Mathematical Patterns formed by the Spheres

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INTRODUCTION

In Graph Theory, the contact graph of a collection of spheres is a graph whose nodes are represented by the spheres, and each edge corresponds to two externally tangent spheres. In this work, we focus on the construction and the pattern design of models, each having a Platonic solid, an Archimedean solid, or a Catalan solid as the contact graph. The (monochrome) spheres in each model are then assigned with k colors obeying the rule that the spheres receiving two different colors are congruent in the Euclidean sense that there exists a Euclidean motion (a translation, a point reflection, a line reflection, a plane reflection, a rotation w.r.t. a line together with their compositions) taking all spheres of one color onto spheres of another. Whenever such a coloring of the model is possible, the pattern is said to have k-color symmetry.

The models/patterns are organized according to the hierarchy: (1) The number of spheres. The possible numbers are 4, 6, 8,12, 14, 20, 24, 26, 32, 48, 60, 62, 120, as tabulated from the number of vertices in all the polyhedra of interest. (2) The contact graphs. Two models having the same number of spheres may have 1, 2, or 3 different contact graphs. (3) Color symmetry. Two k-color patterns are considered distinct if they have non-congruent parts. Distinct patterns are labeled pattern 1, pattern 2, etc.

There is no preference for particular colors so long as the arrangement causes no mathematical ambiguity.

The completed work of the construction and the design in the native *.cg3 and the screen output *.mov files produced by Cabri 3D can be downloaded for further examination. [7]

This paper is meant to be a survey of a large number of possibilities in designing distinct patterns. We do not attempt (1) to explain in full the motivation for each design; (2) to automate the creative process of designing; (3) to make an encyclopedia of sphere model design. By way of illustration, we are to focus on the model having 120 spheres only.

COMPUTING/GEOMETRIC CONSTRUCTION ENVIRONMENT

All geometric constructions faithfully follow the tradition established by Euclid. This work is made possible under the Interactive Geometry environment created by the software Cabri 3D.

This work is completed under the environment provided by Google Workspace for Education Fundamentals (formerly G Suite for Education).

For the benefit of academic discussions, links to the *.cg3 files saved from Cabri 3D, the animations files *.mov save from video capture of Cabi 3D, and the explanation notes *.pdf can be found at [7].

APPLICATIONS

Designing patterns on the sphere model may serve as a challenging exercise in Concrete Geometry [1,2,3,4,6]. The pre-production and the production of the models as video segments lead the Cabri 3D users to appreciate the visual art of counting when the iPad is seen almost everywhere. At the same time, few people use it for mathematical thinking. Technical terms used for the polyhedra have their origin in the study of Mineralogy. Many models studied here are of current interest in Chemistry [5].

RESULTS

CONSTRUCTION OF THE 120-SPHERE MODEL

Only the Great Rhombicosidodecahedron [4] has 120 vertices among the Platonic, Archimedean, and Catalan solids. Therefore, this is the only possible contact graph for any 120-sphere models for a Platonic solid or an Archimedean-Catalan solid. Since all faces of the Great Rhombicosidodecahedron are regular polygons, all edges have the same length. One by one, the required spheres can be constructed by taking a vertex as the center and one-half length an edge as radius. In practice, it suffices to construct (Fig. 1) only one sphere and then construct the other 119 by taking successive planar reflections, line reflections, and the point reflections compatible with the vast symmetries enjoyed by the polyhedron.

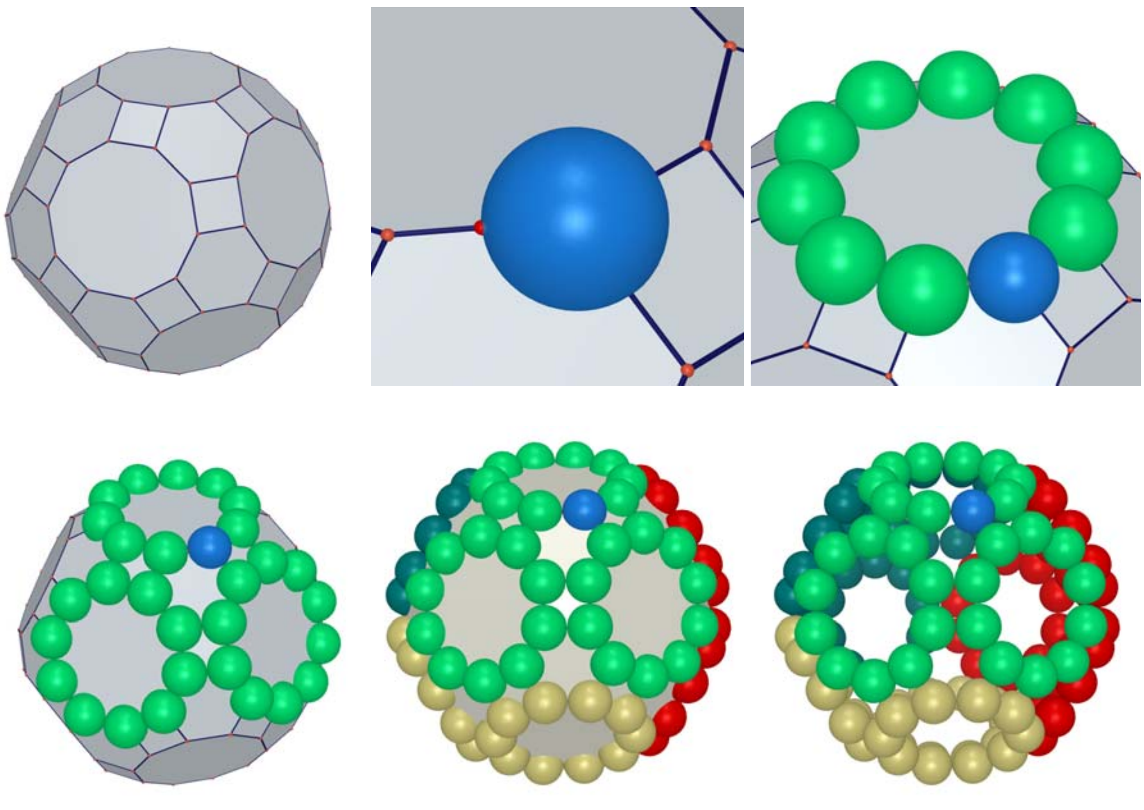

Fig. 1 Construction of the 120-Sphere Model from the Great Rhombicosidodecahedron

PATTERNS ON THE 120-SPHERE MODEL

2-COLOR SYMMETRY

Pattern 1 - Each color consisting of six 10-cycles

The design (Fig. 2) follows the dissection of a baseball into two parts marked by the stitches. The two parts are not symmetric with respect to the center. They are line-symmetric. The appearance suggests the name "The Yin-Yang Pattern."

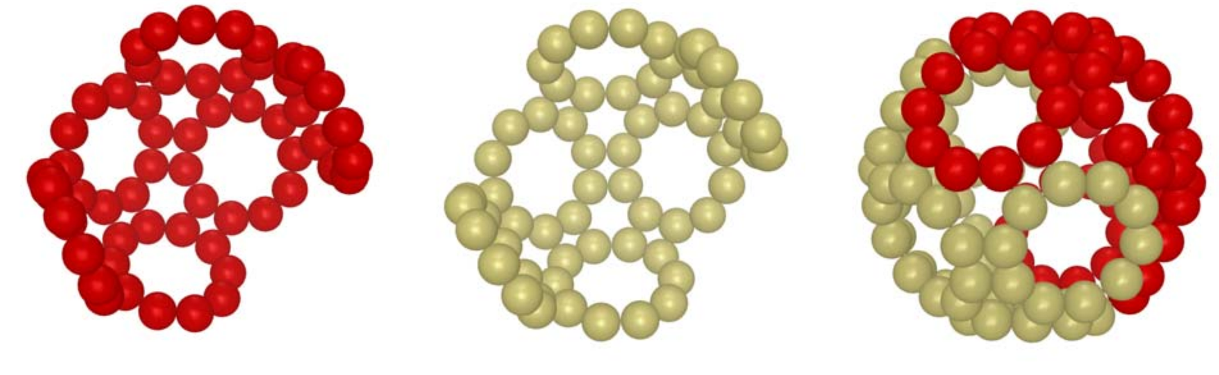

Fig. 2 The Yin-Yang Pattern

Pattern 2 - Each color consisting of 60 connected spheres formed by a single 10-cycle with 5 arms attached

Here is a pattern (Fig. 3, left) having 2-color symmetry, with each color consisting of five connected components formed by 12 spheres. By swapping colors for 10 spheres, we have created (Fig. 3, right) a pattern having 2-color symmetry with each color formed by a single 10-cycle with 5 arms attached (Fig. 4).

