# Discovering New Tessellations Using Dynamic Geometry Software


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**Abstract**: *In this paper we use dynamic geometry software to investigate a class of tilings called k-uniform tilings or tessellations. A tiling consisting of regular polygons whose vertices belong to k-transitivity classes under the action of its symmetry group (vertex-k-transitive) is said to be k-uniform. We also present constructions of tilings consisting of irregular polygons that are vertex-k-transitive.*


## 1. Introduction

A *tiling* (or *tessellation*) of a plane is a countable collection of compact sets, called *tiles*, which fill the plane without gaps or overlaps. In tiling theory, the study of tilings by types of tiles, vertices or edges have always been of interest. Classifying tilings according to their symmetry groups, and/or how the tiles, vertices or edges behave under the symmetries has been addressed in tiling literature. One of the oldest tilings that have been studied are regular tilings. A *regular tiling* is an edge-to-edge tiling by congruent regular polygons. These have been studied since antiquity, as they have been the basis for pattern construction in many traditions of sacred and decorative art. They can be found in Islamic patterns, and in the natural world, they appear as crystal and cellular structures.

Among the well-known tilings by regular polygons are the 1-*uniform* or *uniform* tilings in the Euclidean plane [5], tilings whose vertices form one transitivity class under the action of their respective symmetry groups (Figure 1.1). These are also known as *vertex-transitive* tilings. There have been previous studies on Euclidean $k$-uniform tilings, tilings by regular polygons where the vertices of the tilings form $k$ transitivity classes under their respective group of symmetries; however, the enumeration of these tilings for specific values of $k$ is far from complete. There is the classification of 2-uniform tilings by Krötenheerdt [6], the enumeration of the 3-uniform tilings by Chavey [1], and Galebach's enumeration of the 4-, 5- and 6-uniform tilings. The 4-, 5- and 6-uniform tilings have been found by computer and published on the web by Galebach [2]. The $k$-uniform tilings for $k > 6$ have not been classified. In this paper, we discuss our algorithm and present 8- and 13-uniform Euclidean tilings, which we have formulated and constructed using the aid of dynamic geometry tools.

We also present constructions of *vertex-k-transitive* tilings in the Euclidean and the hyperbolic plane using the Poincare disk model. These tilings consist of polygons that are not necessarily regular. In the literature, there is scarcity of examples on hyperbolic tilings with vertex-$k$-transitive properties.

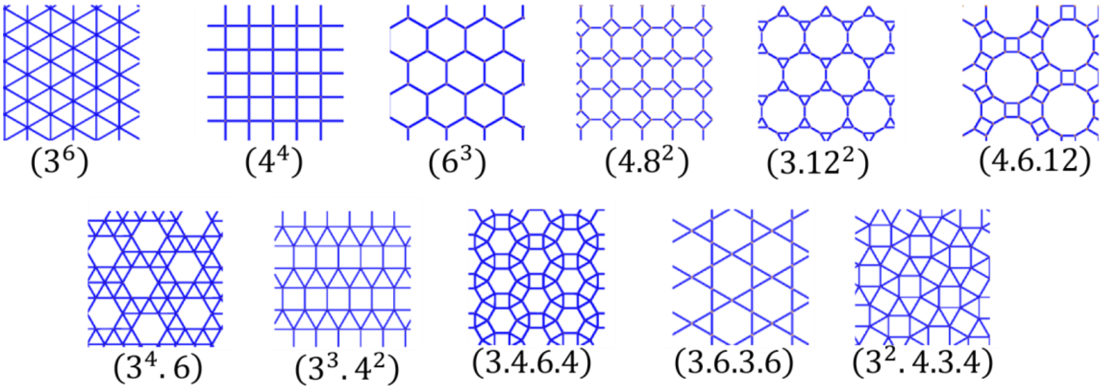



**Figure 1.1** The eleven uniform tilings in the Euclidean plane. The notation denotes the type of *r*-sided regular polygons meeting at a vertex.

## 2. Tilings from regular tilings

In understanding the basic rudiments of tilings, regular tilings are useful examples for studying the elements of tilings: vertices, tiles and edges; and for investigating their symmetries (isometries in the plane that sends the tilings to themselves) and symmetry groups, as well as determining vertex-, tile- and edge-transitivity properties. Aside from being a vertex-transitive tiling, a regular tiling is also tile-transitive and edge-transitive. A *tile* (respectively *edge*)-*transitive tiling* is a tiling whose tiles (respectively edges) form one transitivity class under the action of its symmetry group. Regular tilings can also serve as the basis for the construction of other tilings.

One can construct a regular tiling using dynamic geometry software by utilizing the Construction tools and the Isometry tools corresponding to the isometries of the Euclidean plane (Rotation, Reflection, Translation tools). These tools facilitate distinguishing the different symmetries in the tiling and understanding the effect of the different isometries on the elements of the tiling. With the flexibility of such tools, one can perform explorations by introducing new elements to the tiling or varying existing elements of the tiling. These can lead to the construction of new tilings, with other properties.

