The Role of Technology to Build a Simple Proof: The Case of the Ellipses of Maximum Area Inscribed in a Triangle

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Abstract: *We know that there is a unique ellipse inscribed in a triangle and passing through the midpoints of its sides ([5]). This ellipse is known as the Steiner ellipse. Among the properties of this special ellipse, one states that it is the ellipse of maximum area inscribed in a triangle. The first complete proof of this property was given in 2008 by Minda and Phelps ([4]). Their proof uses lots of properties of complex numbers and especially the complex forms of some transformations. When I read this proof for the first time, I was unpleasantly surprised by its complexicity. From this moment, I worked on an approach of this property using dynamic geometry and Computer Algebra System. My aim was initially to find investigations that could lead to this property. I was successful but what was astonishing is that I could build a proof of this property following the stages of the previous investigations. This paper will discribe first, how the investigations conducted with technology led to the expected conjecture and secondly how a simpler proof could be built in translating with CAS the stages of the investigations ([1']). This process is really unusual because, it is known that there is a gap between the conjecture and the proof in an experimental process of discovery mediated by technology (or not). The story of this research will give an example of bridging the stage of conjecture and the stage of proof ([2']). We will also have the opportunity to show how the possibilities of a software can influence our constructions and the way to conduct our proofs: here we will conduct a backward reasoning which is the core of the simplification provided by my proof. As usual in any research work, we will give some extra results met during our investigations (construction of all ellipses inscribed in a triangle, simple constructions of isoptics…).*

1. Preliminary results

1.1. Parallel projections and affinities (Cabri 3D)

As shown in Figure 1 (left), we can notice that the ratio between the areas of the blue triangles is not changed by a parallel projection (same as the ratio between the purple triangles). As this figure is a screenshot of a Cabri 3D file, it is possible to corroborate this observation by changing the shapes of the blue triangles, or by changing the direction of the parallel projection, or by changing the planes supporting the blue triangles or the purple ones. Such an investigation with a dynamic geometry software aims to suggest the general result stating that the ratio between the areas of two figures of a given plane is kept with a parallel projection on another plane. To prove this conjecture, we will prove that any parallel projection from one plane to another can be considered as an affinity. Knowing that any affinity keeps the ratio between areas, the proof will be complete.
As shown in Figure 1 (middle), M'is the image of M with respect to the parallel projection which direction is (mm'). h is a point that can be dragged along the intersection between the two planes and H is constrained by the conditions (mh) // (MH) and (hm') // (HM').
If we rotate the yellow plane around the intersection line in order to superimpose it with the grey plane and if we drag point h until a position on segment $[mm']$, we obtain a plane figure shown in Figure 1 (right). In this figure, thanks to the properties of similar triangles, we obtain

$$\frac{hm'}{hm}=\frac{HM'}{HM}$$

which means that the the parallel projection can be considered as an affinity. That completes the proof.

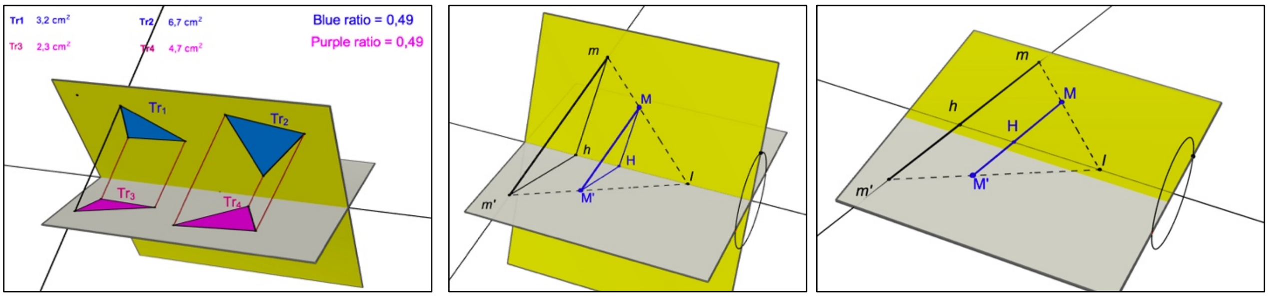

Figure 1: Parallel projection and affinity

1.2. Different dynamic constructions of all possible ellipses (via Geometry Expressions)

1.2.1. Reminder about the bifocal definition of an ellipse (Figure 2 left)

Given a > 0 and 0 ≤ c ≤ a, the ellipse is defined as the set of points M constrained by the condition MF +MF' = 2a. Points F(c,0) and F'(-c,0) are its foci. Points A(a,0) and A'(-a,0) are the vertices of its principal axis and points B(0,b) and B'(0,-b) are the vertices of its other axis where b = $\sqrt{a^2-c^2}$. Its eccentricity e is defined by $\frac{c}{a}$. We know that for an ellipse 0 ≤ e ≤ 1.

1.2.2. Possible choices of the two parameters defining all possible shapes of ellipses (which means all possible eccentricities).

First choice: fix the value of **a** and constrain **c** to be a variable taking all the possible values between 0 and **a**. When **c** = 0 we obtain the circle centered on O and which radius is **a** (***e*** = 0). When **c** = **a**, we obtain the segment [A'A] or segment [F'F] : we can except this case (***e*** = 1). In Figure 2 (middle), **a** = 1 and we have displayed 50 possible ellipses (the case ***e*** = 0 is visible but not the case ***e*** = 1).

Second choice: fix the value of **c** and constrain **a** to be a variable taking all the possible values larger than **c**. When **a** = **c** we obtain segment [A'A] or segment [F'F]: we can also except this case (***e*** = 1). The case of the circle is obtained as a limiting case when **a** goes to infinity (the limit of the eccentricity would be 0). In Figure 2 (right), we have chosen **c** =1 and displayed 100 possible ellipses: we can notice that the ellipses approach the shape of a circle when **a** increases (goes to infinity).

Third choice: fix the value of **a** and constrain **b** to be a variable taking all the possible values between 0 and **a**. In this case **c** must be equal to $\sqrt{a^2-b^2}$ which defines the positions of F and F'. The ellipses generated in changing the values of variable **b** are similar to those obtained for the first choice (Figure 2 middle).

