Teaching of the Graph Construction Techniques using Integer Partitions

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Abstract

In this paper, we propose new ideas to construct graphs using partitions of integer. Partition of Integer is an important area of Combinatorics and very fascinating for mathematicians. Integer partition is the study of integers by partitioning them into smaller integers. The graphs constructed in this paper are based on integer partitions and their graphical representations.

1 Introduction

Integer partition is a very useful area of Combinatorics. It plays an important role in understanding and teaching of mathematics. This subject can be taught from K12 up to the university level. Integer partition is the study of an integer by partitioning it into smaller integers. It relates number theory, graph theory and combinatorics to one another. Leibniz, a german mathematician, introduced it very first time. According to Andrews [7], Leibniz was the first person who asked about partitions. He asked J. Bernoulli about the number of divulsions of integers in 1674. partition theory is based on the concept of breaking down the integer into the sum of integers. The order of the sum must be non-increasing, in other words, this is a sum of weakly decreasing sequence of positive integers. First, we need a few definitions which will be used throughout this paper.

Definition 1 [2] *A partition function P(n), for integer $n \geq 1$, is a non increasing sequence $\{n_1, n_2, n_3,$ ..., n_k $\}$ such that*

$$|P(n)| = \sum_{j=1}^{k} n_j = n$$

where $j \leq k \leq n$, n is finite, and n_j are the parts of the partition $P(n)$, P(0)=1 by default.

Now we give the Mathematica code [9] to find integer partitions of integer 5:

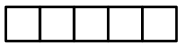
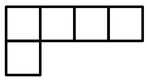
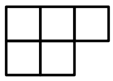
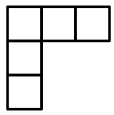
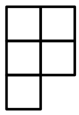
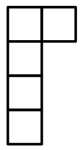
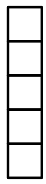
```
In[1]:  Integerpartitions[5]
```

```
Out[1]:  {{5}, {4, 1}, {3, 2}, {3, 1, 1}, {2, 2, 1}, {2, 1, 1, 1}, {1,
1, 1, 1, 1}}
```

The graphical representation of integer partitions is to represent each integer partition by appropriate horizontal boxes or cells and move vertically downward whenever we have the next part of integer partitions. We need the basic concepts of the Young diagram which we use in the construction of various graphs in this paper.

Definition 2 [6] *A Young diagram of $P(n)$, of length h, is a collection of empty cells of h rows of left-justified order where row j containing n_j cells for $1 \leq j \leq h$.*

.

Partitions	**5**	**4+1**	**3+2**	**3+1+1**	**2+2+1**	**2+1+1+1**	**1+1+1+1+1**
Young Diagram							

Definition 3 [3] *The term graph G is typically defined as the set of vertices $V = \{n_1, n_2, n_3, ..., n_k\}$ and edges $E = \{m_1, m_2, m_3, ..., m_k\}$ where $m_1, m_2, m_3, ..., m_k$ are the links betweeen different vertices $n_1, n_2, n_3, ..., n_k$. Two vertices are said to be adjacent if there is an edge connecting them. Adjacent edges are edges that share a common vertex in the graph.*

In the Figure 1, vertices n_1 and n_2 are adjacent as the edge m_1 is connecting them, and m_1 and m_2 are adjacent edges because they have a common vertex n_2.

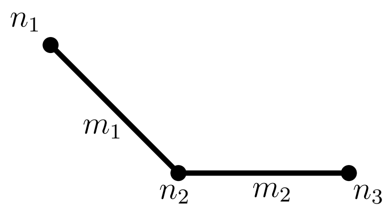


Figure 1: A graph with three vertices and two edges

Definition 4 [12, 8] *The line graph of an undirected graph G is another graph L(G) that represents the adjacencies between edges of graph G.*

Definition 5 [4] *The complement or inverse of a graph G is a graph on the same vertices such that two distinct vertices of the complement of the graph are adjacent if and only if they are not adjacent in graph G.*

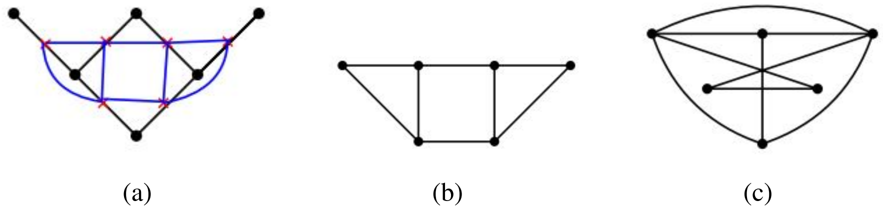


Figure 2: (a) A Graph G with six vertices and six edges, and the blue lines show the adjacencies between the edges of G (b) Line graph of G is obtained using adjacencies (c) Complement of G

