GeoGebra Reasoning Tools for Humans and for Automatons

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Abstract

We present two recent tools, integrated in the dynamic mathematics program GeoGebra, for automated proving and discovering in elementary geometry. First of all, the GeoGebra Discovery module, with the Relation, Prove, ProveDetails and LocusEquation commands. They are for humans because it is a human who must introduce the objects the human person wants to Relate, the thesis the human wants to Prove or the missing hypotheses the human wants to discover with the LocusEquation command. Moreover, we will describe some tools we have developed within GeoGebra for automatons, such as the Discover(X) and the WebDiscovery. We conclude the paper with reflections on the pros and cons and on the potential impact of these reasoning tools in the educational world.

Automaton: from the Greek word *automatos*, meaning "a mechanism that is relatively self-operating. . . Plural: automatons or automata"[1]. On the other hand, a geometer is "a specialist in geometry"[2]. Therefore, a "geometer automaton" is a mechanism that successfully performs, in a self-operating mode, the tasks that human geometers are expected to accomplish: to conjecture, prove, refute, discover. . . geometric properties!

In this context, our main goal is to present here our on-going work towards the implementation of a "geometer automaton" (in short: *GA*) module within the dynamic mathematics program GeoGebra[3], a freely available software with millions of contributors and users at high schools and universities, all over the world, thus conferring our proposal with a potential impact beyond the specific research scientific community limits.

In the next Section we will summarily describe the already implemented commands, allowing the exploration, by humans, of geometric tasks by using GeoGebra as a kind of "symbolic geometry calculator": the user poses a concrete geometric task and GeoGebra provides a mathematically sound answer. This is implemented on GeoGebra Discovery[4] available in two options:

[1] `https://www.merriam-webster.com/dictionary/automaton`
[2] `https://www.merriam-webster.com/dictionary/geometer`
[3] `http://geogebra.org`
[4] `https://github.com/kovzol/geogebra-discovery`, `http://autgeo.online/geogebra-discovery/`

GeoGebra Classic 5, for Windows, Mac and Linux systems; and GeoGebra Classic 6, made for starting it in a browser, available mainly for tablets and smartphones.

Then, Section 2 presents the on-going work towards the construction of GA[5], a self-operating geometer (i.e. without human intervention) within GeoGebra, that is, a module that will allow GeoGebra to automatically formulate, following some heuristics, different geometric conjectures within a given geometric construction, and to confirm or deny them using internally tools similar to those in GeoGebra Discovery. In summary, we are developing automated reasoning tools (in short: ART) on GeoGebra that can be used either by humans or by GeoGebra, standing alone by itself.

The evident potential impact of such tool makes necessary to reflect on the pros and cons, in different contexts, of developing a GeoGebra GA. Thus, as we understand this paper would be read only by (if any) human beings, the second aim of our contribution here is to describe some guidelines about how humans/automatons should interact in this framework, and what could be the potential benefits (for humans) of having at hand a loyal, devoted, friendly geometer automaton, albeit quite introverted. This will be addressed in the final Section, ending with the description of some future research lines, as well as with reflections of the educational relevance of these reasoning tools.

1 GeoGebra automated reasoning tools (for humans)

It all started with... the *Relation* tool! In fact, the current version of *Relation* is a symbolic extension of the command of the same name that has existed, since 2002, in GeoGebra, but in a purely numerical way [19]. That is, in the traditional version of this command, the user first called the *Relation* command and pointed to two specific geometric objects, obtaining in response the existence of certain relationships between them, such as perpendicularity, parallelism, equality or incidence[6], being warned in a message that the conclusion obtained was only *numerically valid.*

On the contrary, in the more recent versions of GeoGebra, the *Relation* command responds by adding an additional button in that same message, labeled "More...". By pressing it, certain symbolic calculations are started via the GeoGebra Automated Reasoning Tools system we have been developing [18], [21], which translates the geometric figure in terms of polynomial equations and then considers, from a programmed collection of possible relationships holding between the two input objects, whether each of the proposed relations holds or not. This is addressed selecting (by using certain heuristics) an automated proof method [5] from the available set of built-in provers[7], the one which is considered as the most appropriate to decide if the thesis holds or not, with mathematical rigor.

Figure 1 shows two instances of the use of the *Relation* tool, analyzing the relationship between the symmetric of vertex A with respect to the midpoint of the opposite side, and the line defined by the circumcenter and the symmetric of the orthocenter O with respect

[5] `http://autgeo.online/ag/`, `https://github.com/kovzol/ag`

[6] See `https://wiki.geogebra.org/en/Relation_Command` for a full list.

[7] The current version of GeoGebra can choose internally between applying a) the Gröbner basis method, b) Wu's method of characteristic set, c) the area method, or d) the so-called "Recio method" of exact verification, but through testing a certain finite number of cases, [34].

