Steiner Ellipse and Marden's Theorem

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Abstract: *In this paper we will see the richness of both algebraic and dynamic approaches of a problem related to the three zeroes of a third-degree complex polynomial and the two zeroes of its derivative. Visualization with DGS in the Argand plane of these five points with the help of CAS will facilitate the exploration leading to the discovery of a very special ellipse (the Steiner ellipse).We will first prove (via a geometric proof) the existence and the uniqueness of such an ellipse (tangent to the triangle defined by the zeroes of the third degree polynomial at the midpoints of these zeroes) and then prove (complex proof) that the foci of this ellipse are the zeroes of the derivative polynomial (Marden's theorem). Connecting easily DGS and CAS within the TI-Nspire environment is a crucial tool of this work as well as the use of sliders to summarize the different stages of the investigations, the conjectures and their corroboration (experimental) or their validation (proof). This paper aims to show ideas of challenging mathematics to everybody at all levels. It aims also to show rich techniques of investigation within DGS and CAS environments. A final problem in relation to Marden's theorem will be investigated within DGS, leading to a very nice conjecture which is proved with a blend of DGS and CAS. An original way to prove that the Steiner ellipse is the ellipse of maximum area inscribed in a triangle will be shown in an important part of this paper : the way this very last work was conducted could be an important resource for techniques of investigation or proofs to know in order to use DGS and CAS relevantly.*

1. Introduction

1.1. Investigations leading to the Steiner ellipse

All the beginning of this paper is focused on polynomials of third degree of the complex variable z with coefficients which are also complex. We will represent in the Argand plane the three points associated to the three zeroes of this polynomial (M, N and P) and the two points F and F' associated with the two zeroes of its derivative. This work is performed within the TI-Nspire environment where we can use the power of dynamic geometry and computer algebra system. It allows us to investigate dynamically such a figure in which the zeroes can be modified with the use of sliders. The investigations conducted will lead us to highlight a very special ellipse connected to triangle *MNP* and to points F and F'.

Consider the polynomial $p(z) = (z - z_1).(z - z_2).(z - z_3)$ where $z_1 = a+i.b$, $z_2 = c+i.d$ and $z_3 = e+i.f$. Let us call $g+i.h$ and $k+i.l$ the two zeroes of the derivative $p'(z)$. In a Graphs page of TI-Nspire, let us create six sliders (for *a, b, c, d, e* and *f*) allowing us to change the values of the real part and the imaginary part of z_1, z_2 and z_3 and by the way changing the position of the vertices of triangle *MNP* constructed such as M(a,b), N(c,d) and P(e,f). Let us call F and F' the points defined by $F(k,l)$ and $F'(g,h)$. **Figure 1** displays triangle *MNP* and points F and F'. The first investigation suggested by this figure is to construct all the possible ellipses admitting F and F' as foci (**Figure 1on the left**). Changing the size of such an ellipse leads us to the first conjecture: « it seems that one of these ellipses is tangent to the sides of *MNP* at the midpoints of its sides ». More than that, it seems that the center of such an ellipse seems to be the centroid of *MNP* (the intersection point of the three medians). Another way to state this conjecture is:

Conjecture 1: given a triangle, there is an ellipse inscribed in this triangle passing through the midpoints of its sides where it is tangent to these sides.

Conjecture 2: if such an ellipse exists, its two foci are the zeroes of the derivative of the third degree polynomial admitting the vertices of the triangle for zeroes.

A visual corroboration of these conjectures could be the following one: construct the ellipse with foci *F* and *F'* passing through the midpoint of [*MP*] and state that this ellipse seems to be tangent to [*MP*], to pass through the midpoints of the two other sides and tangent to these sides (**Figure 1 on the right**). We can state it even if we change the values of the coordinates of points *M*, *N* or *P* with the sliders which will change the positions of *M*, *N* or *P*. It is a G1 Informatique validation (praxeology pointed in [7]).

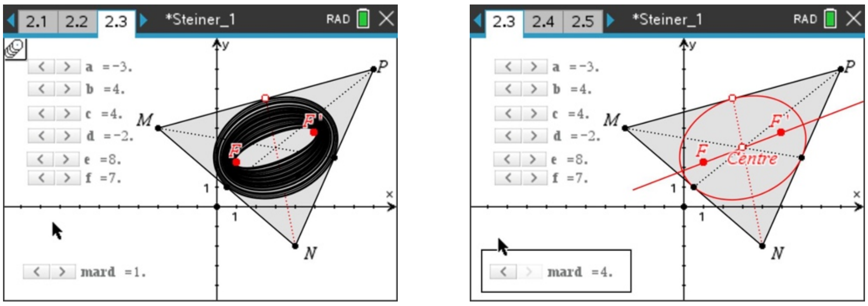

Figure 1: Investigations leading to the discovery of the Steiner ellipse

1.2. Other dynamic investigations for other conjectures

In a research activity like the previous one, the researcher always adopts an attitude of a very curious person. Before trying to prove what he was happy to conjecture, because the nature of a researcher is to be greedy, he tries to investigate some other relationships in this figure related to known configurations. It is what I did twice.

1.2.1. Investigations around the Euler's line (unsuccessful)

If (*OH*) is the Euler's line of triangle *MNP*, I have evaluated the ratio between the area of polygon *FHF'O* and the area of triangle *MNP* (**Figure 2 on the left**). But when we modify the polynomial in changing the positions of *M*, *N* or *P* with the sliders, this ratio changes. Eventually, nothing interesting was deduced from this investigation.

1.2.2. Investigations around a relationship between areas (successful)

Now we evaluate the ratio between the area of the previous ellipse and triangle *MNP* (**Figure 2 in the middle**). When changing the positions of points *M*, *N* or *P* this ratio seems to be constant. An approximate value of this constant could be 0.6046 to four places of decimals.

