Some affordances and opportunities of advanced scientific calculators for learning introductory calculus

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Abstract: *Advanced scientific calculators, developed in recent years, include more sophisticated capabilities than original models, including some of direct relevance to elementary calculus. In this paper, some relevant calculator capabilities are described and their potential use for students and teachers of elementary calculus are illustrated and evaluated, against the framework of a model for learning mathematics with calculators. Despite the name of such devices, calculators derive their educational value from processes other than mere calculation. Key ideas of continuity, convergence, limits, differentiation and integration can be both represented and gainfully explored on an advanced scientific calculator, supporting student learning in powerful ways not otherwise available to many students in Asian countries.*

1. Introduction

It is now around forty years since the use of scientific calculators has become routinely accepted in secondary school mathematics education in many countries, including Australia, the USA and many European countries. The use of calculators in those countries was originally focused on computational use – since the devices afforded ready access to otherwise troublesome computations – but recently attention has shifted to educationally more important aspects. Over that time, other forms of technology – most noticeably computers and the Internet – have been developed and achieved some popularity, leading to the suggestion in [1] that the scientific calculator has appeared to have become a 'forgotten technology'.

In many affluent countries, the scientific calculator seems indeed to have almost become forgotten, as mathematics students routinely use graphics calculators and even graphics calculators with computer algebra system (CAS) capabilities over the last quarter of a century, both in classrooms and in high stakes external examinations. Experience in Asian contexts, however, has been different to date, most likely for economic reasons, and it seems still to be the case that the scientific calculator continues to be the most appropriate tool for many classrooms, as suggested in [2]. Recent developments in scientific calculators have increased their significance for learning, as functionality has increased to match the school curriculum and usability has been improved. Modern advanced scientific calculators like the *CASIO fx-991EX* calculator (*ClassWiz*) have many capabilities shared with graphics calculators – without the graphing capabilities and limited by smaller screens – and thus have significant potential for school and early undergraduate mathematics education.

In this paper, we focus on the productive use of such calculators for the particular context of calculus, drawing in part on earlier work highlighting their educational significance [3]. Although other calculators might be used, for convenience we illustrate the arguments with the *ClassWiz*.

2. Learning calculus with calculators

In part to place calculation in context, [4] proposed a model for the educational use of calculators, identifying four kinds of activity: representation, computation, exploration and affirmation. While many productive uses of calculators in school involve several of these components simultaneously, those that involve *only* calculation are usually of very limited importance. Many activities include computation in some way – in precisely the same way that computer software such as that for

spreadsheets, word processing, or dynamic geometry or access to the Internet via web browsers always involve computation in some way, although the computation is not the key feature for learning, and indeed is often hidden from the direct view of the user.

To illustrate this model, consider the command on *ClassWiz* for determining the numerical value of a derivative of a function at a point. As the command is actually written on the keyboard, and thus always visible to a user, it provides unmistakeable evidence that the calculator has links to the calculus. Figure 1 shows an example of this command in use. This screen reflects, potentially, all four of the educational uses of the calculator, as described below.

$$\frac{d}{dx}(x^2+x+1)\Big|_{x=3}$$
$$7$$

Figure 1: Using the numerical derivative command on *ClassWiz*

Representation: Like many modern calculators, the *ClassWiz* represents mathematical ideas using conventional mathematical syntax, so that the ideas are familiar to users. In this case, the representation of d/dx is shown as a fraction; the function (of *x*) involved is represented conventionally using an expression in which indices are raised, smaller, numbers and variables are shown in italics; the point at which the derivative is evaluated is shown using the conventional notation of a vertical line; the resulting derivative is shown as an exact number, in this case the integer 7. In short, the calculator represents the result in a way that is perfectly acceptable in everyday conventional representation, and thus immediately recognisable.

Computation: The *ClassWiz* has computed the value of the numerical derivative, saving the labour involved for the user of finding a (symbolic) general term for the derivative and then substituting the particular value of $x = 3$. It is not clear at all *how* the calculator has done this, but the result is immediate and the calculation is correct, both of which are helpful for students.

