Connections between mathematics and digital technology in problem-solving: evidences of techno-mathematical fluency

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Abstract: *In this paper we discuss the connections established between mathematical and technological knowledge that emerged from mathematical problem solving with digital technologies in a beyond school competition. We argue that mathematical problem solving with technologies is a synchronous activity of mathematization and expression of mathematical thinking, which leads to the emergence of conceptual models, and that the connections between mathematical and technological knowledge enhance the solving-and-expressing of the techno-mathematical solution of a problem. From an interpretative stance, we present the activity of a middle-school teenager working on a covariation problem and focus our analysis on the key aspects of the simultaneous use of mathematical facts and procedures and the digital tools she resorts to, in developing a conceptual model of the situation. We conclude that the resources available shape her mathematical thinking during problem-solving, that the connections established between these two types of knowledge open way to a progressive mathematization of the situation, allowing the production of a solution that displays a techno-mathematical discourse, which we explain by expanding on the notion of techno-mathematical fluency.*

1. Introduction

Research has been associating the development of mathematical ideas and concepts with the development of technology itself [1], even though the students' digital experiences in everyday life still have a long way towards permeating the school boundaries, namely in mathematics teaching and learning [2]. Moreover, the question of "why students have difficulties in applying mathematical concepts and skills (which they presumably learned at school) outside the classroom - or in other subject areas" [3] points out to the importance of making connections between the mathematics they learn at school and other subjects or the outside world. It also leads to the connections that are made between mathematical concepts and the digital tools that can be used to explore them.

By addressing the use of digital technologies in solving and expressing mathematical problems [4,5], our main research goals are to understand: a) the use of mathematical and digital resources – and their connections – in solving and expressing mathematical problems, and b) the way in which mathematical knowledge is combined with the affordances of digital tools to obtain and express a conceptual model to solve a given problem. In this paper, we address these issues by resorting to the case of Beatrice working with a covariation problem in her home environment.

2. Theoretical Framework

2.1 Solving and expressing mathematical problems with technologies

Solving a problem requires a mathematical point of view in addressing a given situation, and involves actions such as interpreting, describing or explaining, by using mathematical ideas or methods.

According to [6], models of specific situations emerge as a way of giving meaning to a problem, integrating informal and experience-based strategies and allowing the transition from the real world to the world of mathematical ideas (horizontal mathematization). These models evolve as their focus changes to the mathematical objects or relationships (vertical mathematization), including

moving away from the context and placing an emphasis on symbolization, creating formal strategies and relationships that support reasoning. The conceptual model gradually and progressively assumes the characteristics of a mathematical object, changing until it gains "a life of its own" (p. 98).

Conceptual models only work when they are explained through an array of representations (e.g., diagrams, pictures, tables, text, symbols, and dynamic or concrete figures). However, these "explanations and constructions are not mere processes that students use in the course of producing the solution (…) they ARE the most important components of the answer" [7](p. 3). The explanation is, therefore, a central action in the problem-solving activity. However, 'explaining' is much more than presenting, describing or telling, it is rather an endless process of representing the existing connections around the idea to be illustrated [8]. Hence, the expression of mathematical thinking must be seen as an integral part of the problem-solving process: obtaining an answer implies creating an explanation for the solution, making appropriate connections to emerge, so the problem-solving activity includes the answer obtained and also the explanation of the process followed. Considering problem-solving as a synchronous activity of mathematization and expression of mathematical thinking [5] and that the development of conceptual structures is mediated by digital technologies, we seek a further understanding of how the connections between technology and mathematical thinking become effective resources for solving-and-expressing mathematical problems.

2.2 Connections in solving-and-expressing problems with technologies

Expressiveness is so central to mathematical activity that researchers have been stressing the need to understand how mathematical knowledge is shaped by the use of digital tools to express mathematical relationships. Establishing appropriate connections between informal and intuitive mathematical experience and formal and abstract mathematics is a way to minimize the difficulties experienced by students in school mathematics. Indeed, the absence of these connections leads to neglecting intuitive and meaningful problem-solving approaches, impelling the use of memorized procedures that are often not fully understood [9,10].

While curriculum guidelines [11,12] focus the role of the teacher, it is the connections made by the students themselves that best leads to learning especially when solving challenging tasks that allow building a complex network of connections [13]. Intra-mathematical connections link facts or procedures within topics, while extra-mathematical connections relate those with external reality and are important in applying knowledge and procedures in problem-solving [10,14]. Indeed, when solving problems, one may notice a conceptual model that is useful to several situations or identify types of problems that are solved in similar ways. Yet, strategies may emerge, disappear or coexist, which can point out the connections made (or not) between existing and new knowledge [15].

