General triangular arrays of numbers

Hung-ping Tsao

tsaohp.tsao6@gmail.com

1151 Highland Drive

Novato, CA 94929 U.S.A.

Abstract

We present here all foreseeable approaches to derive polynomial expressions for the power-sum of the natural sequence. Throughout, binomial coefficients play as the key role of linking together the product-sums and the power-sums. We take the opportunity to sort out the intricate liaisons among Stirling numbers of both kinds and Eulerian numbers of two orders. We further generalize the related numbers based on the natural sequence to those that are arithmetically progressive sequence-based. As a result, various structures of triangular arrays can be built on top of different underlying bases.

1. Introduction

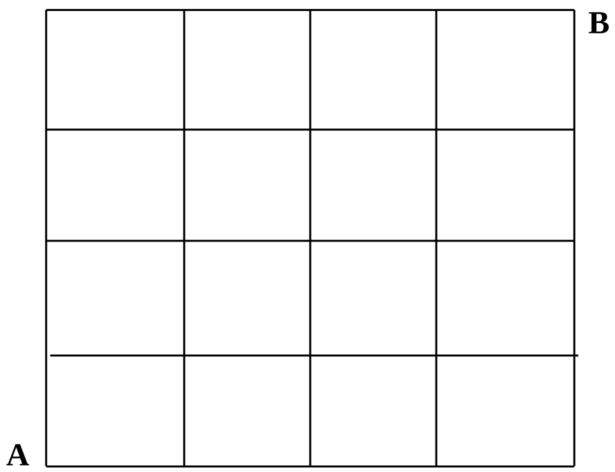

Figure 1.1

Given a figure such as above, people often raise the following two questions.

1) How many squares of all sizes are there?

2) How many different (shortest) paths are there connecting A and B?

The answer to the first question should be $\sum_{i=1}^{4} i^2$, since there are 1^2 4×4 square, 2^2 3×3 squares, 3^2 2×2 squares and 4^2 1×1 squares; whereas the answer to the second question might not be easy to come by. If you use the brute force approach, it could get very messy. A wiser way is to use the combinatorial approach as follows.

Since all possible four vertical (or horizontal) moves out of eight moves would be needed to go from A to B, the answer should be the binomial coefficient $C(8,4)$. So, there ought to be seventy different paths connecting A and B, according to Pascal triangle $P\Delta$ as shown below with $n = 8$ and $k = 4$.

$C(n,k)$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Table 1.1

Pascal triangle was discovered about one thousand years ago by Al-Karaji. In fact, it could trace back to the second century B.C. by Pingala and for the subsequent thousand years there had been documentary evidences that Pascal triangle had been mentioned independently in India, Greece, China and Persia.

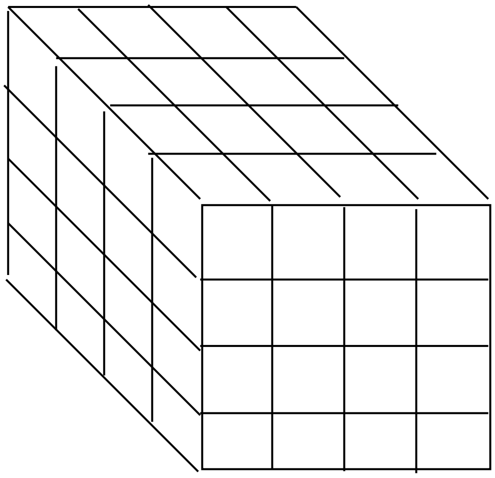

Figure 1.2

How many cubes of all sizes are there in the cubic box shown above?

The answer can be obtained in the similar manner to be $\sum_{i=1}^{4} i^3$. Was there any connection

between two answers? Yes, the recursion $\sum_{i=1}^{4} i^3 = \sum_{i=1}^{4}\sum_{j=i}^{4} j^2$, which was discovered , quite

coincidentally, about one thousand years ago by Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham in the process of solving the famous Alhazen's Problem. His geometric proof could be illustrated by the following display.

	1	4	9	16
		4	9	16
			9	16
+				16
	1	8	27	64

Table 1.2

As a matter of fact, $C(n,k)$ and $\sum_{i=1}^{n} i^k$ got intertwined again in the Eighteen Century by the famous mathematicians such as Blaise Pascal, James Stirling, Leonhard Euler and Jacob Bernoulli. Frankly speaking, I was not familiar with their works when I first started the process of transforming product-sums to power-sums! My goal was using binomial coefficients $C(k, j)$ in Table 1.1 to find general Bernoulli coefficient $b(k, j)$, with $b(k,1)$ denoting Bernoulli numbers, in the following expression

$$S^{(k)}(n) = \sum_{i=1}^{n} i^k = \sum_{j=1}^{k+1} b(k, j) n^{k+1-j} \qquad (1.1)$$

for the power-sum of the natural sequence $(i)_1^\infty$, displayed in the Bernoulli triangle below.