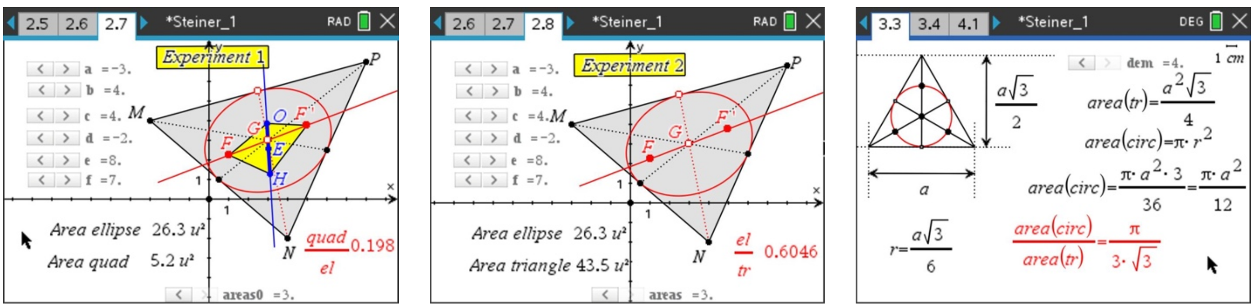

Figure 2: Investigations for other conjectures

At the end of this introduction, I have two directions to follow:

One concerns the Steiner ellipse, in order to prove that such an ellipse exists with the constraints about the midpoints of the initial triangle and the sides of this triangle as the tangents to this ellipse at these midpoints. In addition we will prove the invariance of the previous ratio.

A second direction concerns the fact that points *F* and *F'* are the foci of the Steiner ellipse (Marden's Theorem).

2. The Steiner ellipse

2.1. Reminder

Any triangle can be considered as the image of an equilateral triangle with an affinity. As we know that the ratio between areas is kept with an affinity, that the midpoints are kept, that properties of tangency are kept, this will simplify the proof of our first conjecture.

2.2. Corroboration of the previous conjecture

If the ratio of 1.1.2. is constant, we can evaluate it for an equilateral triangle: let us consider an equilateral triangle and let us evaluate the ratio between the area of the Steiner ellipse which is in this case the inscribed circle and the area of the triangle. A very simple reasoning (**Figure 2 on the right**) leads to a ratio equal to $\frac{\pi}{3\sqrt{3}}$ an approximate value for which is given by 0.604599788078 corroborating the previous invariant.

2.3. Existence of the Steiner ellipse

Let us consider a triangle : it is the image of an equilateral triangle with an affinity $\mathcal{A}$. The image of the inscribed circle of the equilateral triangle with $\mathcal{A}$ is an ellipse passing through the midpoints of the given triangle, the sides of which are tangent to this ellipse. The ratio between the area of the ellipse and the area of the triangle is necessarily equal to the equivalent ratio in the equilateral triangle: $\frac{\pi}{3\sqrt{3}}$. This ellipse is called the Steiner ellipse.

3. The Marden's Theorem

3.1. Bifocal definition of an ellipse and construction with a director circle

Consider a circle of radius 2*a* centered on *F1* and a second point *F2* inside this circle. Let us consider the ellipse defined by the relationship *MF1* + *MF2* = 2*a*. A construction of this ellipse is summarized in **Figure 3 on the left**. From each point V of the circle (called director circle associated to *F1*), we construct a point *M* of the ellipse as the intersection point of *F1V* and the perpendicular bisector of [*F2V*]. Therefore, the ellipse is the locus of *M* when *V* moves along the director circle. Dragging the point *V* along the circle allows those who don't know this property to conjecture that the perpendicular bisector is tangent to the ellipse in *M*. Let us prove this result analytically.

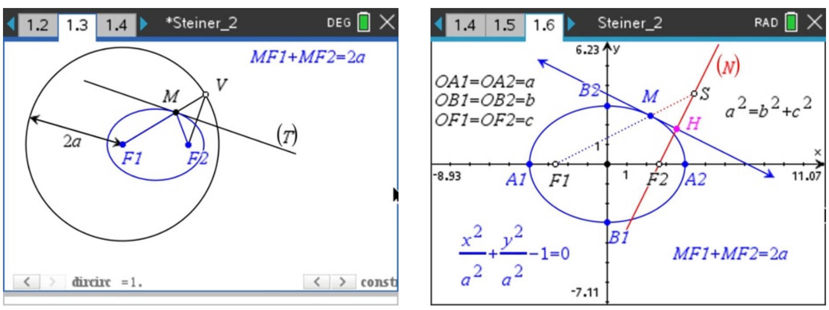

Figure 3 : Ellipse (bifocal definition) and its tangent lines

Our technique is the following one : we consider the tangent line at a point *M* of the ellipse in the coordinate system shown in **Figure 3 on the right**, *S* the symmetric point of *F2* with respect to this tangent ? We already know that $F1M+MS = 2a$, so we only have to prove the colinearity of points $F1$, M and S.

An equation of the tangent line at $M(xM,yM)$ is : $\frac{x.xM}{a^2}+\frac{y.yM}{b^2}-1=0$ and here with the trigonometric coordinates of $M(a.cos(u)\ ,\ b.sin(u))$: $b.cos(u).x + a.sin(u).y - ab = 0$.

As the parametric equations of line (N) (perpendicular to the tangent from $F2$) are:

$x = c + b.cos(u).t$ and $y = a.sin(u).t$, the parameter t_H of point H, is obtained in substituing these coordinates to x and y in the equation of the tangent line:

$(c + b.cos(u).t).\ b.cos(u) + a.sin(u).t.\ a.sin(u) - ab = 0$ or

$bc.cos(u)+b^2.cos^2(u).t + a^2.sin^2(u).t - ab = 0$ or

$t.(\ b^2.cos^2(u) + a^2.sin^2(u))=(ab - bc.cos(u))$ or $t.(\ b^2.cos^2(u) + a^2 - a^2cos^2(u)) = (ab - bc.cos(u))$

And as $b^2 - a^2 = -c^2$, therefore we obtain : $t.(a^2 - c^2.\ cos^2(u)) = b.(a - c.cos(u))$.

