Transfigurations of Polyhedra

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Abstract: *In this study, we will present animations in the form of short video clips showing how polyhedron A can be changed in appearance to polyhedron B, where A, B are particular types of Catalan solids, Archimedean solids, or Platonic solids. Since each such transfiguration is in one to one correspondence with the nets of one specific polyhedron, and since the algorithm for generating all such nets is yet to be developed, the author would welcome your feedback on improvements in making a more efficient video production.*

Introduction

The starting concept for this paper is straightforward. Start with a polyhedron and build a pyramid on each face with the polyhedron's face as its base. Described here operation, we will call a transfiguration of a polyhedron. The result of this operation is again a polyhedron. This simple process brings us to two very interesting questions:
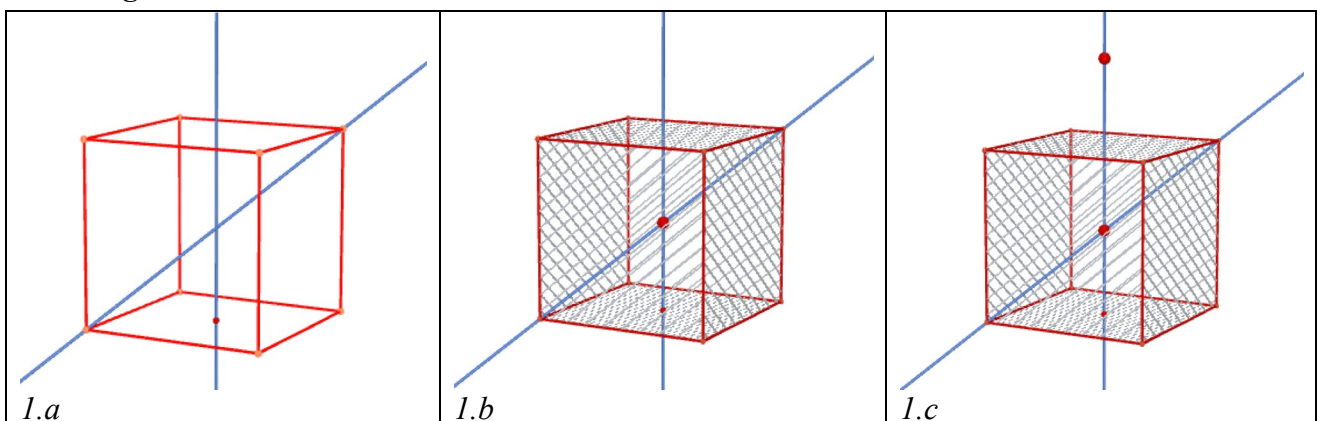
1. How can one create transfiguration of a polyhedron using dynamic geometry software?
2. What will we get depending on the type of starting polyhedron and the way how the transfiguration was achieved?
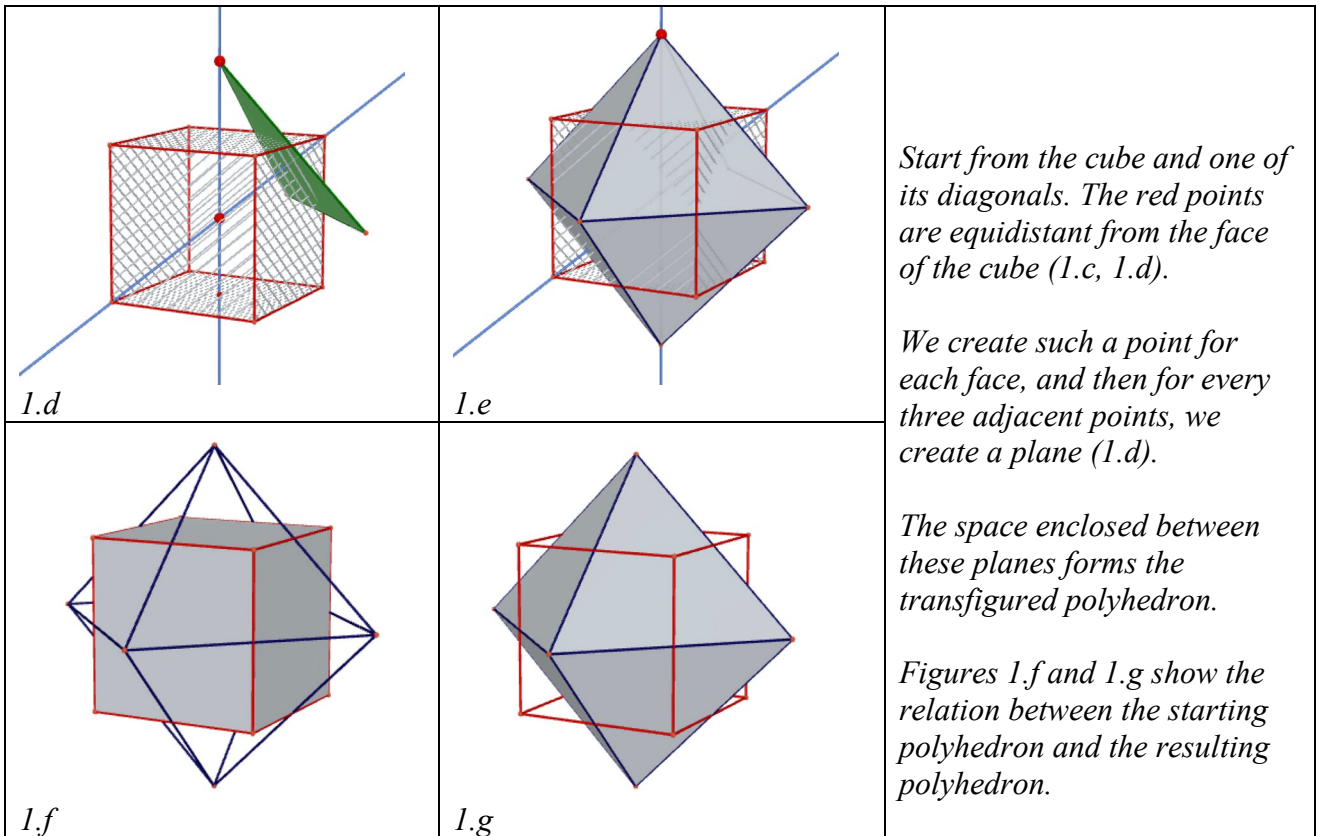
In solid geometry, we have several examples of transfigurations. One such example are the stellations of polytypes. Stellations were researched, and their properties were described in several papers and books (see, for example: [1] and [2]). Wenninger (see [3]) presents a particular and relatively uncommon way of visualizing dual polyhedra, i.e., Archimedean polyhedra and their duals.