Exploration: As suggested later in this paper, a screen like that shown in Figure 1 might have arisen in several contexts, only one of which is the determination of a derivative of this function at this particular point. Student users of the calculator can use this numerical derivative function to determine related derivatives and use these to discover, or confirm, some properties of derivatives. For example, the above screen might have arisen following earlier numerical derivatives, such as those shown in Figure 2, as part of different sorts of student explorations:

$$\frac{d}{dx}(x^2)\Big|_{x=3}$$
$$6$$
$$\frac{d}{dx}(x^2+x)\Big|_{x=3}$$
$$7$$
$$\frac{d}{dx}(x^2+x+1)\Big|_{x=2}$$
$$5$$

Figure 2: Related derivatives to that shown in Figure 1

Affirmation: By its nature a scientific calculator is a responsive device, driven by the user. So it has the potential to affirm (or to contradict) student preconceptions about an outcome, both of which are very helpful to learners. In this sense, it can be regarded as a form of personal hypothesis-testing device, of considerable value to a student thinking about the mathematics involved. To illustrate this

aspect, consider the situation before a student taps the equals key to evaluate the numerical derivative in Figure 1, as depicted below in Figure 3.

Figure 3: *ClassWiz* screen immediately prior to executing the command

Students' preconceptions and expectations will depend on the context and on the student concerned. But to illustrate, a student who has just determined that the derivative of $f(x) = x^2 + x$ is 7 when $x = 3$ (as in the middle screen of Figure 2) might naively expect that the result in Figure 3 to be $7 + 1 = 8$, and find their thinking *contradicted* by the calculator; a fellow student might expect that the result will be the same as that in Figure 2, and find their thinking *affirmed* by the calculator. In either case, the calculator can contribute to student's learning, provided it is used thoughtfully.

In summary, to consider the relevance of the calculator to a student learning calculus, we need to pay attention to the context of the learner, not simply to the technical capabilities of the device.

3. Closer and closer

A major conceptual jump for mathematicians since antiquity and thus for students learning the calculus today is the need to deal with infinite processes, as brilliantly revealed in a remarkable recent book [5]. By definition, we need to think in approximations to infinite processes, and many teachers will be familiar with trying to help students to understand the idea of getting 'closer and closer' to something – without ever reaching it. While technologies can appear to overcome this hurdle (for example, as computer algebra systems generate symbolic results of infinite processes, such as those involved with differentiation or integration), there are several ways in which calculators can help students to see for themselves what is happening behind the symbolism. In this section, we explore briefly some of these connections.

3.1 Continuity and discontinuity

Introductory calculus deals mostly with continuous functions; one way of thinking of these is that small changes in the variable lead to small changes in the function itself. Figure 4 shows an example of this for the continuous function $f(x) = x^3 + x^2$ near $x = 1$. Examining functions in this way is made possible by a tabulate facility available on advanced scientific calculators like *ClassWiz*; such facilities were not typically available on earlier scientific calculator models.
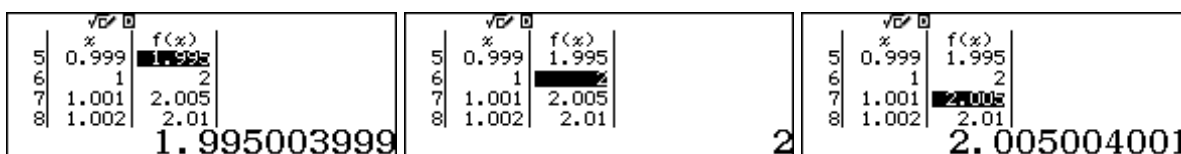

Figure 4: Exploring the continuity of $f(x) = x^3 + x^2$ near $x = 1$

These screens suggest that the function is continuous at $x = 1$. Although the screen is small (and thus the table size small), students can see for themselves that choosing still smaller intervals for x (i.e. those for which the values of x get closer and closer to 1) results in a similar phenomenon for

$f(x)$, which is the essential idea of continuity. The calculator is facilitating an exploration of this kind by permitting fast and efficient calculation – much more so than would be possible without the aid of the technology.

In contrast, when a function is discontinuous, function values can change dramatically with only small changes in the value of the variable. This is illustrated by setting up a table of values as shown in Figure 5.

