Technologies and laboratorial activities for a robust understanding approach: from closed to open laboratories

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Abstract. *The aim of this paper is to show the evolution of a meta-methodology adopted by the authors in some of the activities they proposed over the last 10 years to students and teachers. All the activities, called laboratories, involve the use of technology and are based on laboratorial methodology aiming to a robust understanding of mathematics. We describe, as example, five laboratories and provide a discussion on our meta-methodology as researchers on such laboratories finding a meta-methodological evolution. In particular, by analysing this evolution, we detect an initial attitude towards "closed" laboratories (rigidly structured, letting no freedom of choice for students and teachers) evolving into an attitude to "open" laboratories (letting freedom of choice to students and/or teachers).*

1. Introduction

Mathematics learning in schools is subject to numerous national and international surveys (https://www.oecd.org/pisa/; https://timssandpirls.bc.edu/timss-landing.html) which often reveal difficulties encountered by students in this discipline. In order to deal with these problems, it is advisable to influence the attitude that students have towards mathematics, which consists of three interacting components: emotional disposition, vision of mathematics and sense of self-efficacy [8; 9]. In particular: the *emotional disposition* is the set of emotions (fear, anxiety, frustration, anger, pride, satisfaction, excitement, joy, to name a few) that are awakened by an activity; the *vision of mathematics* is the set of beliefs the person holds about it; the *sense of self-efficacy* is people's beliefs in their ability to organise and carry out the actions necessary to deal adequately with the situations one encounters in order to achieve the desired results. Beliefs of efficacy influence the way people think, feel, find personal motivation and act [3]. The stronger the sense of efficacy is, the more vigorous people are in dealing with problematic situations and the more successful they are in changing them.

The improper or negative manifestation of the various components generates in the student a closure towards the discipline which cannot be removed [8; 9]. The teacher can influence this attitude by preparing specific activities aimed at achieving concrete objectives that respect the abilities of individual students and of the class.

A methodology for fostering a positive attitude towards mathematics that has been developed by the mathematics education research community since the beginning of this century is the *mathematics laboratory* [1], which aims to increase students' sense of self-efficacy by providing a correct view of the discipline and promoting positive emotions.

The mathematics laboratory is a stimulating and engaging environment for the students in which the role of the teacher is fundamental in conveying the activity. We will see in this paper the evolution of the mathematics laboratory in student/teacher training activities, promoted by the Mathematics Education Research Group (MERG) at the University of Catania. These activities are carried out in a STEM (acronym for Science, Technology, Engineering, Mathematics) approach because of a large use of Technology in view of Mathematical learning. In recent years, the activities are more oriented towards a STEAM approach. In the acronym STEAM, evolution of STEM [19; 16], the A stands for "Arts", meaning not only arts themselves (paintings, literature, …) but also an artistic attitude: creativity, openness, observation of reality and self-observation, passion, attention to beauty, ….

In this paper, we ask whether math education researchers can evolve from a STEM to a STEAM approach in designing laboratorial activities in mathematics for teachers and students.

In section 2, we illustrate the theoretical framework, composed of several theoretical lens that frame all the activities: Laboratory of Mathematics and Teaching for Robust Understanding as pedagogical methodology; the Technological, Pedagogical and Content Knowledge as a super-structure which combines pedagogical and content knowledge with technological knowledge; and the Meta-Didactical Transposition, because of the joint and synergistic work of two communities, the researchers and the teachers, in implementing activities for students. In section 3, we describe five of the many activities carried out in the last 10 years, calling them laboratories. Many teaching experiments were done with students and teachers; we chose these laboratories to show the evolution of our methodology as researchers. In section 4 some discussions and conclusions on the laboratories described in section 3 and on their evolution are given. In particular, our considerations involve the methodology adopted by us, as researchers and designers of the laboratories, that we call meta-methodology. It is the result of a reflection that we are able to deduce by considering all the activities that we have been designed over the years. Today, they have undergone a methodological transformation, moving from what we will call *closed laboratories* to what we call *open laboratories*.

2. Theoretical framework

The mathematics laboratory is a "phenomenological space to teach and learn mathematics developed by means of specific technological instruments and structured negotiation processes in which math knowledge is subjected to a new representative, operative and social order to become object of investigation again and be efficaciously taught and learnt" (translated from [6]). At national level, the Italian commission for the teaching of mathematics, back in 2003, suggested: A mathematics laboratory is a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. Such activities involve people (students and teachers), structures (classrooms, instruments, organisation and management), ideas (projects, didactical planning and experiments). We can imagine the laboratory as a learning environment in which students learn by doing, seeing, imitating, communicating with each other, in a word: practising, with the aid of instruments, and interactions between people working together in a collaborative/cooperative modality [1; 23].