Eventually, the parameter of H is $t_H = \frac{b}{a+c.\cos(u)}$ and then the parameter of S is $t_S = 2t_H$.

We now obtain the coordinates of S ($\frac{ac+c^2.\cos(u)+2b^2.\cos(u)}{a+c.\cos(u)}$, $\frac{2absin(u)}{a+c.\cos(u)}$).

When we evaluate the determinant of $\overrightarrow{F1M}$ and $\overrightarrow{MS}$, we easily obtain a result of 0, which completes the proof.

We also obtain this result with the CAS of TI-Nspire.

3.2. A lemma : the Little Poncelet Theorem

With the notations of **Figure 4**, we will prove geometrically the following results which can be used for the proof of the Marden's Theorem:

If $(PT1)$ and $(PT2)$ are the the tangent lines to a given ellipse (with foci $F1$ and $F2$) where P is a given point outside the ellipse and $T1$ and $T2$ the contact points, then $\angle F1PT1 = \angle T2PF2$

Proof : let us use first the following notations

Refl(D) for Symmetry with respect to line D and *Rot(O,a)* for rotation centered in O with an angle of a. We know that

$Refl(PT1)oRefl(PF1)=Rot(P,2.\angle F1PT1)$ and $Refl(PF2)oRefl(PT2)=Rot(P,2.\angle T2PF2)$.

As the image of point $G2$ by these two rotations is the same point $F2$, we can conclude that

$Rot(P,2.\angle F1PT1) = Rot(P,2.\angle T2PF2)$ and eventually $2\angle F1PT1 = 2\angle T2PF2 + k.2\pi$ which leads to $\angle F1PT1 = \angle T2PF2 + k.\pi$. The conclusion is that the angles of lines $(PF1,PT1) = (PT2,PF2)$

The justification visible in **Figure 4** depends of the properties of the tangent lines ; they are detailed below (we use the equalities $F1G1 = F1G2$ and $PG1 = PF2 = PG2$):

$Refl(PF1) : G2 \longrightarrow G1$ and $Refl(PT1) : G1 \longrightarrow F2$ and

$Refl(PT2) : G2 \longrightarrow F2$ and $Refl(PF2) : F2 \longrightarrow F2$

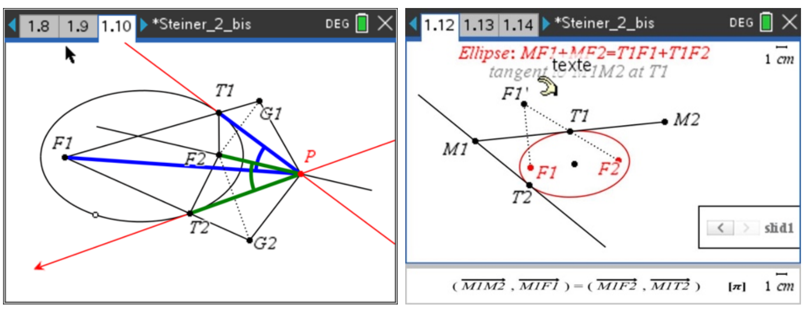

Figure 4 : The Little Poncelet Theorem and the Marden's Theorem

3.3. Marden's Theorem and its proof

Consider $p(z) = (z - z_1).(z - z_2).(z - z_3)$ is our third degree polynomial ; *M1*, *M2* and *M3* are the points of the Argand plane representing its three roots *z1*, *z2* and *z3*. *F1* and *F2* are the points representing the zeroes of *p'(z)*. We will prove the following theorem

Theorem : given a triangle *M1M2M3*, with the previous notations, the ellipse with foci *F1* and *F2*, passing through the midpoint of [*M1M2*] is tangent to the three sides of triangle *M1M2M3* at the midpoints of its sides.

As shown in **Figure 4 on the right**, we have constructed *F'1* the symmetric point of *F1* with respect to (*M1M2)* and *T1* the intersection point of (*M1M2)* and (*F'1F2)*. Then we have constructed the ellipse (*E*) set of points *M* verifying the condition *MF1*+*MF2* = *T1F1*+*T1F2*. According to this definition, *T1* belongs to this ellipse and (*M1M2)* is tangent to this ellipse at *T1*. Thanks to the little Poncelet Theorem, we can state that $(\overrightarrow{M1M2}, \overrightarrow{M1F1}) = (\overrightarrow{M1F2}, \overrightarrow{M1T2})$ where (*M1T2)* is the second tangent to (*E*) from *M1*.

The derivative *p'(z)* can be written 3.(*z - f1*).(*z - f2*) (where *f1* and *f2* are zeroes of *p'*) or (*z - z1*)(*z - z2*)+(*z - z2*)(*z - z3*)+(*z - z3*)(*z - z1*).

Therefore *p'(z1)* can be evaluated with 3.(*z1 -f1*).(*z1 -f2*) or (*z1 - z2*)(*z1 - z3*) from which we get the equality :

$\frac{z2-z1}{f1-z1} = 3.\frac{f2-z1}{z3-z1}$. The equality of the arguments gives

$(\overrightarrow{M1F1}, \overrightarrow{M1M2}) = (\overrightarrow{\mathrm{M1M3}}, \overrightarrow{\mathrm{M1F2}})$ [2π]

$(\overrightarrow{M1M2}, \overrightarrow{M1F1}) = (\overrightarrow{M1F2}, \overrightarrow{M1T2})$ [π] (from the little Poncelet theorem).