For instance, consider the regular tiling by hexagons shown in Figure 2.1(a). By introducing smaller hexagons (with yellow vertices) in the interior of the hexagonal tiles and by connecting their vertices to the vertices of the original tiling, we obtain the tiling by regular hexagons and isosceles trapezoids shown in Figure 2.1(b). This can be also thought of as dissecting the hexagonal tiles of the original tiling by smaller hexagons and trapezoids. The symmetry group of the new tiling is equal to the symmetry group of the original tiling. The vertices of the resulting tiling form two transitivity classes under the action of its symmetry group. This tiling then belongs to the class of *vertex*-2-*transitive tilings*. The tiles of this new tiling also form two transitivity classes under the action of its symmetry group, and so it belongs to the class of *tile*-2-*transitive tilings*. On the other hand, its edges form three transitivity classes under the action of its symmetry group and so it belongs to the class of *edge*-3-*transitive tilings*. One can also vary the edges of the regular tiling by hexagons. In particular, the edges of the tiling are replaced by curves, maintaining only some of the original symmetries. In Figure 2.1(c), we obtain a *non-convex* tiling with curves where the 60º rotational symmetries of the original tiling vanish.

Another example would be to start with a hyperbolic regular tiling by 4-gons shown in Figure 2.2(a). A 4-gon can be dissected into a regular 4-gon and four irregular 5-gons. Consequently, we obtain a new tiling (Figure 2.2(b)) whose symmetry group is equal to the symmetry group of the original tiling. This is an example of a hyperbolic tiling that is vertex-3-transitive.

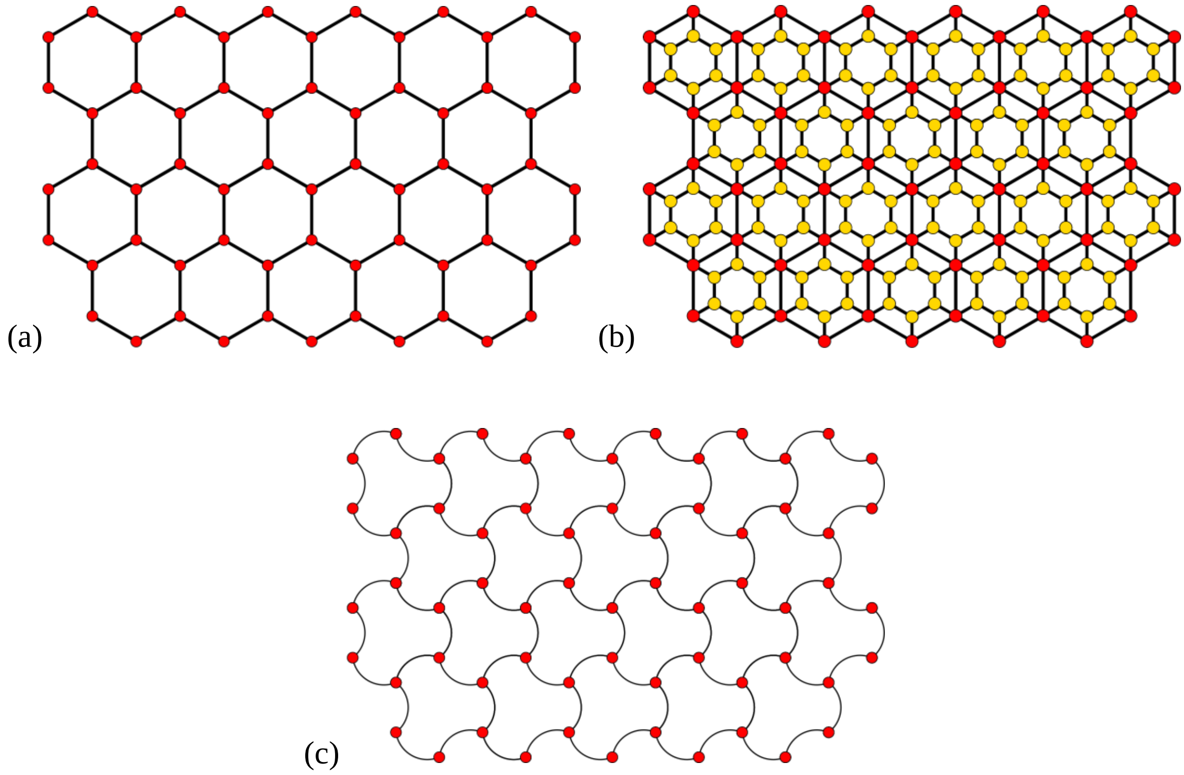



**Figure 2.1** (a) A tiling of the Euclidean plane by regular hexagons; (b) a tiling obtained from (a) by dissecting the hexagons; (c) a tiling obtained from (a) by varying the edges.