In particular, elements (Table 2.1) that characterize an activity in a mathematics laboratory are [22]:

Table 2.1 Mathematics laboratory elements

A Problem to solve (P)	***Instruments that can be used/manipulated (I)***	***Working Method (relationship-interaction) (WM)***	***The role of the Teacher (T)***
The activity in the laboratory starts with a problem. In order to create an atmosphere of research and discovery, the situations proposed must be "new" to the students, they must involve problems that are neither too easy nor too difficult and their resolution must require knowledge they have already acquired.	The activity has to foresee exploration instruments to be manipulated, which, by facilitating the development of thinking, allow to elaborate conjectures, to verify properties, to enhance argumentative and deductive skills, to suggest possible demonstration strategies. These instruments can be either old technology (pen and paper, ruler, compass ...) or new technology (Dynamic Geometry Systems, Computer Algebra Systems, Electronic Sheets, Calculators...).	In the laboratory, students compare ideas, intuitions, arguments, and collaborate/cooperate to achieve results using their critical skills: they explore, formulate conjectures, check their validity and then, eventually, demonstrate them.	It is the teacher's role to guide students to achieve results: he/she validates correct proposals, questions proposals that still need to be improved, encourages students to pursue them, rewards students when they achieve a significant result. He/she also sets the pace, creates a positive setting and is attentive and ready to transform the frustration resulting from failure into a moment of re-evaluation of the objectives set.

In the mathematics laboratory students do not study mathematics, but rather *do* mathematics, i.e. they:

Pose/Deal with a problem, Explore/manipulate, Conjecture, Verify, Prove, Apply.

The mathematics laboratory is therefore suitable for promoting *ambitious* and *robust* teaching, aiming at a deep understanding [25; 26; 27]: in it, students work on a problem (Content), which must be neither too easy because they might get bored, nor too difficult because they might get discouraged (Cognitive Demand), comparing themselves with students from the whole class (Equal Access to Content) and with the teacher (Formative Assessment), affirming and discussing their choices (Agency, Ownership, and Identity). That is, the 5 dimensions of Teaching for Robust Understanding come into play [25; 26; 27], described in Table 2.2.

Table 2.2 The Five dimensions of the TRU Framework (https://truframework.org/)

The Five Dimensions of Powerful Classrooms				
The Content (C)	**Cognitive Demand (CD)**	**Equitable Access to Content (EA)**	**Agency, Ownership, and Identity (AOI)**	**Formative Assessment (FA)**
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers. Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind.	The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conductive to what has been called "productive struggle".	The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.	The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.	The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.

The activities we will present, that foster a Robust Understanding, use technological instruments to manipulate, according to the Laboratory of Mathematics. We strongly believe, in fact, that nowadays the Content and Pedagogical Knowledge have to be joined with a Technological Knowledge, as suggested by Koehler and Mishra [17; 18; 20]. In the Technological, Pedagogical, And Content Knowledge framework (TPACK) they highlight the complex interplay of the three primary forms of knowledge in the learning and teaching process (Figure 2.1).

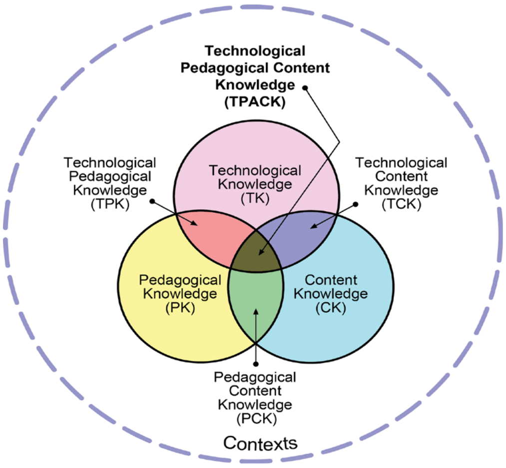

Figure 2.1 TPACK image (from http://tpack.org/)

Here we briefly report the description of the three primary components:

- **Content Knowledge**: "Teachers' knowledge about the subject matter to be learned or taught. The content to be covered in middle school science or history is different from the content to be covered in an undergraduate course on art appreciation or a graduate seminar on astrophysics… As Shulman (1986) noted, this knowledge would include knowledge of concepts, theories, ideas, organizational frameworks, knowledge of evidence and proof, as well as established practices and approaches toward developing such knowledge" [18, p. 63].
- **Pedagogical Knowledge**: "Teachers' deep knowledge about the processes and practices or methods of teaching and learning. They encompass, among other things, overall educational purposes, values, and aims. This generic form of knowledge applies to understanding how students learn, general classroom management skills, lesson planning, and student assessment." [18, p. 64].
- **Technology Knowledge:** Knowledge about certain ways of thinking about, and working with technology, tools and resources. This includes good understanding information technology broadly enough to apply it productively at work and in everyday life, being able to recognize when information technology can assist or impede the achievement of a certain goal, and being able continually adapt to changes in information technology [18].

The activities carried out within our research group involve a constant practice: we are used to working in synergy with teachers. Then, two communities emerge, that of researchers and that of teachers. A theoretical framework that describes and analyses the relationship and reciprocal influence between these two communities – involved in a course in mathematics education for professional development, with respect to their professional practices – is the Meta-Didactical Transposition (MDT), a theoretical model elaborated by Arzarello and colleagues [2], expanding on Chevallard's concept of didactical transposition and praxeology [5].

In the MDT, the researchers have the objective of transposing a certain piece of knowledge, related to the teaching and learning of mathematics, to favour the professional development of the teachers, according to the reference institutions (national curricula, textbooks, ...). In this case, Arzarello and colleagues [2] introduce the notion of meta-didactical praxeologies: they consist exactly of the tasks, techniques, and justifying discourses that develop in teacher education processes. In fact, an educational course generally aims – with the engagement of researchers as trainers – at developing teachers' existing praxeologies, transforming them into new ones, for example targeted to the introduction of new technologies, or teaching practices, or theoretical frames by research in mathematics education, or new curricula, and so on, according to the aims of the programme. This evolution in the praxeologies, if happen, is therefore the result of an interaction between the community of researchers and that of teachers. The tangible result of the evolution of teachers' praxeologies is their application in the classroom, with their students, who benefit from the professional development that their teachers have experienced.

3. Mathematics laboratories at MERG

In this section, we describe some of the laboratories that we, as researchers in mathematics education, have proposed over the last 10 years. We will describe 5 activities, one addressed to students (Laboratory 1) and the others also to teachers (Laboratories 2-5).

The activities are framed in the TPACK as described in Table 3.1, differentiating in Content and Technology.

Table 3.1 Laboratories in the TPACK framework

	Technology Knowledge	**Pedagogical Knowledge**	**Content Knowledge**
Laboratory 1	yEd, Icosien game	Mathematics laboratory; TRU framework	Elements of Graph Theory
Laboratory 2	GeoGebra		Surprising properties of centroids
Laboratory 3	GeoGebra, Excel		Brahmagupta theorem and consequences
Laboratory 4	Moodle platform; MathCityMap		Outdoor math trails
Laboratory 5	GeoGebra classroom		Irrational numbers and connection with philosophy

3.1 Laboratory 1: Elements of Graph theory

Laboratory 1 was part of the *Percorsi per le Competenze Trasversali e l'Orientamento (PCTO)* project, promoted by the Italian Ministry of Education and compulsory for all high-school students, which aims to bring students closer to the world of work.

Several laboratories dealing with graph theory topics have been experimented, from primary school to secondary school [10]. Here we focus on a proposal carried out at secondary school that involved students of the first two years (4 meetings of 2.5 hours each). The course aimed, on the one hand, to introduce basic concepts of graph theory up to Eulerian graphs and Fleury's algorithm and, on the other hand, to develop skills in modelling problems through a graph; in expressing conjectures, arguing them, comparing one's own hypotheses with one's peers in order to reach shared results; in reflecting on the fact that mathematical objects are "hidden" in various situations and objects of everyday life.

Concepts were always introduced by first dealing with a list of problems. For example, problems that can be schematised by means of a graph were initially proposed, before the students knew the mathematical topic of graph (Problem 1), or Eulerian graphs were presented by first solving Problems as Problem 2.