A simple deduction leads to : $(\overrightarrow{M1F2}, \overrightarrow{M1M3}) = (\overrightarrow{M1F2}, \overrightarrow{M1T2})$ [π] which interpretation is that (*M1T2*) and (*M1M3*) are superimposed. Eventually, (*M1M3*) is tangent to ellipse (*E*).

A similar proof can be conducted to state that (*M2M3)* is tangent to this ellipse.

We have at this stage proven the existence of an ellipse tangent to *M1M2M3* tangent to (*M1M2*) at its midpoint.

We have now to prove that *T2* and *T3* are respectively the midpoints of the other sides of triangle *M1M2M3*. Let us prove only that *T2* is the midpoint *J* of [*M1M3*]. *J* is associated to the complex number $jj = \frac{z1+z3}{2}$.

From the Little Poncelet theorem we know that $(\overrightarrow{M1F1}, \overrightarrow{M1M3}) = (\overrightarrow{M1M2}, \overrightarrow{M1F2})$ that can be interpreted in terms of arguments :

$\arg(\frac{z3-z1}{f1-z1}) = \arg(\frac{f2-z1}{z2-z1})$ that can be written $\arg(\frac{\frac{z3-z1}{2}}{f1-z1}) = \arg(\frac{f2-z1}{\frac{z2-z1}{2}})$

Or $\frac{z3-z1}{2} = jj-z1$ and $\frac{z2-z1}{2} = ii-z1$ where $ii=\frac{z1+z2}{2}$. Therefore:

$\arg(\frac{jj-z1}{f1-z1}) = \arg(\frac{f2-z1}{ii-z1})$ which means $(\overrightarrow{M1F1}, \overrightarrow{M1J}) = (\overrightarrow{M1I}, \overrightarrow{M1F2})$

And finally thanks again to the little Poncelet theorem, *J* = *T2*.

A similar proof can be conducted to prove than *T3* (contact point of [*M2M3*] with the ellipse is the midpoint of the third side of *M1M2M3*. That completes the proof.

4. A surprising problem in relation to the Steiner ellipse

4.1. The problem and a dynamic investigation

Given a triangle *ABC* and a point *M* inside this triangle. Let us construct from *M* the parallel lines to each of its sides like in **Figure 5 on the left**. We have defined three triangles (in yellow) and three

parallelograms (in orange). The problem to solve is the following one : **is it possible to find positions of point *M* such that the sum of the areas of the three triangles is equal to the sum of the areas of the three parallelograms ?**
Here is a possible investigation: we drag point M to find a position where the difference between these two sums is close to 0 as shown in **Figure 5 in the middle** where the displayed difference is : -0.097 which was the best we could do. Then, as it is possible in the TI-Nspire environment, we lock this number which means that point *M* can only be dragged on positions where this number does not change. Activating the geometric trace of *M* and dragging it everywhere we can drag it, we obtain what is displayed in **Figure 5 in the middle**. We can state that the trace we get seems to be the Steiner ellipse (G1 level, [7]). To corroborate this conjecture, we can construct the Steiner ellipse, redefine point *M* on it and state that the number -0.097 is changed instantaneously onto -2.E-12 for every position tested on this ellipse With the accuracy of the software and the pixellisation of the screen, it means that the solution points are the Steiner ellipse (G2 level of validation, [7]),that is shown in **Figure 5 on the right**.

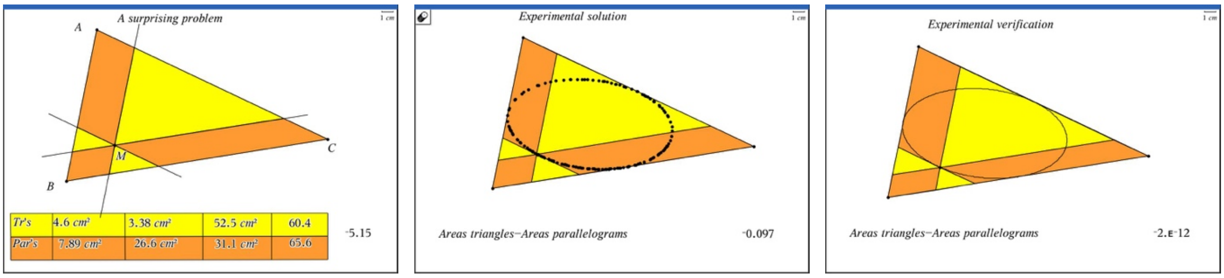

Figure 5 : a surprising problem connected to Marden's Theorem

4.2. Proof of the previous conjecture

We will use the fact that any triangle can be considered as the image of an equilateral triangle with an affinity (or with a parrallel projection in 3D) and that the ratio between areas is kept with such a transformation. If our conjecture is true, as a consequence, in the case of an equilateral triangle the solution must be the inscribed circle of this triangle. We give below a formal proof supported by the CAS of TI-Nspire.
We will prove that the solution of our problem in the case of an equilateral triangle is its inscribed triangle. It will be sufficient to obtain the result in the general case. In order to do that, we work in a coordinate system where the unit circle is the inscribed circle and we use the notations of **Figure 6**.
So, let us express the constraint « **sum of the areas of the three yellow triangles equal half of the area of the given triangle** » to obtain the equation of the solution.
In this coordinate system the coordinates of the vertices of the given triangle are : $A(0,2)$, $B(-\sqrt{3},-1)$ and $C(\sqrt{3},-1)$. Let us call x_M and y_M the coordinates of a point *M* of the plane. We easily obtain the following equations of the sides of triangle *ABC*:
(*AB)* : $3x - \sqrt{3}\,y + 2\sqrt{3} = 0$; (*AC*) : $3x + \sqrt{3}\,y - 2\sqrt{3} = 0$ and (*BC*) : $y = -1$ from which we can get the coordinates of the vertices of the three yellow triangles.
e2 and *g1* are obtained as the intersection points between line $y = yM$ and the two lines (*AB*) and (*AC*) : $e2\left(\frac{(yM-2)\sqrt{3}}{3}, y_M\right)$ and $g1\left(\frac{-(yM-2)\sqrt{3}}{3}, y_M\right)$
f1 and *g2* are obtained as the intersection points between (*f1g2*) (in reality the parallel line to (*AB*) passing through *M*) and lines (*BC*) and (*AC*) :
From (f1g2) : $3x - \sqrt{3}\,y + 3x_M + y_M\sqrt{3} = 0$, we get