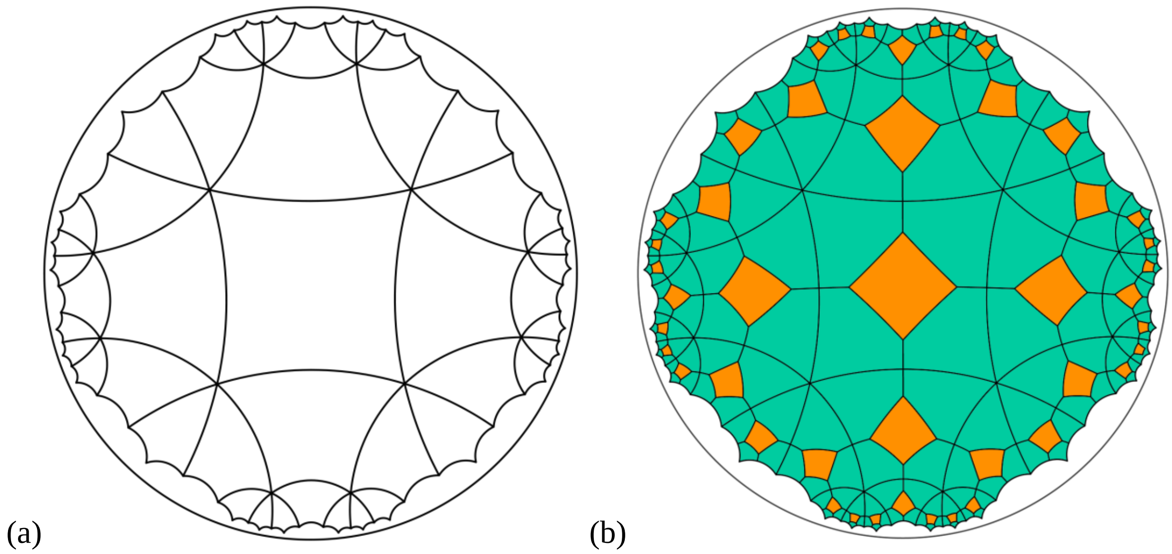



**Figure 2.2** (a) A tiling of the hyperbolic plane by regular 4-gons; and (b) a tiling obtained from (a) by dissecting the 4-gons.

An advantage of using a dynamic geometry software that facilitates hyperbolic constructions is that one manages to make various experimentations of hyperbolic tilings with precision and ease. Studying hyperbolic tilings and their groups of symmetries can be challenging since the hyperbolic symmetry groups are infinite groups consisting of an immense variety of isometries and an infinite number of such groups exist.

## 3. Euclidean tilings from *n*-uniform tilings

In the Euclidean plane, we can dissect some regular polygons into regular polygons. In particular, we can dissect a square into smaller squares, an equilateral triangle into equilateral triangles or a combination of equilateral triangles and regular hexagons, and a regular hexagon into equilateral triangles or a combination of regular hexagons and equilateral triangles. By dissecting the tiles of a $n$-uniform tiling into regular polygons, we can obtain a $k$-uniform tiling, $k > n$.

For example, consider the 7-uniform tiling in Figure 3.2(a) [2]. By dissecting all the hexagonal tiles of the 7-uniform tiling into equilateral triangles we obtain a new tiling by regular polygons (Figure 3.2(b)). The resulting tiling is an 8-uniform tiling. Its symmetry group is equal to the symmetry group of the 7-uniform tiling. The highlighted vertices in Figure 3.2(b) are vertices that belong to different 8 transitivity classes under the action of the symmetry of the 8-uniform tiling.

Given the 8-uniform tiling, we can obtain another interesting tiling. By considering the incenters of the tiles and by connecting using an edge the incenters that belong to two adjacent tiles, we obtain a tile-8-transitive tiling (Figure 3.2(c)). Its symmetry group is equal to the symmetry group of the 8-uniform tiling. Tiles belonging to the same transitivity class under the action of its symmetry group are assigned the same color. The tiles of this tiling are *tangential polygons*, meaning they have incircles. Moreover, the vertices of the tile-8-transitive tiling are regular. A vertex of a tiling is *regular* if the angles formed by consecutive edges incident to the vertex are congruent. See the Geogebra [4] Geometry and Algebra windows in Figure 3.1 corresponding to the construction of Figure 3.2(c), where we use the software to verify and experiment the regularity of vertices.

Conversely, if we consider the incenters of the tiles of the tile-8-transitive tiling, and connect using an edge any two incenters of the adjacent tiles, then we are able to construct the 8-uniform tiling. Actually, experimentations through dynamic tools allow us to formulate the theorem that *an edge-to-edge tile-k-transitive tiling by tangential polygons with regular vertices can be obtained from an edge-to-edge k-uniform tiling and conversely; and both tilings have the same symmetry group* [7,8].

