if we input *ImplicitCurve*$(x^2 + y^2)$, we get just the empty set. Or, if we input *ImplicitCurve*$(x^2 + y^2 - x^3)$, we get a branch of the curve, but not the origin –a point that obviously belongs to the curve. Thus we need to find specific ways to deal with this issue in this case.

To begin with, we know that this equality locus $eq4$ should contain the points $C = (1/2, \sqrt(3)/2)$ and $C = (1/2, -\sqrt(3)/2)$, and this can be easily verified using the *Substitute* command in GeoGebra CAS view, over the above equation $eq4$. It is also easy to imagine (and to check) that the curve contains the degenerate instances $C = A, C = B$. Any other

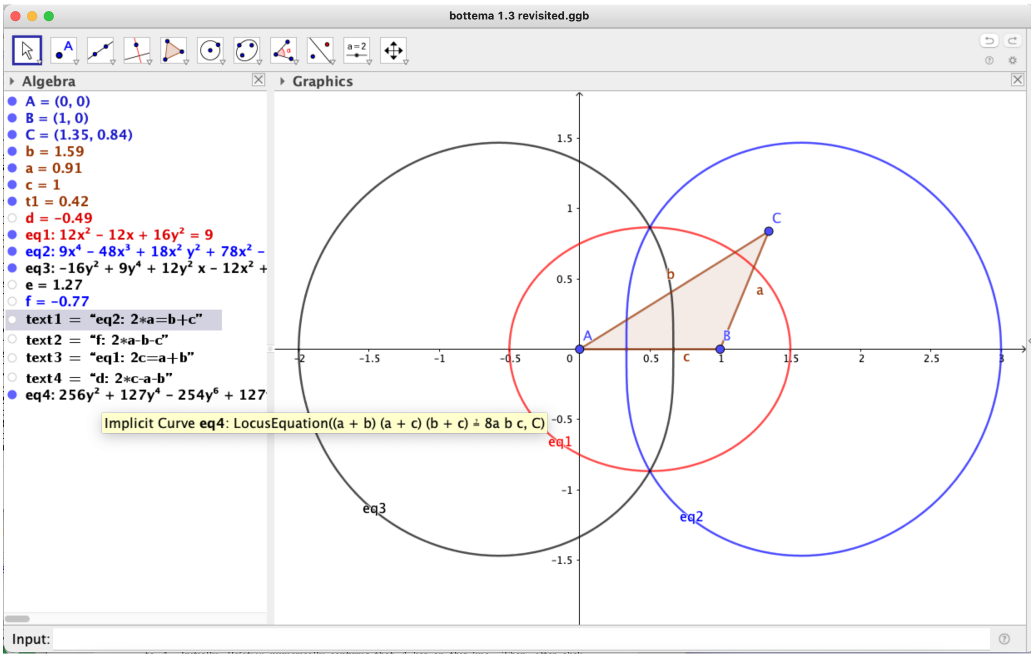


Figure 11: Computing the locus of C for $(a + b) \cdot (a + c) \cdot (b + c) = 8abc$

points? To answer this question we proceed as follows, using some tools from the CAS Maple[5], such as *sturm*, that computes the number of real roots of univariate polynomial on an interval (eventually, the interval $(-infinity, +infinity)$), or *realroot*, that computes isolating intervals for each real root:

- We collect $eq4$ as a polynomial in the variable y with coefficients in x
- We compute the discriminant of this polynomial in y, getting a univariate polynomial in x.
- We compute the different real roots of this discriminant (11 in total), including $0, 1/2$ and 1.
- We compute the real roots of the curve over each of these roots of the discriminant, finding only the already known ones: $y = 0$ over $x = 0$, $y = \sqrt{(3)}/2$ and $y = -\sqrt{(3)}/2$ over $x = 1/2$, $y = 0$ over $x = 1$.
- For each interval conformed by a pair of consecutive roots of the discriminant (or between minus infinity and the smallest root, and between the largest root and plus infinity) assign a test number in the interval. For example, we know that the smallest root is $1/2 - \sqrt{(63618 + 7446 \cdot \sqrt{(73)})}/4$, thus we can choose -90 as a smaller, test number. As a more automatic alternative, use *realroot* for handling intervals enclosing the roots of the discriminant.

[5]https://www.maplesoft.com

- Over each assigned real number between the real roots of the discriminant, compute the number of real roots in the curve (using, again, the *sturm* command). For example, substitute $x = -90$ in $eq4$, yielding a polynomial in y, that has no real roots.