f1($\frac{3xM-(yM+1)\sqrt{3}}{3}$, -1) and *g2*($\frac{3xM-(yM-2)\sqrt{3}}{6}$, $\frac{yM+2-xM\sqrt{3}}{2}$)
e1 and *f2* are obtained as the intersection points between (*e1f2*) (in reality the parallel line to (*AC*) passing through *M*) and lines (*AB*) and (*BC*):
From (*e1f2*) : $3x + \sqrt{3}\,y - 3x_M - y_M\sqrt{3} = 0$, we get
e1($\frac{3xM+(yM-2)\sqrt{3}}{6}$, $\frac{xM\sqrt{3}+yM+2}{2}$) and *f2*($\frac{3xM+(yM+1)\sqrt{3}}{6}$, -1)
We can evaluate now with the help of determinants the areas of the three yellow triangles and we obtain:
Area($Me1e2$) = = $\frac{1}{2}\det(\overrightarrow{Me1}, \overrightarrow{Me2}) = \frac{3xM^2\sqrt{3}-6xM(yM-2)+(yM-2)^2\sqrt{3}}{12}$
Area(*Mf1f2*) = = $\frac{1}{2}\det\left(\overrightarrow{Mf1}, \overrightarrow{Mf2}\right) = \frac{(yM+1)^2\sqrt{3}}{3}$
Area($Mg1g2$) = = $\frac{1}{2}\det(\overrightarrow{Mg1}, \overrightarrow{Mg2}) = \frac{3xM^2\sqrt{3}+6xM(yM-2)+(yM-2)^2\sqrt{3}}{12}$
Adding these three areas, we obtain, the yellow area : yellow area = $\frac{xM^2\sqrt{3}+(yM^2+2)\sqrt{3}}{2}$
We know the area of *ABC* : $3\sqrt{3}$.

The constraint of our problem is :
Area(*ABC*)-2.yellow area = 0 which is$-(xM^2 + yM^2 - 1)\sqrt{3} = 0$ or $xM^2 + yM^2 - 1 = 0$.
This last equation describes the inscribed circle. That completes the proof.

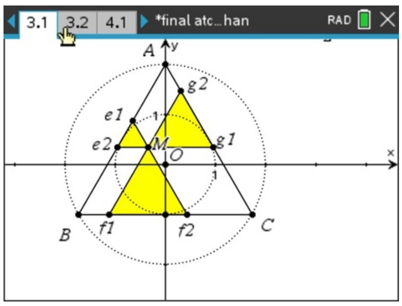

Figure 6 : Analytic proof of the conjecture

5. A property of the Steiner ellipse

5.1. A property of maximum area

This known property states that the Steiner ellipse of a triangle is the ellipse performing the maximum area among the ellipses inscribed in this triangle. The originality of this part is

1. How the investigation is conducted starting from an ellipse and not from a triangle with DGS.
2. How the formal proof can be deducted from the previous investigation with CAS.

The two stages of this proof are as follows :

1. For a given equilateral triangle, locate the positions of its inscribed ellipses of a given shape having a maximum area.
2. For these positions, evaluate the shapes maximizing the area of the ellipse.

The proof will be finished when the final ellipse will be the inscribed circle, because for a random triangle it means that the ellipse maximizing its area is the Steiner ellipse.

5.2. An investigation for a very special problem

The problem I propose to investigate with dynamic geometry is the following one :
Given an ellipse, construct all the equilateral triangles admitting this ellipse as inscribed ellipse and find the one (equilateral triangle) with a minimum area. Solving this problem, at least experimentally, will give us an indication for the ellipse having the maximum area when inscribed in a given equilateral triangle. The proof will conclude such an investigation. During this work, we will have the opportunity to solve the following problem:

For a given equilateral triangle, construct all the inscribed ellipses to this triangle admitting a given shape.

Remark : to obtain all the possible shapes of an ellipse, we need a given circle centered on *F1* and a second variable point *F2* inside this circle. A shape is defined by the ellipse with foci *F1* and *F2* and the given circle as director circle associated with *F1*. All other ellipses of the plane are similar to one of the previous one (use a translation, a rotation and a dilation)

5.2.1. Equilateral triangles admitting an inscribed given ellipse (Figure 8)

The given ellipse is defined by its two foci *F1* and *F2* and the director circle associated with *F1*.
If triangle *MNP* is an equilateral triangle admitting this ellipse as inscribed ellipse (and by the way the three sides (*PM*), (*MN*) and (*NP*) tangent to the ellipse respectively at *T1*, *T2* and *T3*), thanks to the known geometric properties of the ellipse, (*PM*) is the perpendicular bisector of (*F2K1*), (*MN*) is the perpendicular bissector of (*F2K2*) and (*NP*) is the perpendiculare bisector of (*F2K3*). Knowing that the quadrilateral *F2H1MH2* has two right angles (*H1* and *H2*), angles *H2* and *M* are supplementary angles and eventually, $\angle H1F2H2=\frac{2\pi}{3}$. So we know now that $(\overrightarrow{F2H1},\overrightarrow{F2H2})=\frac{2\pi}{3}$.
With the same reasoning we obtain $(\overrightarrow{F2H2},\overrightarrow{F2H3})=\frac{2\pi}{3}$ and $(\overrightarrow{F2H3},\overrightarrow{F2H1})=\frac{2\pi}{3}$.