Figure 5: Setting up *ClassWiz* to explore a discontinuous function

The resulting screens in Figure 6 show a different kind of pattern than that seen in Figure 4. As x gets closer and closer to 3, the function values change dramatically, and the function is not defined when $x = 3$, as indicated by the error message on the screen.
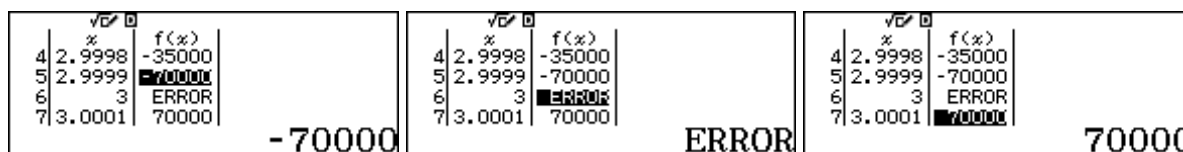
Figure 6: Exploring the continuity of $f(x) = \dfrac{7}{x-3}$ near $x = 3$

The flexibility of the calculator makes it relatively easy for students to explore jump discontinuities like this, and to compare their behaviour with that of continuous functions. Getting 'closer and closer' involves merely changing the interval values for the variable of interest. Elementary functions can be discontinuous in other ways of course, such as the concept of a removable discontinuity, for which a function is continuous everywhere except for one point. A typical example, often studied by beginning calculus students is shown in Figure 7.

Figure 7: Preparing *ClassWiz* to explore the removable discontinuity of $f(x) = \dfrac{x^2-1}{x-1}$

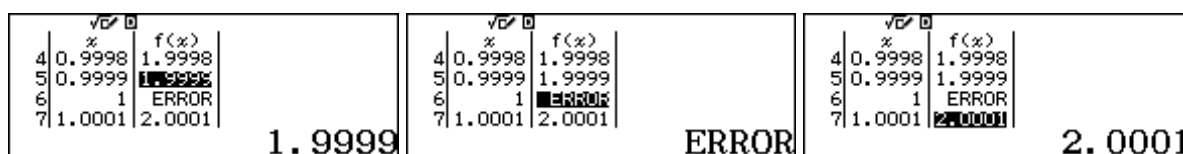Figure 8 shows the function values as x gets closer and closer to 1:

Figure 8: Understanding a removable discontinuity

Introductory calculus students can see that, while the value of the function is undefined at $x = 1$, it seems to behave elsewhere as if it were the continuous function, $f(x) = x + 1$. They can scroll tables like these, and adjust them to get closer and closer to x = 1 (by adjusting the interval appropriately) and see for themselves what is happening: that the value of the function is $f(x) = x + 1$ almost everywhere, except at the point $x = 1$; getting closer and closer doesn't affect this result. Students will of course usually be given more formal treatments of continuity and discontinuity, but the calculator might serve a useful role to build invaluable insight into these formal treatments; it is not suggested that they should replace these, but rather that they should supplement them.

3.2 Limits at a point

The idea of a limit at a point is a critical one, commonly emphasised in introductory calculus courses (although not formally treated by Newton or Leibnitz in their early development of the calculus). Although treatments are often very formal (including proofs with $\varepsilon - \delta$ definitions), an intuitive understanding is important also and a calculator is a useful tool for developing this. Figure 9 suggests how the calculator might be used for this purpose, by constructing a table of values for a function, with an increasingly small interval for the variable x. As x gets closer and closer to 0 – from either side – the tables suggest that the ratio gets closer and closer to 1, and that it is not defined for $x = 0$.
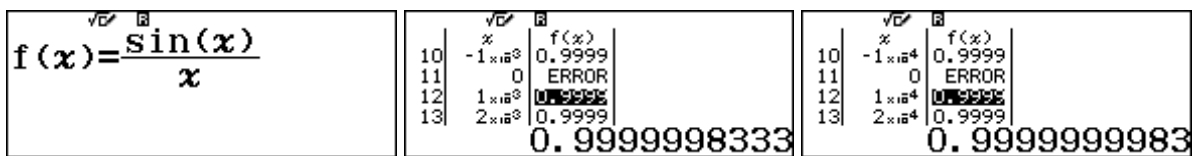