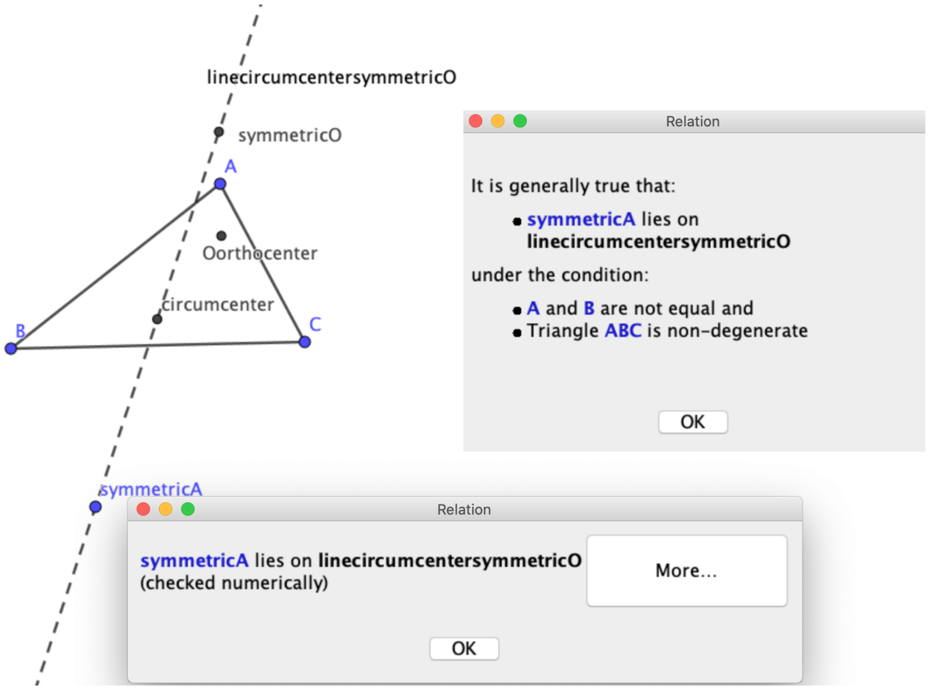

Figure 1: Example 230 of Chou [9], running in GeoGebra Classic 5. *Relation* is asked to examine if there is any geometric property holding between *symmetricA* and the straight line *linecircumcentersymmetricO*. Both the approximate and the exact answer provided by *Relation* are displayed.

to A. Initially, *Relation* numerically confirms that A lies on this line. Then, after clicking in "More...", it carries out the rigorous proof, yielding that *symmetricA* belongs to the *linecircumcentersymmetricO*, except in some intuitively degenerate instances.

Other commands from GeoGebra ART, such as *Prove, ProveDetails*, work in a similar way, but here it is the user who must suggest a conjectured thesis concerning some elements of a construction (e.g., that some point always lies on a certain line), obtaining in response the truth or falsity of the conjecture and, in the affirmative case, providing some additional geometric conditions that must be verified in order for the given statement to be generally correct. These are the so-called "non-degeneration conditions", which usually prescribe that certain input objects (for example, the vertices of a freely defined triangle) should not match or align etc., for the conjectured relationship to be true.

To illustrate these features, Figure 2 deals with the same example 230 from Chou, but here with the command *ProveDetails*, that requires the user to introduce a concrete conjecture (here, that point *symmetricA* belongs to the line *linecircumcentersymmetricO*). The answer that appears in the Algebra View has been enabled in the Graphic View as well: it confirms the truth of the conjecture, except for degenerate cases, such as the collinearity of A, B, C, collapsing the given triangle, etc..

On the other hand, Figure 3 addresses a similar situation but from a very different perspective. Imagine we repeat the previous construction, but forgetting where to place the center of symmetry to compute "symmetric A". Instead, a certain point D has been freely chosen

