Counting Angle Bisection Theorems

Philip Todd
philt@saltire.com
Saltire Software
Brightwood, OR
U.S.A.

Abstract: *Some good geometry proof problems involve angles, angle bisectors, circle chords and isosceles triangles. We identify similarities between elements of such problems and present techniques for generating new problems from old.*

1. Introduction

In order to test symbolic geometry software, it is important to have access to a large number of geometry examples whose results are known. Over the course of the development of Geometry Expressions and its browser based sibling GXWeb, we have developed techniques for generating new geometry problems from old in order to broaden our suite of test cases. These techniques may be useful for educators who need to broaden their suite of test examples for students. It is with this in mind that we present these ideas. In particular, we consider angle theorems involving, in one disguise or another, the angle bisector.

The base of an isosceles triangle is perpendicular to the bisector of the angle at its apex, and hence has the same direction as the exterior angle bisector of that angle. We can, therefore, consider many theorems involving isosceles triangles to be angle bisector theorems. The chord of a circle makes an isosceles triangle with the radial lines at its ends, and hence many angle theorems involving circles can themselves be considered bisector theorems. The axis of a reflection is the external angle bisector of incident and reflected rays, and hence a reflection is simply an angle bisection viewed from a different point of view.

We examine ways of exploiting this perspective to generate new problems from old. We start with a very old problem, from Archimedes' Book of Lemmas [1], in figure 1.

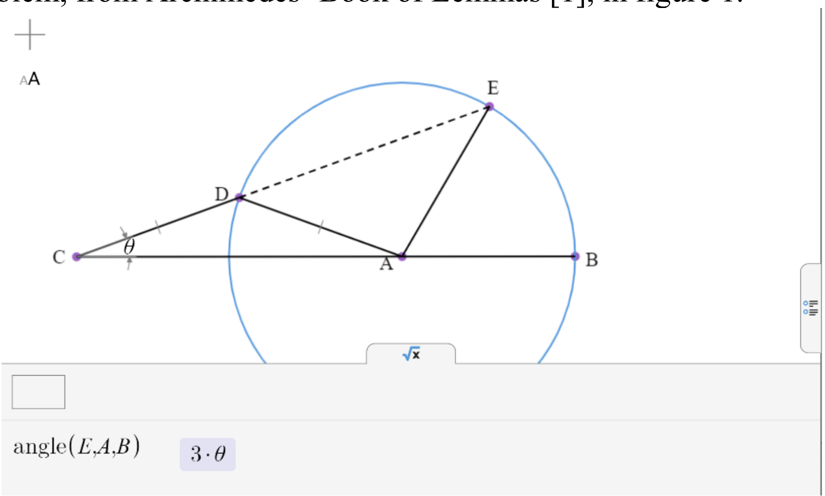

Figure 1: Problem from Archimedes' Book of Lemmas. Given that CD and AD are congruent, we are to show that angle EAB is three times angle DCA.

In the theorem, the center A of a circle with radius AB lies on the line AC. D lies on the circumference of the circle such that CD=AD. E is the second intersection between the circle and CD. The theorem states that angle EAB is 3 times angle DCA. (Archimedes uses this theorem in the opposite direction, as the basis for a method of trisecting the angle. The method involves a 'neusis' or sliding: try to work out how to draw the diagram with BAE given rather than BCE and you'll see where this needs to occur. Try to draw the diagram in GXWeb with BCE specified and it will fail, as GXWeb does not do the neusis either.)

This theorem can be re-imagined in terms of angle bisectors (figure 2a) or even in a form with one angle bisector and one isosceles triangle (figure 2b). The though process to use when doing this sort of re-imagination, is the topic of this paper. Once you have reimagined a theorem, you can use a dynamic geometry system with symbolic theorem proving [2] or a symbolic geometry system [3] to verify that it is correct. This technological safety net allows us to be reasonably loose in our approach, while maintaining confidence in the mathematical integrity of the end result.