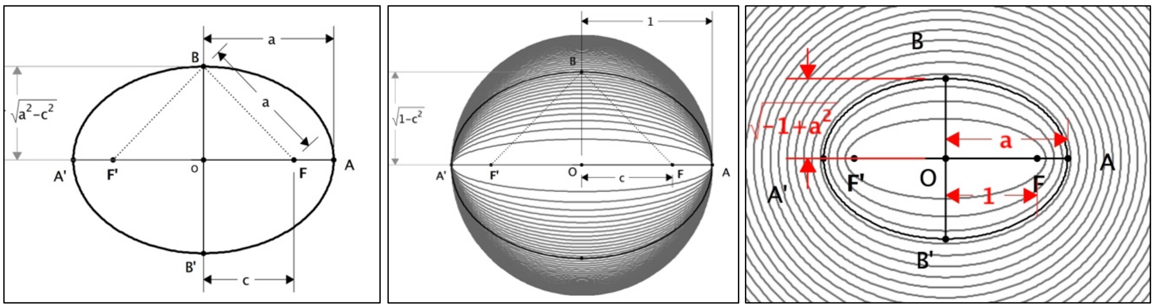

Figure 2: Bifocal definition of an ellipse

1.3. Quick justification of the existence of the Steiner ellipse for any triangle (Cabri 3D)
As shown in Figure 3, any triangle can be considered as the image of an equilateral one with respect to a parallel projection from one plane to the plane containing the given triangle.
In Figure 3, starting from triangle ABC, we have constructed in a plane containing (AC) the equilateral triangle AB'C and its inscribed circle. ABC is the image of AB'C with respect to the parallel projection of direction (B'B). Therefore, the image of the red inscribed circle of AB'C is an ellipse inscribed in ABC, with the same properties of contact, tangency and midpoints. Eventually, we have proven the existence of an ellipse inscribed in ABC which means it is tangential to each side at their midpoints. Moreover, the center of this ellipse is the centroid of ABC and each point of the medians of ABC cutting this ellipse cut it at the third. This ellipse is called the Steiner ellipse.
Another property related to areas can be added. Thanks to the property presented previously, the ratio of the area of the Steiner ellipse to its triangle is equal to the ratio of the area of the inscribed circle of an equilateral triangle to the area of this triangle. It is easy to prove that this ratio is equal to $\frac{\pi.\sqrt{3}}{9}$ ([5])

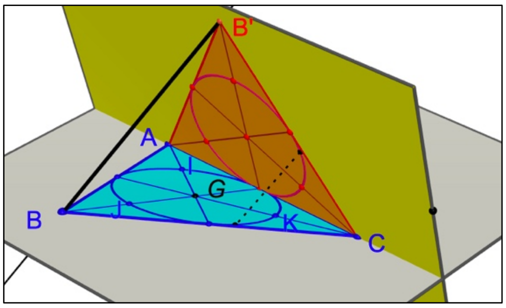

Figure 3: The Steiner ellipse, existence and some properties

1.4. Questions about this special ellipse and a glimpse on the way to solve these questions
The questions we can ask ourselves after proving the existence of such an ellipse could be:
-Are they other ellipses inscribed in a given triangle?
-If yes, which one has the maximum area?
The first trick we will use is to try to solve the problem in an equilateral triangle:
Any ellipse inscribed in a given triangle is the image with respect to a parallel projection of an ellipse inscribed in an equilateral triangle and vice versa.
If an ellipse inscribed in a given triangle is the ellipse of maximum area, this ellipse is the image with respect to a parallel projection of the ellipse of maximum area inscribed in an equilateral triangle and vice versa.
Eventually, we have to consider (really to construct) all the possible inscribed ellipses of an equilateral triangle and find among them the one of maximum area. As we want to prove that the Steiner ellipse is the ellipse of maximum area inscribed in a triangle, it will be sufficient to prove that the ellipse of maximum area inscribed in an equilateral triangle is its inscribed circle.

2. Construction of all inscribed ellipses of a given triangle. Solving a first problem of maximization

2.1. About backward reasoning
Anybody who wants to begin research in order to solve this problem would probably first start from a triangle, equilateral or not and then try to find all the possible inscribed ellipses of such a triangle.

This was not my preference because I like simple approaches which means using more geometry than algebra. It is the reason why, following the advice of one colleague at the beginning of my investigations, I have decided to start from an ellipse of given shape (given eccentricity) and try to construct all the equilateral triangles enveloping such an ellipse (which means that the given ellipse is inscribed in such triangle). I will discribe below this construction which is rather simple. In a second step I have to find the equilateral triangle of minimum area enveloping such an ellipse of given shape. In a last step, I have to see what happens when I change the shape of the given ellipse.
Remark: using this approach, we can notice that the ellipse solution of our problem will be obtained when the ratio of the area of the given ellipse to the area of the enveloping equilateral triangle is maximum.

2.2. Construction of all equilateral triangles enveloping a given ellipse

Our ellipses are constructed with respect to the previous third choice (Figure 4 left).

Initial constraints: A(1,0), B(0,**b**) (with $0 \leq \mathbf{b} \leq 1$), F($\sqrt{1-a^2}$, 0) ; A', B' and F' are constructed with respect to evident reflexions. The ellipse visible in the figure is the ellipse defined by the two foci F and F' and passing through B. As B'' is symmetric to F' with respect to B, the red circle centered on F' passing through B'' is the director circle of this ellipse associated to F' (radius 2).

Other constraints: E(cos(t), b.sin(t)) with t between 0 and 2π and so E generates the whole ellipse, e(-sin(t), b.cos(t)) in order to obtain the direction (Oe) of the tangent line at E called T1.

U is symmetric to F with respect to T1 and belongs to the red circle (known property of ellipses), while V is a point of the red circle constrained by the condition $\angle$UFV = $\frac{2\pi}{3}$. The perpendicular bisector of [FV] is the second tangent to our ellipse I needed to construct because the angle between T1 and T2 is $\frac{\pi}{3}$. The reason for this comes from the shape of quadrilateral FuMv: angles u and v are right angles, then the sum of angles F and U is π . Eventually, from a first tangent line to our ellipse, we have constructed a second one which angle with the first one is $\frac{\pi}{3}$. If we iterate this process, we can construct a third tangent line T3 to the ellipse whose angle with T2 is also $\frac{\pi}{3}$.

These three tangent lines are supporting an equilateral triangle MNP in which our initial ellipse is inscribed. The contact point of (PM) is point E. Dragging point E along the ellipse in changing the values of parameter t between 0 and 2π allows us to construct all the equilateral triangles in which the given ellipse is inscribed. In the Geometry Expressions file created for this construction, as we can change the values of b, we have constructed all the equilateral triangles in which any ellipse is inscribed.

