The Importance of Adopting Evolving Technological Tools to Expand Content Knowledge to 3D

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics
and Statistics
Radford University
Radford, VA 24142
USA

Abstract

It is clear that many uninspired and uninteresting math problems are created mainly to test students' algebraic manipulation skills. In this paper, we shed some lights on how technological tools can be adopted in a classroom to stimulate students' interests in discovering mathematics. It has been demonstrated in [9] and [10] that college entrance exam problems from China sometimes can be daunting to students. However, if technological tools are adopted for explorations, those problems can become more accessible to more students. Furthermore, those explorations can inspire some deep and serious research activities with the help of technological tools. In this paper, we use examples to reaffirm that the first step to attract students' interests in a math problem is to interpret the problem in a more understandable way such as in a real-life setting. Next, we investigate if further 3D or higher dimension extensions are possible while exploring activities with technological tools. Consequently, many boring exercises can be made lively and appealing to broader students again. Moreover, real-life applications in 3D may become possible from these exploratory activities.

1 Introduction

In this paper, we use technological tools to explore and investigate two challenging problems. The first problem is to find the intersecting areas enclosed by a circle and a circle or an ellipse. The problem could have been boring if it involves only algebraic manipulations. It was made more interesting by interpreting it as solving a donkey problem [6]. The problem, however, becomes more algebraic tedious when it is generalized to other scenarios, not to mention if it is generalized to cases in 3D. We present various methods in handling different scenarios in 2D, which involves using the regular coordinate systems, the change of basis and the Green's theorem. Subsequently, we extend the 2D scenarios to challenging scenarios in 3D. In Section 3, we present a problem that was originated from college entrance practice problems from China [9]. To attract students' attentions to explore more challenging tasks with technological tools, we present the problem by first interpret the 2D and 3D problems into real-life settings.

To make our original problems more accessible, interesting and challenging at times, we typically start with a Dynamic Geometry System (DGS) for construction and exploration, and next forming conjectures before verifying the solutions analytically with a Computer Algebra System (CAS). Two problems discussed in 2D in this paper can be extended to respective 3D scenarios once students have knowledge of multivariable calculus and linear algebra. The author also suggests some possibilities for further research or studies and invites readers to imagine more real-life applications on their own. We encourage students and readers to bravely make conjectures while exploring their activities with their favorite DGS. Only when we can expand our abilities to visualize objects in 3D, will we have desire to expand our content knowledge in validating our observations analytically with a CAS.

2 Donkey Problems

Mathematically speaking, our objective is to find the intersecting areas enclosed by a circle and a circle or an ellipse. To make the problem more appealing (see [6]), the problem in 2D is described as follows: A donkey is tethered with a rope at a grass seeded park that resembles the shape of a circle. Find the length of the rope so the area that can be eaten by the donkey is the same as the area of uneaten portion.

2.1 Simple scenario using rectangular coordinates

The first case can be explored by middle or high school students using rectangular coordinates. Although middle school students may not know the concept of integration, but they can approximate necessary areas.

Example 1 *A donkey is tethered with a rope of length r at the $(0,0)$ position at a park that is about the shape of $x^2 + (y-1)^2 = 1$. Find the length of the rope r so that the area donkey can reach is the same as the area of uneaten portion.*

It is easy to see the graphs of $x^2+(y-1)^2 = 1$ and $x^2+y^2 = r^2$ are symmetric to the y-axis, we find the y-value of these two intersections to be $\frac{r^2}{2}$. We observe from the Figure 1 and label proper areas as follows: Area $a = BDCB$, area $c = CEBC$, area $b = BFCB$. With the help of Maple [3] and assume $r > 0$.

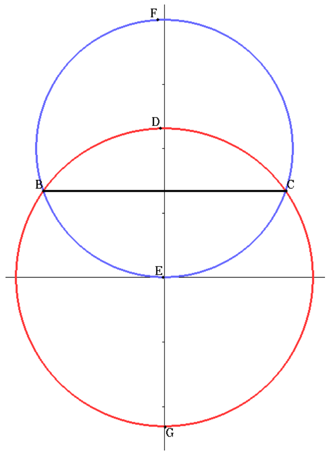

Figure 1. A donkey problem and rectangular coordinates

We obtain

$$
\begin{aligned}
a &= \int_{\frac{r^2}{2}}^{r} \sqrt{r^2 - y^2} dy && (1)\\
&= -\frac{1}{8} r^3 \sqrt{-r^2+4} - \frac{1}{2} r^2 \arcsin(\frac{1}{2} r) + \frac{1}{4} r^2 \pi, && (2)\\
b &= \int_{\frac{r^2}{2}}^{2} \sqrt{1-(y-1)^2} dy && (3)\\
&= -\frac{1}{8} \sqrt{-r^2(r^2-4)} r^2 + \frac{1}{4} \sqrt{-r^2(r^2-4)} - \frac{1}{2} \arcsin(\frac{1}{2} r^2 - 1) + \frac{\pi}{4}, && (4)
\end{aligned}
$$

and

$$
\begin{aligned}
c &= \int_{0}^{\frac{r^2}{2}} \sqrt{1-(y-1)^2} dy && (5)\\
&= \frac{\pi}{4} + \frac{1}{8} r^3 \sqrt{-r^2+4} - \frac{1}{4} r \sqrt{-r^2+4} + \frac{1}{2} \arcsin\left(\frac{r^2}{2} - 1\right). && (6)
\end{aligned}
$$

We thus use Maple [3] to solve for r so that $a + c = b - a$ or, equivalent, $2a = b - c$, which yields

$$r = 1.158728473018121517828233509933. \qquad (7)$$

The preceding simple 2D scenario can be easily extended to the following 3D case.