Figure 9: Using *ClassWiz* to explore $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

Calculators like *ClassWiz* are limited in their display capabilities, with narrow columns, but Figure 9 shows that more precision is available by highlighting a given value.
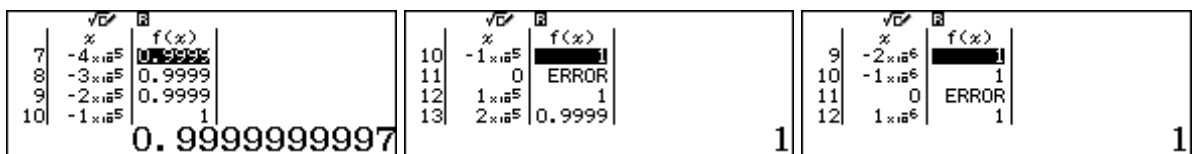

Figure 10: Getting closer and closer with smaller intervals

As students explore limits of this kind, with x getting 'closer and closer' to 0, eventually the calculator gives the impression that the result is 1, as shown in Figure 10 – a good opportunity for a discussion regarding the numerical processes involved and the limitations of a finite display of infinite decimals; although the result is technically incorrect, it is shown by the calculator as the result is rounded to the limitations of the calculator. However, such conversations are an important part of dealing with the many ideas of the infinite that are involved with the calculus.

3.3 Gradient at a point

A major motivation for studying limits in introductory calculus is to deal with the idea of a gradient at a point and thus of the derivative function. The derivative is typically defined as a limit and sometimes interpreted as a gradient of a secant or a tangent:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Advanced calculators like *ClassWiz* allow students to explore suitable gradient functions to see informally the limiting processes involved. An example is shown in Figure 11, to see the gradient of a secant to the curve of $f(x) = x^2$, at various values for x and as x gets closer to $x = 2$ (when h gets closer and closer to 0.)
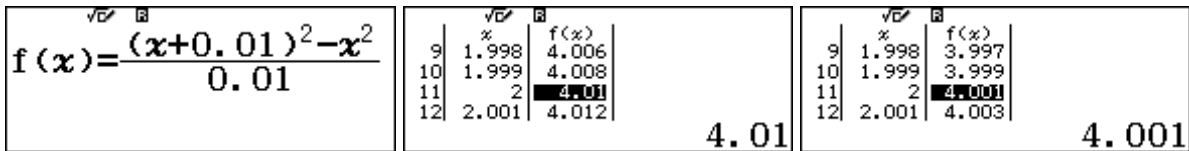

Figure 11: Evaluating gradient functions for $h = 0.01$ and $h = 0.001$

In Figure 11, the first screen shows that the ratio has been evaluated for $h = 0.01$, as shown in the middle screen. A smaller value of $h = 0.001$ is used to generate the third screen. In this way, students can see that the gradient gets closer and closer to 4 as h gets closer and closer to 0. The calculator is used to quickly perform the necessary calculations, allowing students to interpret the results. The first screen in Figure 12 shows the case for a very small positive value of $h = 0.000001$.
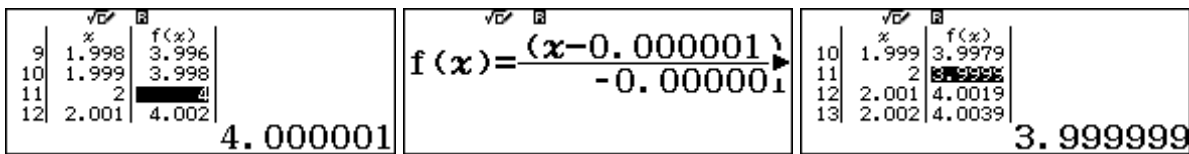

Figure 12: Exploring the gradient function as h approaches zero

Figure 12 also shows the results for a value of h that is close to zero, but is negative rather than positive, to show that the limit approaches the same value (of 4) from both above and below. The middle screen shows the case of $h = -0.000001$ and the third screen shows the resulting table. Using and continuing these kinds of explorations, students can get a strong sense of the idea that the limiting value is 4 in this case, suggesting that the derivative is $f'(2) = 4$.