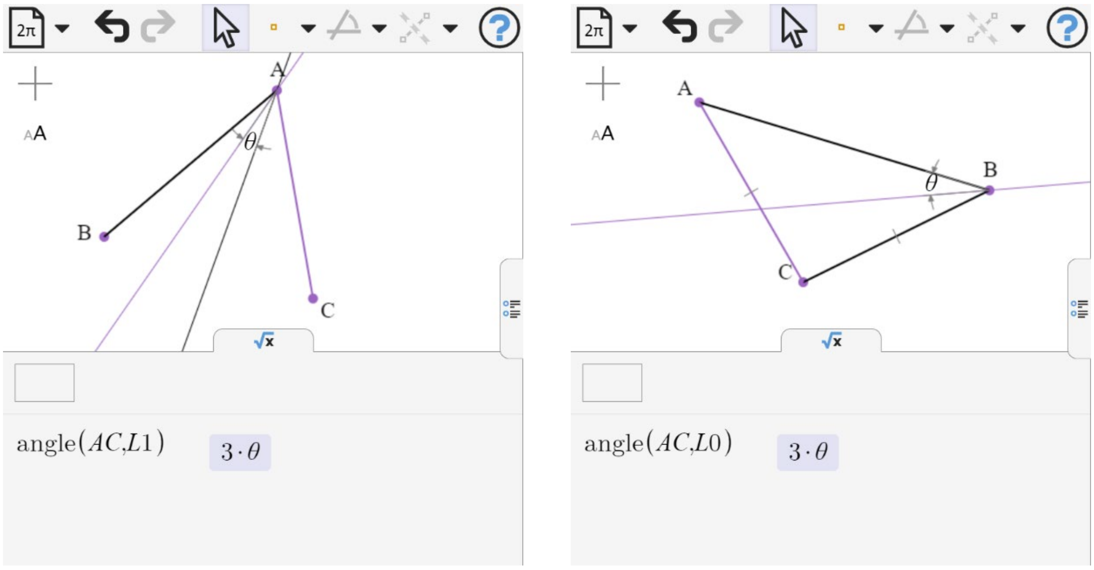

Figure 2: (a) L0 is the angle bisector of AB and AC, L1 is the angle bisector of L0 and AB. (b) L0 is the angle bisector of AB and BC, AC and BC are congruent

2. The Angle at the Center and the Angle at the Circumference

The familiar result that the angle subtended by a chord at the circumference of a circle is half that subtended at the center (fig. 3) is a useful place to start our exploration. In order to explore alternative castings of this theorem, we first make explicit the isosceles triangles inferred by the fact that lines BC and BD are chords of the circle centered at A. To do this, we add the line AB (dashed in figure 3a). A useful next step is to number the lines in the figure. With the numbering in figure 3a, line 4 is the base of an isosceles triangle between lines 1 and 2, while line 5 is the base of an isosceles triangle between lines 2 and 3.

The symmetry of an isosceles triangle implies that the perpendicular bisector of its base is the angle bisector of its apex. Hence the direction of the base is perpendicular to the direction of the angle bisector. It would thus seem reasonable that for any angle theorem involving isosceles triangles, there is a corresponding theorem involving angle bisectors. In figure 3b, we have started with a triangle whose sides are labelled 1,2,3, then created the angle bisector of 1 and 2 (line 5) and the angle bisector of 2 and 3 (line 4). We have added an angle between lines 1 and 3, and measured the angle between lines 4 and 5. We note that, whereas in figure 3a the angle between lines 4 and 5 is $\frac{\theta}{2}$ in figure 3b the angle is $\frac{\pi+\theta}{2}$. Our goal in this endeavor is to find theorems, the fact that the angle's value is given by the inputs is what we claim. A dynamic geometry system (ideally with symbolic components) can be used to establish its actual value.

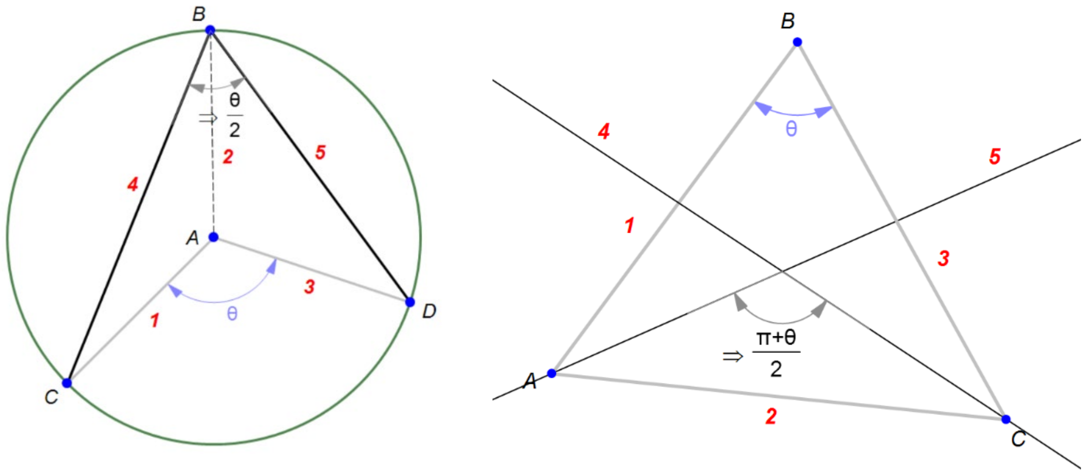

Figure 3: (a) An illustration of the theorem that the angle subtended at the center of a circle is twice that at the circumference. Lines are labelled for reference. (b) A theorem involving the angle at the vertex of a triangle and that between its angle bisectors.