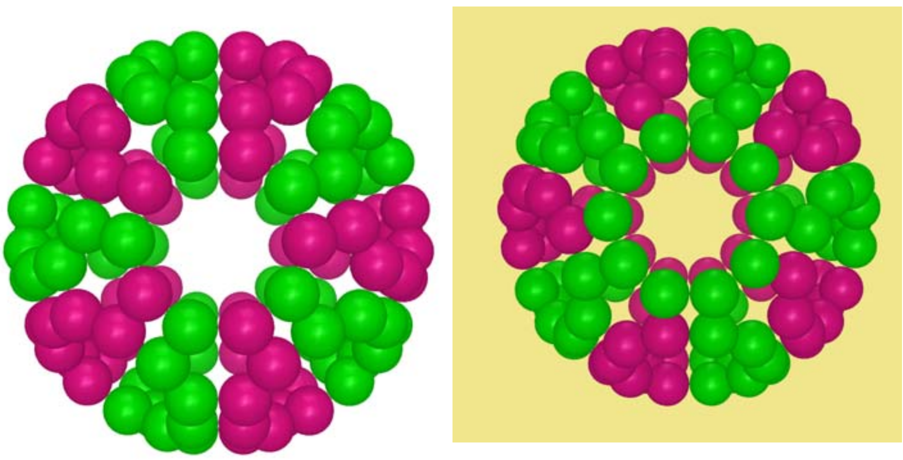

Fig. 3 Swapping colors to create monochrome 10-cycles

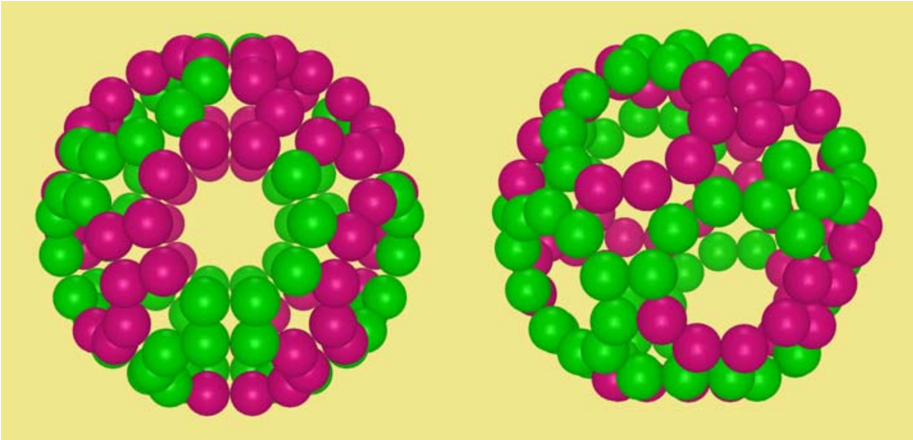

Fig. 4 Each color formed by a single 10-cycle with 5 arms attached

Pattern 3 - Each color consisting of 60 connected spheres with no cycles

Pattern 2 almost depicts a spider, except for the hole at the center. After a slight modification yields a pattern having 2-color symmetry that has no cycles. This pattern suggests the name "the Yin-Yang Spiders" (Fig. 5).

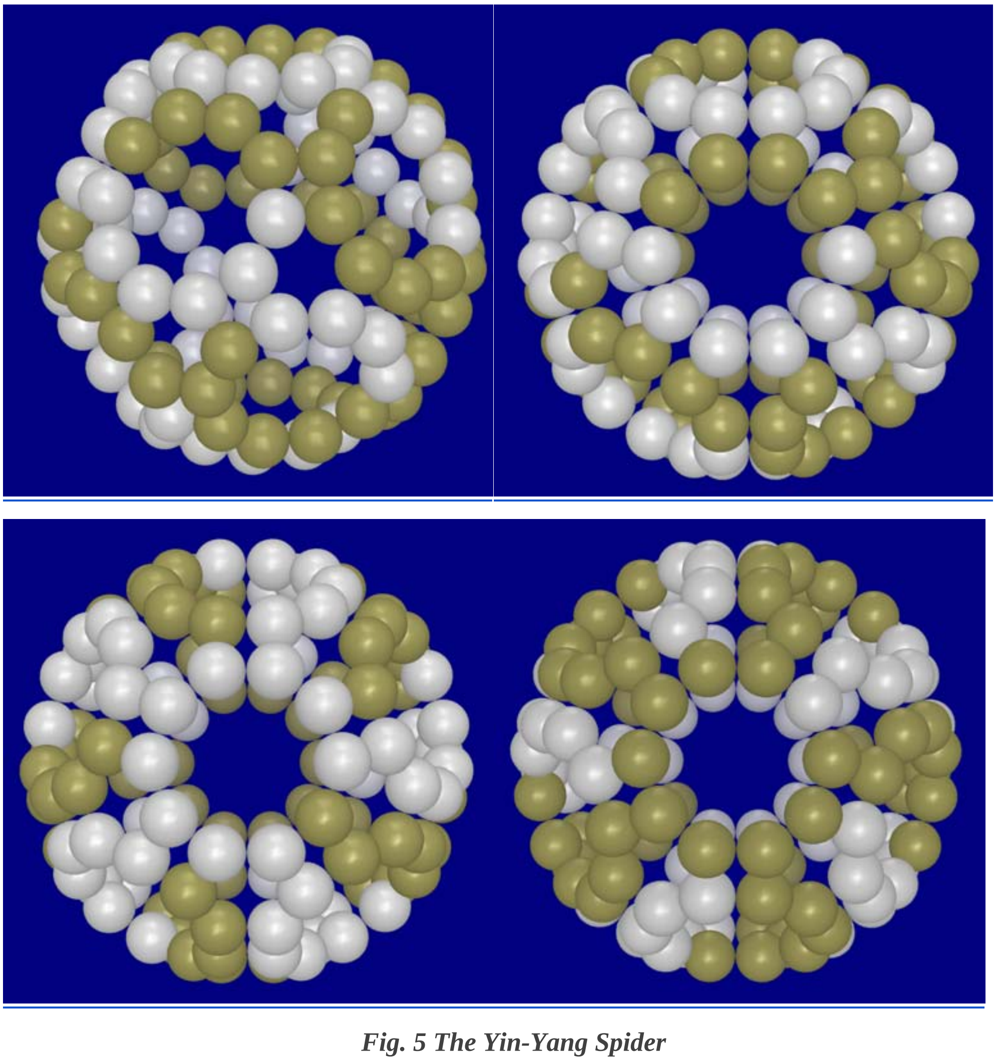

Fig. 5 The Yin-Yang Spider

3-COLOR SYMMETRY

Pattern 1-each color having only a single pair of antipodal 4-cycles

The pattern is suggested by the configuration of the largest cube placed inside a wire-frame dodecahedron.