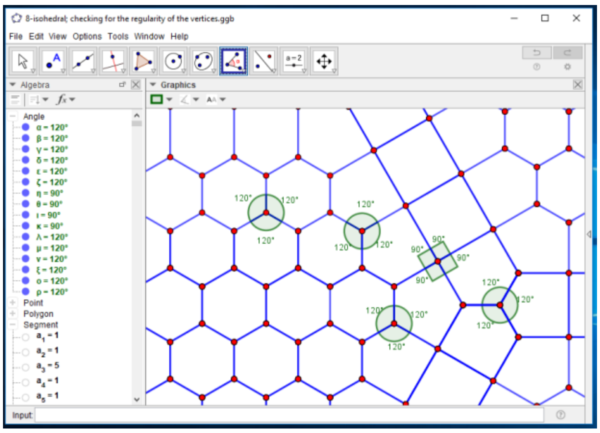
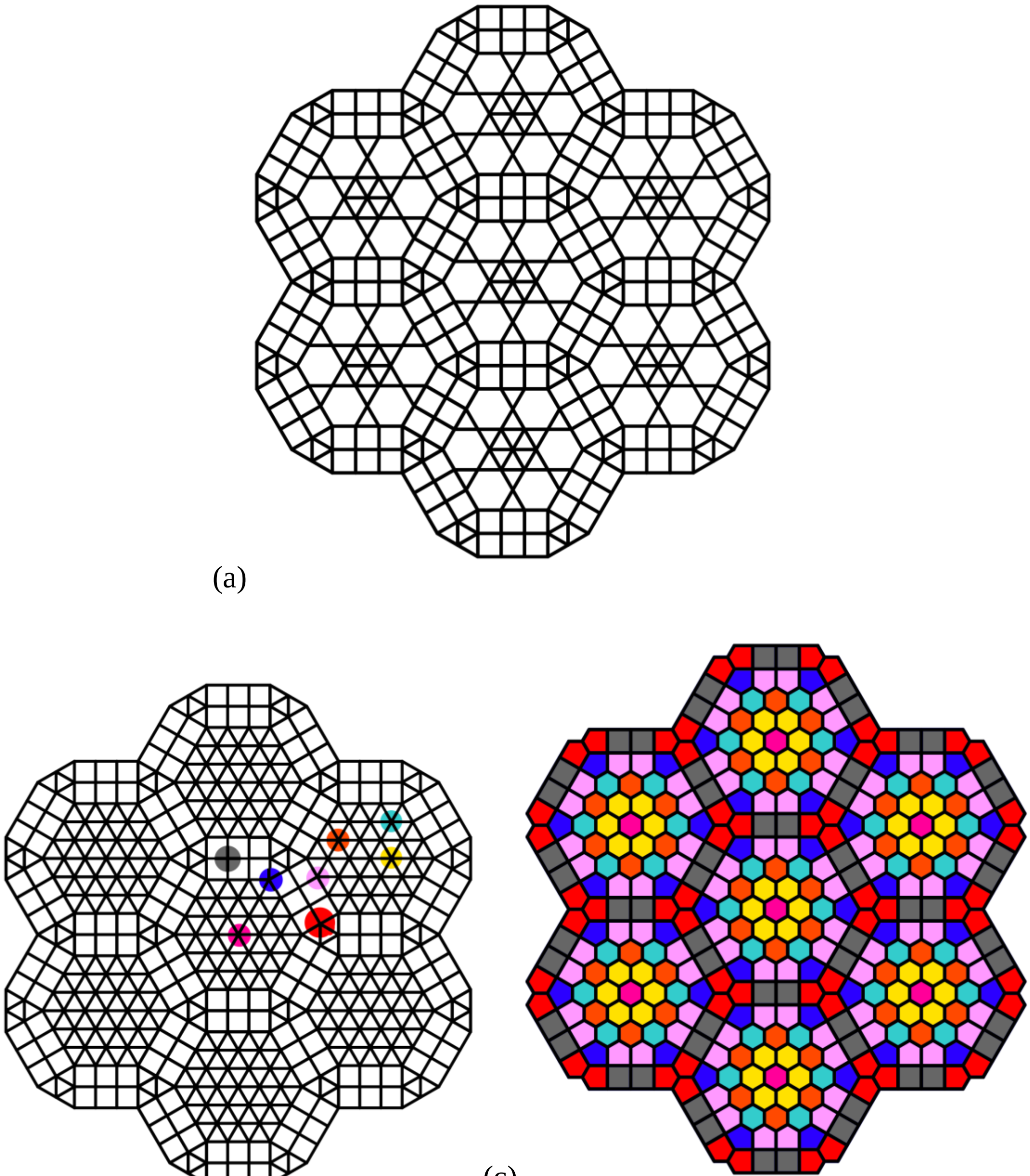