Example 2 *Consider two spheres of the form* $x^2 + (y-1)^2 + z^2 = 1$ *and* $x^2 + y^2 + z^2 = r^2$. *Find* r *so that the volume that is bounded in between is the same as the volume that bounded outside the sphere* $x^2 + y^2 + z^2 = r^2$ *but inside the sphere of* $x^2 + (y-1)^2 + z^2 = 1$.

We consider the graph from 2D and use the disk method of rotating a proper curve around the y axis. First, we write the x values for these two spheres respectively as

$$x_1 = \sqrt{2y - y^2}, x_2 = \sqrt{r^2 - y^2}. \tag{8}$$

Let

$$d = \int_0^{\frac{r^2}{2}} \pi (x_1)^2 \, dy, f = \int_{\frac{r^2}{2}}^{r} \pi (x_2)^2 \, dy \tag{9}$$

and

$$e = \int_{\frac{r^2}{2}}^{2} \pi (x_1)^2 \, dy. \tag{10}$$

We need to solve for r such that $d + f = e - f$ or $2f = e - d$, we use Maple [3] and obtain $r = 1.9129311827723891011991168$.

2.2 Using a slanted line as the new x-axis

Now we consider an alternative way of finding the area enclosed by curves when the rectangular coordinate systems may be complex to use. The method mentioned here has been introduced in [8], which is accessible to those students who may have only knowledge on basic integration but do not know the concept of line integrals. We consider the following

Theorem 3 *Let C be a smooth curve written as $\mathbf{w}(t) = [x(t), y(t)]$ (say for example C is the curve BDA in Figure 2), where $t_1 \leq t \leq t_2$. Let R be the region bounded by C and by a line segment (say for example AB in Figure 2, and is of the form $y = mx + b$). Then the area of R is given by*

$$\frac{1}{1+m^2} \int_{t_1}^{t_2} (-x(t)m + y(t) - b) \left(x'(t) + y'(t)m\right) dt. \tag{11}$$

In short, the formula (11) is derived by using AB as the new $x-$axis and a perpendicular line to AB as its new $y-$axis.

Example 4 *We are given the shape of the park resembles the ellipse of $\frac{x^2}{16} + \frac{y^2}{9} = 1$. A donkey is tethered at the point $(5, 5)$. Find the length of the rope so that the area of the eaten by the donkey that is inside the park is equal to the area of the uneaten portion outside the park.*

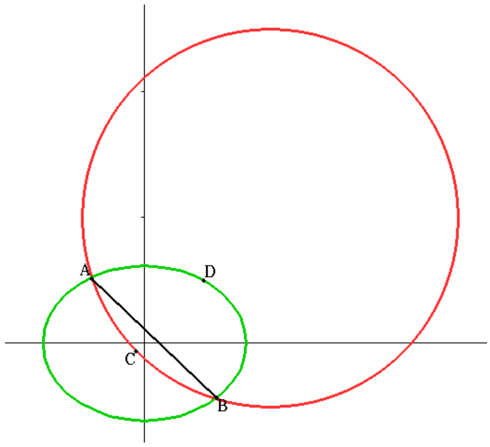

Figure 2. A donkey problem using
a slanted line

We let A_1 be the area of $BDAB$ and A_2 be the area of $BCAB$. In essence, we need to find the radius r so that

$$A_1 + A_2 = \frac{\pi ab}{2}, \tag{12}$$

where $a = 4$ and $b = 3$.We write the parametric equations for the circle and the ellipse respectively as follows $[x_p + r\cos s, y_p + r\sin s]$ and $[a\cos t, b\sin t]$, where $s, t \in [0, 2\pi]$. We proceed to find a proper radius for our needs. In this case, since the point $(5, 5)$ is outside the ellipse, we can form two circles with the center $(5, 5)$ and radii r_1 and r_2, respectively with $r_1 < r_2$, so that two respective circles are tangent to the ellipse. We start with a value for the radius r satisfying the condition $r_1 < r < r_2$. We thus start with a test value say $r = 7$ as a test value and use the bisection method to check the signs for $A_1 + A_2 - \frac{\pi ab}{2}$. If $A_1 + A_2 - \frac{\pi ab}{2} > 0$, then we reduce r. Otherwise, we increase r. After few trials and errors, we see if we set $r = 7.35$ and extract two real intersections between the circle and the ellipse as follows:

$$\begin{aligned}
s_1 &= -2.81154928777274, s_2 = -1.85760686279737, && (13)\\
t_1 &= 2.08096790465439, t_2 = -.752208960479492, \\
A &= (x_1, y_1) = (-1.95330793332386, 2.61798220317826), && (14)\\
B &= (x_2, y_2) = (2.92072548750653, -2.04976010242157). && (15)
\end{aligned}$$

We find the slope m of AB and the y-intercept of AB and apply Theorem 3 to obtain $A_1 + A_2 - \frac{\pi ab}{2}$ to be 0.0622432798052. To obtain even higher accuracy for the answer, we may apply the bisection method from this point on to find appropriate r. We already see that a CAS is needed in this case for computing and a hand-held graphic calculator will be very helpful for visualizing the directions of parametric curves.