In this paper, we will mostly concentrate on a specific type of transfiguration of Archimedean polyhedrons. We will request that, in the obtained polyhedron, adjacent faces of two different pyramids lie on a common plane. This way, we will get Archimedean duals, so-called Catalan solids.
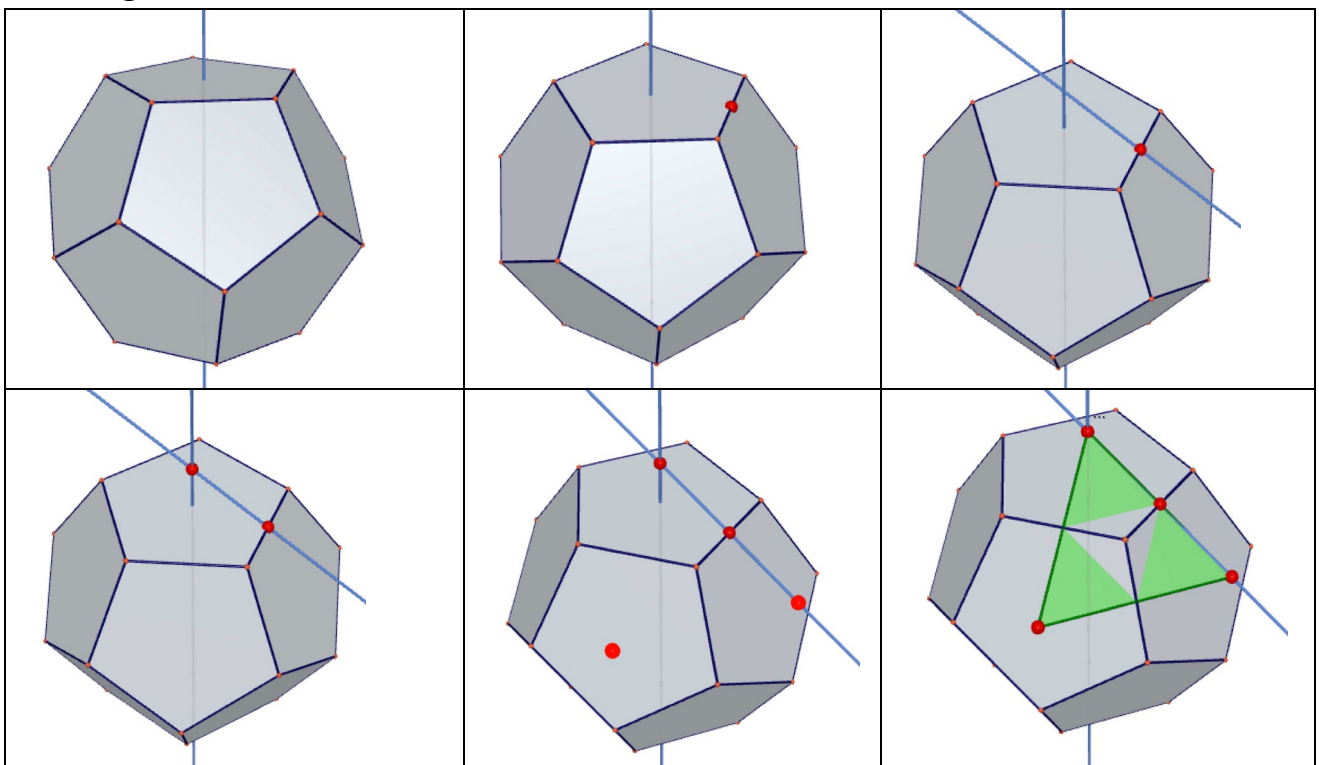
Constructing techniques with Cabri 3D