Problem 1*. The coach in the ball*

Few weeks ago a football tournament between the following schools started: Archimedes, Descartes, Euclid, Fermat, Pythagoras, Thales. The rules of the tournament provide for a single round of matches, i.e. each team meets all the others only once.

The following matches have been played so far: Archimedes-Thales, Euclid-Archimedes, Thales-Pythagoras, Thales-Cartesian.

How can you summarise and clearly express the current situation of the tournament?

Please express the situation at the end of the tournament succinctly and clearly.

Problem 2*. The Pentagon's routes and…*

a) Can you draw a pentagon and all its diagonals without ever lifting the pencil from the paper or going over the same line twice?

b) Can you draw a square and all its diagonals without ever lifting the pencil from the paper or going over the same line twice?

At the end, students are guided to recognise the characteristics of Eulerian/Semieulerian graphs and to use Fleury's algorithm to solve problems. The activity ends with a real problem that deals with routes of airlines companies: Preamble: *Air transport is a complex business. It involves major investments (aircraft and maintenance infrastructure), highly qualified staff (pilots and flight attendants) and precise information in real time (reservation systems, for example). The costs associated with "air traffic" are huge and waste must be avoided. For example, an aircraft on the ground does not provide any revenue, so the amount of time each aircraft is stationary must be reduced. To this end, some airlines identify routes and design circular routes for individual aircraft. A circular route is defined as one that covers all routes, once and only once.* Problem: *Can you help the Eurofly company to organise its routes?* (and a table containing the covered routes was given).

During the activity, students played an online game (Icosien), experiencing the difficulty of solving a problem (in this case finding Eulerian and semi-Eulerian paths) without having a mathematically founded strategy (up to date this game is no longer available). They also used the yEd software (https://www.yworks.com/products/yed), through which graphs can be drawn, manipulated and analysed.

3.2 Laboratory 2: Magic of centroids

Laboratory 2 was part of the *Piano Nazionale Lauree Scientifiche (PNLS)* project, promoted by the Italian Ministry of Education, which plays an important role in teachers training and students' orientation and self-assessment in scientific subjects. The activities of the PNLS are laboratory activities [1], designed and carried out jointly by teachers and university researchers. This joint work of teachers and researchers is the element that characterises the PNLS and promotes the development and strengthening of relations between the school and university systems.

During this Laboratory, some properties of centroids of geometric figures, such as triangles, quadrilaterals and tetrahedra are investigated. In particular, the properties are proved by means of geometric transformations and by introducing extensions of triangles and quadrilaterals, i.e. by adding one, two or three new vertices to the figure [13].

Laboratory 2 is structured in two steps: (1) design of laboratories and preparation of materials necessary for the construction of laboratories; (2) implementation of laboratories in class. Step 1 and 2 were carried out both as mathematical laboratory type activities: the first one for teachers and the second one for students. So, a so-called *double-laboratory* was proposed [11]. This terminology comes from us of the MERG.

Both steps were arranged so that teachers first and students then were just guided, in order to become independent in developing the activity.

We underline that, in Step 1, teachers and researchers decided to structure the work on the use of GeoGebra and through worksheets allowing students to work independently. These worksheets were elaborated from a sample worksheet prepared by the researchers. Worksheet had a tabular layout with two columns: the left column indicated the action that was then made explicit in the right column [12]. Some of the actions that have been used in the worksheets are: Construction, Exploration, Definition, Observation, Control… Most of the actions are carried out on GeoGebra. Technology, and GeoGebra in particular, help teachers in implementing the "explore-discover-test-conjecture-proof" model [12] at every level. The use of Dynamic Geometry Systems "has totally changed the way Euclidean geometry can be studied. With a minimal introduction, students may explore and discover dynamically relevant properties rather than being told about them" [21, p.1]. For example, we ask students to explore the following situation. Let ABCP and ABCQ be two pyramids with the same base ABC. Referring to the Figure 3.2.1, let $P_1P_2P_3$ be the triangle with vertices the centroids of the faces of ABCP, and let $Q_1Q_2Q_3$ be the triangle with vertices the centroids of the faces of ABCQ. Students are asked to explore and conjecture on the two triangles also referring to the medial triangle of ABC (Table 3.2.1).