2.3 Applying the Green's Theorem

Here we introduce the powerful Green's theorem, which uses the concept of line integrals, to find the area enclosed by curves. We note the conclusion of finding the area enclosed by curves is consistent with the Theorem 3 when using the change of basis, which we state as follows:

Theorem 5 *Let C be a smooth curve written as $\mathbf{w}(t) = [x(t), y(t)]$ (for example, say C is the curve BDA in Figure 2), where $t_1 \leq t \leq t_2$. Let R be the region bounded by C and by a line segment (for example it is AB in Figure 2, and is of the form $y = mx + b$). If $P(x, y) = \frac{-y}{2}$ and $Q(x, y) = \frac{x}{2}$, then the area of R is given by*

$$\begin{aligned} \int_{C \cup \overline{AB}} P\,dx + Q\,dy &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \frac{1}{1+m^2} \int_{t_1}^{t_2} \left(-x(t)m + y(t) - b\right) \left(x'(t) + y'(t)m\right) dt. \end{aligned} \tag{16}$$

We apply the Green's Theorem to find the area of enclosed by circle and ellipse when we have four intersections, which we state in Example 6.

Example 6 *We are given the shape of the park that resembles the shape of an ellipse $x^2 + \frac{y^2}{25} = 1$. A donkey is tethered at the point $(-1, 0)$. Find the length of the rope so that the area of the eaten by the donkey that is inside the park is equal to the area of the uneaten portion outside the park.*

Mathematically, we need to consider the intersections between the circle $(x+1)^2 + y^2 = r^2$ and the ellipse $x^2 + \frac{y^2}{25} = 1$. We write the circle as $[-1 + r\cos s, r\sin s]$ and the ellipse as $[\cos(t), 5\sin(t)]$ respectively. After few trials on varying the radius r and use a DGS for visualization purpose, we find that to have a chance for the area of eaten portion is the same as that of uneaten portion, we need to have four intersections between the circle and the ellipse. If we choose $r = \sqrt{7.6}$ and use Maple [3] to find four intersections (see Figure 3).

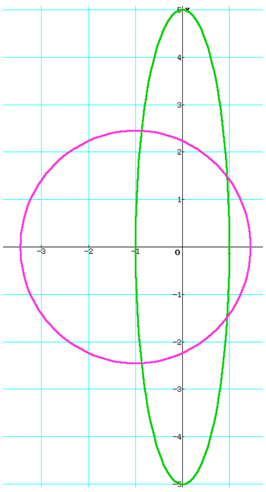

Figure 3. A donkey problem using the Green's Theorem

For the circle, we label the intersections in counter clockwise direction as

$$s_1 = 0.8012307072, s_2 = 1.510879382, s_3 = -1.510879383$$

and

$$s_4 = -0.801230708.$$

On the other hand, we label the intersections for the ellipse, in counter clockwise direction, as

$$t_1 = .4071515203, t_2 := 2.558782282, t_3 = 3.724403025$$

and

$$t_4 = -0.407151521.$$

In view of the Green's theorem and by setting $P(x, y) = \frac{-y}{2}$ and $Q(x, y) = \frac{x}{2}$, we see the following line integral represents the area that can be eaten by the donkey

$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} P(x, y)dx + Q(x, y)dy, \tag{17}$$

where C_1 *is the ellipse from* $t = t_4$ *to* $t = t_1$, C_2 *is the circle from* $s = s_1$ *to* $s = s_2$, C_3 *is the ellipse from* $t = t_2$ *to* $t = t_3$, *and* C_4 *is the circle from* $s = s_3$ *to* $s = s_4$. We thus evaluate the difference between $\int_{C_1 \cup C_2 \cup C_3 \cup C_4} Pdx + Qdy$ and the area of half of ellipse $\frac{\pi ab}{2}$, and yields

$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} P(x, y)dx + Q(x, y)dy - \frac{\pi ab}{2} = 0.017988876. \tag{18}$$

To obtain more accuracy, we may use the bisection method from this point on to find the appropriate radius r. It is clear that Example 6 will become very complicated if one intends apply the earlier two methods to solve it.

Example 7 *Consider the Example 4, we shall apply the Green's Theorem to verify that our answer is consistent with the one by using the slanted line. We refer to Figure 2 and consider the curve* C_1 *to be traveling from* B *to* A *while using the ellipse, and we use* $t_1 = -.7522089605$ *for the point* B *and* $t_2 = 2.080967905$ *for the point* A. *Similarly, we let* C_2 *be the circle traveling from* A *to* B *and we use* $s_2 = 3.47163601917958$ *for the point* A *and use* $s_1 = 4.42557844417958$ *for the point* B. *Now we consider*

$$\begin{aligned} & \int_{C_1} P(x, y)dx + Q(x, y)dy + \int_{C_2} P(x, y)dx + Q(x, y)dy - \frac{\pi ab}{2} \\ = & \int_{t_1}^{t_2} P(x, y)dx + Q(x, y)dy + \int_{s_2}^{s_1} P(x, y)dx + Q(x, y)dy - \frac{\pi ab}{2} \\ = & \ 0.0622432819877, \end{aligned} \tag{19}$$

which is consistent with the answer from Example 4 up to the seventh decimal place.

We remark that we have discussed three methods in solving the donkey problems here: Namely, the rectangular coordinates, the change of basis and the Green's Theorem. We have applied these three methods in solving two and four intersections between a circle and an ellipse. We encourage readers to adopt these three methods appropriately when there are three intersections between a circle and an ellipse. In the next subsection, we extend the 2D scenarios to the corresponding 3D cases and note that we will lose the analogous theorem to the Green's Theorem when we want to find the intersecting surface areas bounded by two respective surfaces.