Construction algorithm of an equilateral triangle starting from the first contact point of tangency *T1* :

-Consider a point *T1* on the given ellipse.
-Construct the ray [*F1T1*) intersecting the given director circle in *K1*.
-Construct *H1* midpoint of [*F2K1*].
-Line (*T1H1*) is the line supporting the first side [*PM*] of *MNP*.
-Rotate [*F1T1*) around *F2* (angle $-\frac{2\pi}{3}$) to get a ray intersecting the director circle at *K2*.
-The line supporting the second side [*MN*] is the perpendicular bisector of [*F2K2*].
-The point of tangency of the second side is point *T2* intersection between this line and segment [*F1K2*].
-Rotate [*F1T2*) around *F2* (angle $-\frac{2\pi}{3}$) to get a ray intersecting the director circle at *K3*.
-The line supporting the third side [*NP*] is the perpendicular bisector of [*F2K3*].
-The point of tangency of the third side is point *T3* intersection between this line and segment *F1K3*.
M, *N* and *P* are the intersection points of the three tangent line constructed.
Remark : dragging point *T1* along the ellipse allows us to obtain all the equilateral triangles admitting this ellipse as an inscribed ellipse

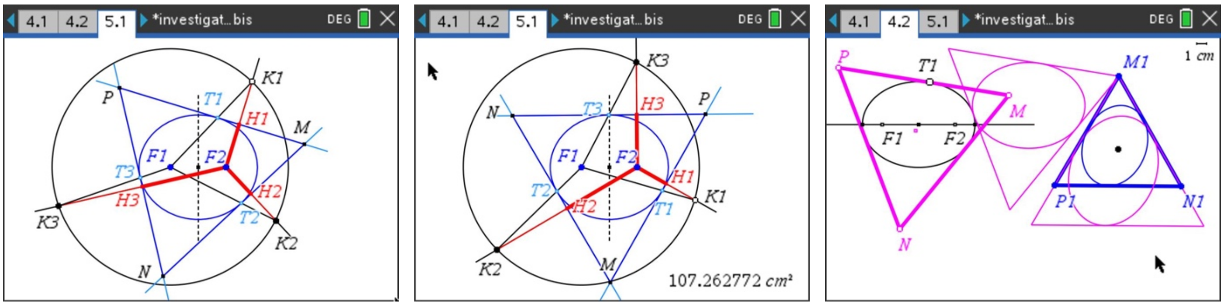

Figure 7 : Equilateral triangles and inscribed ellipses

5.2.2. Equilateral triangles of minimum area admitting an inscribed given ellipse (Figure 7)

We display (to six places of decimals) the area of the triangle *MNP* we obtained previously and we drag point *T1* along the ellipse to state that the minimum of this area is reached each time a base of the equilateral triangle is parallel to (*F1F2*) or which is equivalent, each time one of the three rays [*F2K1*), [*F2K2*) and [*F2K3*) is perpendicular to (*F1F2*) (**Figure 7 in the middle**).

Consequence : given an equilateral triangle, we know now how to construct all the ellipses inscribed in this triangle with a given shape (**Figure 7 on the right**).

Perform the previous construction for any ellipse defined by the director circle centered in *F1* (first focus) and *F2* (second focus inside the circle). We obtain all the equilateral triangles by dragging *T1* on such an ellipse.

Given an equilateral triangle *M1N1P1* on the plane, this triangle is the image of *MNP* by the translation mapping point *M* onto point *M1* followed by the rotation which angle is $(\overrightarrow{MN}, \overrightarrow{M1N1})$ and by the dilation centered on *M1* and with scale factor of *M1N1/MN*.

Eventually, the image of the given ellipse by the composition of these three transformations will give an inscribed ellipse to the given equilateral triangle. And dragging point *T1* along the given ellipse will generate all the inscribed ellipses to the given equilateral triangle.

5.2.3. First result : conjecture and proof

5.2.3.1.An equivalent problem :

To prove it, we return to the initial presentation starting from an ellipse of given shape (**Figure 7 on the left**). Let us evaluate the area of triangle *MNP* with a formula that will help for an analytic proof. This area is equal to the sum of the area of the triangles *MF2N*, *NF2P* and *PF2M* that is to say : $\frac{1}{2}(F2H2.MN + F2H3.NP + F2H1.PM) = \frac{1}{2}(F2H2 + F2H3 + F2H1)a$ where a is the side of the equilateral triangle which depends of the position of *T1*. This area is also equal to $\frac{a^2\sqrt{3}}{4}$. From these two expressions, we get : $a = \frac{2}{\sqrt{3}}.(F2H2 + F2H3 + F2H1)$. Therefore the minimum area will be obtain when a is minimum or when the sum $F2H2 + F2H3 + F2H1$ is minimum or which is equivalent when the sum $F2K2 + F2K3 + F2K1$ is minimum.

5.2.3.2.Investigations leading to the conjecture

On a graph page (**Figure 8**), we represent a circle centered in *F1*, a point *F2* inside this circle and three point *K1*, *K2* and *K3* such as ∠*K1F2K3* = ∠*K3F2K2* = ∠*K2F2K1* = $\frac{2\pi}{3}$. *K1* is commanded by a slider whose boundaries are 0 and the length of the circle. So point *K1* can be dragged from the intersection point between the circle and the positive part of the x axis to this point again after a complete rotation around *F1*. Then we have constructed in an analytic page the point whose coordinates are the length transferred with the slider and the sum of the three distances *F2K1*, *F2K2* and *F2K3*. The locus of this point when the slider point moves on the segment representing this slider is the curve we can see in **Figure 8**. We can conjecture that this curve has six minimum points corresponding to vertical positions of each ray constructed from *F2*. It is easy to conjecture that in these cases the angle between the positive part of the *x* axis and vector $\overrightarrow{F2K1}$, could likely be , $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$ and $\frac{11\pi}{6}$.