The use of digital technologies in solving and expressing problems allows to observe, to explore, to formulate conjectures about possible connections, and to verify those [16]. Hence, digital tools shape the mathematics that students can achieve, as they provide new and unconventional means of expression with a strong connection with the context in which ideas are explored [17], which is why we still need to know how technologies make it possible to build connections between seeing, doing and expressing [10]. Additionally, experiences with a given technology drives conceptual structures that can be incorporated into mathematical thinking processes and, eventually, transferred to other situations [18]. Hence, it is important to discuss the type of competence that allows combining mathematical and technological knowledge effectively to solve and express non-routine problems.

2.3 Techno-mathematical fluency: connecting two worlds

The interplay between an individual's mathematical and technological skills has been studied for quite some time. For instance, an interdependence between the use of technologies and

mathematical resources in work environments that altered the type of mathematical skills needed by the workers has been identified [19], originating the term Techno-mathematical Literacies. More recently, the notion of "mathematical digital competence" has been proposed, referring to the simultaneous activation of mathematical and digital competencies [20].

We have adopted the term Techno-mathematical Fluency [4] in order to capture the interaction mentioned by [19], but also in the sense of someone's ability to produce mathematical thinking through digital tools, and to reformulate or generate new knowledge and express it using technology. Thus, 'fluency' seems to be appropriate to describe "the ability to reformulate knowledge, to express itself creatively and appropriately, and to produce and generate information" through digital technologies [21](p. 83). In this view, digital tools can be regarded as an extension of the individual who is technologically fluent, who is able to think and express himself through a technological and/or digital dialect. Accordingly, techno-mathematical fluency emphasizes being fluent in a 'language' that involves both mathematical and technological knowledge, constantly being intertwined to develop techno-mathematical thinking, as well as to efficiently interpret and express a solution by means of a techno-mathematical discourse.

3. Research Methods

This study aims to examine the connections between mathematical and technological knowledge, put forth while solving-and-expressing mathematical problems with digital tools in a beyond school competition. In that competition, a mathematical problem was posted online every two weeks and students had to develop and submit a solution using the tools of their choice. It was mandatory to submit the solution electronically with a complete explanation of the problem-solving process. Periodically, the organizing committee selected and published online a sample of exemplary solutions, showing different approaches or representations, or the use of different digital tools.

Taking an interpretative stance that involved qualitative techniques for data collection and analysis [22], we developed several cases of participants who usually resorted to a variety of digital tools to solve and express the problems of the competition [4,23]. From a pool of participants in the competition, we selected cases of problem solvers based on the type of digital tools they usually resorted to, their ability in communicating ideas and procedures, as well as their availability and agreement to participate in the study. Data collection involved the gathering of these participants' productions throughout two editions of the competition and the observation of one session working on a mathematical problem at their home environment.

For the purpose of this paper, we examine the data collected during the experimental phase of the research regarding the participant Beatrice (pseudonym), who seemed quite fluent with a diversity of digital tools, although she showed a significant willingness to use the aesthetical affordances of a presentation editor. Beatrice was asked to choose and solve one problem from three possibilities posted at the competition's website, to perform as closely as possible to her usual problem-solving activity within the competition, and to explain out loud her actions and thinking. Besides observation notes and video recording of the participant's problem-solving activity, we held an in-depth interview and video-captured her computer screen using an open source screen recorder software.

The data were anonymized and organized using NVivo10, which supported an interpretative, inductive and descriptive analysis. We focused on identifying extra and intra-mathematical connections established along the activity and key aspects in the development of the conceptual model of the situation, in considering the simultaneity of the use of mathematical and technological knowledge. This allowed us to obtain a descriptive research case that exposes the nature of Beatrice's activity of problem solving-and-expressing with technology, both in terms of the connections established and the conceptual model developed.

4. Data analysis

Beatrice chose to solve the problem "Switching balls" (Figure 1), as she found it "somewhat similar" to the first problem she had solved in the competition. She started to think about a possible approach by establishing connections with that problem she had solved some time ago.