2.4 Description of 3D extensions

The objective here is to find the radius of a sphere such that the surface areas of the portions of the sphere and an ellipsoid that are inside one another are equal. As it turns out, the question addressed is applicable in more important contexts than the above problem. In fact, the intersection of two quadrics is a question that has been studied extensively in various contexts because of its usefulness in computer aided design, solid modeling, and design of mechanical parts. We limit our attention to the question for ellipsoids and spheres and discuss the case when using rectangular coordinates is possible, which has been studied in [7]. For completeness of this paper, we briefly state the theory. Without loss of generality, we assume the ellipsoid is fixed and centered at the origin with the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{20}$$

where a, b, c are fixed positive constants.

We denote by (h, k, l) the fixed center of the sphere and by r the radius of the sphere. The equation of the sphere is then

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \tag{21}$$

It is clear that we need to compute the surface integrals involving

$$\int_{x_1}^{x_2} \int_{\phi_1(x)}^{\phi_2(x)} \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dy\, dx \tag{22}$$

where $f(x, y)$ is obtained from Eqs. (20 and 21) respectively. We remark that the surface integral in Eq. (22) is useful only when rectangular coordinates can be adopted. We see from the following Example 8 that we have no problem projecting the intersecting curve onto $z = l$ so we can apply Eq. (22).

Example 8 *For our first example, we use $(a, b, c) = (2, 3, 4)$, $(h, k, l) = (0, 0, 5)$ and $r = 2$. In this simple case the center of the sphere is on the z-axis. The intersection curve is determined by two real branches $y_1(x)$ and $y_2(x)$ where*

$$y_1(x) = \frac{3}{7}\left(\sqrt{-541 - 21x^2 + 20\sqrt{760 + 35x^2}}\right).$$

and $y_2(x) = -y_1(x)$.

The real domain of $y_1(x)$ and $y_2(x)$ is approximately $(-1.1055, 1.1055)$. The intersection curve is below the equator of the sphere and above the equator of the ellipsoid. The curve intersects neither equator for the quadrics. We refer to [7] to find the intersecting curves and the respective spherical and ellipsoidal surface areas are 5.4 and 5.8. For detailed computational results, we refer readers to [7]. These surfaces are depicted in Figure 4 and again, it is intuitive to see from Figure 4 that there is no problem of using rectangular coordinate to compute the respective surface areas in this case.

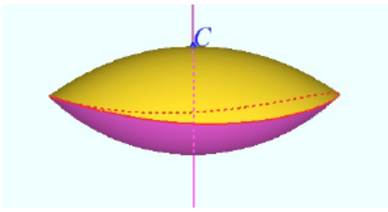

Figure 4. Surface areas and rectangular coordinates

Example 9 *For our second example, we use* $(a, b, c) = (2, 3, 4)$*,* $(h, k, l) = (1, 2, 3)$ *and* $r = 2.2574$*. The surface areas of the portion of the sphere inside the ellipsoid and the portion of the ellipsoid inside the sphere are each approximately equal to* 13.8*, see [7].*

Again, we refer to [7] for details in finding the intersecting curve between these two surfaces and the respective surface areas using rectangular coordinates. The corresponding surfaces are depicted in Figure 5(a). Figure 5(b) depicts the projection of the curve onto the xy-plane and the actual three-dimensional curve.

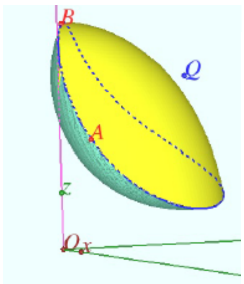

Figure 5(a). Two surface areas and an intersecting curve

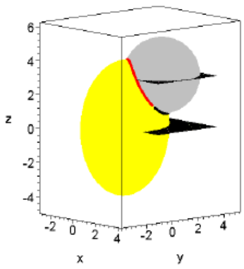

Figure 5(b) Two surface areas and horizotal planes

In Example 9, we start seeing that by projecting the intersecting curve between two surfaces onto the rectangular coordinates may not be always ideal. It is natural to ask if we can find an analogous extension of Theorem 3 and find the respective surface areas by projecting both surface areas onto **a properly selected plane**. We describe the theoretical procedure as follows: We assume the surfaces described as $\mathbf{w}(s,t) = \{[x(s,t), y(s,t), z(s,t)] \in \mathbb{R}^3 : (s,t) \in D\}$ are smooth. Let $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and $\mathcal{B}_2 = \{\frac{\mathbf{p}_1}{\|\mathbf{p}_1\|}, \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|}, \frac{\mathbf{n}}{\|\mathbf{n}\|}\}$ be another orthonormal basis for $\mathbb{R}^3$. If $\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix}$ represents the surface using basis $\mathcal{B}_2$, then we need

$$\begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix} \begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}. \tag{23}$$

Keeping in mind that we would like to find the surface areas enclosed by the intersecting curve C between a sphere and an ellipsoid (say in Figure 5(a)). We describe the surface $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \leq t \leq t_2$, $s_1 \leq s \leq s_2$, representing either the sphere or the ellipsoid, and is bounded by a simple closed curve C. We note that we need to choose a proper plane P and appropriate basis to guarantee the projection of the intersecting curve onto P still remains as a simple closed curve *if possible.* With this in mind, we let $P = \{\mathbf{d} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ be a plane that does not touch the intersecting curve. Furthermore, we assume P does not necessarily pass through the origin and has the z-intercept $\mathbf{d}$. In other words, P is the plane spanned by $\mathbf{p}_1$ and $\mathbf{p}_2$, passing through $\mathbf{d} = (0, 0, d_3)$.