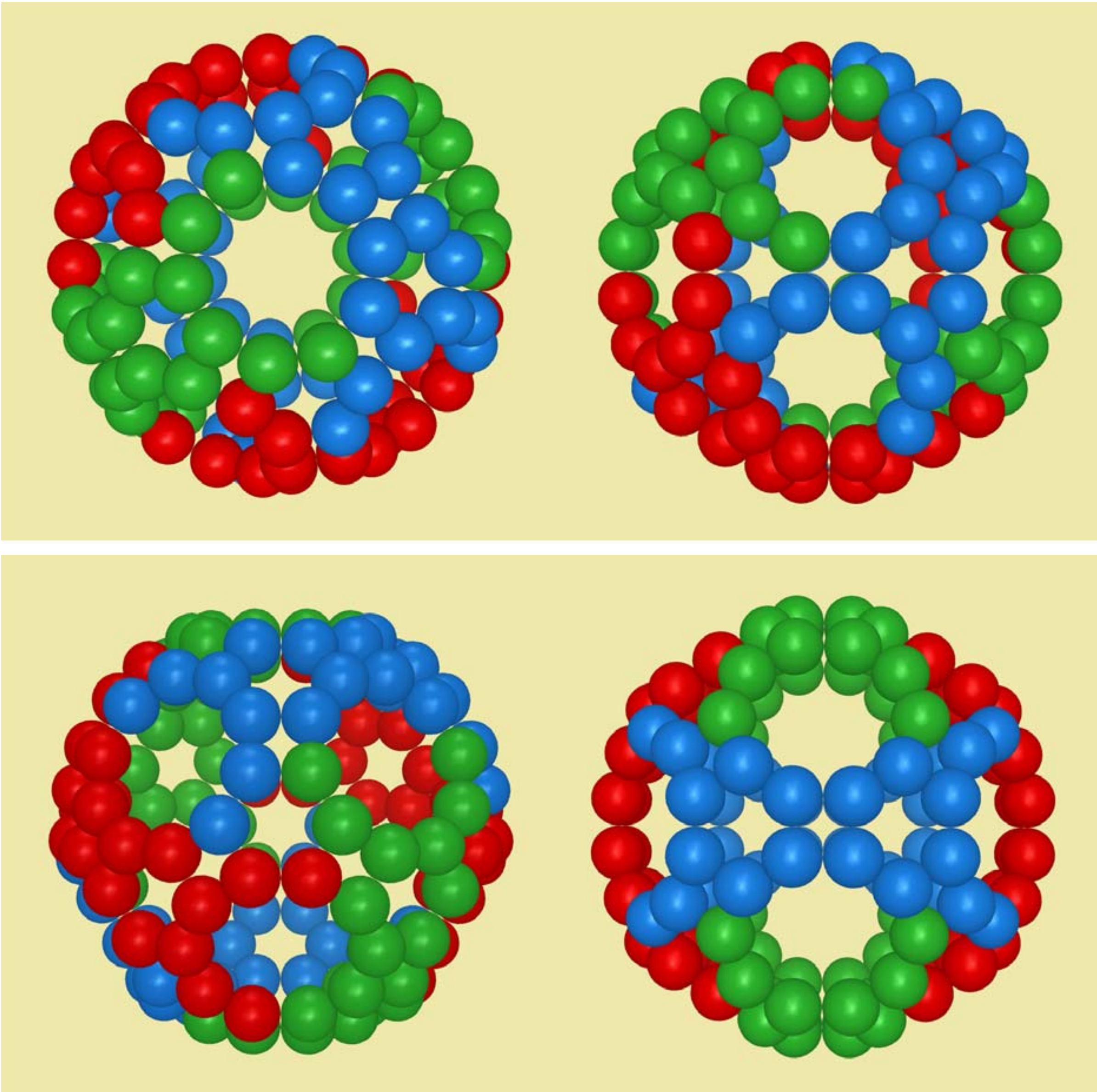

Fig. 6 An octahedral symmetric pattern

Pattern 2- each color consisting of ten 4-cycles

This pattern is a variation of Pattern 1.