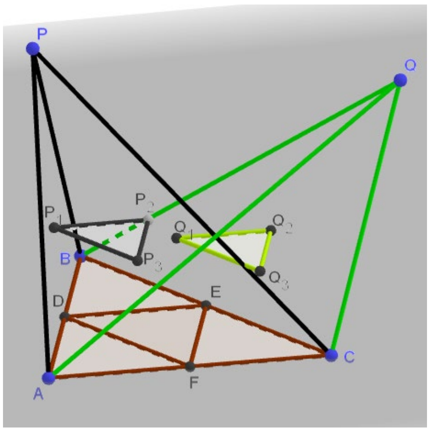

Figure 3.2.1 Centroids' triangles

Table 3.2.1 Exploring centroids' triangles

Exploration	What relationships do you think there are between the sides of $P_1P_2P_3$ and those of $Q_1Q_2Q_3$? To check your assumption, use the appropriate tools in GeoGebra or the Algebra view. Move the points P and Q, the property still holds?
Conjecture	Let us call *centroid triangle* the triangle with vertices the centroids of the faces of a pyramid with triangular basis. The centroid triangles of two pyramids with the same triangular basis are ……… ………………………………………………………………………………..

Table 3.2.1 is only part of a longer activity where students are guided to construct the objects in Figure 3.2.1 and to explore them, in order to conjecture first and prove afterward that

Given a pyramid with vertex P and base a triangle T, the medial triangle of T is the correspondent of the centroid triangle of the pyramid in the homothetic transformation with centre P and ratio 3/2.

3.3 Laboratory 3: Brahmagupta theorem and consequences

Also this laboratory, as Laboratory 2, is a double-laboratory carried out within the *PNLS* project: teachers are first called to work together with researchers in writing the worksheets, and then students work in class. The laboratory activity is aimed at students in their second year of high school and is based on a geometric problem already posed in 600 AD: Brahmagupta's theorem [7]. It aimed to cover the topic in modern terms through the use of GeoGebra, in order to renew students' interest in the study of geometry through an innovative experience.

In the first laboratory, teachers and researchers design and create digital worksheets, implemented in Excel, at various levels of difficulty: *Helps* and *Enhancements* are provided on the worksheet and can be viewed by the student if necessary (Figure 3.3.1a). In this way, each student can customise his/her own learning path, proportionate to his/her skills and abilities. These sheets allow a formative evaluation to be carried out, since the teacher can trace the path taken by each student. The student can also carry out a self-assessment of the course he/she has completed (Figure 3.3.1b).[1]

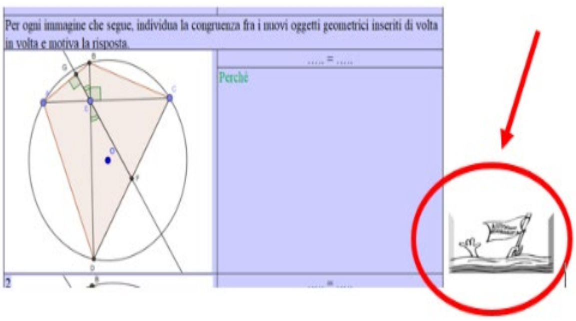

Figure 3.3.1a The help button

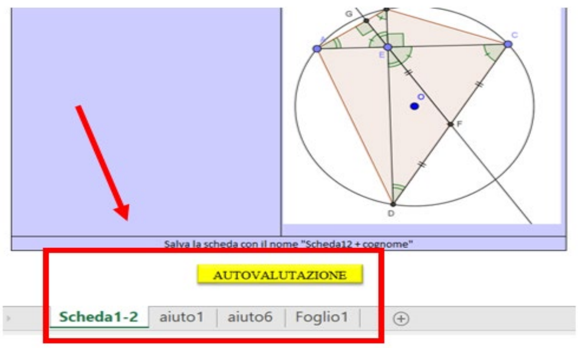

Figure 3.3.1b Assessment

3.4 Laboratory 4: A MOOC for mathematics teacher education

The Laboratory 4 is part of the European project, MaSCE[3]: "Math Trails in School, Curriculum and Educational Environments of Europe" (http://masce.eu/). The project promotes the adoption of math trails, a collection of tasks, located with walking distance, that are useful to discover and solve mathematical problems on real objects [28]. Math trails can be used in the school contexts to offer a real life experience besides textbooks [24]. Nevertheless, they require teachers' preparation and a solid post-processing in the classroom. MathCityMap (MCM, https://mathcitymap.eu/en/), a math trail management system, facilitates this process by the benefits of technology: on MCM users can create tasks and trails and share them among themselves or with the public [15]. In fact, it is already