Using the same notation as we did previously, we replace $\mathbf{w}$ with $\mathbf{w} - \begin{pmatrix} 0 \\ 0 \\ d_3 \end{pmatrix}$ if necessary (by doing this, we are shifting the surface $\mathbf{w}$ down vertically by d_3). In other words, we have

$$\begin{bmatrix} p(s,t) \\ q(s,t) \\ r(s,t) \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{p}_1}{\|\mathbf{p}_1\|} & \frac{\mathbf{p}_2}{\|\mathbf{p}_2\|} & \frac{\mathbf{n}}{\|\mathbf{n}\|} \end{bmatrix}^{-1} \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) - d_3 \end{bmatrix}. \tag{24}$$

If we set

$$\begin{aligned} E &= \left(\frac{\partial p}{\partial s}\right)^2 + \left(\frac{\partial q}{\partial s}\right)^2 + \left(\frac{\partial r}{\partial s}\right)^2, \\ G &= \left(\frac{\partial p}{\partial t}\right)^2 + \left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial r}{\partial t}\right)^2, \text{ and} \\ F &= \frac{\partial^2 p}{\partial u \partial v} + \frac{\partial^2 q}{\partial u \partial v} + \frac{\partial^2 r}{\partial u \partial v} \end{aligned} \tag{25}$$

Then the **surface area** bounded by the parametric surface $\mathbf{w}(s,t) = [x(s,t), y(s,t), z(s,t)]$, $t_1 \leq t \leq t_2$, $s_1 \leq s \leq s_2$, with respect to a general plane $P = \{\mathbf{d} + a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ is

$$A = \int\int \sqrt{EG - F^2} \left| \begin{bmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{bmatrix} \right| dsdt. \tag{26}$$

We remark that we may choose the plane P of $x + 2y + 3z = 0$ in Example 9, where the normal vector $\mathbf{n} = (1, 2, 3)$ for P is chosen by connecting the center of the sphere and the origin, see Figure 6 below. In addition, we pick $\mathbf{p}_1$ and $\mathbf{p}_2$ to be any two perpendicular vectors on the plane P so that $\{\mathbf{p}_1, \mathbf{p}_1, \mathbf{n}\}$ forms an orthogonal basis.

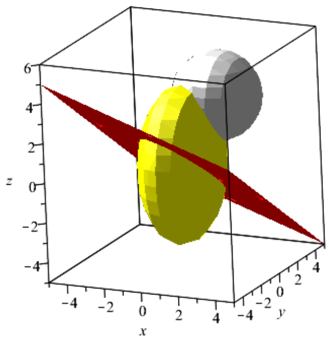

Figure 6. Two surfaces
and a slanted plane

We now apply Eq.(26) and leave detailed calculations in this case to future studies.

Remarks:

1. We invite those readers, who are familiar with the Stokes and Divergence Theorems, to investigate further why we do not have an analogous theorem as Theorem 5 to find the surface area bounded by an intersecting curve, such as shown in Figure 6 above, unless both surfaces are collapsed onto a plane and enclosed by a plane curve.

2. We also invite readers to explore interpreting the 3D extensions as real-life applications.

3 When Should A Slower Runner Start Speeding Up?

There are two long distance runners B and C competing at a track course that whose shape resembles a circle or an ellipse (circle shown in Figure 7(a)) or an ellipse (Figure 8). The coach for the runner B sees that runner B is running behind C at a fixed angle from the point A where the coach sits. At which point do you think the coach for the runner B should signal runner B to start speeding up so B has a chance to pass C? This question is derived from a college entrance practice exam in China, which has been discussed in details and we refer readers to [9] for solving the problems analytically. Instead, we discuss 'what if' scenarios when a Dynamic Geometry System (DGS) is available for students to explore in this section. Indeed, author did conjecture where the possible solutions could be by first exploring with the help of a DGS before validating the results with a CAS.

3.1 Runners for the circle case

Consider the following Figures 7(a)-(c):

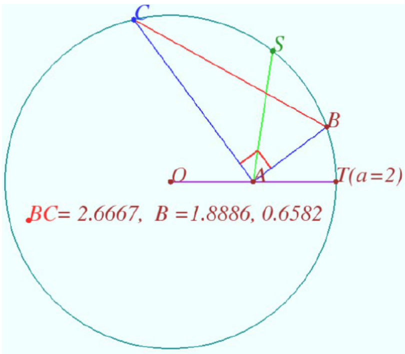

Figure 7(a). Circle and two runners

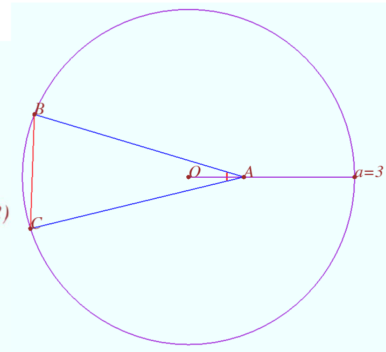

Figure 7(b). Longest distance between two runners

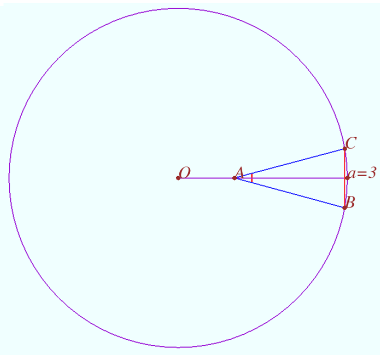

Figure 7(c). Shortest distance between two runners

The observation point $A = (a, 0)$ is located on the horizontal axis of the given circle, which is centered at $O = (0, 0)$ and of radius a. The angle $\measuredangle BAC$ is kept at a fixed angle β_0. The original Chinese university entrance practice is to find where the extreme lengths of BC could be when $a = 2$ and angle $\measuredangle BAC = 90°$. Using your favorite DGS for construction, we make the moving points B and C on the circle and observe when the length of BC might reach either maximum or minimum. We invite readers to further explore the followings:

1. What scenario will make these two runners running at a constant gap in distance?

2. By changing the fixed angle $\beta_0 \in [0, \pi]$, can you conjecture when will the distance between B and C be the longest and the shortest respectively?