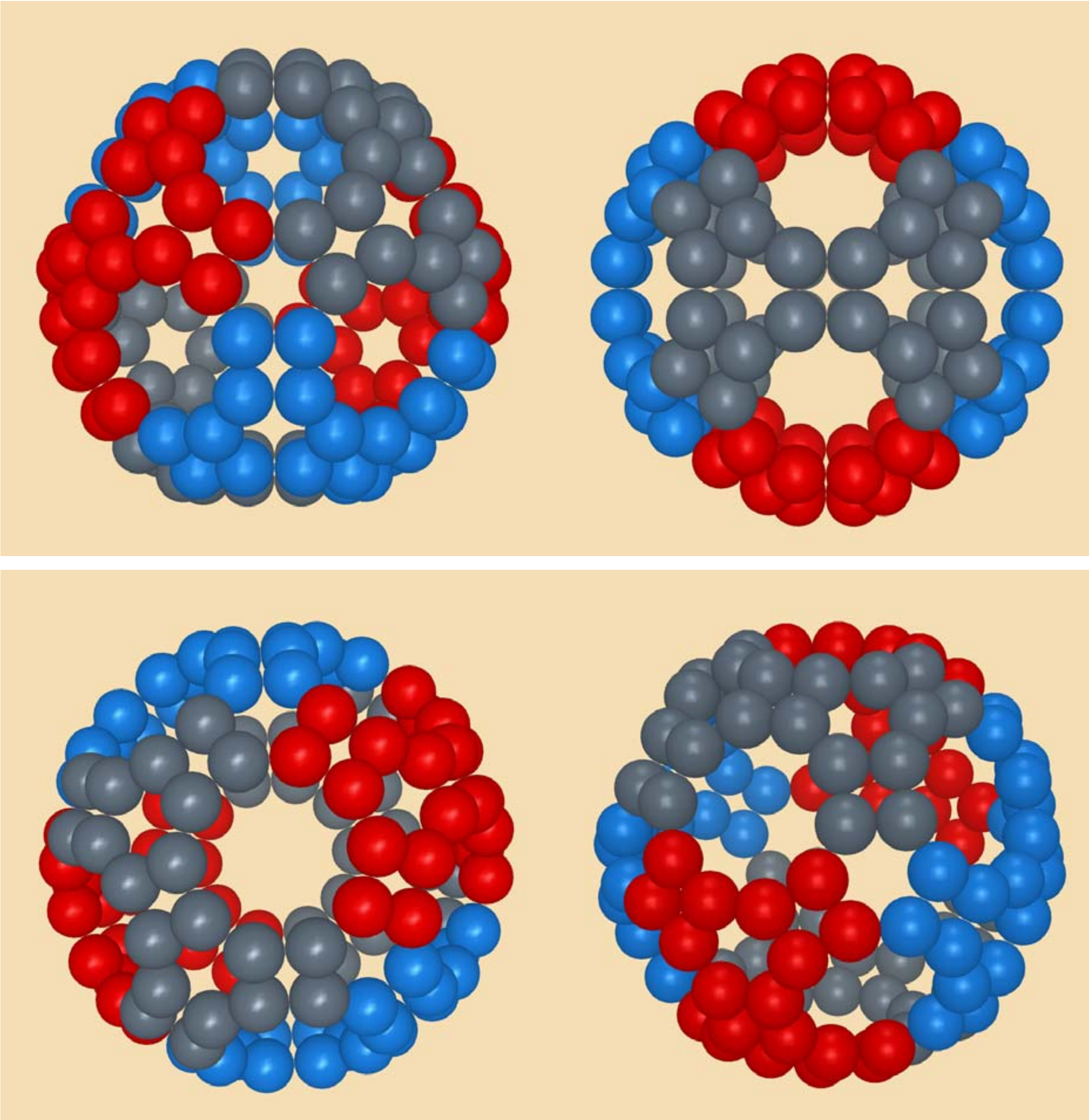

Fig. 7 Another octahedral symmetric pattern

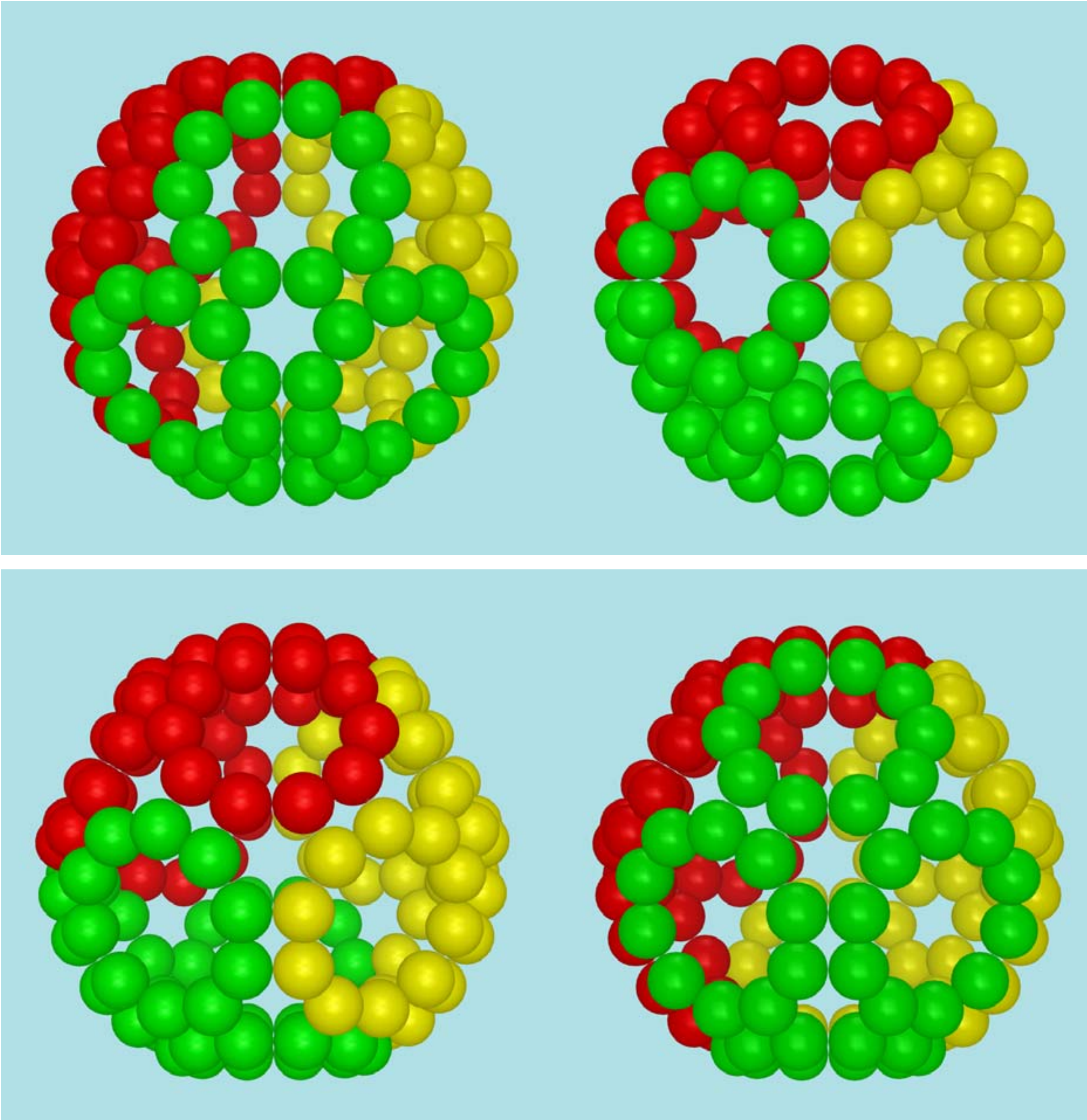

Fig. 8 Each color consisting of four 10-cycles

4-COLOR SYMMETRY

Pattern 1- Spheres of the same color have the graph structure of a tree

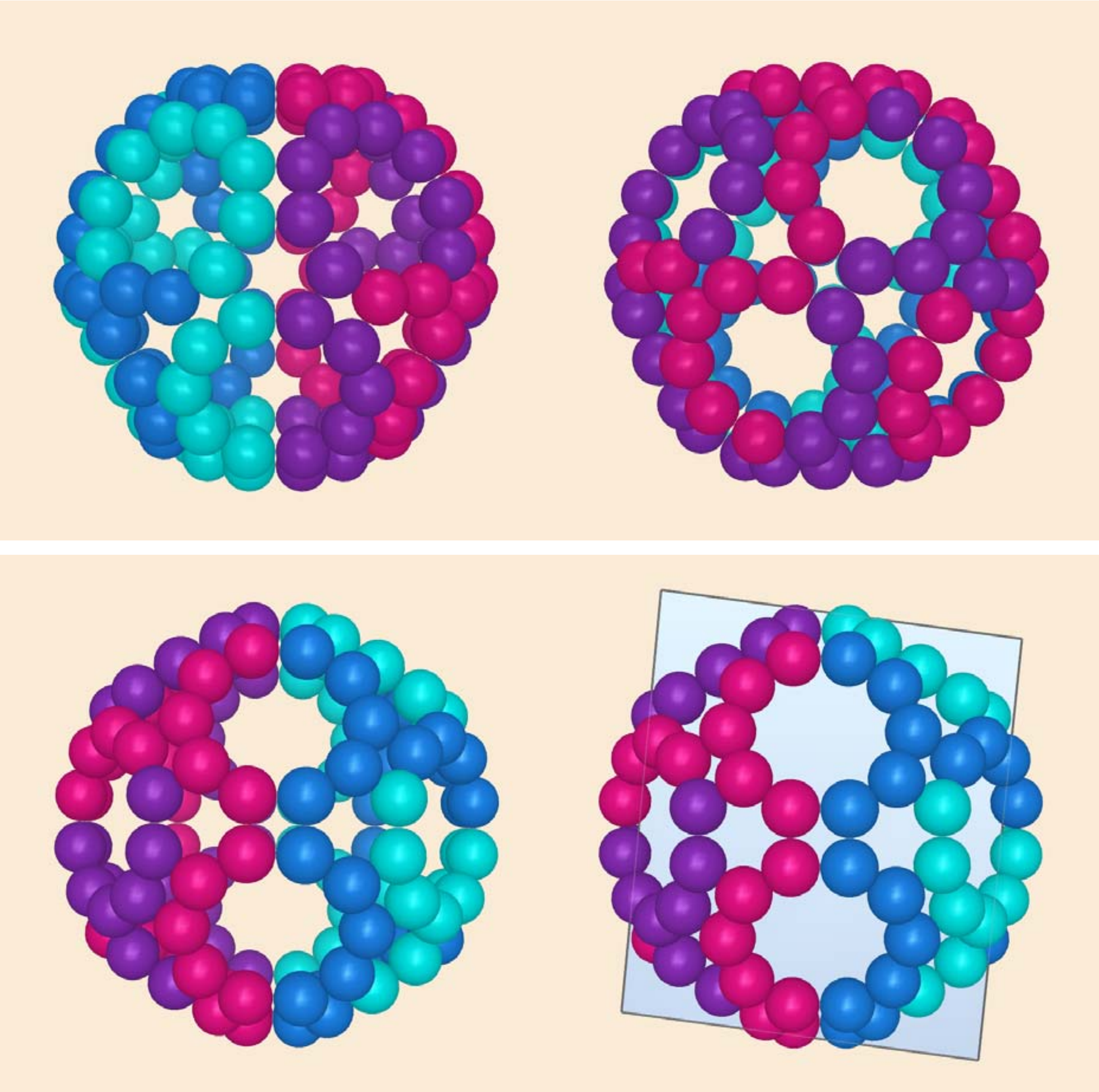

Fig. 9 Each color containing no cycles