- As there are no real roots over each one of these test points, we conclude that the only points in the curve are $\{(0,0),(1/2,\sqrt(3)/2),(1/2,-\sqrt(3)/2),(1,0)\}$. In fact let us recall that the number of real roots of $eq4(x,y)$ does not vary as x moves on an interval between consecutive roots of the discriminant. So, since in the test points of the intervals we have selected there are no roots, it follows the same happens all over the interval. So the only roots are the ones we have already previously found.

Of course, the obtained computation confirms Bottema's assertion that Cesàro's inequality holds as an equality only in the equilateral case, leaving aside the degenerate instances when C coincides with A or B.

5 Conclusions

The last two sections show, in our opinion, a way to approach geometry learning that combines empiric experimentation and formal reasoning, in which man-machine interaction is fundamental, not just auxiliary, following the declaration of Corless [5]: "Any tool should always be used to expand the users capabilities, and not as a crutch to prop up weak skills". See [8] for another recent example in the same direction.

We could say that "computer mediated thinking" (again, a formulation from [5]) should have a parallel status to the traditional "writing mediated thinking": how could we think of developing a sound geometry reasoning without using symbols and writing skills? How can we nowadays think of developing geometry reasoning without using "personal geometry assistants" such as *GeoGebra Discovery*? The evolution of our approach to the proof of Cesàro inequality (and equality) shows well –at least in this particular example– the great advantages (and specific difficulties) of having this tool at our disposal. It would have been very difficult to replicate our way of proving these statements without the concourse of GeoGebra. Of course, it is possible that there is an alternative way, but ...will it be relevant in mathematics education in the digital era?

We are aware that this is just one isolated example, and it is one addressed to persons with some skills on higher mathematics. But it also intends to support the urgent need for "Opening a discussion on teaching proof with automated theorem provers" [9], the title of a very recent paper by Hanna –one of the world most reputed experts in the topic– and Yan. In that paper there is a section specifically dealing with GeoGebra's automated reasoning tools, and the authors conclude that

> It is perhaps too early for empirical studies of classroom experience using the enhancements to GeoGebra. In this respect the situation of GeoGebra is similar, but not identical, to the proof technology in general. While it is reasonable to expect proof technology to foster students proving abilities, and there is certainly supporting anecdotal evidence, its potential advantages have not yet been systematically assessed.
>
> [...]

> We know that automated proof assistants are designed to provide a guarantee of correctness, and indeed they are very good at establishing the validity of a proof. The question, then, is to what degree these tools can also be helpful in explaining why it is that a theorem is true.

We agree about the need to seriously start investigating these issues about how the use of *GeoGebra Discovery* can improve mathematics learning, not of the traditional curriculum, but of one that already takes into account the existence of new possibilities associated to the 'discovery function' of computer-enhanced proof processes.

It is also true that this research must go in parallel, involving, on the one side, to curriculum decision makers, teachers, students [1] and, on the other, proof assistant developers. As stated in [9]

> Proof assistants ... will never be developed in the absence of initiative on the part of mathematics educators and a demonstrated demand fueled by increased use. Secondly, success also requires new and effective teaching strategies. These two efforts stand in a reciprocal relationship, so that the full benefit of proof assistants will be seen only over time as new teaching strategies effect the demand for new tool features and vice versa. The responsibility for both efforts rests squarely on the shoulders of educators. The key is to make a start, beginning with exploratory studies of the potential of these new tools at both the secondary and post-secondary levels.

Indeed, our verification of Clough's conjecture, see Figure 1, shows the need (and the involved mathematical, algorithmic and user-interface difficulties) to improve proof assistants to output some answer that could be clearly understood by most users. Here let us just succinctly state that 'true on parts, false on parts' refers to the fact that the algebraic translation of the construction involves different components (but, for a standard user, there is only one, the one that is graphically and intuitively perceived), because the idea of 'length' of a side is, in the complex geometry algorithmic background for *GeoGebra Discovery*, some square root that can take positive or negative values. And, of course, the involved conjecture is true for the component where these roots are positive, and fails for the others.

The option to associate signs to the lengths of segments involves real algebraic geometry and it is on-going work [17]; but it is much less efficient at this moment, so it would be more useful to develop –with the cooperation of teachers, experimenting with students, etc.– some user-interface modifications to avoid such confusing answers for the expected users of the program in the educational world, perhaps implementing two kind of versions: for 'experts' and for 'students'.

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