We now have a theorem with two isosceles triangles and another theorem with two angle bisectors, it is natural to look for a theorem with one of each. In figure 4a, we start with triangle ABC whose sides are labelled 1, 2, 3. We use a circle centered at A to place D on BC such that ABD is isosceles and label AD as line 4. We then create the angle bisector of ACB, labelling it 5. We then specify the angle between lines 1 and 3, and measure the angle between lines 4 and 5.

If we have three lines L0, L1, L2 such that line L2 is the image of line L1 under reflection in line L0, then L0 is the angle bisector of L1 and L2. Hence, we could expect to be able to rephrase an angle bisector theorem as a reflection theorem. Figure 4(b) illustrates this in the context of the theorem of Figure 3(b). Lines 1, 2 and 3 show the path of a billiard ball which bounce in turn off lines 4 and 5. Whereas in the formulation of figure 3(b) it was natural for lines 1, 2 and 3 to be given and 4 and 5 derived, it is more natural here for lines 4, and 5 to be given (along with line 1) and line 2 and 3 to be derived. Therefore we have specified the angle between lines 4 and 5, and derived the angle between lines 1 and 3.

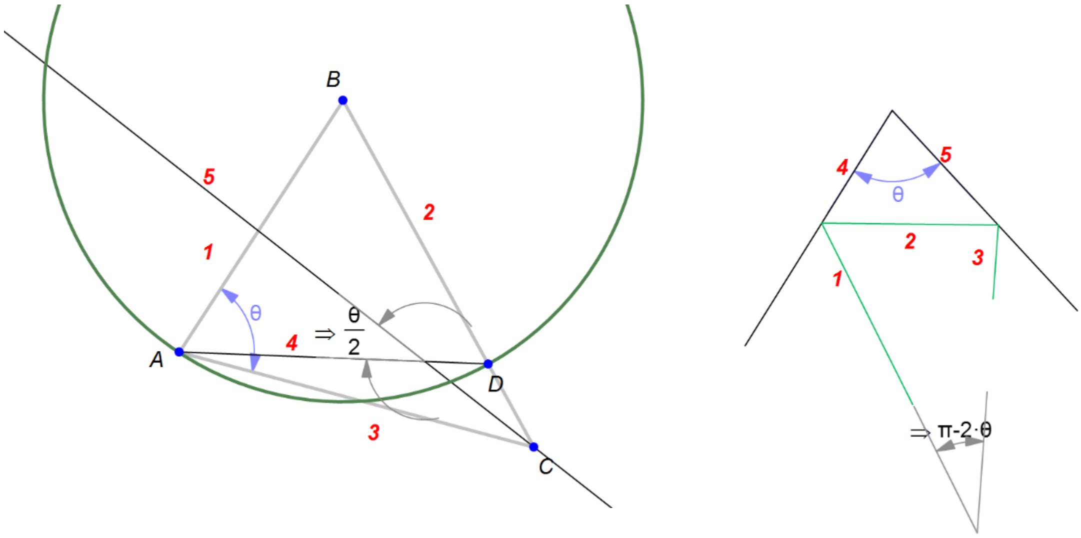

Figure 4: (a) A theorem relating a triangle angle with that between an angle bisector and the base of an isosceles triangle. (b) A theorem relating the angle of the initial and final paths of a billiard ball which bounces off two faces in succession.

3. Cyclic Quadrilaterals

Figure 5(a) shows a cyclic quadrilateral, with explicit radial lines numbered 1,2,3,4, and chords numbered 5,6,7,8. The theorem that the opposite angles are supplementary is illustrated as a relation between the angles between chords 5 and 6 and between chords 7 and 8.