Remark: if we want to evaluate algebrically the ratio of the area of the ellipse to the area of each constructed equilateral triangle, with respect to t and b, we need the area of the ellipse which is equal to $\pi . b$ (known result) but Geometry Expressions gives a strange output:

$$z_0 \Rightarrow \frac{\pi \cdot |b| \cdot \left|2 \cdot \sqrt{1-b^2}\right|}{\left|-2 \cdot \sqrt{1-b^2}\right|}$$

More disappointing is that Geometry Expressions is unable to give a result for the area of the triangle. So we have to modify our approach of the problem of maximization of the area of the ellipses of given shapes inscribed in a given equilateral triangle

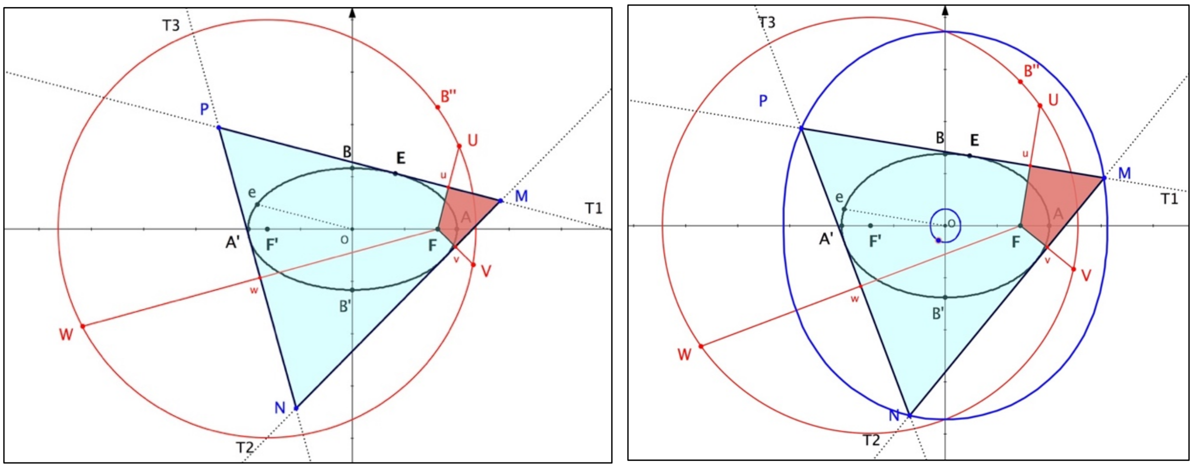

Figure 4: Equilateral triangles enveloping a given ellipse

But before, let us give some extra results that we can obtain with such a figure:

Result 1: in Figure 4 on the right, we have displayed the locus of point M for all equilateral triangles or when t varies from 0 to 2π . It is the largest blue curve which represent the locus of point M viewing the ellipse under an angle of $\frac{\pi}{3}$ which is known as the isoptic of the ellipse corresponding to the angle $\frac{\pi}{3}$. Evidently, the loci of N and P are superimposed to the previous one. This isoptic is a circle when the ellipse is a circle (very elementary proof).

Result 2: in the same figure we have displayed the locus of the centroids of the equilateral triangles (little blue curve). We can notice that this curve never contains the center O of our initial ellipse (except when **b** = 1)

2.3. First conjecture on the way to the solution

As we were unsuccessful in using the algebraic mode of Geometry Expressions, let us use the numerical mode for a more classical Dynamic Geometry Software approach. As shown in Figure 5 (left), we have displayed respectively the numerical areas of the ellipse and the triangle in variables called AreaEllipse and AreaTriangle and calculated and displayed in red the ratio between these two areas. Then we rotate the equilateral triangle in changing the values of t to obtain the following conjectures:

First conjecture: the minimum area of the ellipse which is equivalent to **the maximum of the previous ratio** (or the maximum area of the ellipses of given eccentricity inscribed in a given equilateral triangle) **is reached when any of the sides of the triangle is parallel to the principal axis of the ellipse** or **when one of any vertex of the triangle belongs to the second axis (and by the way to the director circle associated to F' (Figure 5 left).**

Second conjecture: the maximum area of the ellipse which is equivalent to **the minimum of the previous ratio** (or the minimum area of the ellipses of given eccentricity inscribed in a given equilateral triangle) **is reached when any of the sides of the triangle is perpendicular to the principal axis of the ellipse (Figure 5 middle).**

A more accurate investigation leads, in the previous case to conjecture that any of the vertex of the triangle belongs to the principal axis of the ellipse and more precisely to the circle centered in O which radius is 2a-3c (Figure 5 right) equal to 3-2c in our figure.

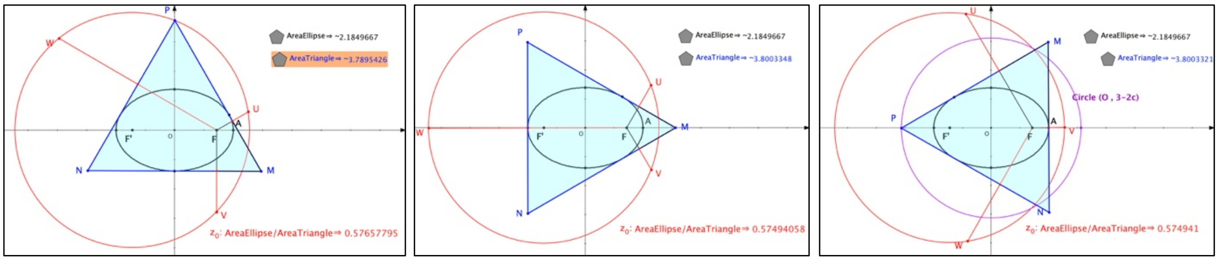

Figure 5: Positions of the equilateral triangles conjectured to be solutions

Watching accurately at the values of the ratio and the possible values of angle AFU, we can conjecture the following table of variation:

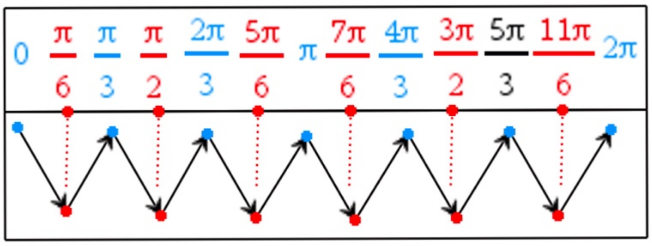

Figure 6: Possible variations of the area of the equilateral triangle with respect to angle AFU

We will see the important role of this angle to choose a system of axes in which our algebraic proof will be conducted successfully. But before, let us show how our technique of construction can be used to construct dynamically all the ellipses of given eccentricy inscribed in a given triangle: equilateral to begin with and then in any triangle.