**Figure 3.1** Geogebra [4] screenshot showing the regularity of the vertices of the tiling in Figure 3.2(c).

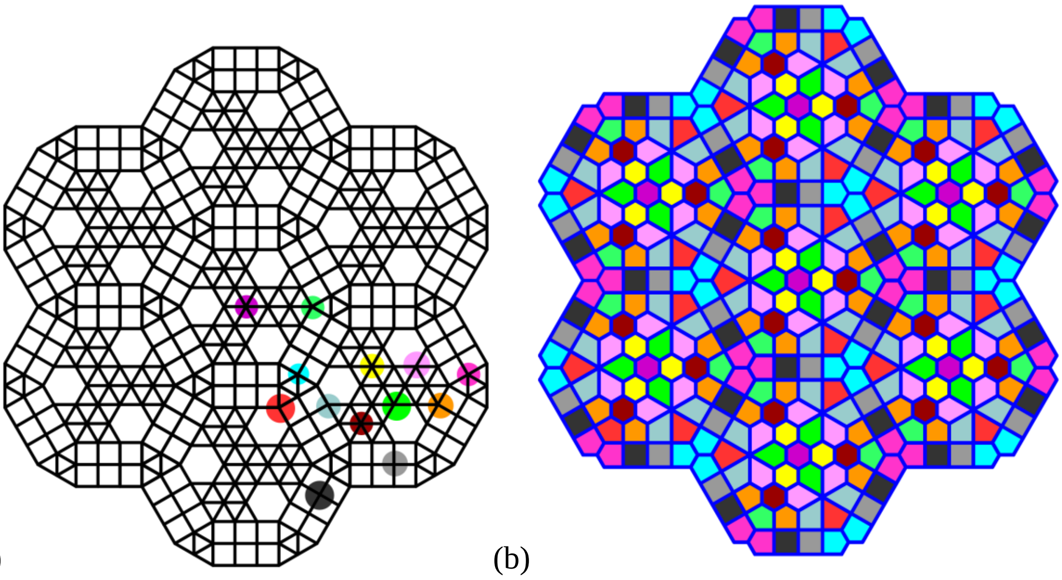


(a) (b) (c)

**Figure 3.2** (a) A 7-uniform tiling [2]; (b) 8-uniform tiling obtained from (a); and (c) tile-8-transitive tiling obtained from (b).

In Figure 3.3(a), we show another tiling by regular polygons obtained from Figure 3.2(a). This time not all hexagons are dissected into triangles. The symmetry group of the resulting tiling is a subgroup of the symmetry group of the 7-uniform tiling. The new tiling has no 6-fold rotational symmetry. It is a 13-uniform tiling. The highlighted vertices shown in Figure 3.3(a) belong to 13 different transitivity classes under the action of its symmetry. From this 13-uniform tiling, we can also obtain the tile-13-transitive tiling by tangential polygons with regular vertices shown in Figure

3.3(b). The tiles belonging to one transitivity class are given the same color. This tile-13-transitive tiling has symmetry group equal to the symmetry group of the 13-uniform tiling.



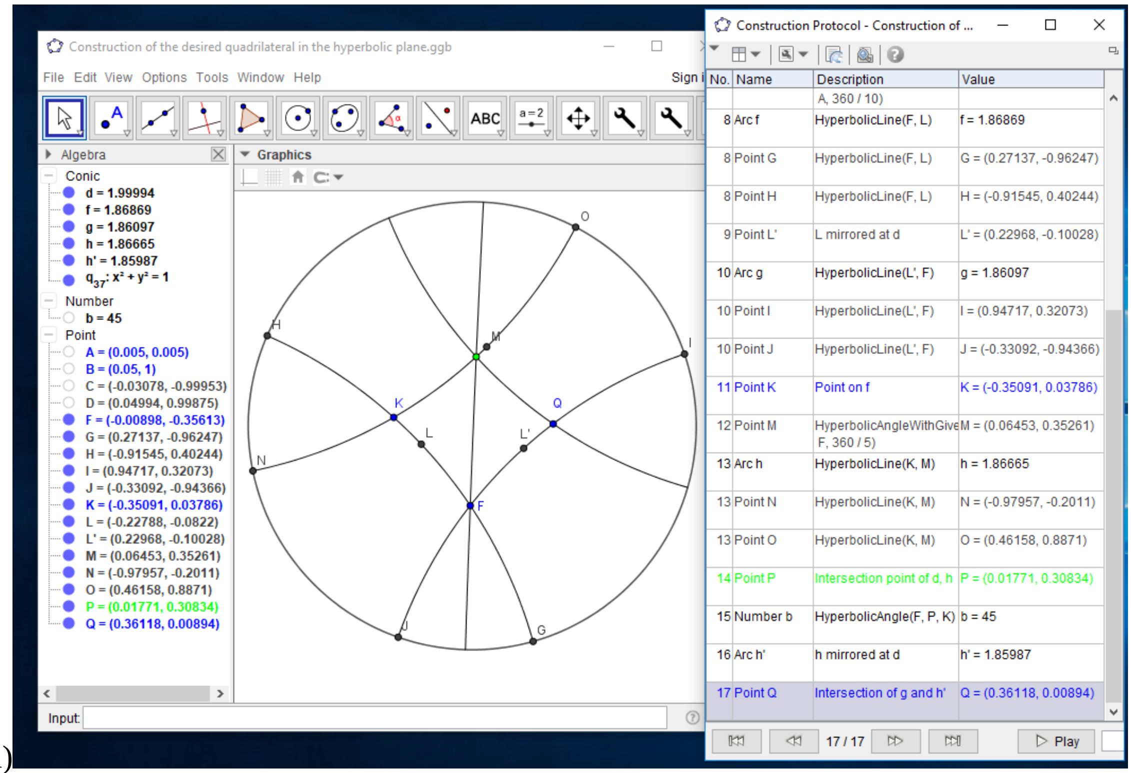
(a) (b)