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[1] The oval in Figure 3.3.contains the picture of the Help button. The rectangular box in Figure 3.3.1b specifies the number of the worksheet (1-2 in this case) and the Helps that have been used (1 and 6 in this case).

possible for teachers to create trails in a web portal and for students to run them on a smartphone app. Studies show that MCM math trails have a positive effect on student motivation and learning when run regularly [4]. However, math trails are still used sporadically by teachers and not systematically within learning curricula. Within the MaSCE[3] project, in order to make teachers autonomous in the use and creation of math trails with their own students, a 12 weeks MOOC was delivered online, in English, via the DI.MA. platform (http://dimamooc.unict.it/) managed by the University of Catania.

During the MOOC, the enrolled teachers, by means of specific digital resources (videos, tutorials, ...) received indications on how to design math trail tasks on MCM. In particular, in addition to the types of format of a math task (for more details see: https://mathcitymap.eu/en/the-mathcitymap-task-formats-2/), specific design criteria were provided that had to be respected to produce a suitable MCM task. The criteria indicated to the teachers were as follows [14]:

- *Clarity:* For each task, a picture must be created that allows the clear identification of the situation or the object the task is about.
- *Presence:* The task can only be solved on site, i.e. the task data must be collected on site. This also means that the picture or the task description must not be sufficient to successfully solve the task.
- *Activity:* The person who solves the task must be active and do something (e.g. measuring or counting).
- *Multiple solutions:* The task should be solvable in different ways.
- *Reality:* The task should be application-oriented, realistic and not too contrived.
- *Graduated hints:* At least two hints should be added to each task.
- *School mathematics and "tags":* The task should have a clear relation to school mathematics: Use the prepared tags or add new terms. The task should also be assigned to a class level**.**
- *Solution formats:* The solution of the task should be presentable as a solution interval (good and medium interval), as an exact number, as multiple choice or as a GPS task.
- *Tools:* No special tools should be required to solve the task.
- *Sample solution:* One should offer a solution and hints (only visible in the portal) for teachers.

The design of the tasks took place on the MCM web portal, filling in a template structured in the light of the design criteria. For the final homework, teachers had to run their own math trail with their students and report back on this experience.

3.5 Laboratory 5: Order and Disorder.

Laboratory 5 is part of the *Liceo Matematico* project. This project aims to enhance mathematical skills in high-school students, through laboratorial activities, pointing mathematics as a glue among different subjects. In particular, here we refer to a module entitled "Order and Disorder", aiming to underline connections between mathematics and philosophy. The laboratory we present here is again a double-laboratory. The teachers' laboratory (4 meetings of 2 hours each) was held by the researchers at distance on an online collaboration app, because of the COVID-19 pandemic situation. The mathematical content was irrational numbers, and aimed to make students owners of *reductio ad absurdum* proof. The students' laboratory started with a philosophical text, orienting students to the questions "Who am I? Am I ordered or disordered?" and "When can we define that a number is disordered?". The laboratorial methodology in class was used not only for the mathematical part, but also for the philosophical one, because classes were set as research communities on the Socratic model, typical of the ancient mathematical and philosophical Greek schools. There was not a telling of history of philosophy, but rather students philosophized themselves on the topic of Order and Disorder, and thought about irrational numbers, originally seen (by Pythagoras) as disordered numbers. The initial task of searching a "definition" of Disorder was faced by analysing several situations, from paintings (Velàzquez' and Picasso's "Las Meninas") and geometry in Arabian ceramic (a mosaic in Samarcanda, where many irregular polygons are arranged for a final regular structure), to nature (irrational numbers useful in plants to guarantee light to all the leaves). Then, the attention was focused on the geometric construction of the square root of two, by using GeoGebra to manipulate the diagonal of a square, in phase of conjecturing that the number $\sqrt{2}$ has an infinite number of digits without a repeating sequence. Students worked at distance in the *GeoGebra classroom* platform (implemented by the GeoGebra developers during the spring 2020 in view of distance teaching). Thanks to this environment the teacher can see in sync how students are working. While working with GeoGebra, students had ideas exchanges with classmates and the teacher. Then teachers and students read an excerpt of Plato's "Menone", in which a maieutic process is used by Socrates to prove that everyone (even the slave of the famous dialogue) can learn something originally unknown (the irrationality of $\sqrt{2}$, this time). Finally, students were guided to the famous proof of irrationality of $\sqrt{2}$, made by *reductio ad absurdum*, again in the *GeoGebra classroom* platform. During the activity, students were invited to express their own thoughts and feelings on the theme of Order and Disorder with poems, drawings or whatever they wanted. We note that teachers gave the possibility to use both technological and classical tools, but students preferred to use paper. In Figure 3.5.1 and Figure 3.5.2 two of their works.