2.4. Construction of all ellipses inscribed in a given triangle

Case of given equilateral triangle: here is the algoritm of construction (shown below in Figure 7)

Step 1: Given a red equilateral triangle M1N1P1 and a black ellipse of given shape (eccentricity changes when **b** varies between 0 and 1): Our aim is to construct all the ellipses inscribed in the given triangle.

Step 2: Start from a point E of the ellipse to construct the equilateral triangle MNP enveloping the ellipse as done in 2.2.

Step 3: Translate the ellipse and triangle MNP with respect to the translation mapping P on P1. Obtain ellipse with F2 and F2' as foci inscribed in triangle M2N2P1.

Step 4: Rotate previous ellipse and triangle around P1 which angle is N2P1N1. Obtain ellipse with F3 and F3' as foci inscribed in triangle M3N3P1.

Step 5: Dilate previous ellipse and triangle; the center of the dilation is P1 and it maps N3 onto N1. Obtain ellipse with F1 and F1' as foci inscribed in triangle M1N1P1. This ellipse is the solution of our problem.

Step 6: Hide the last construction. At this stage, if we move point E along the given initial ellipse in changing the value of t from 0 to 2π we obtain all the ellipses inscribed in the given equilateral triangle whose eccentricity is equal to the eccentricity of the given ellipse.

Remark: if we change the shape of the initial ellipse in changing the value of **b**, the final ellipse in the given equilateral triangle is refreshed. Our problem is completely solved.

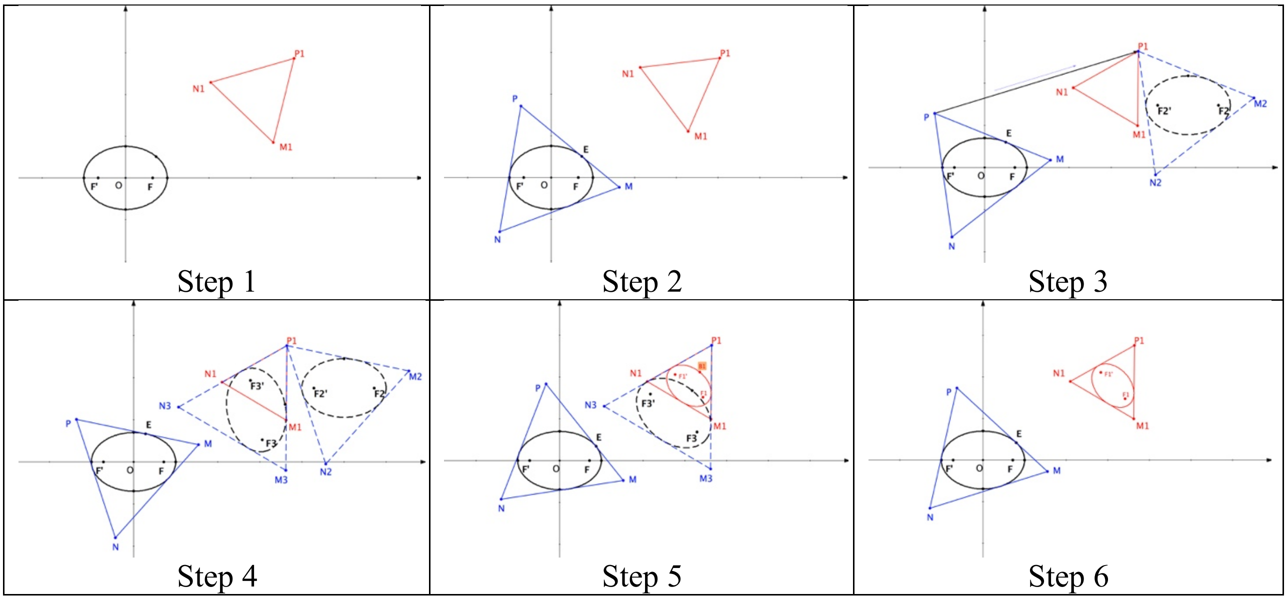

Figure 7: Construction algorithm of all ellipses of given eccentricity inscribed in a given equilateral triangle

General case (case of an arbitrary triangle)

In this case, let us give a red arbitrary triangle M1N1P1 and a black ellipse of given shape (eccentricity changes when b changes between 0 and 1): Figure 8 left. Our aim is to construct all the ellipses inscribed in this given triangle. Angles M1 and N1 are given as variables θ and φ.

We construct first the triangles (with two angles equal to θ and φ) enveloping the given ellipse. The technique of their construction is similar to the one given in 2.2. but here $\frac{2\pi}{3}$ is changed onto $\pi - \theta$ for the construction of the second tangent line and onto $\pi - \varphi$ for the construction of the third tangent line. (Figure 8 left)

Then, we use the same technique as the one used in 2.4. to construct all the ellipses inscribed in the given red random triangle (Figure 8 right). Each position of E on the given black ellipse generates a different ellipse inscribed in the given red random triangle.