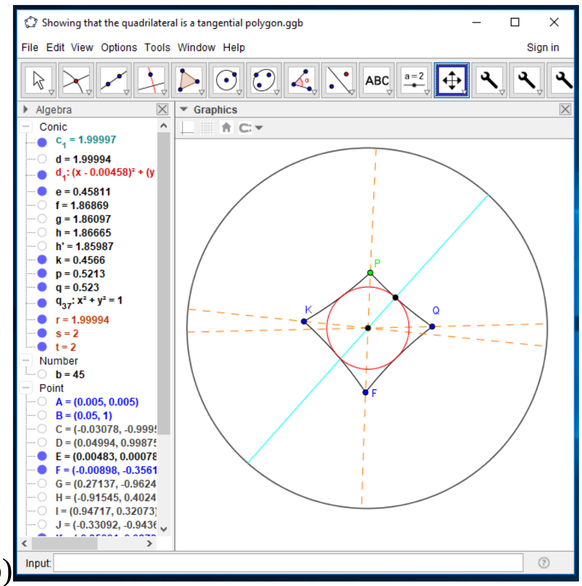
**Figure 3.3** (a) A 13-uniform tiling obtained from Figure 3.2(a), and (b) tile-13-transitive tiling obtained from (a).

## 4. Hyperbolic tilings from tile-*n*-transitive tilings

In this section, we show how we obtain a hyperbolic 2-uniform tiling consisting of 4-gons and 5-gons. We first construct a tile-2-transitive tiling by tangential polygons with regular vertices. In particular, we construct a tiling by quadrilaterals with interior angles $\frac{\pi}{2}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$. The construction of the desired quadrilateral using Geogebra's Construction Protocol Panel [10] is shown in Figure 4.1(a). Figures 4.1(b) and (c) show snapshots of the incenter of the quadrilateral, and the investigation of the regularity of the vertices. Next we apply the isometries, namely, the $180^0$ rotations, $72^0$ rotations and reflections to two quadrilaterals to form the tile-2-transitive tiling in Figure 4.2(b). Centers of 2-fold rotations (red circles) and 5-fold rotations (black hexagons) and axes of reflections are illustrated in Figure 4.2(a). In Conway's notation the tile-2-transitive tiling has symmetry group type **2*52**. A fundamental region or fundamental tile of the tiling is shown in Figure 4.2(b). The notation denotes the presence of 2- and 5-fold rotations lying on a reflection axis (thus the 2 and 5 follow the asterisk) and another 2-fold rotation not on a reflection axis. Tiles belonging to the same transitivity class under the action of its symmetry group are given the same color.

Consider the incenters of the tiles in the tile-2-transitive tiling. By connecting two incenters that belong to two adjacent tiles, we obtain a 2-uniform tiling in the hyperbolic plane (Figure 4.2(c)). The highlighted vertices in Figure 4.2(d) show vertices belonging to two different transitivity classes. It is interesting to note that in this tiling the combinatorial types of polygons incident in each vertex are congruent. This is an example of a *monogonal* tiling.

(a) 


(b) 

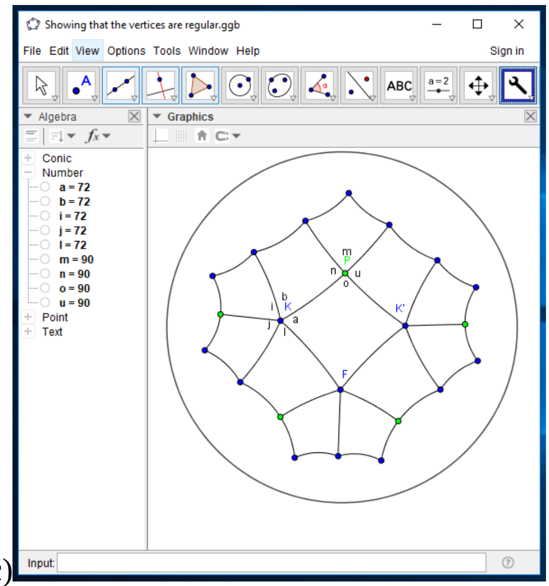

(c) 


**Figure 4.1**. (a) Construction of a quadrilateral using Geogebra [10] with interior angles $\frac{\pi}{2}, \frac{2\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$; (b) incenter of the quadrilateral; and (c) regularity of the vertices of the quadrilateral.