> Afonso and Bernardo live in the opposite ends of the same street. Afonso had a football borrowed from Bernardo and Bernardo had another football belonging to Afonso. They both left their homes, at the same time, to switch the footballs. Bernardo's speed was twice the speed of Afonso until they met on the street. As soon as they switched the balls, they went back to their homes, but Afonso's speed was then twice the speed of Bernardo.
>
> When Afonso got to his house, Bernardo was still 120m away from home. How long is the street?
>
> **Do not forget to explain your problem-solving process!**

Figure 1 Statement of the problem "Switching balls"

When asked about what she was planning to do, Beatrice answered, laughing, "Drawings!" and, while looking for a paper and a pen, she repeated "Drawings, yes! I'm going to make a house… very cute!" While reading slowly, Beatrice began to draw a first 'diagram' with paper-and-pencil, scribbling two houses, one on the top left and one on the top right of the sheet, as if to bring the situation to a concrete form, but quickly abandoned it and built another diagram (Figure 2).

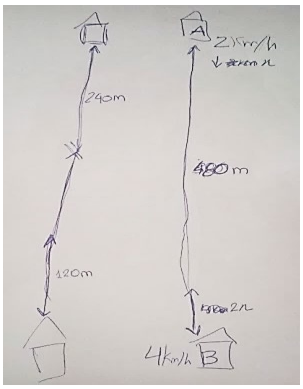

Figure 2 Two diagrams produced by Beatrice

4.1Testing with numerical resources

In order to understand the situation, Beatrice started experimenting with plausible pairs of numbers for the friends' speeds, in order to maintain the stipulated relationship: "as they don't tell me what the speed was... at least for one of them, I'll decide one of the speeds, to see (…). I don't know! But I'll try something (…). What is our average walking speed?". She decided to try out particular speed values, showing that she was making sense of the first condition but still thinking out loud: "I only know that the speed (…) the speed of Bernardo is twice the speed of Afonso. And when they go back home it was the other way around. I don't know anything more". By then, Beatrice was already connecting the two speeds, but she didn't yet fully understand what "the other way around" meant.

Focusing on the diagram (Figure 2), she then considered that Afonso travelled 120m until he met Bernardo, who had walked 240m along the street, which would mean that the street would be 360m long. She was experimenting with a pair of chosen values, according to a given relationship, to see if it made sense in the diagram. Realizing that these values did not work, though not being able to explain why, she kept struggling to go beyond the first condition and considered the speeds of the friends on their way back home, independently from the third condition.

4.2 Looking for clues in a similar problem

Beatrice also looked for realistic values when looking for a solution. Hesitant about the use of the speed of 4km/h, she wondered if time was important and referred to a similar problem that she solved previously.

> B: [Time] is important if it were as in the other [problem]. If it asked how many minutes they would take (…) one walks twice as fast as the other, right? Well, it would be cool to know the time. Actually time is important. Oh my, I don't know where the other problem is. I saved it…

She kept insisting on the first condition of the problem, deepening her understanding about the displacements in the same time interval. As she could not find her own solution to the similar problem mentioned, she decided to analyse other students' solutions published online. Although showing some anxiety, as the problems were rather different, she recalled having used the least common multiple of the speeds given, and tried to use it in this new situation.

> B: I'm now thinking of something else. In that problem, the first one (…) I used the least common multiple of the speeds (…) It was 5 and 4, wasn't it? I calculated the least common multiple of 5 and 4, which is 20. It could work now. I don't know… $20km$ is quite a bit for a street, so I think I will use meters instead.

The least common multiple emerged as an important mathematical resource that Beatrice seemed eager to use. She tried the pairs 20 and 40 and after some time observed that the street could be 480 meters long, but in that way, Bernardo would not be 120m away from home when Afonso got to his house. About to give up, she analysed the possibility of the street being 480m long, concluding that Afonso would walk $2 \times 80m = 160m$ while Bernardo would travel $2 \times 160m = 320m$ until they meet. However, she was unable to establish a relationship between that and the expected result, so she continued to pursue another hypothesis. At some point, she explained "I am doing the same as always, that is, when there is the least common multiple, I double and add them, and that is the [street's] length". These are evidences that a conceptual structure of the situation is being generated, although at a very early stage due to the lack of coordination of the two parts of the journey.

Up to this point, Beatrice made connections: a) with previous experiences in problem solving (approaches, mathematical facts, relationships between pairs of numbers), b) between mathematical facts she had resorted to in a familiar problem (use of the least common multiple, comparing conditions), c) with everyday knowledge, to make sense of the situation and judge de feasibility of the values selected (speeds and distances). The conceptual model is taking shape from the construction of a diagram and the testing of pairs of numbers following particular chosen criteria.