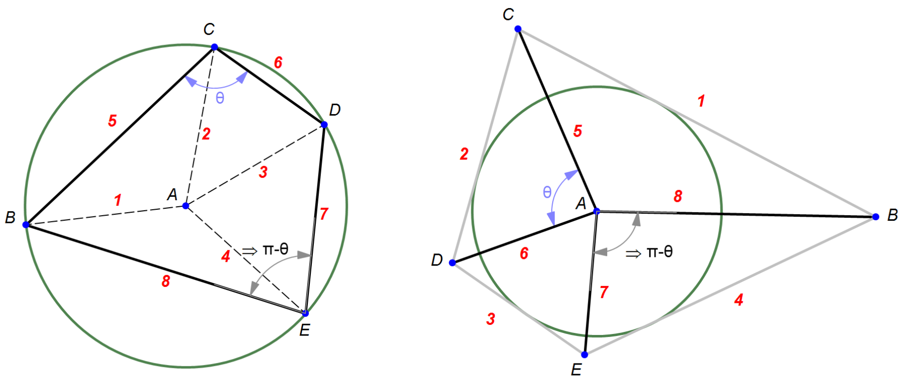

Figure 5: (a) A cyclic quadrilateral (b) A quadrilateral with inscribed circle..

Given arbitrary directions 1,2,3,4, the requirement that BCDE is cyclic imposes the conditions that line 5 is the base of an isosceles triangle with sides 1 and 2, that line 6 is the base of an isosceles triangle with sides 2 and 3, that line 7 is the base of an isosceles triangle with sides 3 and 4 and that line 8 is the base of an isosceles triangle with sides 4 and 1. We can't have just any directions for lines 5, 6, 7 and 8. In fact if the directions of 5, 6 and 7 are chosen, the direction of line 8 is determined, as the theorem attests.

To create a comparable diagram where angle bisector conditions are used rather than isosceles triangles, we can start with a quadrilateral which has an inscribed circle (figure 5b). In this diagram the sides of the quadrilateral are numbered 1,2,3,4. The angle bisectors meet at the center of the inscribed circle. We number the angle bisectors 5,6,7,8. The angle between 5 and 6 is supplementary to the angle between 7 and 8.

4. Two Cyclic Quadrilaterals

A theorem in [4] extends two sides of a cyclic quadrilateral to form a second cyclic quadrilateral (fig 6a). The new cyclic quadrilateral shares three sides with the original. The theorem states that the fourth side of the new quadrilateral is parallel to the fourth side of the original quadrilateral. This theorem may be reimagined in a number of ways.

First, and most simply, we can phrase the theorem as follows: two cyclic quadrilaterals have three sides parallel in pairs, prove that the fourth sides are parallel.

If we cast one of the cyclic quadrilaterals as a quadrilateral with inscribed circle, we get the theorem of figure 6b. Whereas in figure 6a the fact that EFCB is cyclic is given and the parallelism of lines EF and AD is to be proved, in 6b the parallelism is given and the fact that AEDG is cyclic is to be proved.

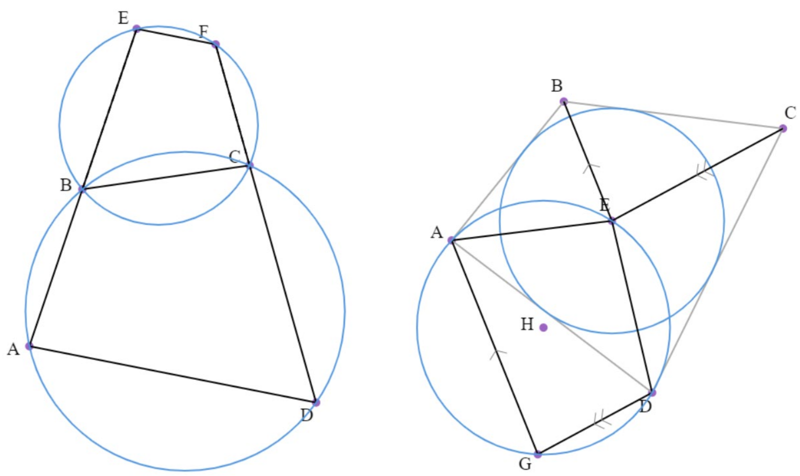

Figure 6: (a) ABCD is a cyclic quadrilateral. E lies on AB extended and F lies on DC extended such that EFCB is cyclic. EF is parallel to AD. (b) ABCD is a quadrilateral with inscribed circle center E. Let G be the intersection of the parallel to EB through A and the parallel to EC through D. AEDG is a cyclic quadrilateral.

Reimagining the theorem with angle bisectors for one quadrilateral (figure 7a), we have the succinct statement that the intersections of the angle bisectors of any quadrilateral form a cyclic quadrilateral.