Transfiguration of the cube

1.a   1.b   1.c

1.d

1.e

Start from the cube and one of its diagonals. The red points are equidistant from the face of the cube (1.c, 1.d).

We create such a point for each face, and then for every three adjacent points, we create a plane (1.d).

The space enclosed between these planes forms the transfigured polyhedron.

Figures 1.f and 1.g show the relation between the starting polyhedron and the resulting polyhedron.
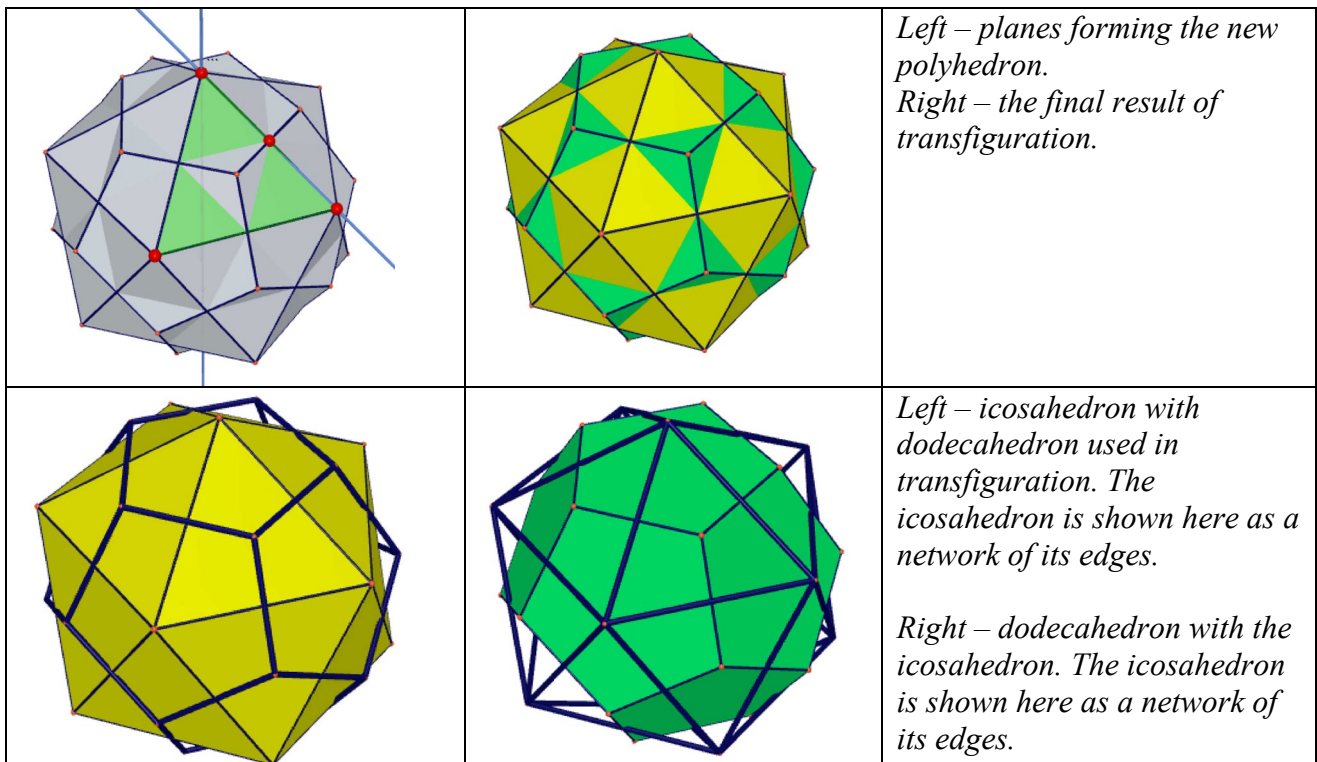
1.f