After exploring with a DGS, one can easily make educated guesses that the maximum length occurs when BC is perpendicular to OT and BC is on the opposite side of A (see Figure 7(b)). Similarly, the minimum length occurs when BC is perpendicular to OT and BC is on the same site as A (see Figure 7(c)). We refer readers to [9] and see how these observations can be verified analytically with the CAS Maple [3].

3.2 Runners and the ellipse case

Now we assume two runners are on a track field which resembles an ellipse (see Figure 8).

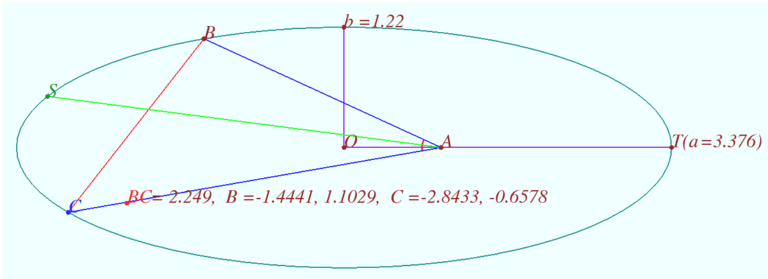

Figure 8. Two runners and an ellipse

Let the coach for the runner B sit at the observation point $A = (d, 0)$ and is inside the given ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $d < a$. The runners B and C are on the ellipse such that the angle $\beta_0 = \measuredangle BAC$ is kept as a fixed angle and the runner C is ahead of the runner B (see Figure 8). We want to know when the coach for the runner B should signal runner B start speeding up so the runner B has the best chance to catch up on the runner C. Mathematically, we would like to find the minimum of the length BC. For completeness purposes, we discuss the extreme lengths for BC in this case. As we have done for the circle case, we first make use of GInMA [2] and ClassPad [1] to conjecture where the solutions might be. We draw the given ellipse and the fixed point A. We place moving point B on the ellipse curve, rotate AB by a fixed angle β_0 around point A, construct ray AC and define the point C as the point of the intersection of ellipse and ray AC (See Figure 8). We again emphasize that a DGS will allow us to make challenging problems more accessible to more students. For example, we may place the point B at any position on the ellipse and record the length of the chord BC. For example, the green curve in Figure 5 shows the scattered plot when we use the x-coordinates of the point C as inputs (x) and the lengths of BC as the outputs (y).

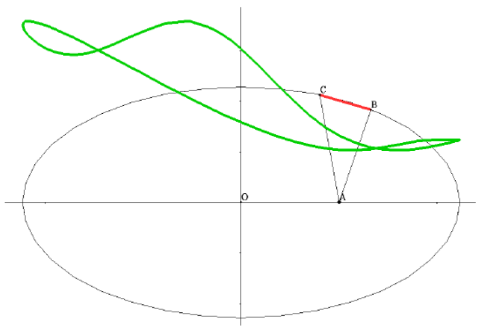

Figure 9. Scatter plot and the
extreme length BC

Learners can conjecture where the position of C will correspond to the extreme lengths of BC by moving the point C. We refer readers to the video clip [4] to see how we use GInMA [2] and ClassPad [1] for exploring the extreme lengths of BC. Consequently, when the length of BC is at its minimum, it is the best chance for the runner B to catch up with the runner C if the runner B starts speeding up. We observe from the computed extreme lengths for BC (see Figure 9), when $a = 2.24, b = 1.15, d = 1$ and $\measuredangle BAC = 30°$.

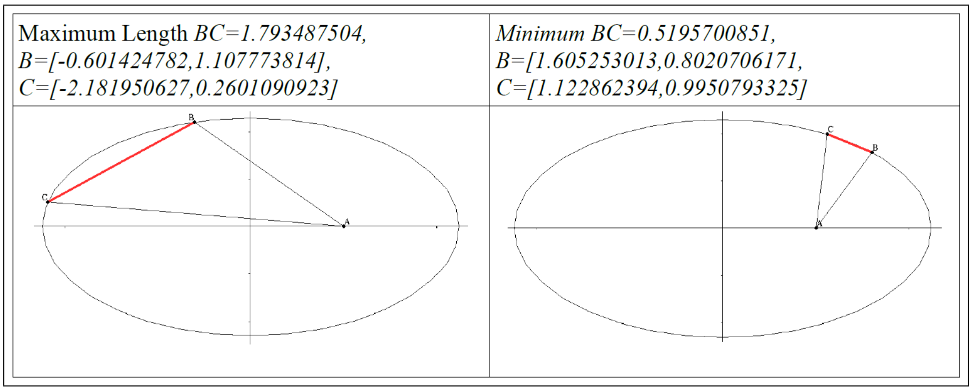

Figure 9. A case for the extreme lengths of BC

Indeed, it follows from Figure 9 and further computed results stated in [9] that we do not see a pattern how we can achieve the extreme lengths for BC in general for a given a, b, d and $\measuredangle BAC$. However, if a DGS is available for exploration, it is not hard to make the following observations, which of course need to be verified analytically in future studies.

Discussions. Here we consider the observation point is at $A = (d, 0)$:

1. When $d \to 0^+, \measuredangle BOC = 90$ degree, the largest length BC occurs when BC is when B is close to the point $(0, b)$ and C is close to the point $(-a, 0)$, see Figure 10 below.