Definition 6 [3] *A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V.*

Definition 7 [3] *The graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are isomorphic if there is an one-to-one and onto function θ from V_1 to V_2 with the property that a and b are adjacent in G iff $\theta(a)$ and $\theta(b)$ are adjacent in H, for all a and b in V_1. Such function is called an isomorphism.*

In Figure 3, both graphs G and H are isomorphic as $\theta(a) = 1, \theta(c) = 2, \theta(e) = 3, \theta(b) = 4, \theta(d) = 5$ and $\theta(f) = 6$.

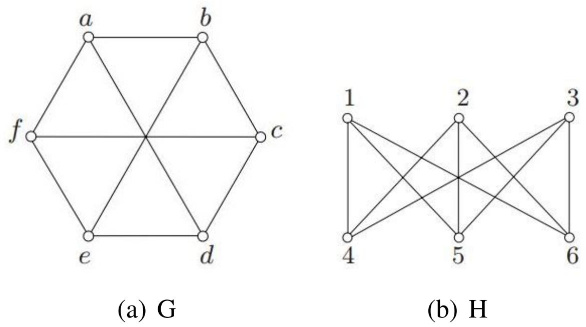


Figure 3: Isomorphic graphs and bipartite graphs

Definition 8 [4] *Let G be a graph and $\{H_1, H_2, H_3, ..., H_n\}$ be the subgraphs of G such that H_i and H_j are isomorphic to each other, where $1 \leq i, j \leq n$, then H_i is the copy of H_j in graph G.*

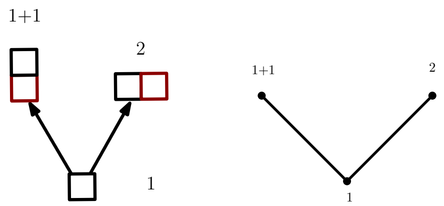
In Figure 8 (a) there are four subgraphs H of W_3 given in Figure 8 (b), all four subgraphs are isomorphic to each other and W_3 is consist of four copies of H.

2 Construction of the graphs using Integer partitions

Now we construct some families of the graphs which are completely based on integer partitions, their corresponding Young diagrams, and the connection between the diagram of the parts of integer n with the diagrams of the parts of integer n+1. The construction starts with the Young diagram of integer

1, whose diagram is represented by a cell □ and from the perspective of the graph the diagram □ is represented by the vertex (•).

For integer 2, integer partitions are 2 and 1+1 and their corresponding Young diagrams are □□ and □ respectively. There is a strong relationship between the integer partitions and the diagrams of integer 1 and integer 2, given in Figure 4. By the rules and properties of the diagrams add a box at the bottom and right-hand side of the given box for integer 1. Each young diagram is replaced by a vertex and two different vertices are adjacent by an edge shown in Figure 4(b) in graph G_2.



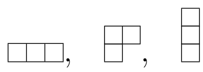
(a) (b)

Figure 4: (a) The red boxes are the possible ways to add a box to the previous Young diagram (b) the corresponding graph G_2

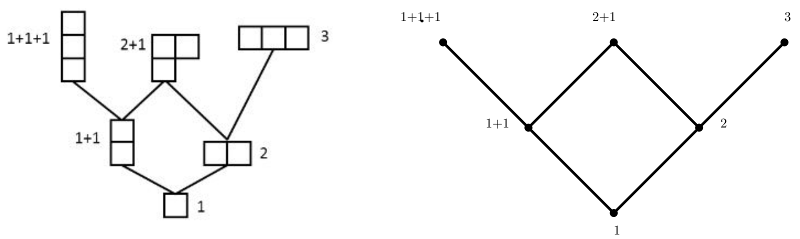
Similarly, by following the same procedure we move on to the construction of G_3 graph which is originated by G_2. To construct G_3 we use integer partitions of integer 3 :