Pattern 2- Each set of the same color has 3-fold rotational symmetry

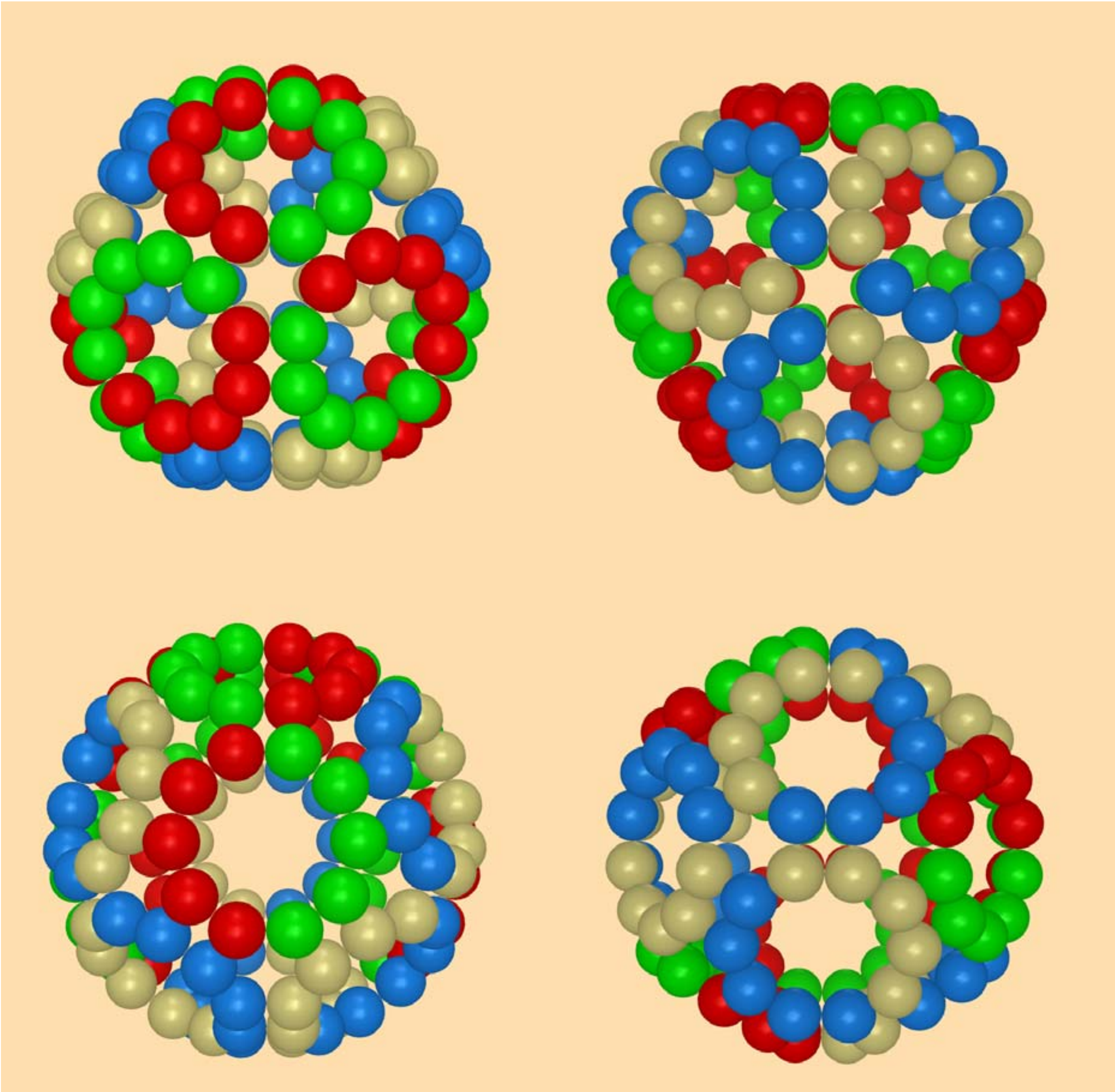

Fig. 10 One-Halves of 10-cycle distributed evenly

Pattern 3- Each set of the same color has no forks

Among 30 spheres of the same color, each touches two others except the two at each end. The author proposes to call it an ATCM Snake.