3.4 Infinite limits
Limits to infinity are involved when a variable increases without bound. In such situations, 'closer and closer' usually means in effect 'larger and larger', so can be explored by comparing results as the value of a variable increases. An example is shown in Figure 13, to evaluate

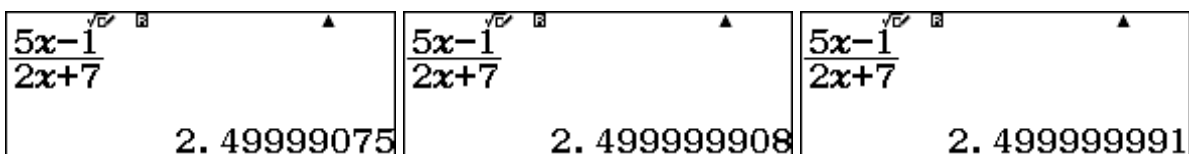$$\lim_{x \to \infty} \frac{5x - 1}{2x + 7}$$

Figure 13: Exploring an infinite limit as x gets larger and larger

Although this sort of process can be explored on any scientific calculator, efficient explorations are facilitated by modern calculators like *ClassWiz*; the screens in Figure 13 evaluate the expression for successively larger values of *x*: respectively 1 million, 100 million and 1 billion, using a calculator command for easy substitution of values. Again, the calculator rounds results to its level of precision, so that eventually what might seem to be an incorrect result is provided, as shown in Figure 14. While this result makes clear that the limiting value seems to be 2.5, as suggested by the succession of screens in Figure 13, it is of course only a (very good) numerical approximation to the value of the expression for *x* = 100 billion.

Figure 14: The limiting value of an expression suggested for *x* = 100 billion

3.4 Convergence of a series

Some infinite series can also be explored fruitfully on advanced scientific calculators, using special commands. A good example involves the infinite series for the exponential function, *e*:

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

As more terms of the series are included, the series can be seen to converge (in this case, very quickly) to a good numerical approximation to *e*, rather than the exact value.
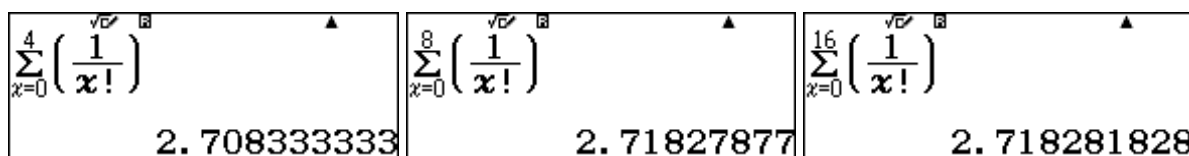

Figure 15: Exploring the limit of an infinite converging series for *e*

The *ClassWiz* environment makes it easy for students to edit the parameters of the (finite) series to see the effects of taking more and more terms, so that attention is readily focused on the output. Figure 15 shows a typical progression of series values as the number of terms increases, showing a rapid convergence of the result to the well-known value of the transcendental number ***e***.

4. Using numerical calculus commands

One of the distinguishing features of advanced scientific calculators like *ClassWiz* is the presence of numerical calculus commands for differentiation and integration. While these are numerical in nature – and thus do not permit the user to determine either a derivative function or an indefinite integral, unlike computer algebra system (CAS) calculators – they can be used to support student learning, in addition to the computational task of determining numerical answers. There are a number of ways in which this can be done, some of which are illustrated below.

4.1 Patterns for differentiation

When a calculator is used to determine a derivative at many points, not just a single point, the underlying patterns can be seen by students, and help them to understand the idea of a derivative *function*. An example is shown in Figure 16, where the derivative of $f(x) = x^2 + 3x$ is tabulated and shown at a succession of points. (To do this, note that the tabulated function determines the derivative at each of the chosen table points, using the symbol $x = x$, rather than just a single point).
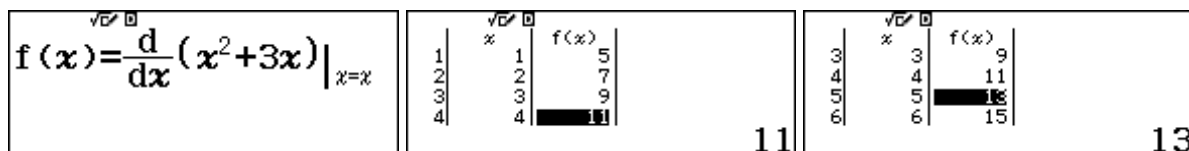