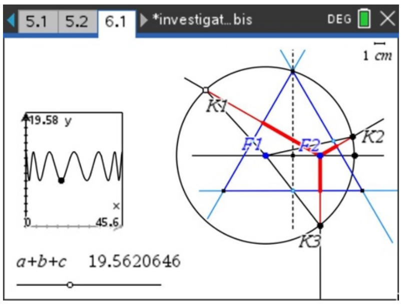

Figure 8 : Investigations for a first result

Conclusion1 : for a given equilateral triangle and an ellipse of given shape, there are three ellipses (of this given shape) of maximum area inscribed in the triangle, those which principal axis is parallel to one side of the given equilateral triangle.

5.2.3.3. CAS supported solution

Eventually, consider a circle centered on *F1* (radius 2), *F2* a point inside the circle ($F1F2 = 2c$ where $0 < c < 1$) and three points on the circle *K1*, *K2* and *K3* seen from *F2* with the same angle (necessarily $\frac{2\pi}{3}$). To obtain all the possible shapes of this ellipse, we only have to change the position of *F2*. Let us evaluate the three distances with respect to angle t ($0 \leq t \leq 2\pi$) as shown in the coordinate system of **Figure 9 on the left**.

If we call *k1= F2K1*, *k2=F2K2* and *k3=F2K3*, the coordinates of *K1* are $k1\cos(t)+2c$ and $k1\sin(t)$. Here the parameters *a*, *b* and *c* of the initial ellipse are such as $a = 1$ and $a^2 = b^2 + c^2 = 1$. In order to evaluate *k1* we have to express the constraint « *K1* belongs to the director circle which equation is « $x^2 + y^2 - 4 = 0$ » which means : $(2c+k1.\cos(t))^2 + (k1.\sin(t))^2 - 4 = 0$ (which is a quadratic equation) easily solved by the CAS of TI-NSpire. The CAS provides two solutions : the non negative one is the one we expect : $k1 = -2.c.\cos(t) - 2\sqrt{1-c^2.(\sin(t))^2}$; from this formula we obtain the following ones : $k2 = -2.c.\cos(t-\frac{2\pi}{3}) - 2\sqrt{1-c^2.(\sin\left(t-\frac{2\pi}{3}\right))^2}$ and

$k3 = -2.c.\cos(t+\frac{2\pi}{3}) - 2\sqrt{1-c^2.(\sin\left(t+\frac{2\pi}{3}\right))^2}$.

In a graph page of TI-Nspire we have represented function *f* (called *f4* on the screen of **Figure 9 in the middle**) defined by :

$f(t) = k1 + k2 + k3$ for a special value of *c* commanded by the slider **ccc** *(*0< ***ccc*** <1*)*.This function seems to reach its minimum six times on the interval $0 \leq t \leq 2\pi$. The values we have guessed geometrically for the vertical positions of one of the three rays drawn from *F2*, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$ and $\frac{11\pi}{6}$ are corroborated by **Figure 9 in the middle**. These values seems to be appropriate for every values of *c* because the observation does not change even if we change the values of *c*.

Using the CAS, we confirm this property by evaluating $f\ '(t) = 0$ for t equal to each of these six numbers and by evaluating $f(t) = 2\sqrt{1-c^2} - 2\sqrt{4-c^2}$ for t equal to each of these six numbers.
That completes the proof « for me ».

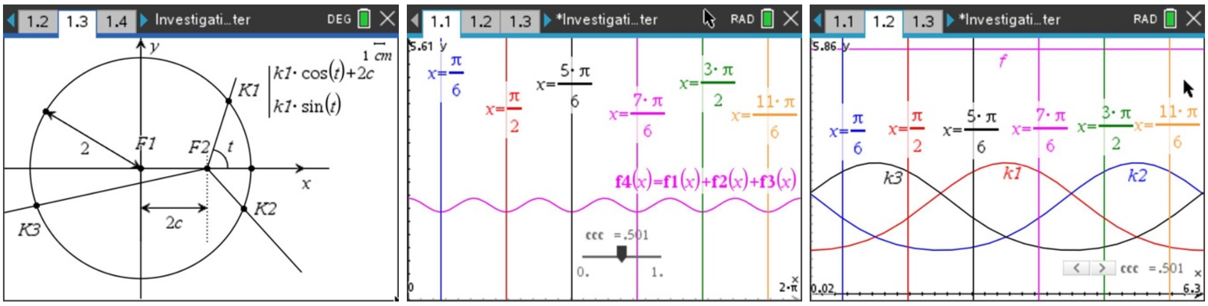

Figure 9 : The sum of the distances from a point inside a circle to three of its points

Remark : the CAS of the software is unable to evaluate the zeroes of f.