Figure 3.5.1 My Order and Disorder

Figure 3.5.1 My Order and Disorder

The whole materials (considerations about paintings, mosaic, irrationality in nature, reads, *GeoGebra classroom*'*s* activities) were prepared by the researchers. But teachers (both mathematics' teachers and philosophy's teachers), who attended the first laboratory of the double-laboratory were free to adapt the contents to their classes, given only the indication to use laboratorial methodology. During the first laboratory, teachers were open and interactive, and proposed some changes and improvements of the second laboratory.

4. Discussion and conclusions

The proposed activities, when experimented in the classroom, from a methodological point of view, are based on the *mathematics laboratory*: Students *work together* (WM in Table 2.1) on *problems* (P in Table 2.1), discover concepts by themselves, experiment and manipulate instruments (I in Table 2.1), guided by the *teacher* (T in Table 2.1). All laboratories are oriented to a *robust teaching*: the *Contents* (C in Table 2.2) are rich because either the mathematical topic is not trivial (as in Laboratories 1, 2 and 3), or it is standard but connected with reality (Laboratory 4) or to other disciplines (Laboratory 5). Moreover, all the contents are chosen to develop productive disciplinary habits of mind, such as modelling (Laboratories 1 and 4), exploring, conjecturing and proving (Laboratories 2, 3 and 5). The activities have task difficulty ranging from moderate to demanding in such a way the *Cognitive Demand* (CD in Table 2.2) is challenging. Anyway, thanks to the support of the teacher and the classmates, given the working together modality, no student is left behind, guaranteeing an *Equitable Access to Content* (EA in table 2.2). With the laboratorial methodology, by seeing mathematics in real-life objects (as in Laboratory 4) and/or thanks to the use of technological tools (yEd, GeoGebra, MathCityMap), able to make students "see" and "manipulate" mathematical objects, students feel themselves within the mathematical environment, perceiving graphs, geometric figures and numbers as "alive" objects next to them. This causes engagement, *Agency, Ownership* over the content and the development of a positive *Identity* (AOI, in table 2.2) as thinkers and learners. Teachers do not evaluate students, but rather help them to adjust or refine their reasoning, when they are wrong. In such a way, a *Formative Assessment* (FA in table 2.2) takes place. Mistakes are not judged, but rather taken as a pretext to discuss and argue, helping the comprehension of the content.

Moreover, we observe how, even in a totally online context (such as Laboratory 4), the two communities are present in most of the proposals illustrated: the community of the researchers and that of the teachers. As we have already pointed out, working in synergy with teachers is a strength for us at MERG. In each of the workshops (except Laboratory 1 that is directly intended for students), the researchers were engaged in transposing meta-didactical praxeologies to teachers. These praxeologies concerned new mathematical contents that were not strictly curricular (properties of centroids, Brahmagupta's theorem, ...), new teaching methodologies (e.g. conscious use of laboratory methodology, outdoor mathematics, connection with other disciplines ...) and new technological tools that teachers may use in their own teaching (e.g. yEd, GeoGebra, Excel, MathCityMap, ...). The teachers who took part in the laboratories worked in an active way: they not only followed the training sessions, but then implemented these proposals in class with their students. Sometimes they faithfully reproduced the activities (e.g. Laboratory 2, 3), sometimes they also customised the activities in terms of the choice of contents to be proposed (e.g. Laboratory 4) or the way in which they were proposed in class (e.g. Laboratory 5). These are, therefore, testimonies of evolution in the teachers' didactical praxeologies, as they have benefited from training and modified their usual teaching practices.