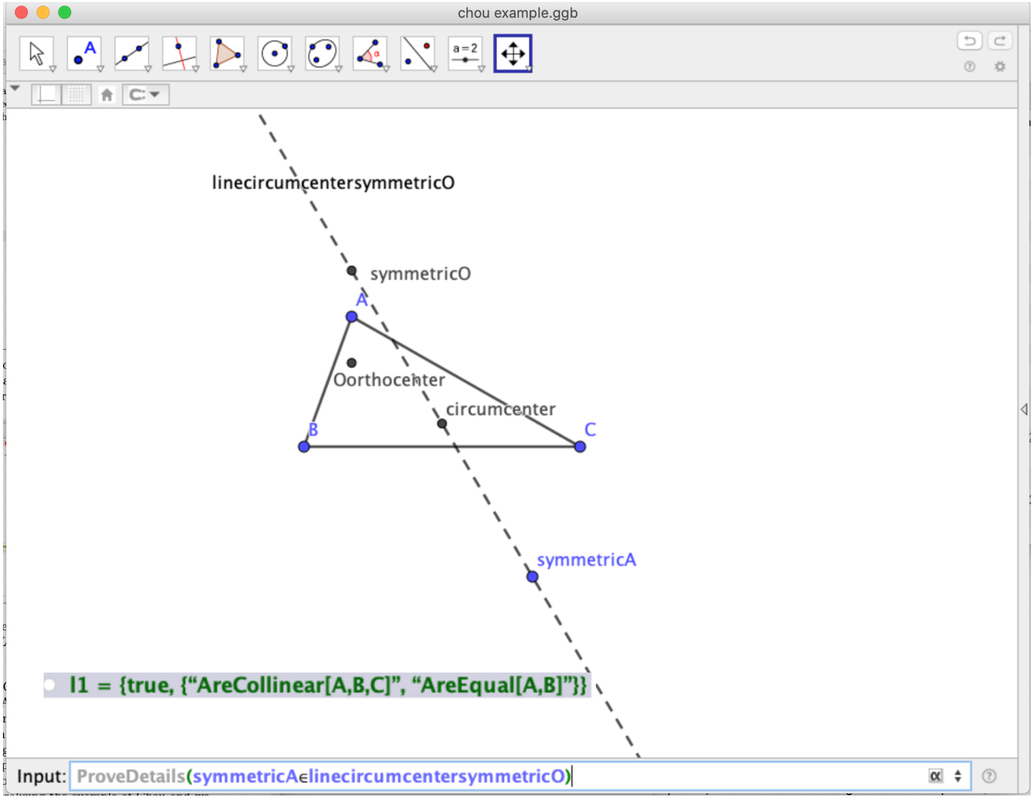

Figure 2: Chou's Example 230 again, but here with the command *ProveDetails*, after introducing the conjectured thesis in the Input bar. It outputs the truth of the statement except for some degenerate cases.

and *symmetricA* in this case is the symmetric of vertex A with respect to point D (not with respect to the midpoint of side BC as in the previous two Figures). Obviously, for an arbitrary D, the point *symmetricA* is no longer aligned with the *circumcenter* of the triangle and the symmetric of the orthocenter O with respect to vertex A. We could say, then, without imposing some constraint on D, that the statement is generally false.

To deal with such cases (a given thesis, but some missing hypothesis for the statement to hold true), GeoGebra ART provides the *LocusEquation* command, that allows us to find that it is necessary to locate D on a certain line a (the parallel to *linecircumcentersymmetricO* through the midpoint of BC) for the alignment of *symmetricA*, *symmetricO*, *circumcenter*. In this way *LocusEquation* generalizes the example of Chou (that placed point D just on the middle of side BC), allowing us to actually discover a new theorem, as illustrated in Figure 4.

Finally, we recall that in the current version of GeoGebra the user does not obtain a visible or readable proof of the validity of the statement that has been verified using the ART tools. GeoGebra Discovery, in some sense, behaves as a kind of "oracle" that provides no further information except the certainty of correctness of the response. However, it must be emphasized that the result obtained by GeoGebra Discovery is based on a mathematically correct proof, which is philosophically a higher-level truth and completely different from that achieved by *numerically approximate* verification of the *frequent* validity of a thesis on a collection of