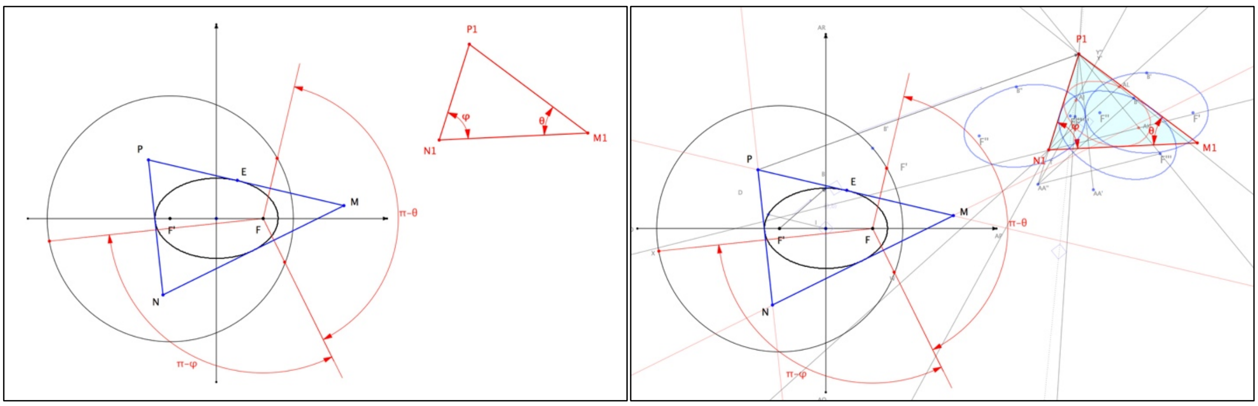

Figure 8: Construction of all ellipses inscribed in a given random triangle

2.5. Formal proof of the first conjecture

We will use a property of equilateral triangles that states that the height of such triangles is equal to the sum of the distances between any point interior of the triangle and its three sides.
So, with the notations of Figure 4 (left), the height of triangle MNP is equal to Fu+Fv+Fw. As the area of any equilateral triangle is proportional to its height, minimizing or maximizing the area of triangle MNP is equivalent to minimizing or maximizing Fu+Fv+Fw or its double FU+FV+FW.
Eventually, we have to find the positions of M minimizing FU+FV+FW (which means the position of an ellipse of maximum area inscribed in an equilateral triangle). To solve this problem, as shown in Figure 9 (left), we consider two points F1 and F2 such that F1F2 = 2**c** (**c** varies between 0 and 1), a rectangular system of axes centered on F1, a red circle centered on F1 with radius of 2 and a variable point U on this circle defined by angle AF2U equal to t between 0 and 2π, Points V and W defined by the common value of angles UF2V and WF2V ($\frac{2\pi}{3}$). If we ask Geometry Expressions to evaluate formally the three distances F2U, F2V and F2W, we obtain very complicated expressions that I haven't tried to export in any CAS software (Figure 9 right).

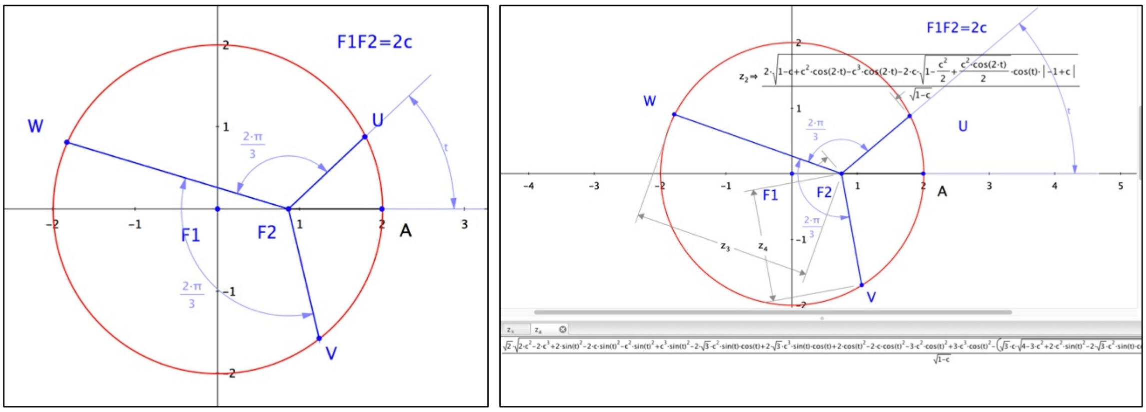

Figure 9: chosing parameters for the formal proof

In a previous paper, we have obtained the following results ([5])
F2U = $-2(c.\cos(t) - \sqrt{1 - c^2.sin^2(t)})$ from where we can deduce
F2V = $-2(c.\cos\left(t - \frac{2\pi}{3}\right) - \sqrt{1 - c^2.sin^2(t - \frac{2\pi}{3}}))$ and
F2W = $-2(c.\cos\left(t + \frac{2\pi}{3}\right) - \sqrt{1 - c^2.sin^2(t + \frac{2\pi}{3}}))$ and then we have to minimize the following function between 0 and 2π.

f(x) = 2$\boldsymbol{(\sqrt{1 - c^2.sin^2(x)} + \sqrt{1 - c^2.sin^2\left(x + \frac{2\pi}{3}\right)} + \sqrt{1 - c^2.sin^2(x - \frac{2\pi}{3})})}$ because
$c.\cos(t) + c.\cos\left(t - \frac{2\pi}{3}\right) + c.\cos\left(t + \frac{2\pi}{3}\right) = 0$.
This function is periodic (with period $\frac{\pi}{3}$) and it is an even function. The variations of this function are known as soon as we know them on the interval $[0, \frac{\pi}{6}]$. Eventually, we have only to prove that this function is decreasing on this interval to prove our conjecture. None of the CAS I have used were able to prove such result. I tried a lot of techniques by hand, unsuccessfully. The best I could do to justify this result was to export this function in a Note page of TI-NSpire to evaluate its derivative, and display the curve of the sign of this derivative in a Graph page on this interval. We can check that for any value of **c** (between 0 and 1) commanded with a slider, this sign is negative on this

interval and positive on $[\frac{\pi}{6}, \frac{\pi}{3}]$ because the graph displayed is a horizontal segment at y = -1 on $[0, \frac{\pi}{6}]$ and a horizontal segment at y = 1 on $[\frac{\pi}{6}, \frac{\pi}{3}]$ (Figure 10). I consider this proof as a valid proof supported by technology but it is a challenge for all of us to provide the usual expected formal proof. By the way, these variations justify our conjecture about the minimization of the area of the equilateral triangles enveloping a given ellipse.

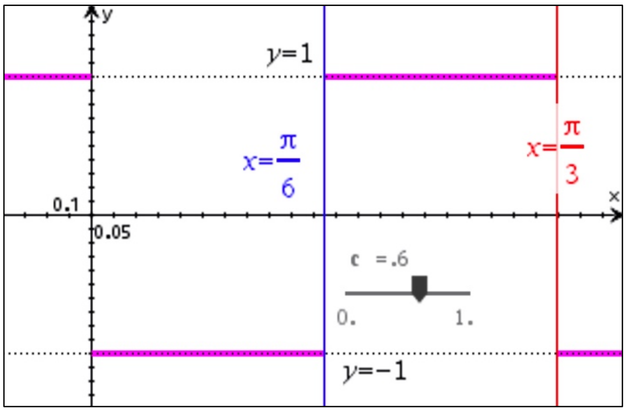

Figure 10: Sign of the derivative

2.6. Summary of this first part and bridge to the next part

At this stage of our work, we know that for any given number ***e*** between 0 and 1, and for any equilateral triangle, there is an infinity of ellipses of eccentricity ***e*** inscribed in this triangle. Those of maximum area are the three ellipses with principal axis parallel to one side of the triangle.
In the next part, we will consider all the ellipses inscribed in an equilateral triangle with principal axis parallel to a given side of a given equilateral triangle and we will investigate which value of the eccentricity maximizes the area of these ellipses.