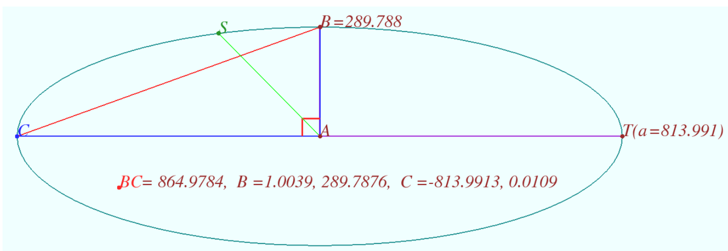

Figure 10. The largest length of BC when $d \to 0^+, \measuredangle BOC = 90$ degree

2. When $d \to 0^+$, BOC is 90 degree, the shortest length BC occurs when the angle bisector

of $\measuredangle BOC$ is perpendicular to the minor axis, see Figure 11.

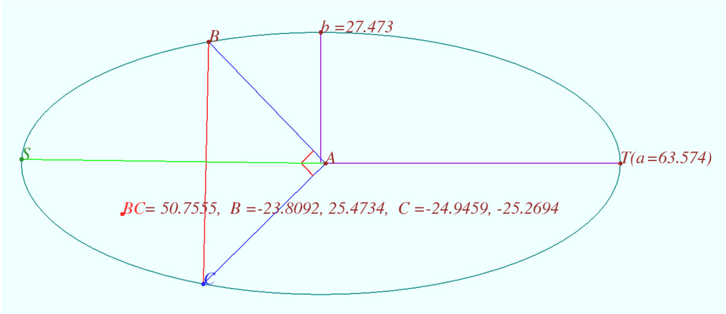

Figure 11. The shortest length of BC when $d \to 0^+, \measuredangle BOC = 90$ degree

3. For $A = (d, 0)$ with $d < a$ and a fixed angle $\measuredangle BAC$. If we denote the extreme length for BC as $F(d)$, then F is a continuous function. In other words, the extreme lengths for BC will not vary too much if we vary d within a specified range.

3.3 Where can we find the largest or smallest intersecting surface area?

With the help of technological tools, we discuss the scenarios when we extend the 2D problems in the preceding section to corresponding ones in 3D. It is not difficult to visualize and extend this problem from a circle to a sphere in 3D. We describe how the software GInMA [2] gives a rich visual intuition to our conjectures. First, we describe the 3D setting mathematically as follows: Given a sphere of the form $x^2 + y^2 + z^2 = r^2$ and pick the point $A = (d, 0, 0)$, where $d \in [0, r]$. Let B be a point on the sphere and rotate AB with a fixed angle β to form a cone, see Figure 12(a) or 12(b). We want to find, respectively, the maximum and minimum intersecting surface areas between the cone and the sphere. After exploring with GInMA [2], which we refer readers to the video clip [5], it is not difficult to conjecture that the maximum intersecting surface area occurs when the normal vector at B is parallel to the vector OA and A is on the opposite of B (see Figure 12(a)). The minimum intersecting surface area occurs when the normal vector at B is parallel to the vector OA but A is on the same side of B (see Figure 12(b)). We note that the results are consistent with our answers from the corresponding

2D scenarios.

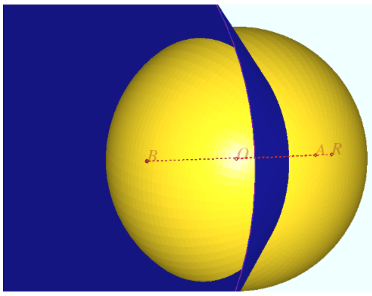

Figure 12(a). Maximum intersecting surface area

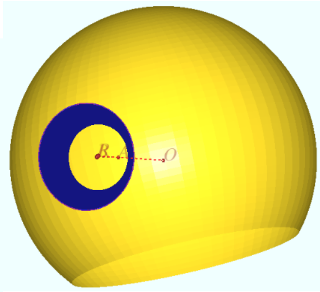

Figure 12(b). Minimum intersecting surface area

Real-life application. We may interpret the 3D scenario as following: There is a radar beam resemble the shape of a cone, which starts at the point A and its beam center points at a moving point B on the sphere. We want to find the point B on the sphere that will result in achieving the largest or smallest intersecting surface areas between the cone and the sphere.

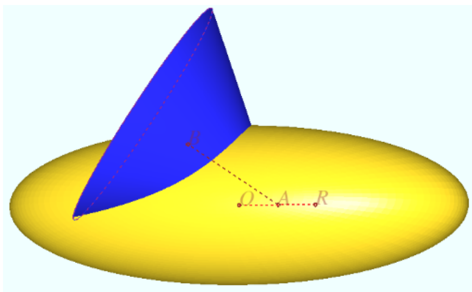

Figure 13. An ellipsoid and a cone

The problem becomes even more challenging and **unsolved** as mentioned in [9] if we replace the sphere by an ellipsoid. For example, given an ellipsoid of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and let $A = (d, 0, 0)$ with $d \in [0, a]$ be fixed. Pick B be on the ellipsoid and consider the cone that is determined by rotating the axis AB with a fixed angle β, see Figure 13. Find the point B so that the maximum or minimum intersecting surface area between the ellipsoid and the cone (see Figure 13) can be achieved respectively. Just like ellipse cases, we expect the answers vary depending on the shape of ellipsoid, the fixed angle β and the position of the point A. The video clip of this ellipsoid and a cone problem can be seen in the latter part of [5]. Furthermore, we observe the following

Discussions:

1. As we have seen in section 2.4 that we may apply Eq. (26) to find the intersecting surface area between a cone and an ellipsoid, by using a projection method. In this case, we may use the vector AB as the normal vector for the plane P and let the plane P pass through A and note consequently that the intersecting curve between the cone and the ellipsoid does not intersect the plane P.