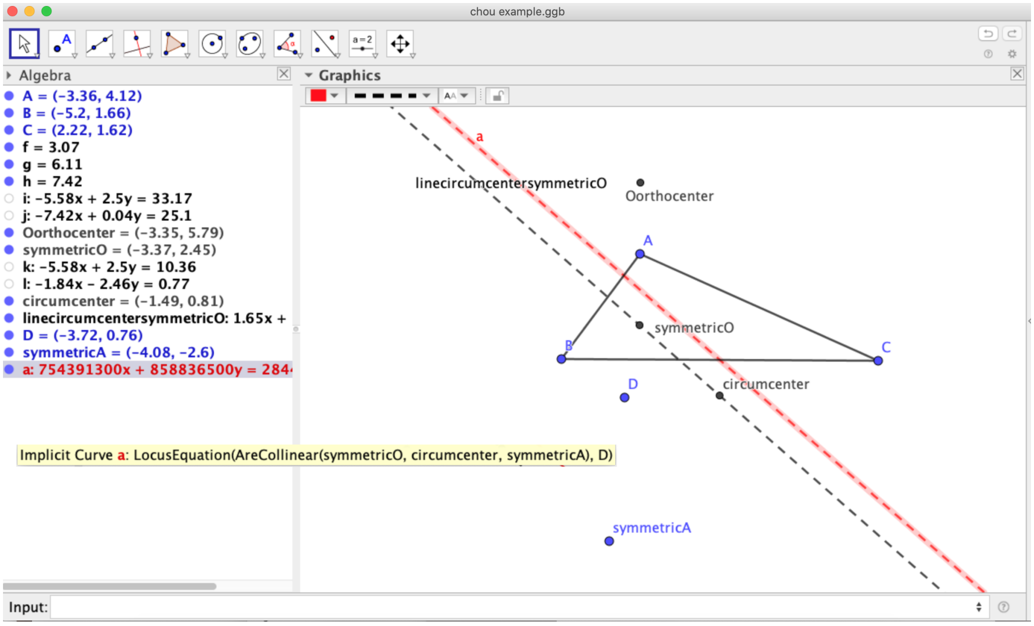

Figure 3: Discovering a new theorem: Point D, the center of symmetry for A, has been freely chosen to make the construction. Then, the *LocusEquation* command suggests us how to generalize Chou's example 230, requiring that D is positioned on line a for Chou's example thesis to hold.

particular cases[8] obtained by dragging the given construction to various positions, which has been the classic way of working in dynamic geometry systems to date.

2 GeoGebra reasoning tools (for automatons)

As we have indicated, in a certain sense, the ART tools we have described so far are a kind of wise tutor, an oracle, but one who must be interrogated by the human user. On the other hand, a *geometer automaton* should be able to autonomously observe and investigate the geometric properties of a figure. This is already done in GeoGebra Discovery, through the *Discover* command, but only around a point chosen to center the discovery task.

An example in this direction is shown in Figure 5, which presents three screenshots of a mobile phone. On the top, the initial construction, a simple square is shown, on which the command *Discover(B)* is executed. This command searches, automatically and combinatorially, within a large series of possible geometric relationships between the elements of the figure in which point B is included, and then verifies the truth or falsity of each of them. In the center of the figure a message with the result after launching the command is shown. Finally, at the bottom, the visualization—also produced automatically by GeoGebra—of the properties is obtained.

[8]It should be emphasized that the traditional method has two essential differences with the test method through a finite number of cases that we have previously mentioned in footnote 7: in the traditional approach the cases are chosen randomly, and the verification of each case is approximate.

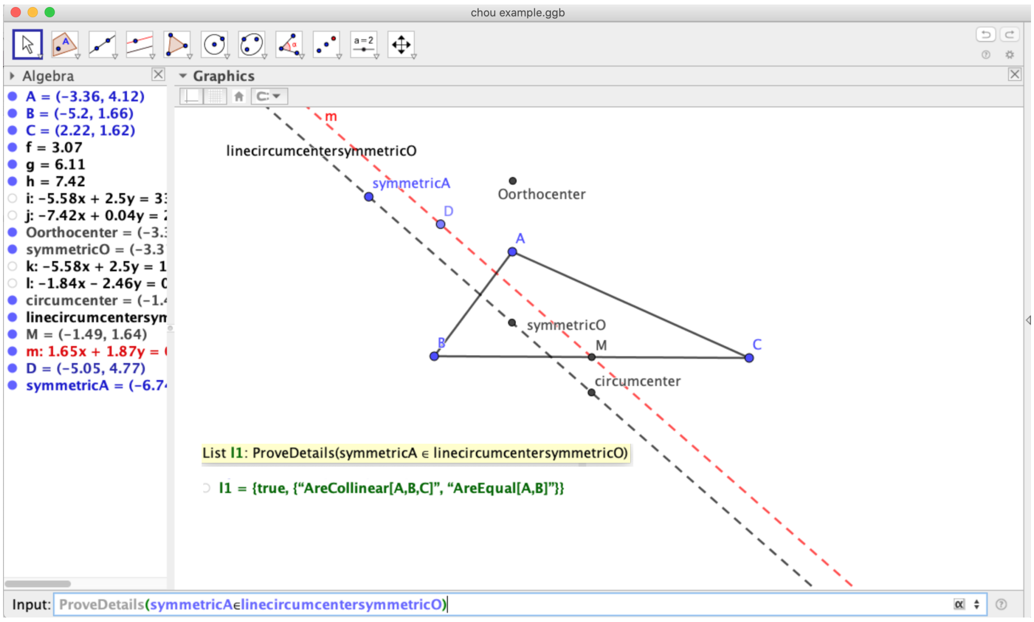

Figure 4: Proving the new theorem: If point D is placed on the line parallel to the line $linecircumcentersymmetricO$ passing through the midpoint M of side BC, then the symmetric of vertex A with respect to D is on the $linecircumcentersymmetricO$.

But we have already implemented in GeoGebra a much more general way for a truly *geometer automaton* to obtain relationships in a geometric construction. The tool *GA* starts from a given figure to which, in a systematic and programmed way, various elements will be automatically added. For example, if the given figure is a triangle with the midpoints of the sides, then the added elements are the lines that connect the vertices with these midpoints[9]. After this the *geometer automaton* produces a collection of propositions related to collinearity, parallelism, etc., between the various existing elements (originally or added) in the figure—this collection is created in a combinatorial way—but the list of possible propositions can be controlled by the user, to avoid exhausting resources, e.g. computer memory. This conjecture generation protocol has been inspired by the heuristics collected in [16, p. 44]. Finally, the mechanical geometer applies the tools of automatic reasoning to verify the certainty or falsity of the obtained propositions.