If we take a cyclic quadrilateral and exscribe a quadrilateral such that it forms isosceles triangles with three sides, then it forms an isosceles triangle with the fourth (figure 7b)

If a billiard ball bounces off the four sides of a cyclic quadrilateral in turn (figure 7c) then its final path is parallel to its initial path.

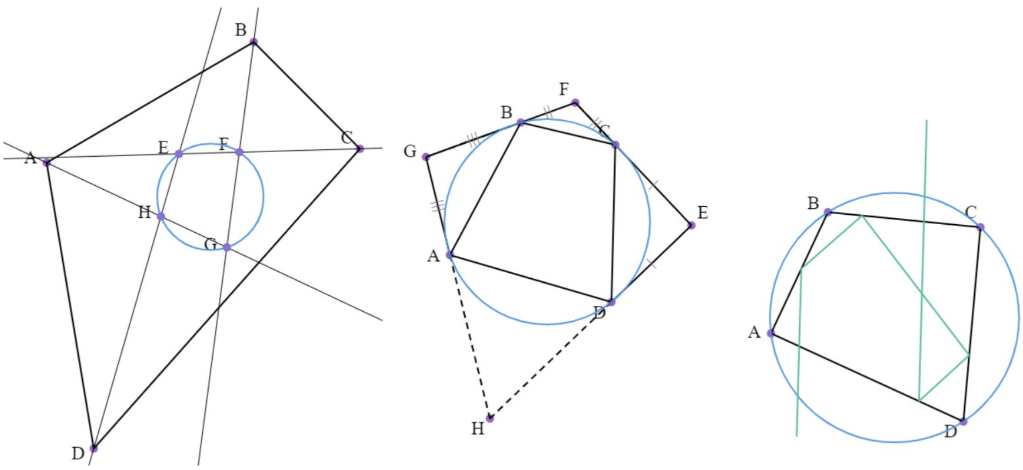

Figure 7: (a) The intersections of the angle bisectors of a quadrilateral are cyclic. (b) EFGH is an exscribed quadrilateral to cyclic quadrilateral ABCD such that DEC, BFC and BGA are isosceles, then ADH is also isosceles. (c) A billiard ball boounces off the four sides of a cyclic quadrilateral in turn. Its final path is parallel to its initial path.

5. Coalescing Lines

Another way of creating new theorems from old is by coalescing lines. For example, in figure 8(a), we have reproduced the diagram of figure 6(a) but drawn all the implicit radial lines, and numbered them (we see that our five lines and two circles have become thirteen lines!). Imagine what would happen if we allowed lines 1 and 8 in the diagram to get closer and closer to each other. Line 12 would remain parallel to line 13 and lines 1 and 8 would approach the perpendicular bisector of FG, which is perpendicular to line 13. Once they have coalesced, we have diagram 8(b) and this theorem: let F be the intersection of line AB and CD of cyclic quadrilateral ABCD; let G be the circumcenter of triangle BCF; then GF is perpendicular to AD.

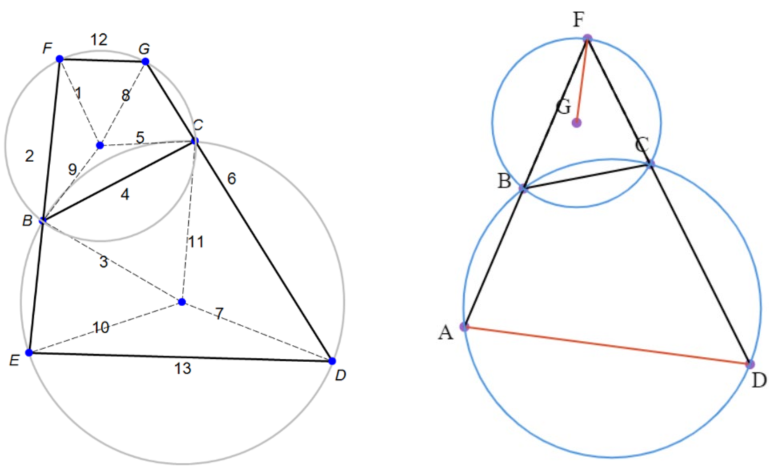

Figure 8 (a) The theorem of figure 6(a) with all the implied radial lines present and numbered. (b) Coalescing lines 1 and 8 leads to a new and distinct theorem.

A precise statement of this argument would involve calculus, however having stated the theorem, we can readily prove it geometrically. Our goal here is the discovery of theorems, the proof to be left to our geometric technology or to our geometry students. Hence the use of imprecise limits as part of the discovery process seems justified.