4.3 Working with a diagram and extending connections

After 1h30 of struggle with this problem, Beatrice was about to give up. Then the interviewer, reassuring her that it would be alright to end the session, asked her to analyse another diagram, similar to the ones she had been experimenting with. Beatrice appeared quite surprised by the simplicity of the diagram (Figure 3), drawn with a pencil on a sheet of paper with a grid: "it is more or less what I was doing, isn't it?... only…there are no numbers, right?"

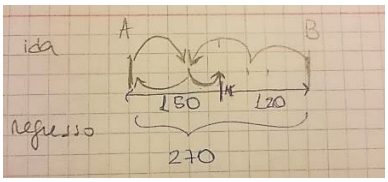

Figure 3 Diagram presented (pencil) with values registered by Beatrice (blue ink)

By looking attentively to the diagram, she identified A and B as the houses of Afonso and Bernardo, realized that the arrows above the squares referred to their displacements in the first part of the journey, and identified the 120m of the street that Bernardo had to walk at the end of his journey. Surprisingly, however, Beatrice was not able to recognize this distance as ½ of the street's length, hence she still couldn't see the solution.

> B: Bernardo walked twice as fast as Afonso, right? On the outbound. When coming back, it was the other way around, but Afonso kept walking at the same pace […] the distance and the speed of Afonso maintained, they were the same […] But now how I get to the numbers, I don't know.

She was making sense of the second condition, recognizing that Afonso kept the same speed as in the outbound journey. After a while she realized that if each arrow corresponded to a displacement of 3, on the inbound journey, Afonso's travel was represented by one arrow and Bernardo's by an arrow with half of that size. However, she kept struggling to give meaning to "one displacement and a half" and decided to resort to experimentations with numbers (Figure 3, in blue ink), after which she built an enlargement of the diagram, leading her to a more inquiring approach.

In this new diagram (Figure 4) she found a flaw when trying to explain the result of 270m: Afonso's inbound journey corresponded to two arrows to the left, while Bernardo's was not completely represented. By completing the diagram, she observed that the length to go, 120m, was "the same as three times", that is, it corresponded to three units in a total of six, thus understanding that the units were not relevant. After this experiment, where the displacement unit corresponded to three squares and the speeds were 2 and 4, Beatrice went back to the previous diagram (with the relation 1:2) and was now able to tell that when Afonso arrived at home, Bernardo was exactly on the middle of the street: "Yes, yes! It's 240!" She went back to confirm that the conceptual model obtained from the enlarged diagram was still accurate in that previous diagram. Thus, she realized the possibility of generalizing the considered values, since they were defined upon the relationships and not on particular speeds or displacements, contrarily to her initial approaches.

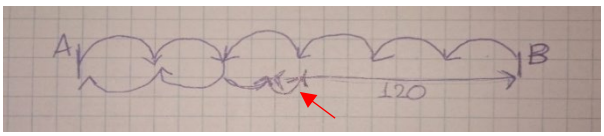

Figure 4 Larger diagram built by Beatrice (the arrow drawn afterwards is signalled)

It is timely to note the connections made between the different diagrams, in an attempt to make sense of them, and between the diagrams and the experimentations with numbers while building the enlarged diagram with the purpose of achieving a better understanding of the solution. The conceptual model is moving away from the experimentations with numbers and is now grounded on the diagrammatic representation of the journeys, showing that Beatrice is in control of the several conditions of the problem and that she recognizes the possibility of generalization of the solution. However, this is still a horizontal mathematization, based on a *model of* that particular situation.

4.4 Constructing a digital solution

In the construction of a digital solution, Beatrice used PowerPoint and resorted to the Internet and a text editor. She started by explaining that what she was about to do "is not very different from the diagram… I'll do about the same [as with paper-and-pencil]", that is, she intended to replicate the solution constructed previously.

The diagram (Figure 5) reproduced the displacement of the two friends while walking towards each other according to the initial conditions: given that Bernardo walks twice as fast as Afonso,

Beatrice inserts two arrows with equal direction and orientation from B to A; and another one with the same length and direction but from A to B. As the two friends get together to switch the balls where the green arrow meets the blue one, Beatrice placed the beginning of the inbound journeys, approximately at that point. Afonso reaches his house with a single displacement (black arrow) but his speed is twice of Bernardo's, which means that Bernardo's speed is actually half of Afonso's, so in the same amount of time his displacement was half of the previous one (smaller black arrow). Beatrice inserted a dashed line to represent the 120m that Bernardo still had to travel.