An example use of *GA* is the “9-point circle”, which can be seen in Figure 6. An arbitrary triangle ABC is given with midpoints of the sides, denoted by D, E and F, and the orthocenter G, and the feet of the heights H, I and J. In addition, the midpoints between the vertices and the orthocenter, denoted by K, L and M are included. Finally, the circumcenter N is defined as the intersection of two perpendicular bisectors of the sides.

The idea is that *GA* should find—without further human help—the theorem that establishes the equalities

$$ND = NE = NF = NH = NI = NJ = NK = NL = NM, \tag{1}$$

[9]These steps are currently fixed in the program, but in a future version they may be controlled by the user.

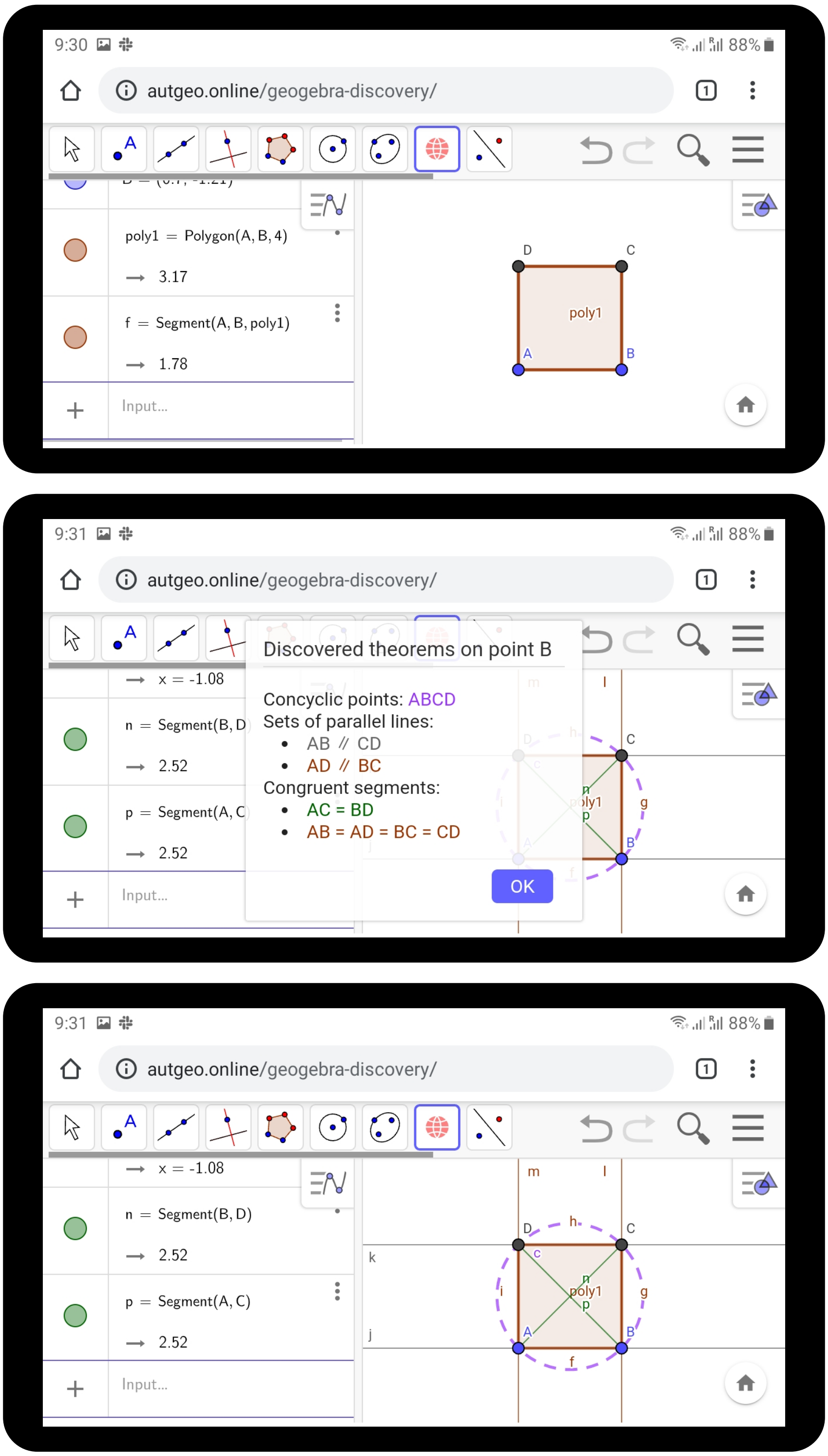

Figure 5: Output of command *Discover(B)* in *GeoGebra Discovery*.

that is, that points D, E, F, H, I, J, K, L and M are concyclic and the center of the corresponding circle is N. This well-known theorem was discovered by Brianchon and Poncelet in 1821, and is called *nine-point circle* or Feuerbach's circle[10].