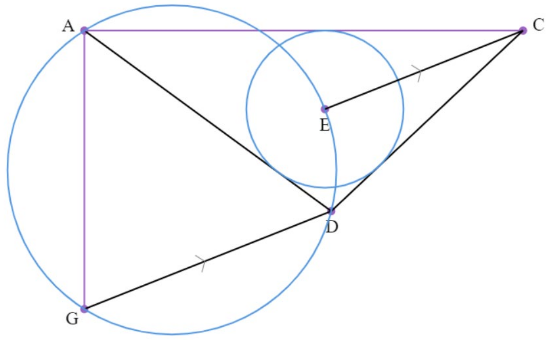

Figure 9. Coalescing the lines AB, BC in figure 6b yields this diagram and the theorem: Let E be the incenter of triangle ADC. Let G lie on the circumcircle of ADG such that DG is parallel to EC. Angle GAC is right.

A second example of this approach is given in figure 9. In this figure, we have coalesced the lines AB and BC of figure 6b. Line EC in figure 6b is the bisector of angle ABC and by definition parallel to line AG. In the limit the coalesced line ABC is perpendicular to the angle bisector. Hence in our new diagram (figure 9) , we have AC perpendicular to AG.

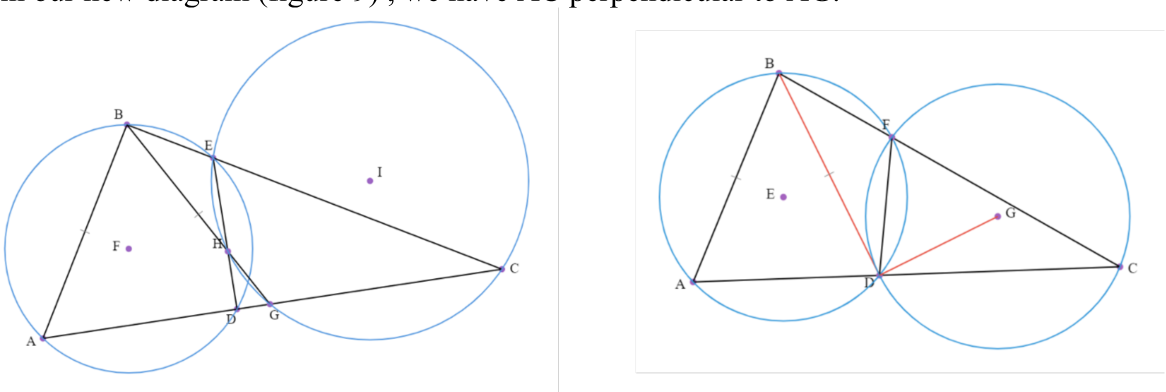

Figure 10: (a) D lies on side AC and E on side BC of triangle ABC such that ABED is cyclic. G lies on side AC such that ABG is isosceles. H is the intersection of ED and BG. EHGC is cyclic. (b) D lies on side AC of triangle ABC such that ABD is isosceles. F lies on side BC such that ABFD is cyclic. G is the cicrumcenter of CDF. GD is perpendicular to BD.

Figure 10 illustrates a third application of this technique. Coalescing lines IG and IH of figure 10a leads to points H, G and D of the figure coalescing into the single point D of figure 10b and a right angle postulated at GDB. This can readily be confirmed geometrically.

6. Cyclic Polygons

In section 4 above, we were able to generate a number of different theorems from the theorem that two cyclic quadrilaterals with three parallel sides have a fourth parallel side. A natural question is this: do we have a similar theorem for cyclic pentagons? Do we have a similar theorem for cyclic hexagons? The answers to these questions are no and yes.

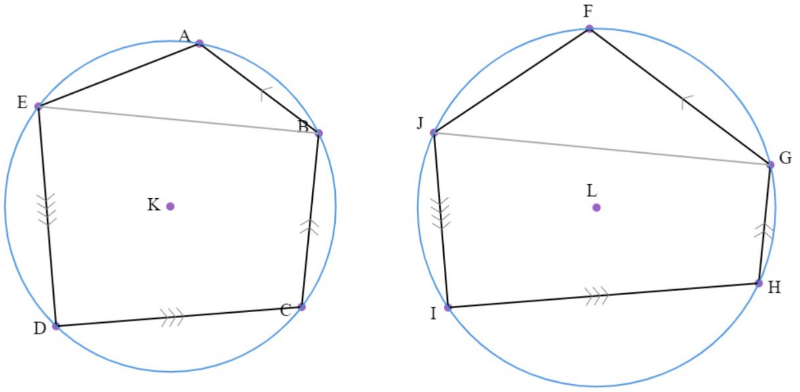

Figure 11: Cyclic pentagon FGHIJ has sides FG, GH, HI, IJ parallel to sides AB, BC, CD, DE of cyclic pentagon ABCD.