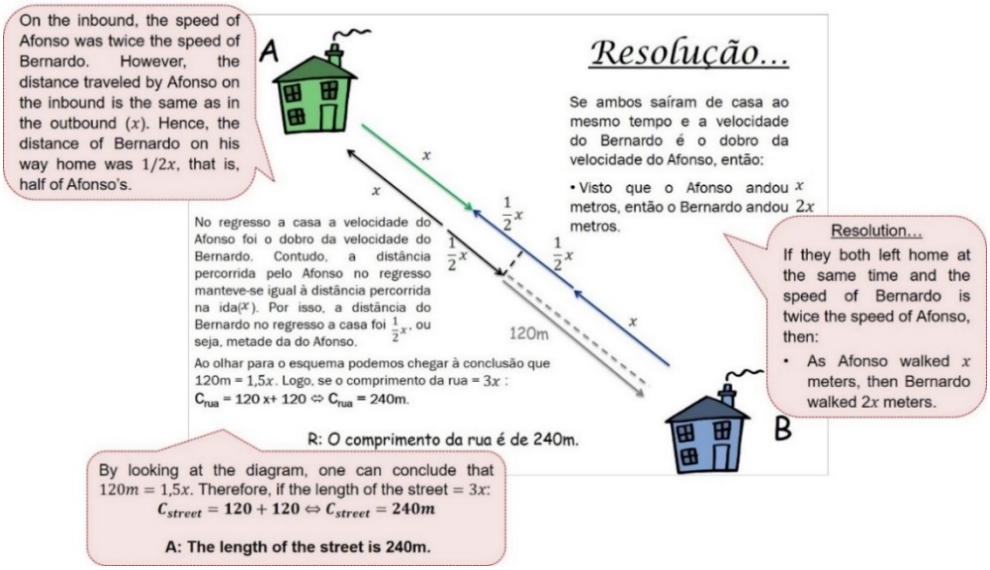

Figure 5 Solution submitted with translation of the written explanations

She also decided to label the lines and arrows in the diagram using the equation editor tool (from Word) to insert the appropriate mathematical symbols. After using the unknown x in each of the equal displacements, she included a label on the line corresponding to half of that length: $\frac{1}{2}x$. When asked about the fact she was doing some things differently from before, she explained:

> B: As I work, sometimes I use other things to explain better. I was thinking that it might be best to explain with an x, for instance, if Afonso walked x… then Bernardo walked $2x$. When they go back home, the speed of Afonso was twice the speed of Bernardo, so… if Afonso walked x, hummm, Bernardo walked half x. [If] by walking x the other one reached home… after Bernardo walked ½ of x it was missing $120m$, this part, right.

Beatrice has embraced the paper-and-pencil diagram to such an extent that she was already able to use it as a model to explain how she obtained the solution. The digital diagram, containing technological entities that convey a mathematical meaning (e.g., lengths, colours), was becoming part of her techno-mathematical discourse.

On the textbox on the left she types her reasoning, using symbols from the Equation Editor. As she writes, she decides to include a dashed line, vertically to the displacement of Bernardo in his inbound journey to show that it matches half of his second displacement in the outbound journey, thus arriving at the middle of the street when Afonso reached his house.

Beatrice was developing her solution digitally, in the sense that this was not a mere reproduction of the approach rehearsed with paper-and-pencil; instead, she was coordinating technological and mathematical resources to visually communicate powerful ideas: the oriented

segments with a particular length, direction and orientation represent the displacements of the two friends on the outbound and inbound journeys, mathematized by associating labels containing algebraic expressions that exhibit their relationships (x and $\frac{1}{2}x$). These techno-mathematical entities are new knowledge objects that unveil her way of thinking and her conceptual model of the situation.

She continued by combining text with algebraic expressions. From the analysis of this diagram she noted that $120m = 1{,}5x$ and that the total lengh of the street is $3x$. The fact that she did not found the value of x from the first expression may be related to the strong visual perception afforded by the paper-and-pencil experiment that led her into recognizing the $120m$ left to walk as half of the street. She then added 120 with 120, concluding the total length was $240m$. With this process of composing a digital solution, using mathematical resources, Internet, PowerPoint and Word resources, Beatrice explained her reasoning through a diagram that evolves into a visual argument supported by the written justification. After almost 3 hours of work, Beatrice still insisted in formatting text and googling images related to the context to make her presentation more elaborate.