1.g

Transfiguration of the dodecahedron

This series of drawings shows step by step of creating one of the planes used in the transfiguration of dodecahedron into an icosahedron.

		Left – planes forming the new polyhedron. *Right – the final result of transfiguration.*
		Left – icosahedron with dodecahedron used in transfiguration. The icosahedron is shown here as a network of its edges. *Right – dodecahedron with the icosahedron. The icosahedron is shown here as a network of its edges.*

These two examples are enough to understand how transfiguration works. More examples and discussion will be presented during the ATCM 2020 lecture.

Complete list of resources for this lecture

The enclosed link is the entry point to all works related to this lecture:

https://drive.google.com/drive/folders/1OfxWNpb9JQDq-Laee_pk8Ly0SsjsDnFM?usp=sharing

Everything under it is open to the public, i.e., anyone can download files there.

The subdirectory "sorted" has the link:

https://drive.google.com/drive/folders/1WzwFKF67hdHwNMNCeKTaEwio-vsGPo4S?usp=sharing

In the future, I'll try my best to collect anything related to the same .mov file in the same subdirectory. For example, "Transfigurations" has the link

https://drive.google.com/drive/folders/1I6u_653520TcSpWzGHBOzqAqUqZrXtWG?usp=sharing

The directory "pdf" has the link:

https://drive.google.com/drive/folders/16wWA46ouAEukzxeILeJ_caf7npgRZQLS?usp=sharing

All content presented there a visual index to the keywords. If viewed under Android or under any browser (iPad, iPhone, Chrome...), the link to the .mov files is seen as picture icons. For those who only want to see something moving quickly, this is the direct link.

Final comments

There are a few terms important while presenting and discussing this paper.

Linkages

This is important for historical reasons. It was a big challenge that puzzled many people since the steam engine's invention: to design a mechanical device transforming circular motion to a straight line one. It turned out that the simple solution was based on the concept of inversion, a new direction in geometry at that time.

This term may be necessary for this age of robotics, although I know little about it.

Nets of polyhedrons

This is the engine for producing animations in Transfigurations. For the cube and for the octahedron, there are 11 each. So working by hand is not difficult. But for the dodecahedron and icosahedron, there are over 40K of them. I only pick the symmetrical ones and still don't know the number of symmetric nets of each. I attempted to work with the nets of the rhombic dodecahedron. I don't know how many are there and don't know the right kind of symmetry to take advantage of. In general, finding an algorithm enabling the generation of all nets appears to be beyond the mind. (Trial and error could be fastest...)

Stellations

The term "stellation" could be essential to attack the problems of dissections.

Convex hull

The term convex hull is essential for the constructions in this paper. I first learned the term in Functional Analysis, where the Hahn-Banach Theorem is fundamental to everything there. Funny that such math is essential theoretically, but it depends on the Well-Ordering Principle, Zorn's Lemma, etc. But I find applying convex hull a kind of divide-and-conquer approach, a beneficial and practical approach. By the way – back in Fortran days, there was already a library devoted to "taking the convex hull."

References

[1] Coxeter, H.S.M.; Regular complex polytopes (1974).

[2] Coxeter, H.S.M.; Du Val, P.; Flather, H. T.; and Petrie, J. F. The Fifty-Nine Icosahedra, 3rd Edition. Stradbroke, England: Tarquin Publications (1999).

[3] Wenninger, M.J. Dual Models, Cambridge University Press, Cambridge (1983).

[4] Kempe, A.B.; How to Draw a Straight Line: A Lecture on Linkages, London (1877).
https://archive.org/details/howtodrawstraigh00kemprich/page/n7/mode/2up

[5] R. Kanayama, Bibliography on linkage, Tohoku Mathematical Journal, First Series (1933).
https://www.jstage.jst.go.jp/article/tmj1911/37/0/37_294/_article