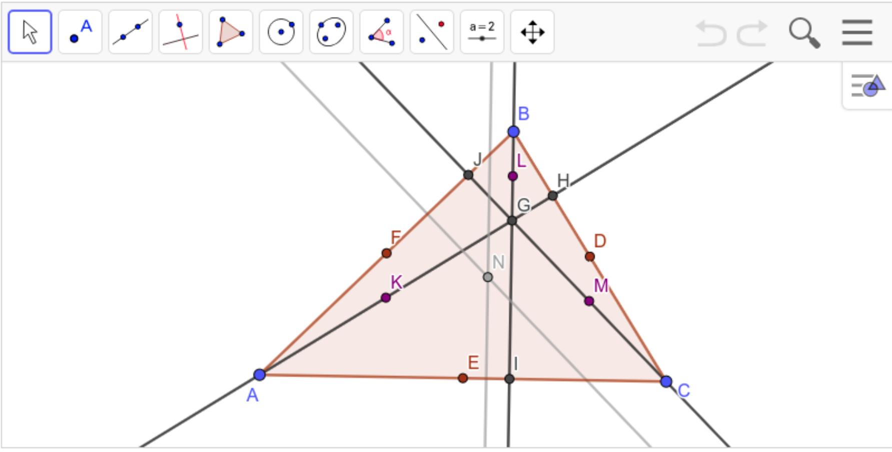

Figure 6: Input of GA to obtain the nine-point circle theorem.

Let us select this theorem in the catalog of sample examples that are shipped with GA, after loading the application with the web link `http://autgeo.online/ag/automated-geometer.html?offline=1`. Now Figure 6 appears on the screen. At this point the user has to select some relationships from a list of options in a menu of possibilities that appears below the construction—these relations will be explored by GA. Let us assume, for example, that we choose *equality of distances between two points* and that we press the *Start discovery* button. GA then produces, in less than 26 seconds, on a laptop[11], a list of 129 "theorems", among which are the equalities of (1), which are part of the results 64–71, 84–90, 100–105, 114–118, 120–123, 124–126, 127–128 and 129, as listed in Figure 7.

Note that GA automatically represents those segments with the same color that have the same length. This makes it easier for the user to discover some other, perhaps unexpected results like the equalities

$$AK = GK = IK = JK, \tag{2}$$

which mean that the points A, I, J and the circumcenter G are concyclic!

3 The future...

Clearly, the list of results provided by GA, as in the Figure 7, is repetitive, requiring $8+7+6+\ldots+1 = 36$ equalities to express the 8 statements of formula (1). Simplifying the output of GA is, thus, one of the pending work topics for the immediate improvement of our "geometer automaton".

[10] `https://en.wikipedia.org/wiki/Nine-point_circle`

[11] For example, on a Lenovo ThinkPad with 8 cores, Intel(R) Core(TM) i7-8550U CPU @ 1.80 GHz, with 16 GB RAM (of 2018).