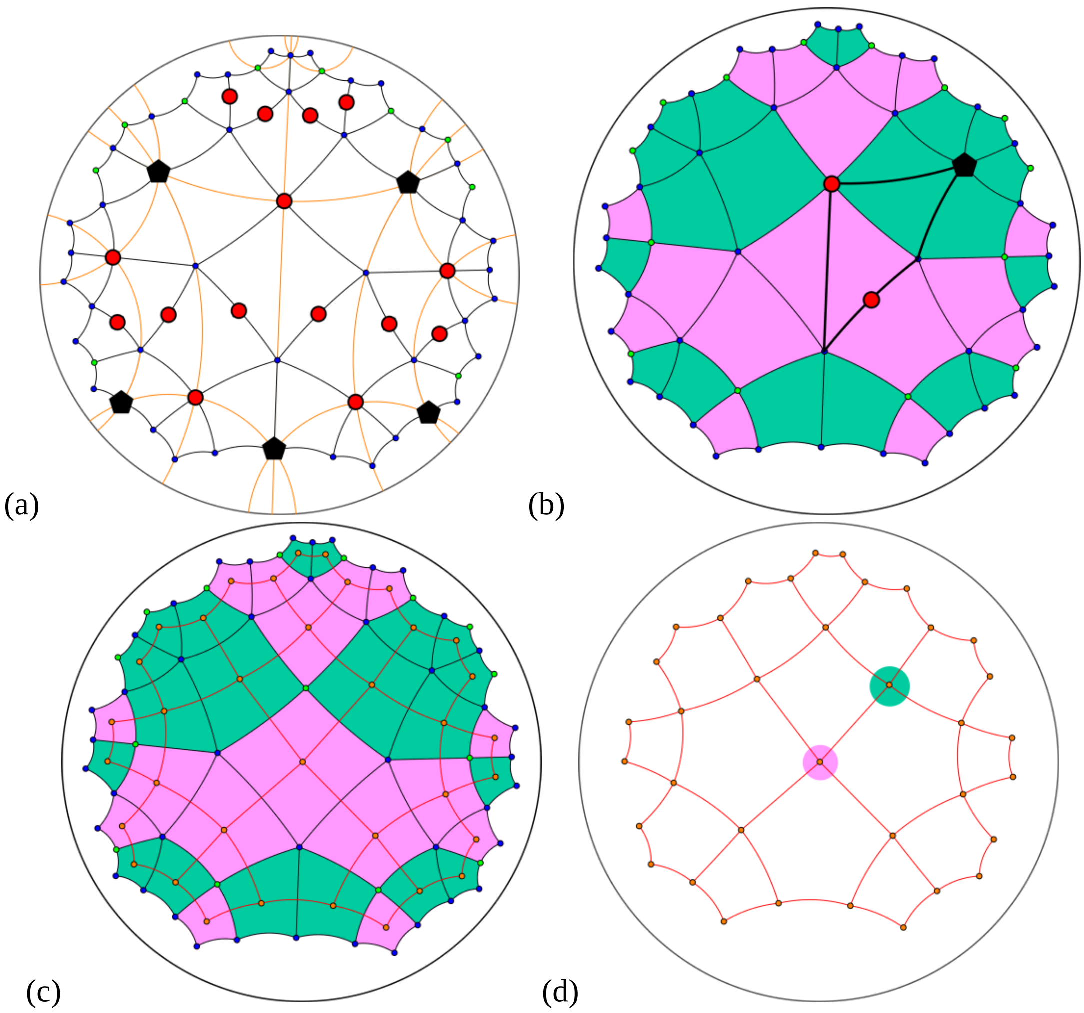


**Figure 4.2** Construction of a 2-uniform tiling in the hyperbolic plane from a tile-2-transitive tiling.

As a final example, we present the construction of a hyperbolic isocoronal tiling, An *isocoronal tiling* is a tiling whose vertex coronae belong to one transitivity class under the action of its symmetry group. The *vertex corona* of a vertex of a tiling is the vertex together with the tiles incident to it. An isocoronal tiling is necessarily a vertex-transitive tiling.

We start with a tile-transitive tiling by quadrilaterals with interior angles $\frac{2\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{4}$ shown in Figure 4.3(a). This tiling has symmetry group $G = < P, Q, R >$ where $P, Q, R$ are reflections with axes shown. In Conway notation, $G$ is denoted by ***832**; there are 8-, 3- and 2-fold rotations with centers lying on reflection axes. Consider a point, which we will denote as $x$ in the interior of a tile in the given tile-transitive tiling. Then we consider a subgroup $H$ of $G$ such that the action of $H$ on $x$ forms orbit of points where each tile in the given tiling contains exactly a point (Figure 4.3(b)). In this case we consider $H = < QR, PQRP >$ where $QR, PQRP$ are 3-fold rotations. Connecting by an edge points that belong to two adjacent tiles results in an edge-to-edge isocoronal tiling shown in Figure 4.3(c). Its vertex corona consists of a regular 3-gon, an irregular 8-gon, a regular 3-gon and another irregular 8-gon.