As said, all activities share the mathematical laboratory approach aimed at a Robust Understanding of mathematics, but they differ in the way of proposing the laboratory. In Laboratory 1 students work solving a series of problems in order to introduce concepts and apply acquired knowledge. In Laboratory 2 students start using a two columns worksheet, where they are invited to explore, observe, manipulate, conjecture. Worksheets are quite guided and they are the same for all students. In Laboratory 3 the 2 columns worksheets become "flexible": students can personalise it, freely deciding whether to use helps or not. A new opening appears in Laboratory 4, the MOOC. In fact, on the MCM web portal, on the one hand, the teachers had to follow the "rigid" structure of the template, but on the other hand, they had a lot of freedom in the choice of contents to be inserted. In fact, although a precise typology of the task had to be respected, the choice of the mathematical object to be considered and the related mathematical problem to be associated with it, together with hints and solution strategy, were freely chosen by the teacher. If we wanted to draw a parallel with the worksheets adopted in the laboratories described above, we could say that here the worksheet (i.e. the MCM template) is a container and there is openness towards teachers, while there the openness is towards students. In Laboratory 5, in the end, there is an opening towards both, students and teachers: the laboratory is open for teachers, because in the first laboratory the researchers introduced the contents and teachers not only proposed changes valid for the whole activity, but also could decide how to guide their own classrooms' activities. The only materials equal for all the involved students were the GeoGebra classroom activities, agreed with all the teachers. The laboratory is open for students because the questions were related to students' lives (Are you ordered?) and their points of view (What is order?), rather on fixed topics, in such a way the classroom discussion was free to flow, and no classroom had the same discussion of another one. Moreover there is openness for students because they were free to produce their thoughts and feelings toward the theme with every way they preferred (discussion, draw, poems, …).

Then, the flow of the evolution of our approach over the years, as researchers, that we call meta-methodology, is summarized in the following table:

Table 4.1 From closed to open mathematics laboratories

Activity	Meta-Methodology
Laboratory 1: *Elements of Graph Theory*, addressed to students. Worksheets: list of problems.	Closed
Laboratory 2: *Magic of centroids*, double laboratory addressed to teachers and students. Worksheets: tabular two columns layout.	Closed for students; Open for teachers (that work on the construction of the worksheets).
Laboratory 3: *Brahmagupta's theorem and consequences*, double laboratory addressed to teachers and students. Worksheets: tabular two columns layout, with help options.	Flexible for students (because of the help options in the two columns worksheets); Open for teachers (that work on the construction of the worksheets).
Laboratory 4: *MOOC on Math Trails*, laboratory addressed to teachers. Worksheets: MCM container.	Open for teachers (that can choose contents and problems).
Laboratory 5: Order and disorder, double laboratory addressed to teachers and students Worksheets: GeoGebra Classroom Activities	Open for students (that were free to express themselves); Open for teachers (that decide how to bring the content in class).

We are working in the direction of what we can call *open Labs*: the path taken by the students and by the teachers is not fixed, students can take different paths and teachers are free on how to present the topics. In the introduction of the activities, many hints are shown and the students choose which aspect to take care of. This is the direction of a laboratory proposed last year on *Mathematics and reality* (mathematical content: *The golden ratio*). Awareness of the strong link between mathematics and real life is unfortunately often lacking in students. The activity intends to make the connection between mathematics and reality more evident. In particular, the ideas offered are aimed at promoting a more lively and attractive view of mathematics, directed at meaningful learning of mathematics which helps students to develop skills adapted to the demands of society. The activity consisted of a double laboratory: teaching materials on "Mathematics and reality" with particular reference to the golden ratio and its countless applications were prepared and presented, together with some worksheets, at the first laboratory with teachers. Teachers and researchers elaborated together the final version of the material to be presented in class for starting the activity. Digital worksheets, implemented in Excel with the possibility of *asking for helps*, were used so that each student could customise their use, thus working at various levels of difficulty. Subsequently, taking their cue from the initial presentation, the students were free to deepen or integrate certain aspects of it, freely choosing their own learning path to follow. Changing the curricular teaching approach to more engaging paths and methodologies linked to the use and integration of digital technologies also means taking into account the important role that emotions and motivation play in learning, to develop a positive attitude towards mathematics.

Then, a STEM approach is slowly evolving mutating to a STE**A**M approach, where the A, as said in the introduction, is not just arts, but it is meant in a broad sense of *creativity*: on the one hand, teachers are "granted" freedom to customise and re-adapt the content or materials offered by researchers, on the other hand, students are encouraged to be systematic and experimental, as well as to use their imagination and make new connections among ideas. Students can play with concepts of aesthetics and with sensory and emotional engagement, in the context of critical thinking, logical inquiry, or creative production about the world around them.

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