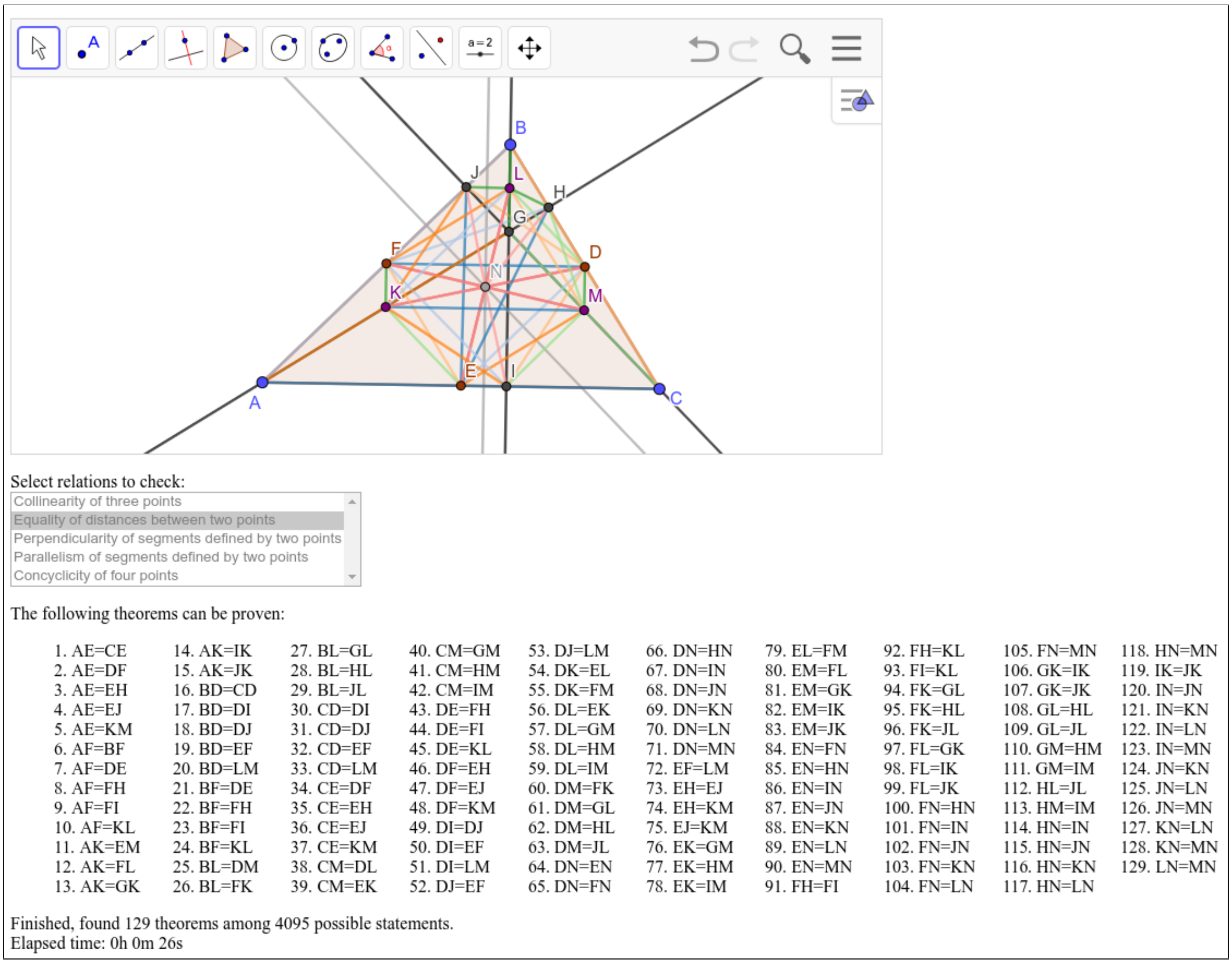

Figure 7: The nine-point circle, discovered by *GA*.

It is also apparent that several of the *GA* results are simple trivial reformulations of the starting construction data. For example, the first equality in the Figure above, $AE = CE$, is evident, since, by definition, E is the midpoint of segment AC. Other results, such as the second equality $AE = DF$ (which expresses that the length of the mean parallel is half of the corresponding side) can be considered more or less obvious, depending on the user's level of geometric knowledge.

Thus, another pending task is to find an algorithm that quantifies the interest of a result for a certain category of users—this may be based on some measure of complexity (perhaps specific to the context of dynamic geometry, [37]) of the hypotheses and theses, and of their comparative weight. This work has already been started by other research groups in the field of *Big Data*, using the results of our geometer automaton *GA* as a source of examples whose relevance must be discriminated (see the discussion in [13], [14], [15]).

There is further on-going work on improving other automatic reasoning tools in GeoGebra. For instance, the Classic 5 version of GeoGebra already offers a way to make conjectures on geometric inequalities, and to prove them. In general, we are working towards implementing new tools to prove/discover aspects of elementary geometry that make sense or hold only over the real numbers. Implementation is currently in an advanced stage of development—it connects GeoGebra with the free version of Mathematica on Raspberry Pi, [1, 38].

But perhaps the potentially most complex pending problem, as well as the one with the greatest impact, is the study of the application and consequences of having automatic reasoning tools, a *GA*, of our "symbolic geometric calculator", of our intelligent geometry book, a "i-geometrybook" (using the terminology of [11]), ready to be used in the educational context.

When, in the above paragraph, we refer to the human or automaton geometer using GeoGebra Discovery as dealing with a "symbolic geometry calculator" we mean, first, that we are handling an electronic device, like a calculator. And as we argue below, we consider there is an analogy, for the good or for the bad, between the performance of these GeoGebra Discovery commands and the performance of a pocket "calculator". On the other hand, we emphasize that the output of these automated reasoning commands can be labeled as close to those of a "symbolic" calculator, as the answer they provide is not numerically approximate or probabilistic, but mathematically accurate, since they are based on algebraic elimination techniques, following the pioneer work of Wu [40] and others, described in the foundational book of Chou [9]. These methods are not only fully rigorous but yielding answers that are also valid on the synthetic geometry realm, as shown in [8], describing the formalization of the arithmetization of Euclidean plane geometry, and stating that

> . . . The arithmetization of geometry paves the way for the use of algebraic automated deduction methods in synthetic geometry. Indeed, without a "back-translation" from algebra to geometry, algebraic methods only prove theorems about polynomials and not geometric statements. However, thanks to the arithmetization of geometry, the proven statements correspond to theorems of any model of Tarski's Euclidean geometry axioms.