On the other hand, if we choose $x$ such that there are three collinear points contained in a chain of three tiles, we obtain a non-edge-to-edge isocoronal tiling by regular polygons (Figure 4.3(d)). Its vertex corona consists of 3-gon, 4-gon, 3-gon and 4-gon (in cyclic order). Using various subgroups of $G$ and applying the isometries in each subgroup in the manner described here can facilitate the construction and characterization of families of isocoronal tilings in the hyperbolic plane. We rely on the computer software Groups, Algorithms and Programs (GAP) [9] in listing the subgroup structure of hyperbolic symmetry groups. See Figure 4.4 for a sample GAP output of a list of subgroups of $G$ of a given index.

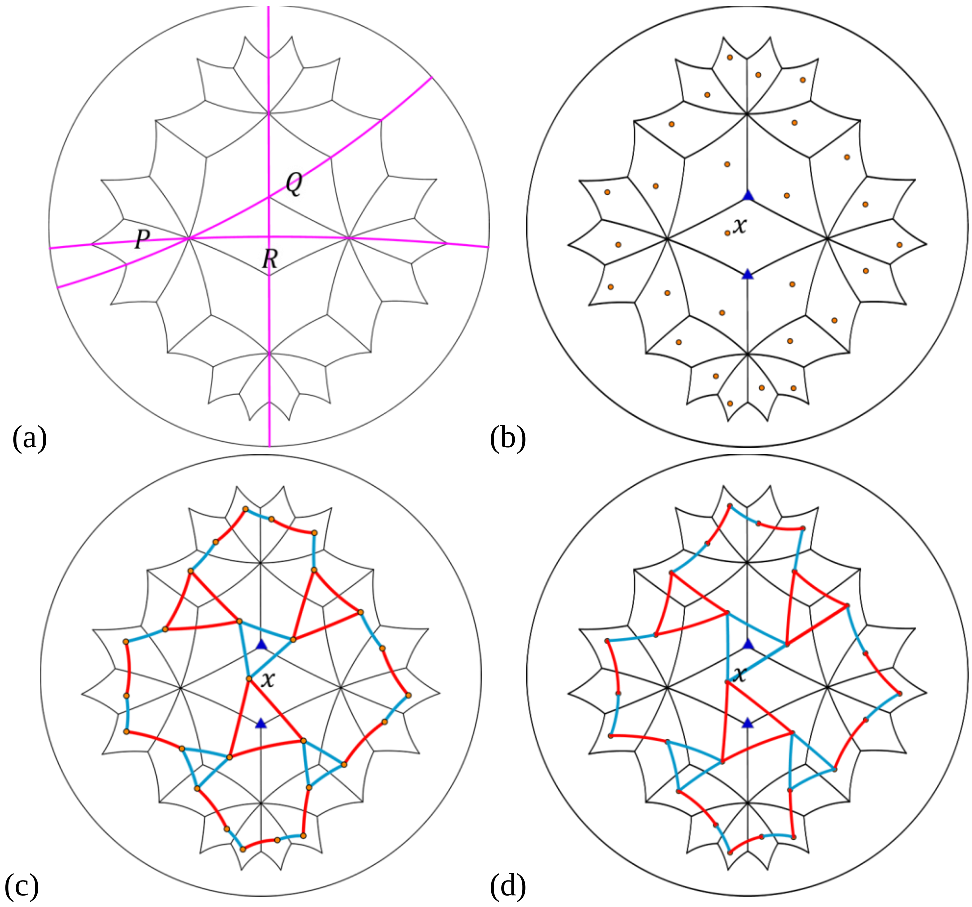



**Figure 4.3** Construction of hyperbolic isocoronal tilings from a tile-transitive tiling.

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Number of Subgroups of Index at most 4 is 7

Subgroups of index : 1
  (1) Generators: [ P, Q, R ]

Subgroups of index : 2
  (2) Generators: [ P, R*Q^-1 ]
  (3) Generators: [ Q, R, P*Q*P^-1 ]
  (4) Generators: [ Q*P^-1, R*P^-1 ]

Subgroups of index : 3
  (5) Generators: [ P, Q, R*Q*P*Q^-1*R^-1 ]

Subgroups of index : 4
  (6) Generators: [ Q, R, P*Q*R*P^-1*Q^-1*P^-1 ]
  (7) Generators: [ R*Q^-1, P*R*Q^-1*P^-1 ]
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**Figure 4.4** Sample GAP output of a list of subgroups of $G$ of index at most 4.

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