3. Second problem of maximization

3.1. Construction for the proof

Here is the algorithm of our construction (Figure 11).
-Let us chose the first choice of 1.2. to construct our ellipse of given shape (**a** = 1 and **b** parameter between 0 and 1).
-Construct Point E(0,3) and segment [EE’] such as angle OEE’ = $\frac{\pi}{6}$.
-From F’ and F, construct parallel lines to (EE’)
-Construct intersections of this lines with the given ellipse: i and j for the first one and k and l for the second one
-Construct line (mm’) where m and m’ are respectively the midpoints of [ij] and[kl]
Point C which is the left intersection between this line and the ellipse
-From C, parallel to (EE’) which is the tangent line to the ellipse at C with angle with the y axis of $\frac{\pi}{6}$.
-Construct reflected line of this tangent with respect to the y axis which is the second tangent line to the ellipse which angle with the y axis is $\frac{\pi}{6}$.
-From B’, construct parallel line to the x axis which is a horizontal tangent line to the ellipse

-Construct triangle MNP supported by these three lines which is the equilateral triangle, with one side parallel to the principal axis of the given ellipse and enveloping this ellipse
-Display with Geometry Expressions in algebraic mode, the areas of the given ellipse (depending on b) and the area of the constructed triangle.

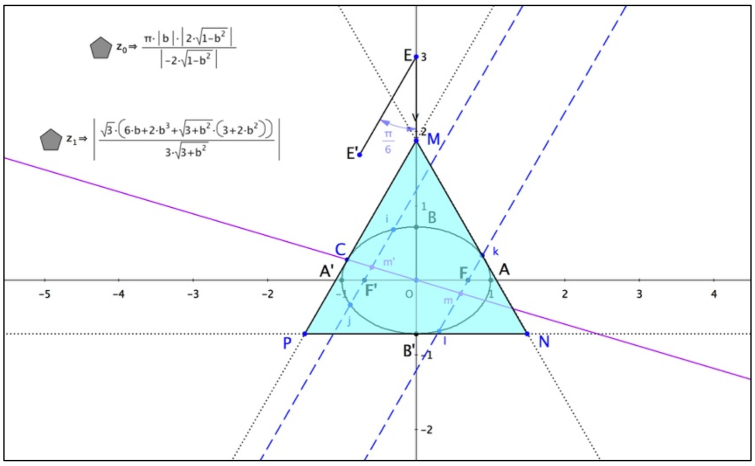

Figure 11: Construction algorithm of all equilateral triangles enveloping a given ellipse, the principal axis of which is parallel to one side of the triangle

3.2. Proof using Geometry Expressions and TI-NSpire

Our aim with this figure is to find the ellipse of maximum area inscribed in an equilateral triangle with the constraint: the principal axis of our ellipse is parallel to one given side of the triangle. To use this figure, we need again to conduct backward reasoning: as the ellipse is given, we have only to find b in order to get the equilateral triangle of minimum area or which is equivalent to get the maximum of the ratio between respectively the area of the ellipse and the area of the triangle constructed for the given value of b.
These two areas are visible in Figure 11 (enlarged in Figure 12), z_0 for the area of the ellipse and z_1 for the area of the triangle.

$$z_0 \Rightarrow \frac{\pi \cdot |b| \cdot \left|2\cdot\sqrt{1-b^2}\right|}{\left|-2\cdot\sqrt{1-b^2}\right|}$$

$$z_1 \Rightarrow \left|\frac{\sqrt{3}\cdot\left(6\cdot b+2\cdot b^3+\sqrt{3+b^2}\cdot\left(3+2\cdot b^2\right)\right)}{3\cdot\sqrt{3+b^2}}\right|$$

Figure 12: Areas obtained with Geometry Expressions

We can copy these two expressions as TI-NSpire inputs. So we paste them in a Note page of TI-NSpire CAS to use the CAS of this software to solve easyly our problem: see Figure 13 left.
The area of the ellipse is pasted as **areaell**(b) and the area of the triangle as **areatr**(b). Then we display the expressions of **areaell**(x) and **areatr**(x) to visualize them as TI-NSpire functions: you can

notice that we have obtained simple expressions (Figure 13 left). We store then the ratio of these two expressions depending on x in $\mathbf{r}(x)$. As this expression contains an absolute value, we store in $\mathbf{re}(x)$ the expression of $\mathbf{r}(x)$ evaluated for $x > 0$ (Figure 13 right). Its derivative is stored in $\mathbf{dre}(x)$; the sign of $\mathbf{dre}(x)$ is the sign of $\sqrt{x^2+3}-2.x$. The software says that this expression is positive for $x<1$ which is easy to prove by hand. Eventually we have proven that the derivative of $\mathbf{re}(x)$ is positive on the interval $[0,1]$ and therefore that function $\mathbf{r}(x)$ is an increasing function on this interval. Finally, the maximum of the ratio we have considered at the beginning of the reasoning is reached for x=1, evaluated as $\frac{\pi\sqrt{3}}{9}$ (Figure 14). As x =1 (**b** =1) corresponds to the circular ellipse, among the ellipses with principal axis parallel to one side of the given equilateral triangle the one of maximum area is the inscribed circle of this given triangle.

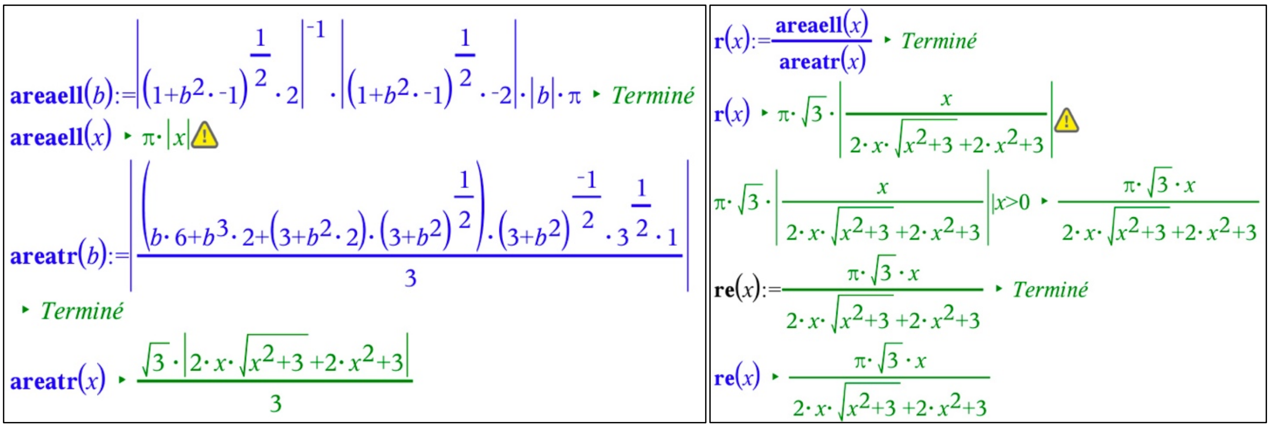

Figure 13: Formal proof with TI-NSpire (beginning)