It is well known that such methods are extremely successful in that they can deal with highly difficult geometry problems, as recognized by authors that work towards similar goals, but with a different approach (logic, Artificial Intelligence, machine learning, etc.), declaring (see [2]):

> Automated geometry theorem proving (consisting of several techniques such as Wu's method, Gröbner basis method, . . .) is one of the most successful areas of automated reasoning.

In particular, our GeoGebra ART reasoning commands are also based on this algebraic approach, using some specific algorithms that have been developed by the authors of this paper and their research team colleagues along the past twenty years (e.g. [35], [10], [36], [23], [24]), and that have already shown their ability to deal with highly complicated problems. See, for instance, the solution—using the underlying algorithms, but with human interaction, before its implementation in GeoGebra—to a problem posed by the Spanish Royal Mathematical Society with the occasion of its centennial celebration, [25], or the diverse generalizations of the Steiner-Lehmus Theorem ([26], [27], [28], [30]).

Moreover, the already implemented GeoGebra Discovery commands have already successfully addressed many complicated statements, far above the standard of math graduate students, cf. a large benchmark here: `https://tinyurl.com/provertest`. As an impacting example, we can highlight the automated solution of a problem from the Spanish Civil Service examination to become a Math Teacher at a public High School [22]. In our ATCM presentation we will show how GeoGebra is able to solve problems of explicitly declared high level of difficulty, such as the ones collected in the Appendix of the recent Master thesis on "Automated Generation of Planar Geometry Olympiad Problems" [4]. As stated in [31], IMO (International Mathematical Olympiad) problems provide a clear challenge for automated provers:

> ... Writing an automated theorem prover (ATP) that could solve a large portion of IMO problems is a challenge recognized in the field of artificial intelligence and could potentially lead to strong ATPs in general.

This bright panorama we have summarily sketched, on the behavior of GeoGebra towards achieving a "geometer automaton", has some dark counterparts. It partially accomplishes and partially diverges from the declared goals and methods (close to AI) in the invited talk by H. Fu, J. Zhang, X. Zhong, M. Zha and L. Liu at the past 2019 Asian Technology Conference in Mathematics [12]: "We aim to develop a mathematics robot for automated solving of elementary mathematics problems just like human being, which can automatically answer both standard and non-standard questions *with the input of text in natural language and the output of human readable solutions*[12]." Unfortunately, this is not the case for GeoGebra automated reasoning commands.

Indeed, as already remarked in their article, our approach is based on what the authors of [12] describe, first, as a consequence of the work of "... pioneering mathematicians (who have) made profound progress", mentioning Tarski, Wen-Tsun Wu and Z. Zhang, and then concluding that:

> Stimulated by them, many new methods appeared such as the resultant elimination method, Gröbner basis and so on. These methods are called algebraic elimination methods, *they are however hardly comprehensible and not intuitive enough for humans to understand*[13].

In fact, in close similarity to a pocket calculator, GeoGebra ART provides quickly answers that are little informative about the way they have been elaborated. Indeed, GeoGebra ART, very roughly speaking, provide usually little bit more than a yes/no response to the posed question (e.g. "it is true that this line is parallel to this other one, except in this list of degenerate cases...") or a list of relations between two geometric objects pointed out by the user (e.g. "these two lines are parallel") or the formulation of an extra hypothesis needed for the truth of a user-conjectured property (e.g. "place point C in this circle if you require these two segments to be perpendicular").

But..., who cares, who understands, how a calculator finds out that 1234 times 5678 is equal to 7706652? This conflictive reflection deserves to be argued in some specific paper and, more important, to be analyzed through different experiments, such as the a basic one we

[12]The italics are ours.

[13]The italics are ours.

have conducted in [33], on the use of GeoGebra ART with future math teachers. See also the Conclusion section at [17] for further details, references and thoughts on this issue!

For this reason, following the reflection that we have recently discussed in [17] or [39], we believe that it is essential to make a call to the educational community about the need to seriously consider the role of these new tools of automatic reasoning in the teaching of geometry. Could they contribute, embedded in some kind of tutoring–agent program that provides the didactical interface, helping students to develop reasoning skills? Will the dissemination and accessibility of these tools mean that students will be no longer consider elementary geometry as the most relevant field in mathematics for developing such skills, getting tired about competing with a "pocket calculator" in the discovery of geometric facts? Some of the foreseeing changes could be similar to those that the emergence and dissemination of pocket calculators have (or should have) already implied for mathematics education, not only about how certain problems have to be solved, but, more generally, concerning what type of techniques and problems should be considered as the true objective of mathematics education in the middle of the Digital Age.

In summary: this is the challenge we believe is most urgent and important to ask the reader to think about!

4 Acknowlegements

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