2. Although we do not expect to find a pattern for finding a general answer for the ellipsoid and cone scenario, we leave it to readers to explore and verify if we have analogous phenomenon to 2D scenarios when A is getting closer to the center of the ellipsoid $(0,0,0)$, and the rotating axis AB for the cone is rotating at the angle of $45°$. More precisely, we make the following observations:

   (a) When $A \to (0,0)$ and the angle of rotating axis AB is 45 degree, we conjecture that the resulting minimum intersecting surface area between the cone and the ellipsoid happens when AB is parallel to OB. We depict these scenarios in Figures 14(a) and 14(b):

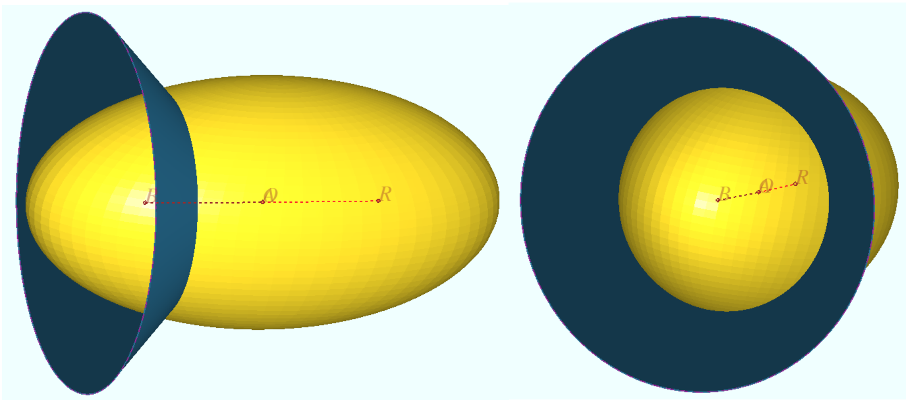

Figure 14(a). Special case when A is close to the center of ellipsoid

Figure 14(a) Different view when AB is parallel to OB

   (b) When $A \to (0,0)$ and the angle of rotating axis AB is 45 degree, we conjecture that the resulting maximum intersecting surface area between the cone and the ellipsoid happens when AB is lying on the xz plane and the angle between BO and the $z - axis$ is 45 degree. Moreover, we shall get the 2D scenario depicted in Figure 10 if we project the ellipsoid and the cone onto the xz plane.

4 Conclusions

It is known that any form of college examinations is an inevitable policy for many countries to control who can get into which colleges. Consequently, examiners find it necessary to create harder and harder problems in their efforts to identify top students. Naturally students are scared away by those algebraic manipulation intensive problems quickly and we are losing potential students as a result. We all know that teaching to the test can never promote creative thinking skills, which make educators concerned about the nature of assessment methods does not reward exploration. And yet it is common sense that innovation and creativity do not come from drills or rote-type learning, but from exploration. Therefore, we should recognize the importance of implementing a curriculum which involves stimulating the discussion of mathematics and its applications through timely use of technological tools. Furthermore, we encourage readers to continue creating innovative examples by adopting technological tools for teaching and research and to influence their colleagues and communities and the decision makers in their respective countries. Allowing users to drag and view figures from different perspectives definitely assists us before we attempt to set up complex algebraic equations. Through exploring these two examples mentioned in this paper, readers are challenged to think more and do more because of evolving technological tools. Those examples in 2D and 3D explored in this paper indeed can be explored from middle to high schools, university levels, or even beyond. Only when we can explore and visualize objects in 3D with proper tools, will we have motivation to expand our content knowledge accordingly. Access to evolving technological tools has definitely motivated us to rethink how mathematics should be presented more interestingly, and how mathematics can be explored as a cross disciplinary subject.

References

[1] ClassPad Manager, a product of CASIO, https://edu.casio.com/freetrial/freetrial_list.php

[2] Geometry Interactive Mathematical Arts (GInMA): A Dynamic Geometry System, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[3] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[4] A video clip for a 2D circle conjecture: https://mathandtech.org/eJMT_June2015/circle_conjecture.mp4.

[5] A video clip for a 3D sphere, 2D ellipses and 3D ellipsoid conjecture: https://mathandtech.org/eJMT_June2015/sphere.mp4.

[6] A donkey problem found from The Math Forum, http://mathforum.org/dr.math/faq/faq.grazing.html.

[7] Thompson, S, Yang,W.-C. and Shelomovskii,V., The Area of the Solid of Intersection of a Sphere and an Ellipsoid, a First Approach, *The Electronic Journal of Mathematics and Technology*, Volume 7, Number 3, pp. 221-231, 2013.

[8] Yang, W.-C., and Lo, M-L., Finding Signed Areas and Volumes Inspired by Technology, *The Electronic Journal of Mathematics and Technology*, Volume 2, Number 2, pp. 133-149, 2008.

[9] Yang,W-C., Shelomovskii,V. and Thompson, S, Extreme Lengths for Chords of an Ellipse, *The Electronic Journal of Mathematics and Technology*, Volume 9, Number 2, pp. 180-191, 2015.

[10] Yang, W.-C., From Static Locus Problems to Exploring Mathematics with Technological Tools, *The Electronic Journal of Mathematics and Technology*, Volume 9, Number 2, pp. 67-88, 2017.