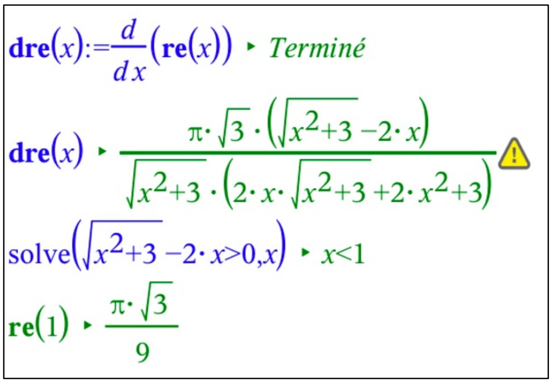

Figure 14: Formal proof with TI-NSpire (end)

3.3. Conclusion for the initial problem of maximum area

Grouping the results obtained in **2.6.** and **3.2.**, we have proved that the ellipse of maximum area inscribed in a given equilateral triangle is its inscribed circle. We have also proved that the maximum of the ratio between the area of an inscribed ellipse and the given equilateral triangle is $\frac{\pi\sqrt{3}}{9}$.
Grouping these previous results with **1.1.** and **1.3.**, we have eventually proved this property of the Steiner ellipse stating that for any triangle this ellipse is the ellipse of maximum area inscribed in the given triangle. We have also proved that the ratio between the area of the Steiner ellipse of a triangle and the area of this triangle is equal to $\frac{\pi\sqrt{3}}{9}$.

3.4. Other possible approaches for the proof of this second problem of maximization

Thanks to the chosen construction of the ellipse at the beginning of our process of proof with Geometry Expressions, this software was able to generate a very simple expression of the area of the equilateral triangle enveloping the given ellipse with respect to parameter b. It is why this last three-line proof is a lot simpler that the one I proposed in my previous paper ([5]). We will see now, two other approaches depending on two other different constructions, one successful and the other unsuccessful which is strange but not really because the algorithm of computation of the software depends on the equations of the different objects of the figure and by the way on the construction chosen for the equilateral triangle.

Second approach: We construct the given ellipse according to the second choice of 1.2.2. and if we conduct a similar reasoning to the one conducted in **3.2.**, we obtain with Geometry Expressions the following expressions for the area of the ellipse and the area of the triangle (Figure 15). Here we have fixed F at (3,0) and the parameter is **b** > 0 (where B(0,b)). It was among the possible values to be fixed, those allowing us to reach the conclusion

$$\mathbf{areaell}(b) := |b| \cdot (9+b^2)^{\frac{1}{2}} \cdot \pi \quad ▸ \ \textit{Terminé}$$

$$\mathbf{areaell}(x) \ ▸ \ \pi \cdot \sqrt{x^2+9} \cdot |x|$$

$$\mathbf{areatr}(b) := \left| \frac{\left(b \cdot 54 + b^3 \cdot 8 + (27+b^2 \cdot 5) \cdot (27+b^2 \cdot 4)^{\frac{1}{2}} \right) \cdot (27+b^2 \cdot 4)^{\frac{-1}{2}} \cdot 3^{\frac{1}{2}} \cdot 1}{3} \right| \quad ▸ \ \textit{Terminé}$$

$$\mathbf{areatr}(x) \ ▸ \ \frac{\sqrt{3} \cdot \left| 2 \cdot x \cdot \sqrt{4 \cdot x^2+27} + 5 \cdot x^2 + 27 \right|}{3}$$

$$\mathbf{r}(x) := \frac{\mathbf{areaell}(x)}{\mathbf{areatr}(x)} \quad ▸ \ \textit{Terminé}$$

$$\mathbf{r}(x) \ ▸ \ \pi \cdot \sqrt{3 \cdot (x^2+9)} \cdot \left| \frac{x}{2 \cdot x \cdot \sqrt{4 \cdot x^2+27} + 5 \cdot x^2 + 27} \right|$$

Figure 15: Beginning of the proof with the second choice of construction of the given ellipse

The end of the proof is rather different from the previous one (Figure 16). The derivative is more complicated than the previous one but easy to manipulate. The sign is positive for $x > 3$ and eventually function $\mathbf{r}(x)$ is increasing for $x > 3$ and by the way the maximum of the ratio is reached when x goes to infinity which mean that the ellipse is a circle where F and F' can be considered as superimposed. Eventually, the final result is obtained by a different path: the ellipse of maximum area we want to find is the inscribed circle of the given equilateral triangle.

$$\mathbf{dr}(x):=\frac{d}{dx}(\mathbf{r}(x)) \blacktriangleright \textit{Terminé}$$

$$\mathbf{dr}(x)|x\geq 0 \blacktriangleright \frac{9\cdot\pi\cdot\sqrt{3}\cdot\left((x^2+27)\cdot\sqrt{4\cdot x^2+27}-2\cdot x^3\right)}{\sqrt{(x^2+9)\cdot(4\cdot x^2+27)}\cdot\left(2\cdot x\cdot\sqrt{4\cdot x^2+27}+5\cdot x^2+27\right)^2}$$

$$\text{solve}\left((x^2+27)\cdot\sqrt{4\cdot x^2+27}-2\cdot x^3\geq 0,x\right) \blacktriangleright \text{true}$$

$$\text{expand}\left(\left((x^2+27)\cdot\sqrt{4\cdot x^2+27}\right)^2-(2\cdot x^3)^2\right) \blacktriangleright 243\cdot x^4+4374\cdot x^2+19683$$

$$\lim_{x\to\infty}(\mathbf{r}(x)) \blacktriangleright \frac{\pi\cdot\sqrt{3}}{9}$$

Figure 16: End of the proof with the second choice of construction of the given ellipse

Third approach: if we construct now the given ellipse according to the first choice of 1.2.2. and if we use similar reasoning to that used in **3.2.**, we fail because the expression of the area provided by Geometry Expressions is really too complicated to be treated by TI-NSpire. We find an expression containing ***i*** ($i^2 = -1$) and even if we simplify the expression by hand, the derivative of the ratio is really complicated. See the following screenshot to agree with this remark (Figure 17)

$$\frac{d}{dx}(\mathbf{r}(x))$$

$$\blacktriangleright \frac{-5.4414\cdot x\cdot\left|\sqrt{x^2-1.}\right|-\frac{2.7207\cdot x\cdot(x^2-4.)\cdot\text{sign}(x^2-1.)}{\left|\sqrt{x^2-1.}\right|}}{\left|\sqrt{4.-x^2}\cdot\sqrt{1.-x^2}-x^4+6.5\cdot x^2-10.\right|}-\frac{10.8828\cdot x\cdot\left((x^2-4.)\cdot(x^2-3.25)\cdot\sqrt{-(x^2-1.)}+0.5\cdot(x^2-2.5)\cdot\sqrt{4.-\cdots}\right.}{\sqrt{-(x^2-1.)}\cdot\left(\sqrt{4.-x^2}\cdots\right.}$$