Figure 16: Exploring numerical derivatives at a point to generate a derivative function

For an exploration of this kind, students generating the table of derivatives shown in the middle screen of Figure 16 can be encouraged to look for a pattern in the results. Most students studying introductory calculus will readily see the linear function $f(x) = 2x + 3$, and can be further encouraged to predict the next terms (as the table shows only the first four values). So, an activity of this kind offers an opportunity for affirmation of the result, readily checked by scrolling down the table, as the third screen in Figure 16 shows. The transition from the idea of the derivative at a particular point to the derivative at all points (i.e., a derivative function) is a critical one, and the calculator offers students a way to understand the concepts involved. Of course, it is an easy matter for students to explore other functions in a similar way.

4.2 Patterns for integration

In a similar way, the numerical integral commands can be used to generate a pattern of results to understand integration better. While a singular numerical integral allows a student to determine the area under a curve between two points, a succession of integrals allows for a powerful pattern to be seen. An elementary example is shown in Figure 17, using the numerical integral command to generate a table of values.
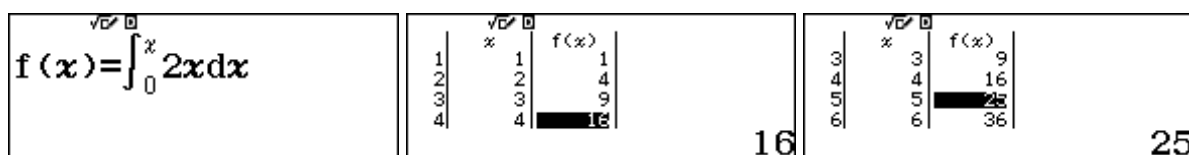

Figure 17: Exploring numerical integrals

Again, the pattern of results from this exploration will be familiar to most students studying elementary calculus: it seems (from the middle screen in Figure 17) as if the integral is well represented by the function $F(x) = x^2$. Again, the calculator environment offers an easy way to confirm this hypothesis by scrolling down a little further, as the third screen in Figure 17 shows. While this is not of course sufficient, and it is assumed that students will encounter the Fundamental Theorem of the Calculus in other ways as well, the calculator provides an opportunity to see important connections in a fresh way. Students can, of course, explore further by considering other functions to integrate, and to strengthen their sense of the important relationships involved.

4.3 Properties of derivatives

When ready access to numerical derivatives is available, an advanced calculator allows for derivatives to be compared and for some properties of derivatives to be highlighted. This is relatively easy for learners to do when a calculator like *ClassWiz* allows a pair of tables to be produced, rather than just a single table. To illustrate this, Figure 18 shows how the tables can be used to compare derivatives of one function, $f(x)$ with those of another function $g(x) = 5f(x)$.
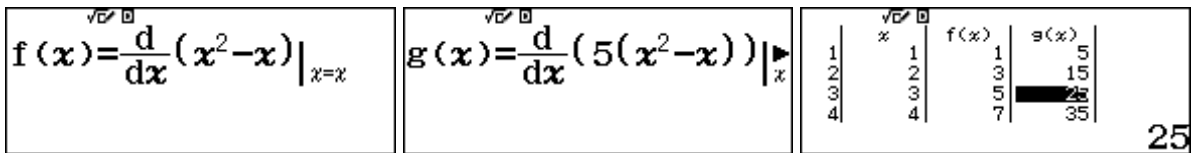

Figure 18: Comparing derivatives to illustrate a multiplicative property

In this case, users will be able to see that multiplying a function by a constant will result in a derivative that has been similarly multiplied. The calculator environment makes it easy for students to check this relationship with other constants or even other functions. Of course, such work does not mathematically establish a result, but helps students to appreciate it by considering some examples of their own choosing.