Beatrice's mathematical problem solving-and-expressing is characterized by the resourceful use of diagrams along the activity. The digital solution portrays her ability in incorporating descriptive elements, organizing graphical aspects using images, arrows, lines, colours, and combining them with written notes, meaning that the visual representations allow to materialize the dynamic nature of the situation and to construct an expressive "assemblage" of the solution. She made connections between mathematical knowledge (variable, equation, solving an equation) and technological knowledge (text editing, search, select and insert images, use arrows and lines, formatting colours and dashes, use mathematical symbols). The conceptual model evolved into a mathematical model of the situation, that comprises two equations: $140 = 1{,}5x$ and $C = 3x$, that she then solved. These are features of a vertical mathematization and hers is a *model to* explain the solution.

5. Discussion of results and conclusion

This study aimed, firstly, at offering a deeper understanding of the connections established between the use of technological and mathematical resources during problem solving-and-expressing. Our second purpose was to characterize the intertwining of mathematical knowledge and the affordances of digital tools that occurs while a conceptual model of the solution is in development.

Several types of connections were identified. Beatrice made extra-mathematical connections with everyday knowledge and past problem-solving experiences, useful to a global understanding of the situation. The intra-mathematical connections comprised, on the one hand, connections between mathematical facts and the paper-and-pencil technology, leading to deeper understanding of the three main conditions in the statement and how they relate to each other – in a horizontal mathematization; on the other hand, the connections between mathematical facts and the digital technology provided robustness and formalism to the model, changing it into a formal mathematical model, typical of vertical mathematization. This leads us to conjecturing that different types of connections have also distinctive roles and purposes in the activity of solving-and-expressing problems with technology.

The conceptual model developed is characterized by the diagrams built at each moment and their corresponding function: the initial diagram worked as an embryo of the conceptual model where informal ideas and experimentations with multiples are tested; the diagram analysed was used to understand the relation between the speeds, displacements, and the coordination of the journeys; the enlarged diagram allowed the solution to emerge from a particular case (model of) and the observation of the possibility of generalization of the solution; the digital diagram integrated a model for explaining the solution and contains a mathematical model expressed by means of equations. This means that the diagrams supported a progressive mathematization of the solution, from a *model of* into a *model for*, from an informal into a formal model [6].

The problem solving-and-expressing activity that took place brought up the need to produce a techno-mathematical solution, thus emerging the techno-mathematical fluency of Beatrice in using several technological and mathematical resources to develop her own conceptual model of the covariation situation. She perceived useful ways of combining knowledge about the technology, such as insert images, draw and format arrows, lines, use labels, and colours, with knowledge about mathematics, such as testing with multiples, least common multiple, using variables, algebraic relations, or solving linear equations. Their simultaneous and connected use supported the understanding of the conditions in the problem in order to generate a conjecture about the solution and also to rehearse a mathematical justification. Although the role of the digital tool in the selected case is particularly evident in communicating the solution, the backbone of this activity rests in the notion of solving-and-expressing – as solving and expressing occur alongside – which leads us to consider that the tools support the mathematization of the situation throughout the problem solving-and-expressing processes. Elsewhere [24], we have presented and discussed other instances of techno-mathematical fluency in problem-solving-and-expressing with GeoGebra.

This case reveals that mathematical problem solving with technology can be a challenging context where students make different types of connections, very often, spontaneously. Thus it can be used, intentionally, in the classroom. Yet, the teacher must be aware of the importance of making connections with everyday knowledge and previous mathematical knowledge and experiences; that a diagram is a 'personal resource', thus it is necessary to pay attention to the way students individually accommodate it or not; that the development of a conceptual model is a progressive mathematization activity, which collides with 'ready-made' unintelligible models; and that the metaphor of techno-mathematical fluency entails the skills that must be developed to face the 21st century challenges that require mathematization with digital tools.

While the connected use of technological and mathematical resources throughout the problem solving-and-expressing activity influence the depth of a conceptual model, the introduction of digital technologies in the activity affords a re-evaluation of the situation from a techno-mathematical point of view (seeing), combining mathematical concepts and procedures with technological knowledge (doing) to create a solution that entails a techno-mathematical discourse (expressing).

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