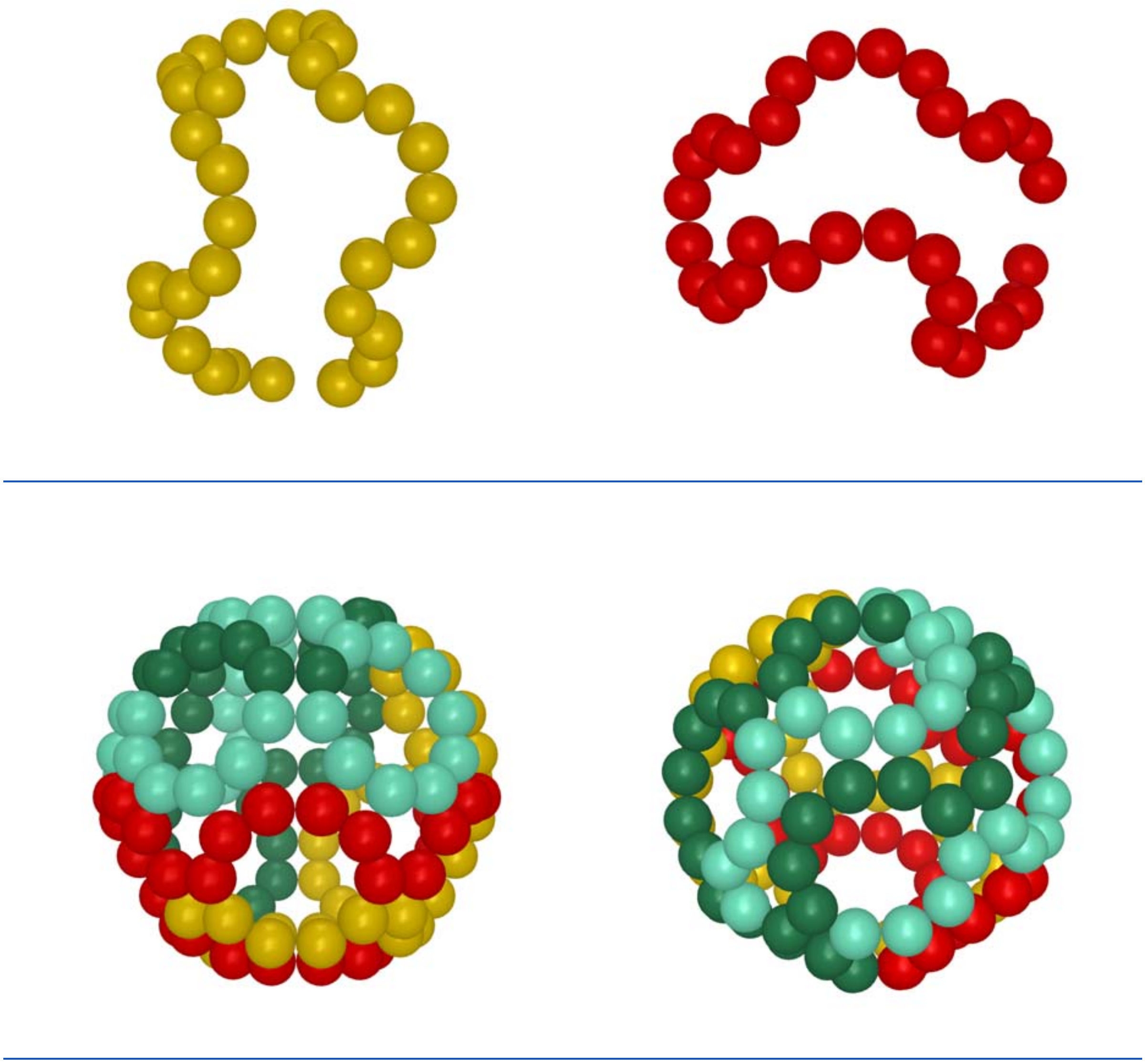

Fig 11 Four snakes covering all

5-COLOR SYMMETRY

Pattern 1-each color formed by 24 connected spheres

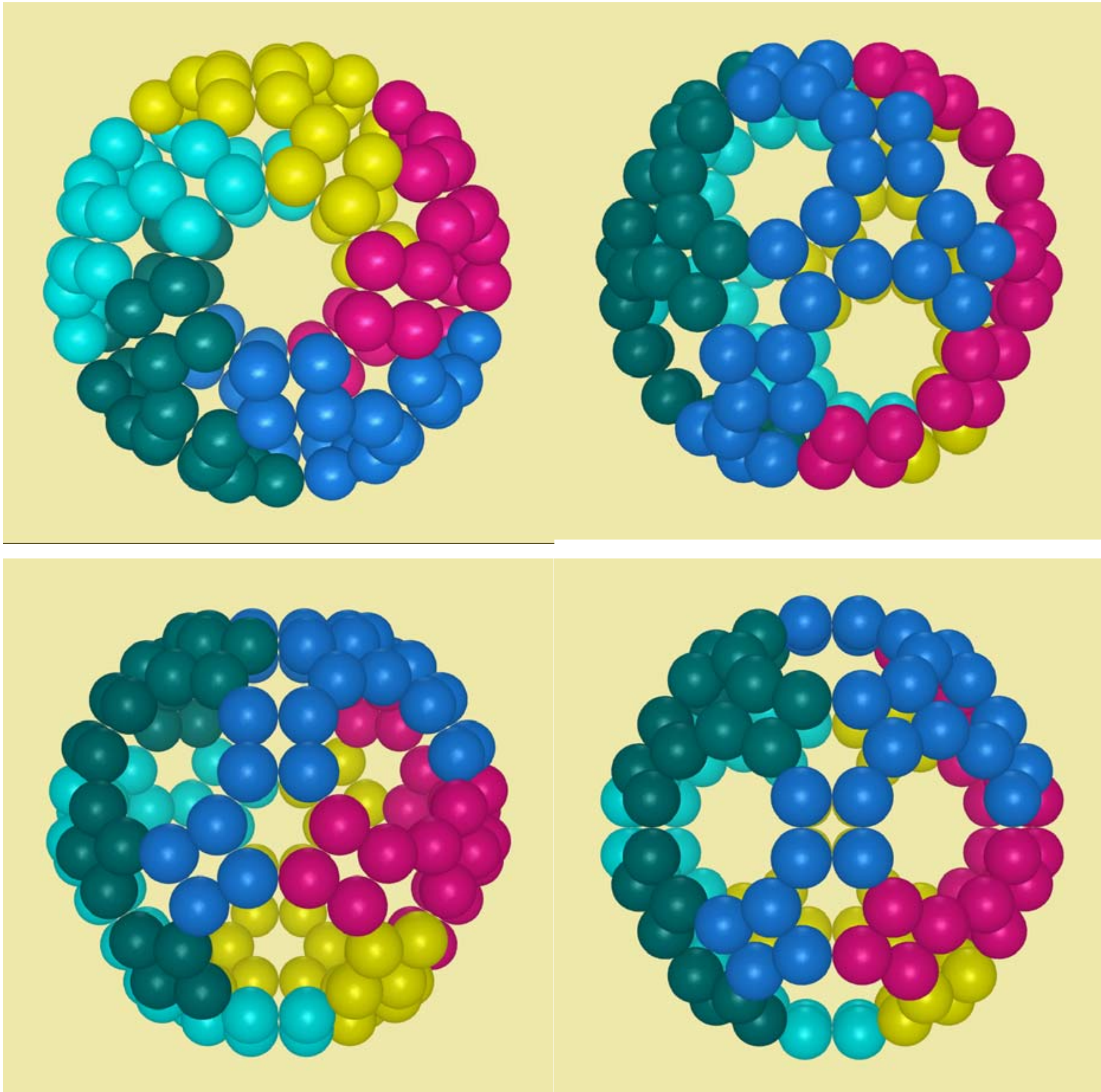

Fig. 12 4-cycles covering all

Pattern 2-each color consisting of an antipodal pair of 3 connected 4-cycles

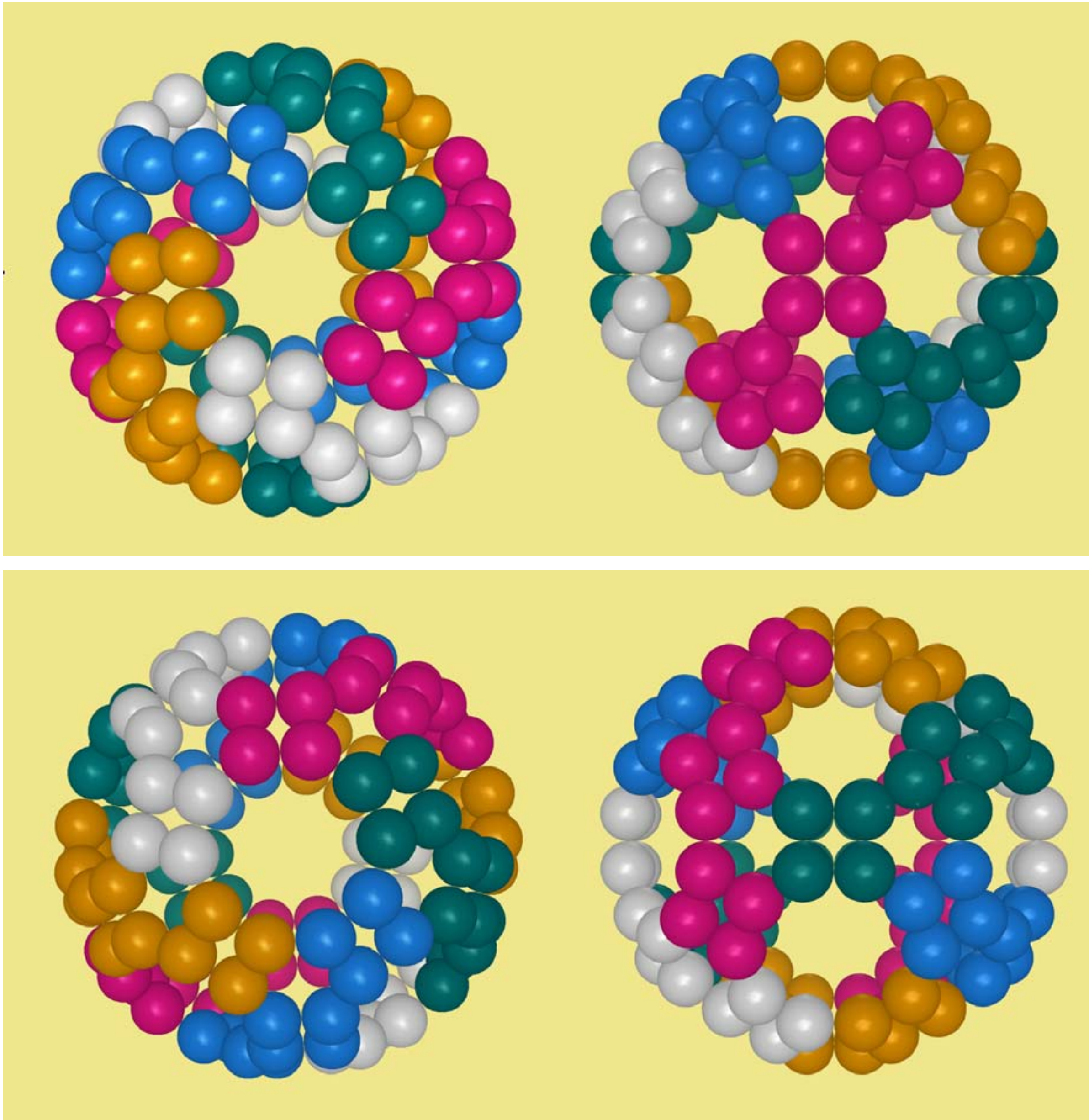

Fig. 13 Art of counting 4-cycles

Pattern 3-each color consisting of four 6-cycles

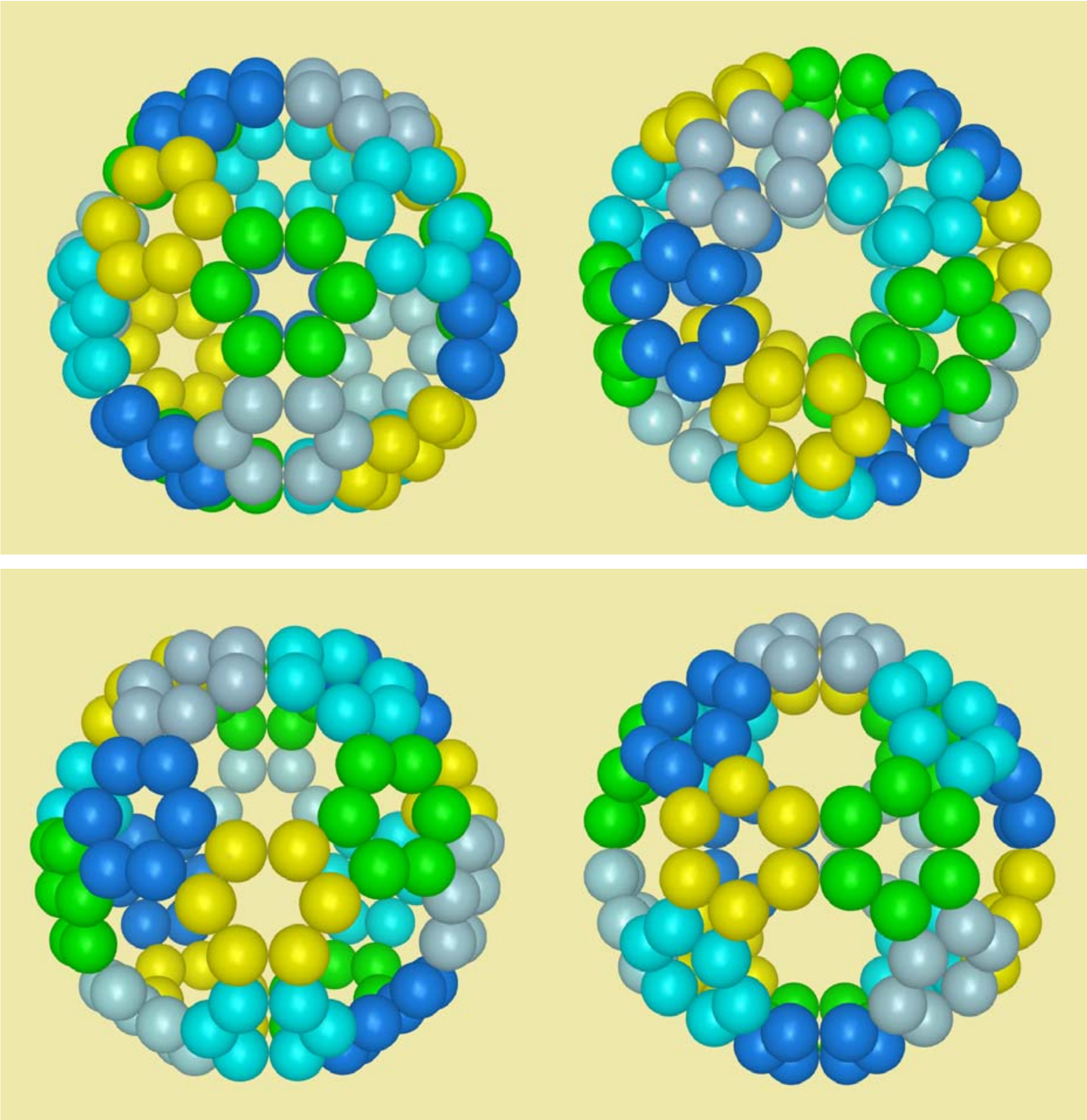

Fig. 14 6-cycle of the same color are centered at vertices of a tetrahedron 5-compound