In Figure 11, two cyclic pentagons ABCDE and FGHIJ have parallel sides AB with FG, BC with GH, CD with HI and DE with IJ. The not-necessarily-parallel sides are EA and JF. As BCDE and GHIJ are cyclic quadrilaterals with 3 pairs of parallel sides, the fourth sides, EB and JG are also parallel. As FG and AB are also parallel, EA is only parallel to JF when angle EAB is equal to angle JFG. This is only true when the angle subtended at the center by EB is equal to the angle subtended at the center by GJ: that is when the two pentagons are similar.

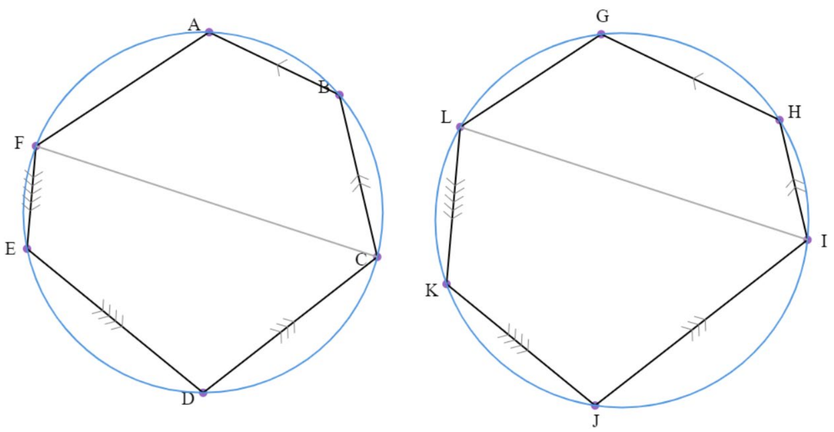

Figure 12: Cyclic hexagon GHIJKL has sides GH, HI, IJ, JK, KL parallel to sides AB, BC, CD, DE, EF of cyclic hexagon ABCDEF.

The situation with cyclic hexagons is different. In figure 12, cyclic hexagons ABCDEF and GHIJKL have parallel sides GH with AB, HI with BC, IJ with CD, JK with DE and KL with EF.

As IJKL and CDEF are cyclic quadrilaterals, IL is parallel to CF. Now GHIL and ABCF are cyclic quadrilaterals with three pairs of parallel sides, hence AF is parallel to GL.

This argument can be adapted in a straightforward way to show that this property holds in general for cyclic polygons with an even number of sides, and does not hold for cyclic polygons with an odd number of sides.

7. Three Cyclic Quadrilaterals and Napoleon's Theorem

In section 4, we looked at theorems involving two cyclic quadrilaterals. In section 6, we generalized by looking at theorems with two cyclic polygons with more than 4 sides. In this section, we consider a different generalization axis and consider theorems involving more than two cyclic quadrilaterals.

In figure 13, the lines AD, BD, CD are not shown, however the cyclic quadrilaterals ADBH, BDCI and CDAJ play comparable roles to the two cyclic quadrilaterals in figure 6a. The fact that the sum of BDA, ADC and BDC is 2π implies that the sum of angles AHB, BIC and CJA is π and hence HA is parallel to AJ, from which we deduce that the line HJ passes through A.

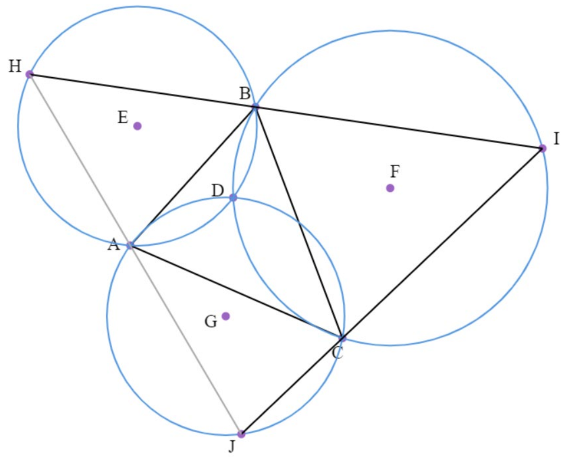

Figure 13: Given triangle ABC and point D, let H lie on the circumcircle of ADB, let I be the intersection of HB and the circumcircle of BDC and let J be the intersection of IC and the circumcircle of ADC. HJ passes through A

The key to this pattern is that each cyclic quadrilateral shares two edges with its neighbor: ADBH and BDCI share BD and HI, BDCI and ADCJ share DC and IJ, ADCJ and ADBH share AD and JH.