$$3, \quad 2+1, \quad 1+1+1$$

and the Young diagrams of these parts are



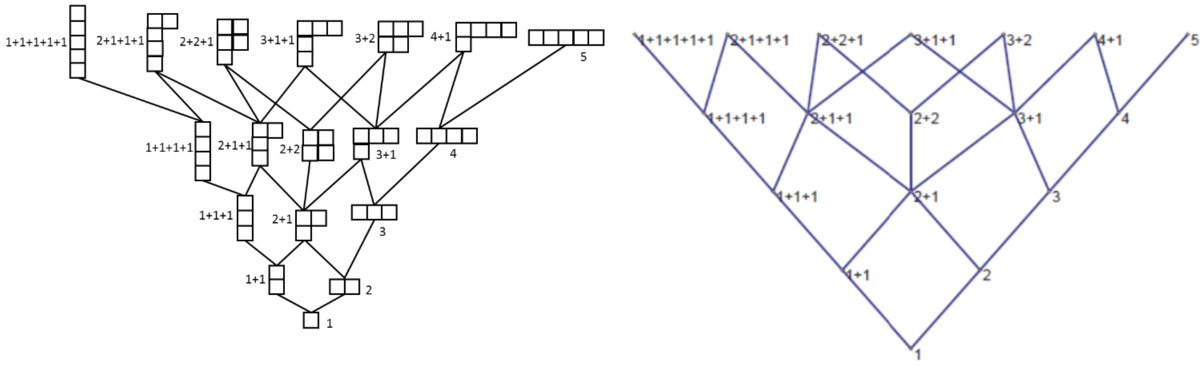
To relate these diagrams to the previous integer's diagrams we have to follow the properties of the Young diagram.



(a) (b)

Figure 5: (a): Young diagrams for integer up to 3 (b): the corresponding graph G_3

Continuing this procedure we can obtain G_5 so on G_n for all n.



(a) (b)

Figure 6: (a) Young diagrams for integer up to 5, (b) Corresponding graph G_5

By using this technique we can construct more fascinating graphs.

2.1 Young Spin Graphs

In order to construct more graphs, we use the above-constructed graphs given in Figures 4, 5and 6.

Definition 9 *The Young spin graphs S_n is defined as the graph which is obtained from the graph G_n by adding a copy of G_n on the left or right side of G_n in which the most right/left part will be overlapped , edges and vertices coalescenced.*

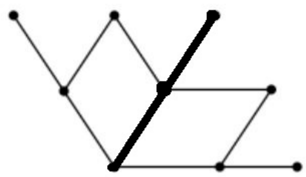


Figure 7: Young spin graph S_3 for integer 3 with 2 copies of G_3

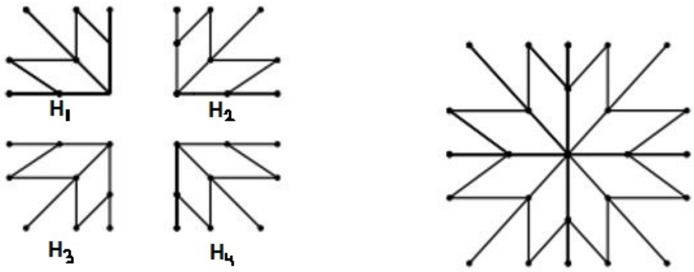
See Figure: 7, using the same approach we can obtain a very amazing family of graphs.

2.2 Young Wheel Graphs

If we add some number of copies of Young spin graph S on the left side of the graph S_n until we complete the circle of 360^o, we obtain another family of graphs which are called Young Wheel graphs W_n, see Figure 8. A question arises here, How many copies we need to add on the left of the graph S to reach 360^o? For this, we characterize Young wheel graphs to five different categories of graphs by distributing them to different degree's $30^0, 45^0, 60^0, 90^0$ and 180^0. For Youngs Wheel Graph, we use formula $360/\theta =$ number of copies to add, where θ is the angle we chose to construct the graph, See Table 1.

Table 1: Number of copies with corresponding angles

Distribution of Angles	30^0	45^0	60^0	90^0	180^0
Number of Copies	360/30=12	360/45=8	360/60=6	360/90=4	360/180=2



(a) H_1, H_2, H_3 *and* H_4 are the isomorphic copies of Young Spin graph S_3 of integer 3

(b) W_3

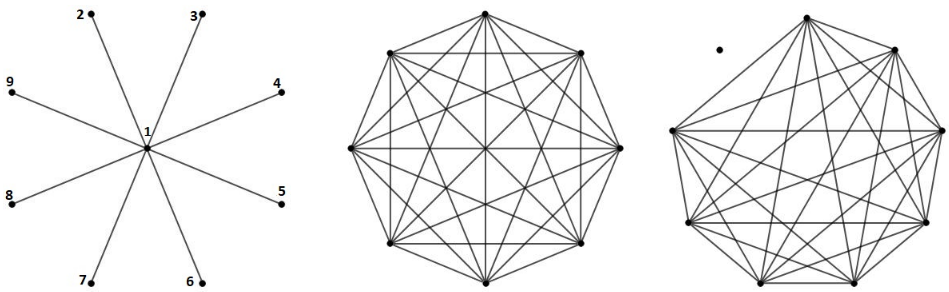
Figure 8: Young wheel graph for integer partitions of 3 at 90^0