Figure 17: A really very complicated form of a derivative

4. The Fermat point of triangles enveloping a given ellipse

4.1. Reminder about the Fermat point (of a triangle where all angles are less than $\frac{2\pi}{3}$)

The Fermat point of triangle MNP (Figure 18 left) is the point minimizing the sum of the distances to M, N and P. The construction used here is the intersection point of the three circumcircles to the three equilateral triangles out of MNP and based respectively on each side of the triangle.

Why, during the previous work, have I the idea to focus my attention on the Fermat point? The answer is simple: during the construction of all equilateral triangles enveloping a given ellipse, I have noticed the role of angle $\frac{2\pi}{3}$. In Figure 4 on the left, for any triangle of minimum area, focus F could be interpreted as the Fermat point of triangle UVW where U, V and W are the symmetric points of F with respect to each side of the equilateral triangle enveloping the given ellipse. Known as the L'huillier problem ([3]), we know the ratio between the area of triangle UVW and the equilateral triangle and especially we know when these areas are equal: when F is the centroid of the equilateral triangle or when F belongs to the circle centered at the centroid and whose radius is the radius of its circumcircle multiplied by $\sqrt{2}$ which is here never the case.

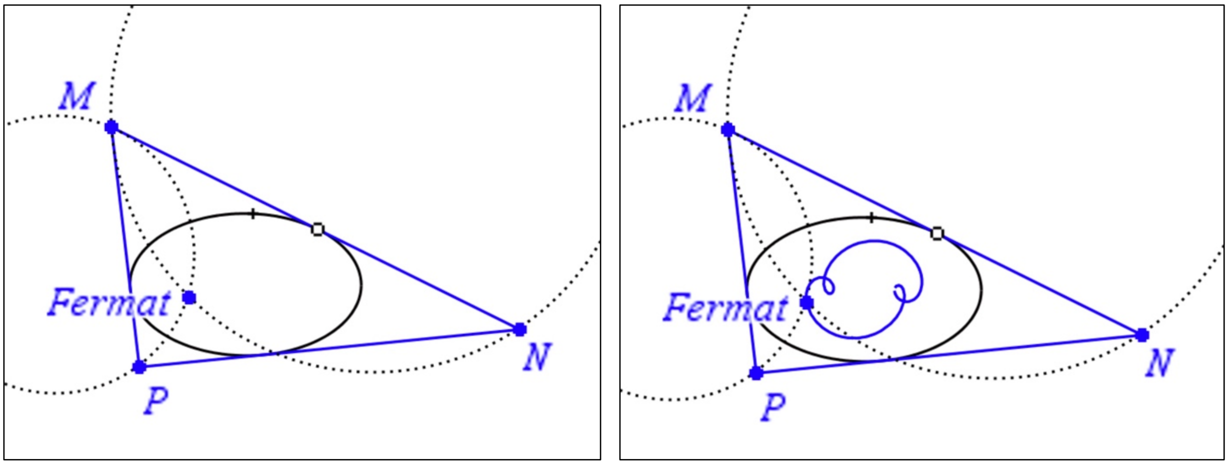

Figure 18: Relation with the Fermat point

4.2. Loci in relation with the Fermat point

When the construction is performed under TI-Nspire and when the triangles MNP are the triangles of given shape enveloping a given ellipse we obtain the blue curve displayed in Figure 18 right which is the locus of the Fermat points of the triangles enveloping the given ellipse It is possible to conduct an interesting investigation in changing the shape of triangle MNP.

Another paper could be focused on the possible links between the Fermat point and the Steiner ellipse.

5. Conclusion

This paper proposed a simpler proof than the last known proof of a special property of the Steiner ellipse. The originality of this proof supported by technology consists in three points:

-The use of backward reasoning to tackle the problem.

-The use of Geometry Expressions which is a formal dynamic geometry software to generate simple equations and the use of the CAS of TI-NSpire in which we have pasted and treated these equations.

-The use of the most appropriate construction of all ellipses of given shape to simplify the expressions generated by Geometry Expressions.

Above all, the proof I have built was built in following step by step the constructions of the investigations conducted to reach the expected property. It means that we were successful in bridging the experimental stage to the conjecture and the stage of proof as usually, it is known that there is a gap between these two stages.

As said in the conclusion of my previous paper, one of the powers of digital tools used for this paper (especially Geometry Expressions and TI-NSpire) is to allow us to revisit known problems and

known results to investigate differently and get new proofs and sometimes new results. It is exactly what we did in this work.
A little part of my proof was under the responsibility of the CAS (in 2.5.) and another work in the future would aim to give a formal proof by hand of this part.

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[3] DAHAN J.J., 2005, *La démarche de découverte expérimentalement médiée par Cabri-géomètre en mathématiques,* PhD thesis, Université Joseph Fourier, Grenoble, France http://tel.archives-ouvertes.fr/tel-00356107/fr/
[4] MINDA D., PHELPS S., 2008, Triangles, Ellipses and Cubic Polynomials, in The American Mathematical Monthly, Volume 115, 2008-Issue 8, pp 679-689.
[5] DAHAN J.J., 2019, Steiner Ellipse and Marden's Theorem, in Proceeding 24th Asian Technology Conference in mathematics, pp 1-13

YouTube videos (links)

[1'] Playlist of YouTube channel « jjdahan »: PRESENTATION JJ DAHAN T3 DALLAS 2020
https://www.youtube.com/watch?v=P-5v4RPQPEk&list=PLOIs4xavv0zEJgGxY-_EqJ32PCr9CImug
[2'] Playlist of YouTube channel « jjdahan »: Une propriété de l'ellipse de Steiner
https://www.youtube.com/watch?v=SwQ8iFdA5YE&list=PLOIs4xavv0zFCmU_ANuf2IPCdIQfoV3Ob

Software

Cabri 3D by Cabrilog at http://www.cabri.com
Geometry Expressions by Saltire Software at https://www.geometryexpressions.com
TI-Nspire™ CX CAS Premium Teacher Software at https://www.ti.com