Figure 14 contains a diagram where this relationship is more obvious. ABCD is a cyclic quadrilateral. CGFE is another cyclic quadrilateral, with CE and CG the same lines as BC and DC. AKLJ is another cyclic quadrilateral with AK and AJ the same lines as DA and BA. Points L and F are put arbitrarily on the circumcircles of AKJ and GCE. The four lines KL, LJ, EF and FG, when extended form a cyclic quadrilateral.

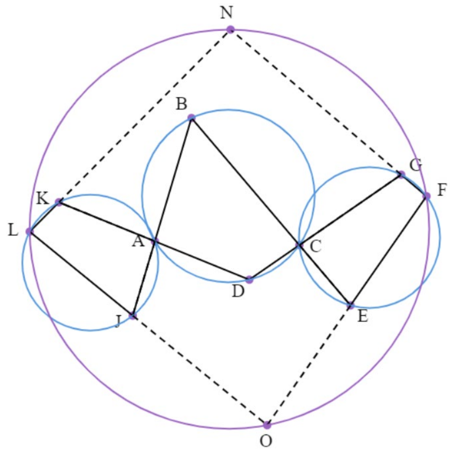

Figure 14: ABCD is a cyclic quadrilateral. G lies on DC extended, E lies on BC extended, J lies on BA extended and K lies on DA extended. CGFE and AKLJ are cyclic quadrilaterals. N is the intersection of LK and FG, O is the intersection of EF and JL. LNFO is a cyclic quadrilateral.

We leave the reader with the challenge of reimagining this pattern with different manifestations of the bisector and illustrate the power of the configuration with a proof of Napoleon's Theorem.

Napoleon's theorem states that the triangle formed by joining the centers of equilateral triangles constructed on the sides of a triangle is itself equilateral. A proof based on the above configuration takes the following form. Let D in figure 13 be the intersection of the circumcircles of equilateral triangles drawn on sides AB and BC of triangle ABC. Angles ADB and BDC (and hence CDA) are 120 degrees. Hence HIJ is equilateral. To complete the proof, we relate a particular HIJ to the Napoleon Triangle.

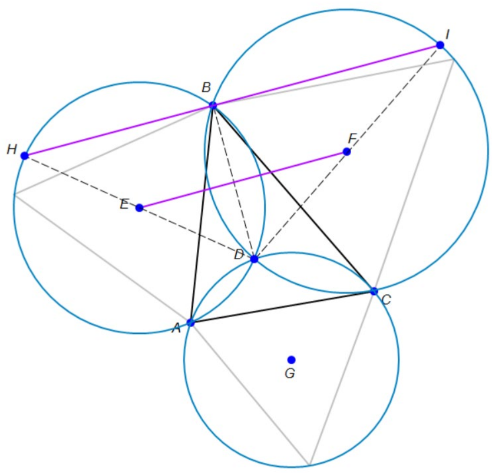

Figure 15: D is the Fermat-Toricelli point of triangle ABC, EF is one side of the Napoleon triangle, HI its dilation by a factor of 2 centered at D.

In Figure 15, E F and G are the centers of the circumcircles of equilateral triangles drawn on the sides of triangle ABC. D is the intersection of the circles. Napoleon's triangle is EFG. If we dilate Napoleon's triangle by a factor of 2 centered at D, its vertices lie on the three circumcircles and its edges pass through points ABC, and hence the dilated triangle is an instance of HIJ in figure 13.

The diagram of Figure 13 can be generalized so that instead of triangle ABC, we have a general polygon $A_1A_2 \cdots A_n$ and point D. In Figure 14, we could have a chain of n cyclic quadrilaterals rather than 3.

8. Conclusion

We encounter the angle bisector, in disguise, in theorems which feature reflections and in diagrams which contain isosceles triangles. As a circle chord and its two radial lines form an isosceles triangle, this latter category includes many diagrams containing circle chords. In this presentation we have described two ways of creating new angle bisector theorems from old. One method involves starting with a theorem which involves only circle chords and replacing some or all of these with angle bisectors. Another method involves coalescing lines of the original model and making an informal limit argument. The availability of software to check the validity of a theorem absolves us from the need to do calculus, rather we can use our limit argument to show us a candidate theorem, then check it using technology, or geometry.

Acknowledgements The author would like to thank the referees for suggesting a sensible restructuring of this presentation.

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