2.3 Some mazing Graphs using Young Wheel Graphs and Mathematica

After learning different techniques to construct graphs using Integer partitions, we can move to many other families of graphs using some Mathematica codes.
The Mathematica Code for graph in 9(a) is:

```
In[2]= Graph[{1 ↔ 2, 1 ↔ 3, 1↔ 4, 1 ↔ 5, 1 ↔ 6, 1 ↔ 7, 1 ↔8, 1 ↔
9}]
In[3]= LineGraph[% 2]
In[4] = GraphComplement[%2]
```

The output from these codes is in Figure 9.



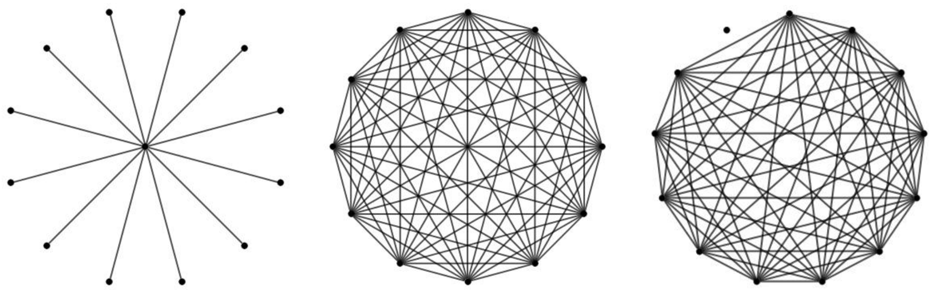
(a) Young wheel graph W_2 for integer 2 at 90^0 with four copies of S_2

(b) Line graph for W_2

(c) Complement of W_2

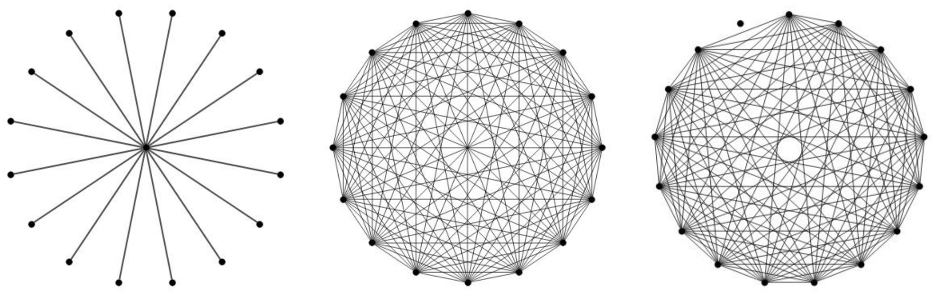
Figure 9: The output from the Mathematica codes

There is an isolated vertex which is shown in Figure 9(c), which shows that this vertex has no adjacent edges/vertices because in the graph W_2, there is a vertex that is adjacent to all the other vertices of the graph. Some more graphs obtained from Mathematica are given in below.



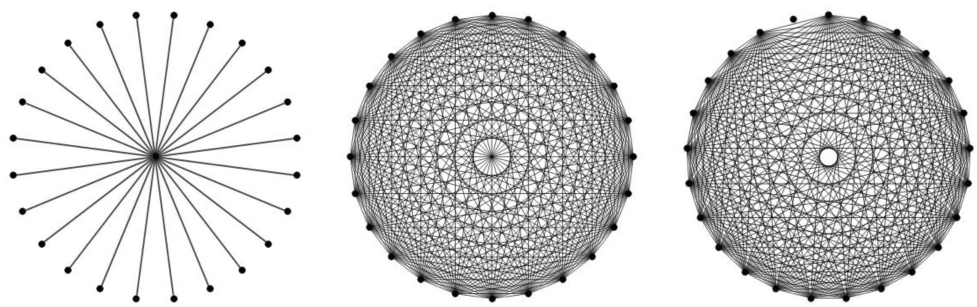
(a) Young wheel graph W_2 for integer 2 at 60^0 with six copies of S_2 (b) Line graph for W_2 (c) Complement of W_2