Pattern 4-each color consisting of six 4-cycles

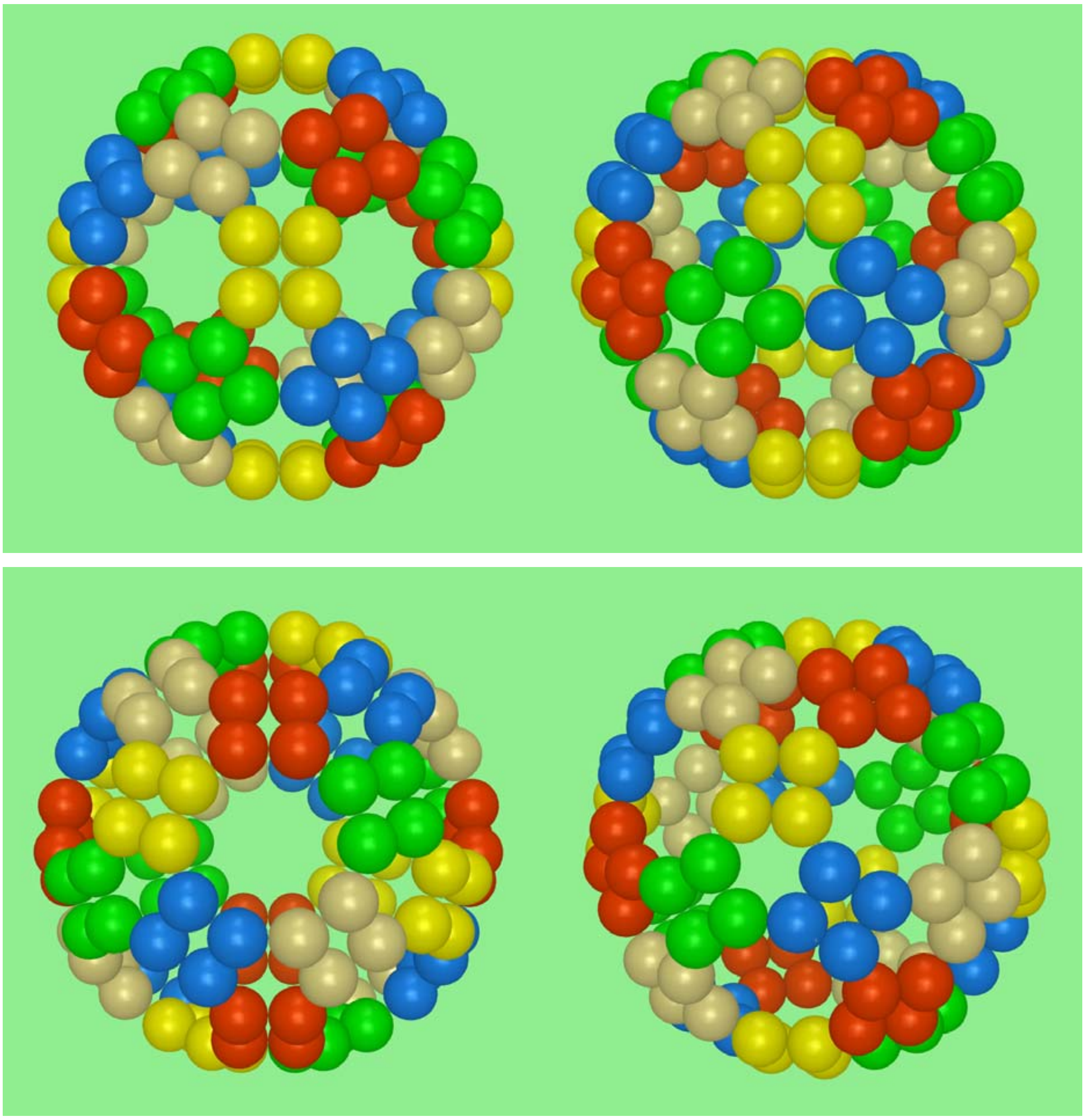

Fig. 15 4-cycles of the same color are centered at a vertex of an octahedron 5-compound