5.2.4. Second result : conjecture and proof

We work again as in paragraph 4.2. in the same coordinate system, centered at the center of the inscribed circle of the given equilateral triangle (radius 1). In this coordinate system we know the equations of the three sides of the given triangle. We will consider all the inscribed ellipses of this triangle whose principal axis is parallel to the horizontal side of the triangle. We will compute their equations to get their areas.
At last we will see which ellipse maximizes its area.
Our reasoning is based on **Figure 10**. If I is the center of such an ellipse, if H is the midpoint of $[BC]$, we chose for the coordinates of I, $I(0,h)$. As I can only be located between H and A and as the symmetric of H with respect to I must be below A, therefore $-1 < h < 0.5$.
The equation of such set of ellipses is given by : $\frac{x^2}{a^2} + \frac{(y-h)^2}{(1+h)^2} - 1 = 0$. Let's evaluate a in order to obtain ellipses tangent to (AB) and by symmetry tangent to (AC). The technique is simple with the CAS or by hand : determine the intersection between such an ellipse and (AB) which equation is $3x - \sqrt{3}\,y + 2\sqrt{3} = 0$. The screenshot of the Note page of TI-NSpire displayed in **Figure 10 in the middle** shows that we obtain in the general case two intersection points. The condition to obtain only one point is : $a^2 + 2h - 1 = 0$ or $a = \sqrt{-2h+1}$. As we want the principal axis to be horizontal, we have the constraint $\sqrt{-2h+1} \geq 1+h$ which is equivalent to $h \leq 0$. Eventually the area of those ellipses is given by the formula $\pi.(1+h)\sqrt{-2h+1}$. We can check it in displaying some of these ellipses until the last one for $h = 0.5$ (**Figure 10 on the right**) : for $h = 0.5$, the ellipse reaches the position of the inscribed circle with area of π.

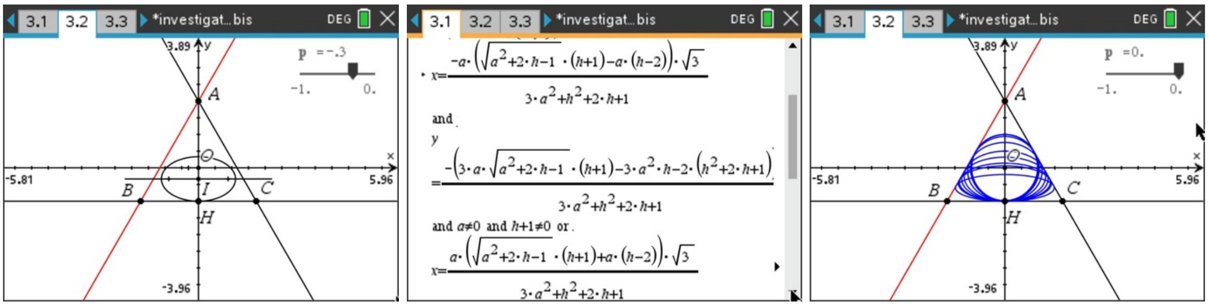

Figure 10 : Maximizing the area of a set of ellipses

Now on a graph page, let us display the function $f(x) = \pi.(1+x)\sqrt{-2x+1}$ and the horizontal line $y = \pi$ (**Figure 11**). We obtain an increasing curve for $x \leq 0$ the maximum of which seems to be reached for $x = 0$ and with $f(0) = \pi$. The software corroborates that it is really the maximum. But let us notice that the abscissa of the maximum is evaluated by $-1.29.10^{-7}$. Nevertheless the CAS used in a Note page gives $f'(0) = 0$ and $f(0) = \pi$.

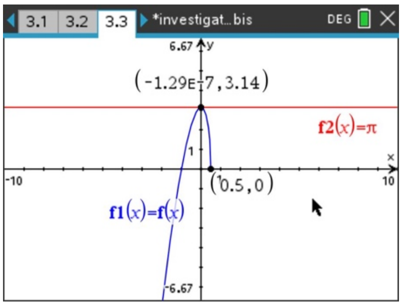

Figure 11 : Areas of horizontal ellipses tangent to an equilateral triangle

The formal proof is now given with the expression of $f'(x)$ obtained with the CAS : $\frac{-3x\pi}{\sqrt{-2x+1}}$ which is positive when $x \leq 0$. Now we are sure that the ellipse of maximum area is the inscribed circle.
Conclusion2 : among the inscribed ellipses to an equilateral triangle with principal axis parallel to one side of the triangle, the one with the maximum area is the inscribed circle.

5.2.5. Final conclusion

The interpretation of conclusion 1 and conclusion 2 is : the ellipse of maximum area inscribed in an equilateral triangle is its inscribed circle and therefore the ellipse of maximum area inscribed in a triangle is its Steiner ellipse (the one which is tangent to the triangle at its three midpoints).

6. Conclusion

After writing such a paper, telling the story of a research principally in paragraph 5 (or some moments), my first impression is that nothing would have been possible without geometric knowledge (properties of affinities to solve a problem in a particular case and obtained by the way this property in the general case), the communication between the members of my group of research enriching my way of investigation by DGS or CAS, my skills in analytic geometry allowing a pertinent use of the CAS which boundaries are very disappointing, the fact that calculations by hand are crucial to sort ideas that seem interesting and very often disappointing. Proving that the inscribed ellipse of maximum area of a triangle is the Steiner ellipse was the final and very challenging part of this paper and eventually the most important even if at the beginning I thought that this proof could not be interesting : it gave me the oppurtunity to use an original technique suggested by one of my colleagues. In order to use this idea I had to solve the problem of constructing all the ellipses (of any shape) inscribed in a random triangle. The first part of this paper aimed principally to show to teachers some ways to present mathematics to their students in using the power of visualization and animation of digital tools (paragraphs 1 to 3). The second part aimed to give an example of problem with an unexpected link with the previous work : in mathematics never forget to consider unexpected connections (paragraph 4). Finally the aim of paragraph 5 was to show to teachers that research is not only based on a unique skill especially on technical skills but on a lot of necessary knowleledge and techniques (by hand or DGS or CAS techniques). One of the power of digital tools used for this paper (especially the TI-NSpire environment) is to allow « us » to revisit known problems and known results to investigate differently and get new proofs and sometimes new results.

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YouTube videos (links)

[1'] Playlist of YouTube channel « jjdahan » : T3 SAN ANTONIO 2018 JJ DAHAN
https://www.youtube.com/playlist?list=PLOIs4xavv0zFNELpk0S7SkQ9RWPasFvdx

Software

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