Figure 10: For W_2 with 60^0



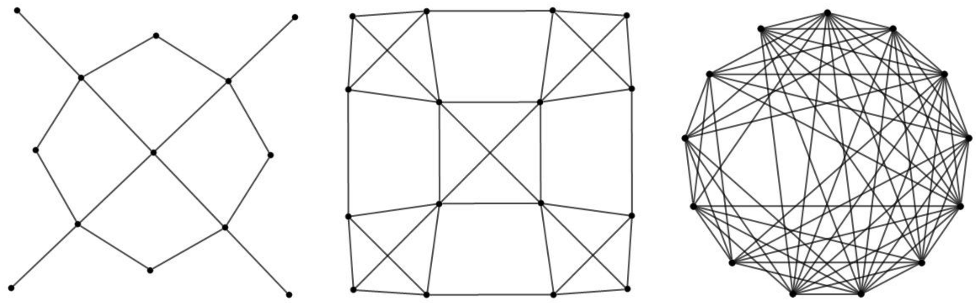
(a) Young wheel graph W_2 for integer 2 at 45^0 with eight copies of S_2 (b) Line graph for W_2 (c) Complement of W_2

Figure 11: For W_2 with 45^0



(a) Young wheel graph W_2 for integer 2 at 30^0 with twelve copies of S_2 (b) Line graph for W_2 (c) Complement of W_2

Figure 12: For W_2 with 30^0

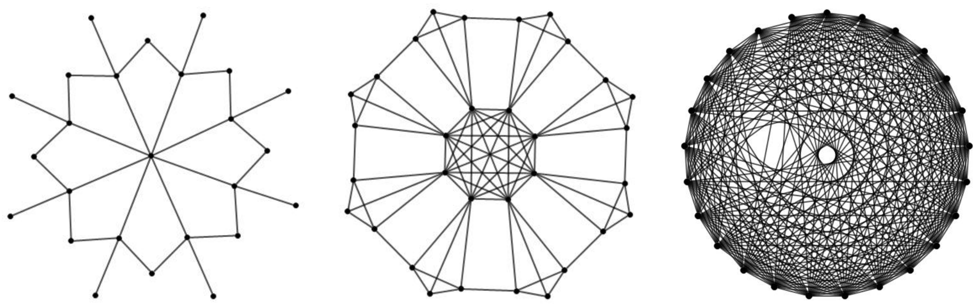


(a) Young wheel graph W_3 for integer 3 at 180^0 with two copies of S_3

(b) Line graph for W_3

(c) Complement of W_3

Figure 13: For W_3 with 180^0

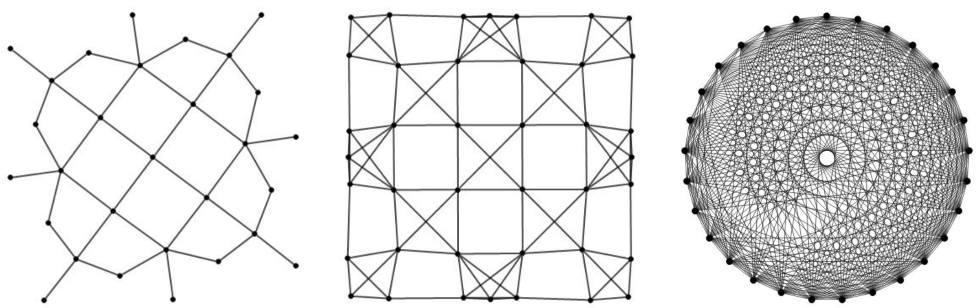


(a) Young wheel graph W_3 for integer 3 at 90^0 with four copies of S_3

(b) Line graph for W_3

(c) Complement of W_3

Figure 14: For W_3 with 90^0



(a) Young wheel graph W_4 for integer 4 at 180^0 with two copies of S_4

(b) Line graph for W_4

(c) Complement of W_4

Figure 15: For W_4 with 180^0

Due to the constraints on the number of pages we have not given other graphs which are also based on integer partitions for integers $n \geq 5$ [10, 13]. All the graphs G_n, S_n and W_n are bipartite but all the line graphs and complement of G_n, S_n and W_n graphs are not. We can also work on many properties of graphs such as spectral and structural [5, 11].

3 Conclusions

All the graphs G_n, S_n and W_n are constructed using integer partitions and Young diagrams. To teach integer partitions we do not need very heavy mathematical concepts. Mathematica software [9, 14] plays a very crucial role in finding the properties of the graphs which have been given in this article. For the graphs, G_n, S_n and W_n, which are discussed in this paper, have many properties including line graphs, complement graphs, and they can be easily obtained using Mathematica.

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