4.4 Special derivatives

While access to numerical derivatives can provide both practical results as well as opportunities for insight for any functions, students can use calculators to explore two special functions with surprising derivatives – at least surprising to novice students. One of these is the exponential function $f(x) = e^x$, for which the derivative at a point is the same as the value of the function at the same point. Figure 19 shows that this unexpected result is easily verified on the calculator.
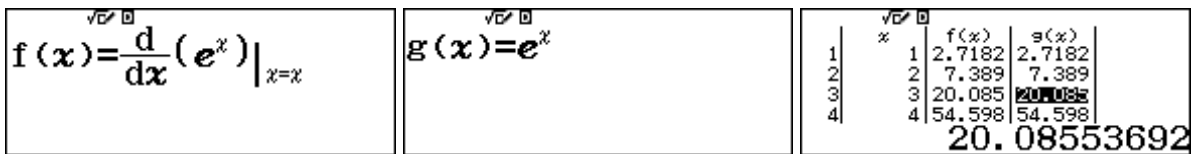

Figure 19: Affirming a special property of the exponential function

Again, a capacity to tabulate a pair of functions is key to this kind of exploration. Students can readily see for themselves that similar results do not hold for powers of numbers other than e, by tabulating them for themselves. As an illustration, Figure 20 shows that the derivatives of $f(x) = 2.7^x$ are generally lower than the corresponding values of the function, helping to make clear how special is the transcendental number e.
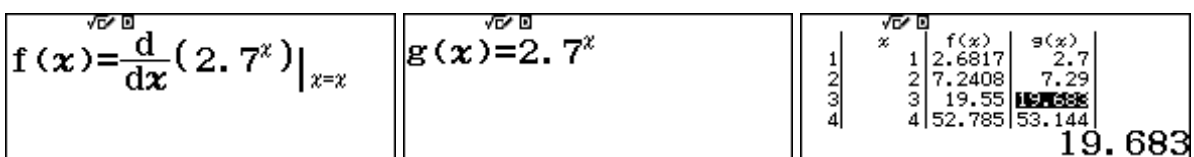

Figure 20: Exploring another exponential function to reveal a special property of $f(x) = e^x$

Another surprising result, readily affirmed with numerical derivatives, concerns the derivative of the natural logarithmic function. Again, students can use a suitable table of values to see for themselves that the derivative of the function is merely the inverse of the associated number.
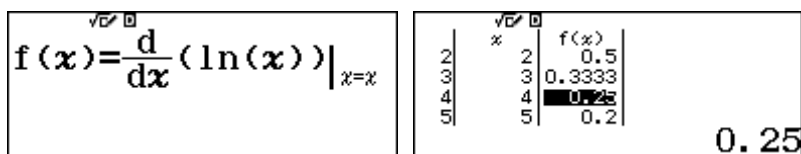

Figure 21: Demonstrating a special property of the natural logarithmic function

At this level of study, students would hopefully recognise the reciprocals in their decimal forms, but they could of course include them as a second column to the table if necessary.

5. Conclusions

The paper demonstrates that it is too limiting to regard calculators as merely devices for computation. The proposed model [4] for an educational role of calculators suggests that representation, exploration and affirmation are also important processes for student learning. Some examples of these processes are provided for the case of calculus, indicating two important consequences of advanced scientific calculator capabilities, using the CASIO *ClassWiz* as an example.

Firstly, they permit the exploration of the idea of 'closer and closer', regularly used by mathematics teachers as a helpful idea for both teaching and learning calculus. Consequently, they offer opportunities to attain further insight into the infinite processes involved than would otherwise be possible, as generations of teachers can attest.

Secondly, the availability of commands for numerical differentiation and integration offer some opportunities for the exploration of important calculus ideas, unburdened by the requirement for by-hand computation. The calculator provides a way for students to explore for themselves some properties and relationships associated with differentiation and integration.

In most introductory calculus courses, students will be expected to learn more formal and more symbolic aspects of the subject. The paper does not suggest that these are to be replaced by numerical opportunities provided by advanced scientific calculators; rather it is suggested that the calculator offers fresh opportunities to develop insight into the ideas involved with an affordable and available technology, well-suited to many classrooms across the region served by ATCM.

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