6-COLOR SYMMETRY

Pattern 1- each color consisting of 10 connected couples

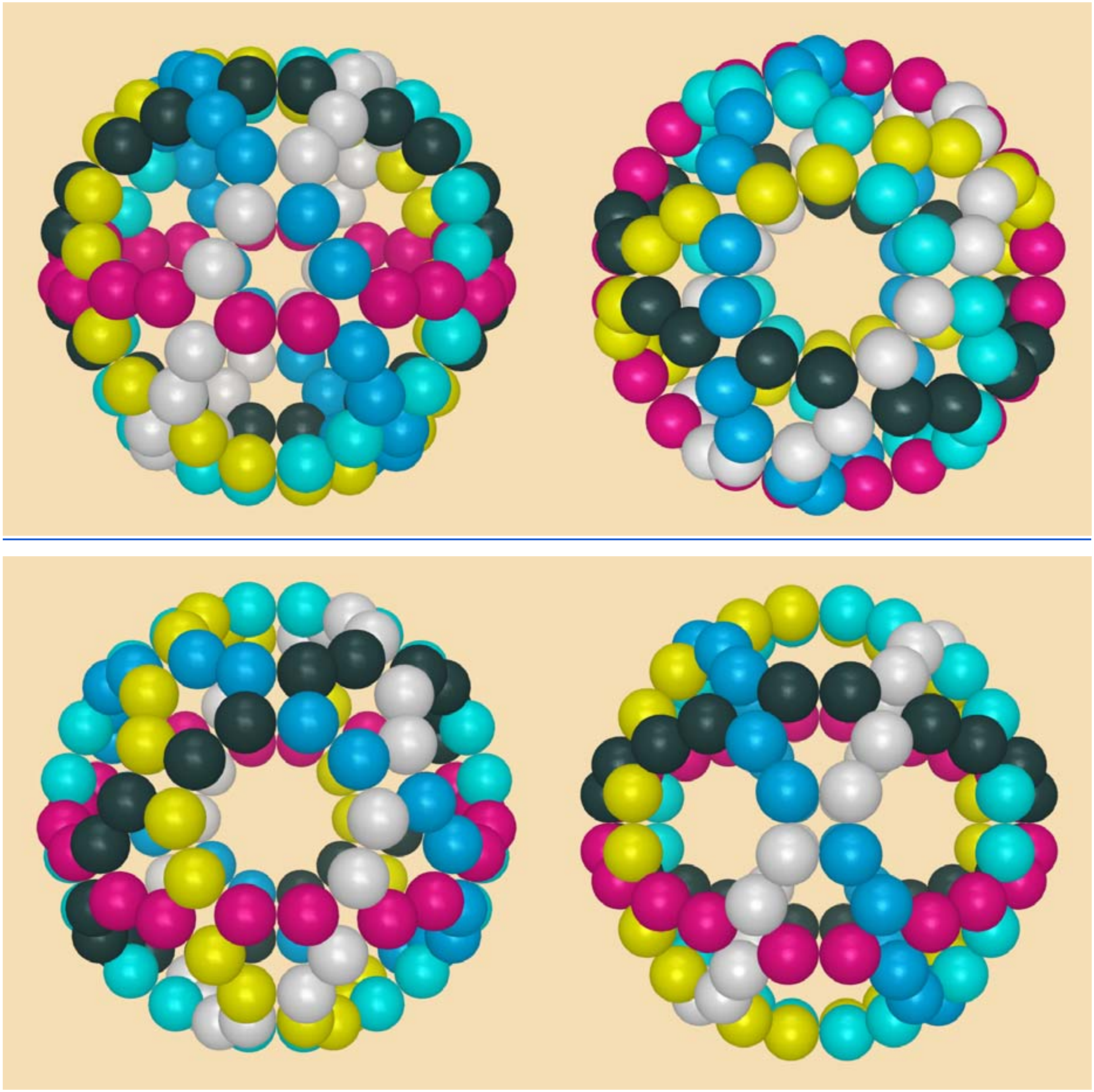

Fig. 16 Ten pairs of kissing spheres near a great circle

Pattern 2 - each color consisting of a pair of connected 10-cycles

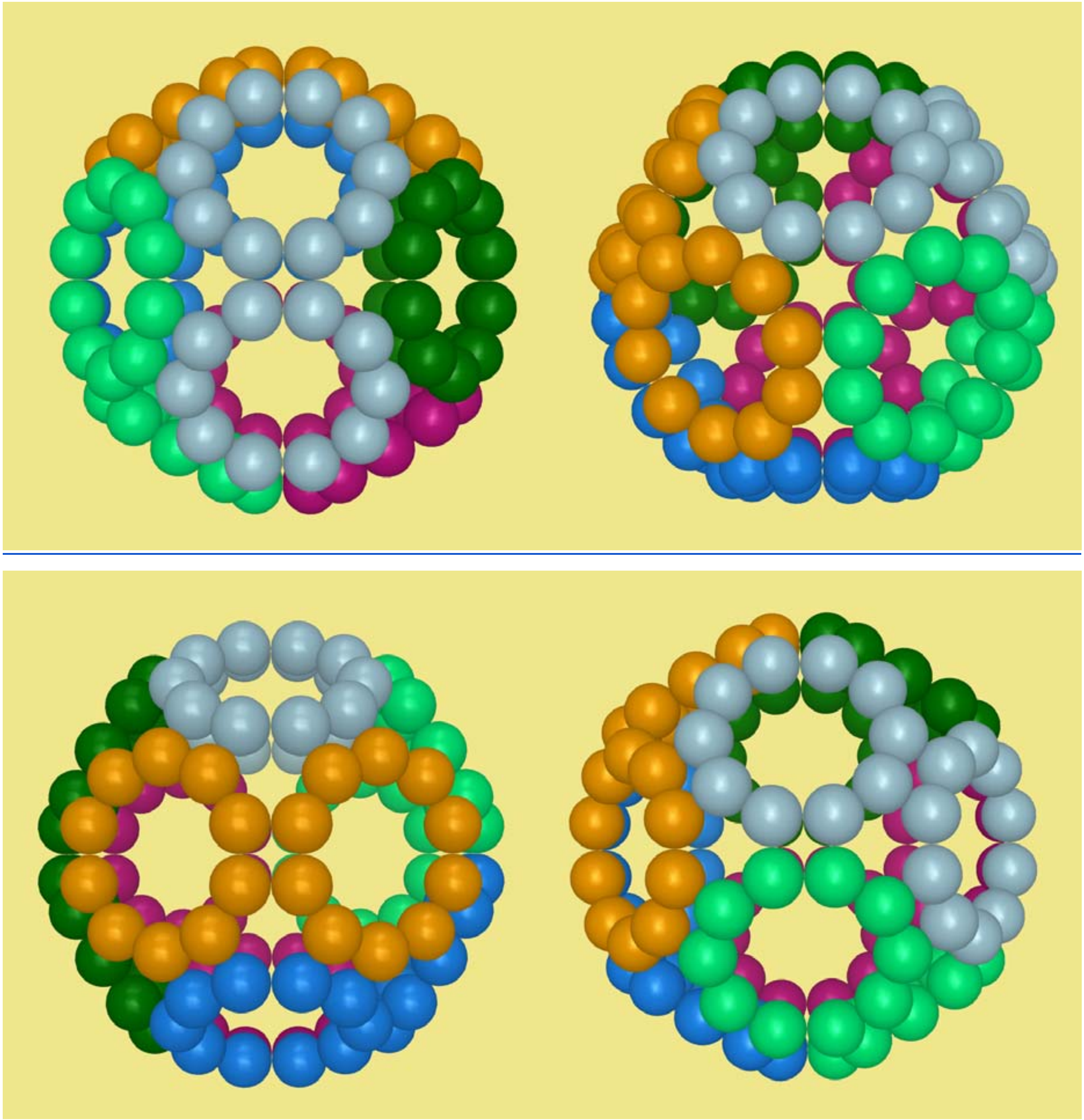

Fig. 17 Asymmetric coloring of 6 pairs

Pattern 3 - each color consisting of two antipodal 10-cycles

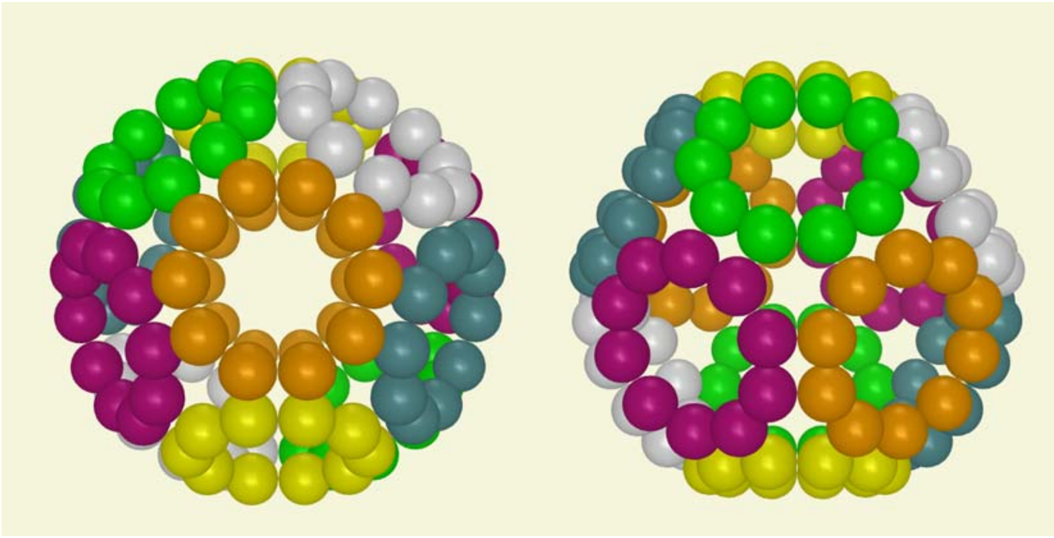

Fig. 18 Six antipodal pairs of 10-cycles

Pattern 4 -120 spheres formed by 6 snakes

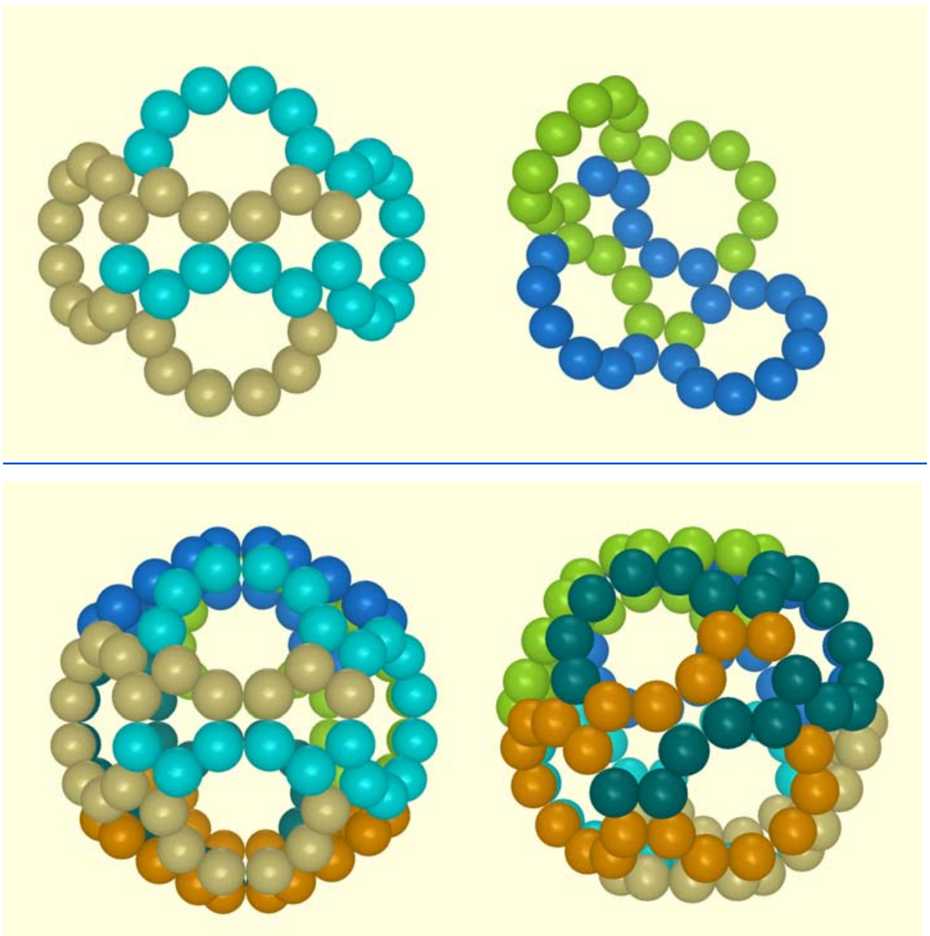

Fig.19 120 spheres formed by 6 snakes

8-COLOR SYMMETRY

Patent 1-coloring by Northants

This appears to be the only interesting pattern, created by dividing 120 spheres into 8 Northants:

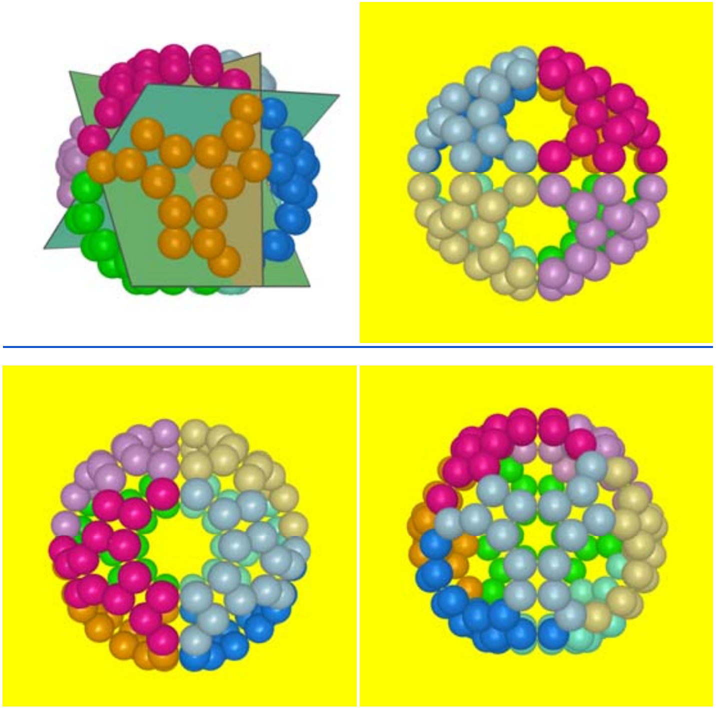

Fig. 20 Same Northants, same color

CONCLUSION

G.H. Hardy once said: "A mathematician, like a painter or a poet, is a maker of patterns." The reader is invited to download the cg3 file of the monochrome 120-Sphere model as a template to make many more interesting patterns under the Cabri 3D environment.

REFERENCES

[1] Nathan Altshiller-Court, Modern Pure Solid Geometry, Macmillan, 1935.

[2] John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss, The Symmetries of Things, A K Peters-CRC Press, 2008.

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[5] Ryan L. Marson, Erin G. Teich, Julia Dshemuchadse, Sharon C. Glotzer, and Ronald G. Larson, Computational self-assembly of colloidal crystals from Platonic polyhedral sphere clusters, The Royal Society of Chemistry, Soft Matter,15, 6288--6299, 2019.

[6] David G. Wells, The Penguin Dictionary of Curious and Interesting Geometry, 1992.

[7] This work is completed under the environment provided by Google Workspace for Education Fundamentals (formerly G Suite for Education): https://drive.google.com/drive/folders/1z9C6IibpW8EvSNT024_Eu5WpKdMX76Gs?usp=sharing