$b\Delta$	1	2	3	4	5	6	7	8
1	$1/2$	$1/2$						
2	$1/6$	$1/2$	$1/3$					
3	0	$1/4$	$1/2$	$1/4$				
4	$-1/30$	0	$1/3$	$1/2$	$1/5$			
5	0	$-1/12$	0	$5/12$	$1/2$	$1/6$		
6	$1/42$	0	$-1/6$	0	$1/2$	$1/2$	$1/7$	
7	0	$1/12$	0	$-7/24$	0	$7/12$	$1/2$	$1/8$

Table 1.3

2. Intuition

An intuitive approach is to equate the coefficients of the like terms in the expansions of both sides of the identity $(n+1)^k = \sum_{i=1}^{n+1} i^k - \sum_{i=1}^{n} i^k$ for $j = 0,1,2,,...,k$, then use the identity

$$(n+1)^k = \sum_{j=0}^{k} C(k, j)n^{k-j} \tag{2.1}$$

to obtain $C(k,i) = \sum_{j=0}^{i} C(k+1-j, i+1-j)b(k, k+1-j)$. Likewise, for the power-sum $S_{a;d}^{(k)}(n) = \sum_{i=1}^{n}[a+(i-1)d]^k = \sum_{j=1}^{k+1} b_{a;d}(k, j)n^{k+1-j}$, we can obtain

$$a^i d^{k-i} C(k,i) = \sum_{j=0}^{i} C(k+1-j, i+1-j)b_{a,d}(k, k+1-j). \tag{2.2}$$

When $i = 0,1,2$ in (2.2), $d^k C(k,0) = C(k+1,1)b_{a;d}(k, k+1)$ gives $b_{a;d}(k, k+1) = d^k \dfrac{1}{k+1}$; $ad^{k-1}C(k,1) = C(k+1,2)b_{a;d}(k, k+1) + C(k,1)b_{a;d}(k.k)$ gives $b_{a;d}(k,k) = d^{k-1}\left(a - \dfrac{d}{2}\right)$ and $a^2 d^{k-2} C(k,2) = C(k+1,3)b_{a;d}(k, k+1) + C(k,2)b_{a;d}(k.k) + C(k-1,1)b_{a;d}(k, k-1)$ gives

$$b_{a;d}(k, k-1) = d^{k-2}\left(a^2 - ad + \frac{d^2}{6}\right)\frac{C(k,1)}{2}.$$

3. Iteration

Alternatively, we can derive

$$\begin{aligned} S_{a;d}^{(k)}(n) &= \sum_{i=1}^{n}[a+(i-1)d]^{k-1}[a+(i-1)] = \sum_{i=1}^{n}[a+(i-1)d]^{k-1} d\left[\left(n+\frac{a}{d}\right) - (n+1-i)\right] \\ &= d\left\{\left(n+\frac{a}{d}\right)\sum_{i=1}^{n}[a+d(i-1)]^{k-1} - \sum_{i=1}^{n}(n+1-i)[a+d(i-1)]^{k-1}\right\} \\ &= d\left\{\left(n+\frac{a}{d}\right)S_{a;d}^{k-1}(n) - \sum_{k=1}^{n}(n+1-i)[a+d(i-1)]^{k-1}\right\}, \end{aligned}$$

the last term of which can further be derived as

$$\sum_{i=1}^{n}(n+1-i)[a+d(i-1)]^{k-1} = S_{a;d}^{(k-1)}(n) + \sum_{i=1}^{n-1}[(n-1)+1-i][a+d(i-1)]^{k-1},$$

$$\sum_{i=1}^{n-1}[(n-1)+1-i][a+d(i-1)]^{k-1} = S_{a;d}^{(k-1)}(n-1) + \sum_{i=1}^{n-2}[(n-2)+1-i][a+d(i-1)]^{k-1}, \ \ldots$$

$$\sum_{i=1}^{n-(n-2)}\{[n-(n-2)]+1-i][a+d(i-1)]^{k-1} = S_{a;d}^{(k-1)}(2)+\sum_{i=1}^{n-(n-1)}\{[n-(n-1)]+1-i\}[a+d(i-1)]^{k-1}$$

so that $S_{a;d}^{(k)}(n)=d\left[\left(n+\frac{a}{d}\right)S_{a;d}^{(k-1)}(n)-\sum_{j=1}^{n}S_{a;d}^{(k-1)}(j)\right]$, from which and

$S_{a;d}^{(2)}(n)=\frac{d^2}{3}n^3+d\left(a-\frac{d}{2}\right)n^2+\left[a(a-d)+\frac{d^2}{6}\right]n$, we can obtain

$$S_{a;d}^{(3)}(n)=\frac{d^3}{4}n^4+d^2(a-\frac{d}{2})n^3+\frac{3d}{2}(a^2-ad+\frac{d^2}{6})n^2+a(a-d)(a-\frac{d}{2})n. \qquad (3.1)$$

4. Induction

We define the first Stirling numbers $s(n,k)$ with $s(n,0)=1$ and $s(n+1,k)$ being the product-sum of all k numbers in row n of the natural triangle $N\Delta$, where $N(n,k)=k$ as displayed in the first Stirling triangle $s\Delta$:

$n \backslash k$	0	1	2	3	4
1	1				
2	1	1			
3	1	3	2		
4	1	6	11	6	
5	1	10	35	50	24

Table 4.1

Likewise, we can define $C(n,k)$ to be the product-sum of all k numbers of row n of the unity triangle $U\Delta$, where $U(n,k)=1$. Note that $s(n,n-1)=(n-1)!$ and

$$s(n,k)=(n-1)s(n-1,k-1)+s(n-1,k),\quad k\le n-2. \qquad (4.1)$$

By way of $\sum_{i=0}^{n}i^k=\sum_{i=1}^{k+1}S(k+1,j)C(n,j)$, we define the second Stirling number $S(k+1,j)$ as follows. Since $\sum_{i=1}^{n}i^0=n=C(n,1)$ and $\sum_{i=1}^{n}i^1=\frac{n(n+1)}{2}=C(n,1)+C(n,2)$, we have $S(1,1)=1, S(2,1)=1$ and $S(2,2)=1$. Then, we have $S(3,1)=1$, $S(3,2)=3$ and $S(3,3)=2$, since $\sum_{i=1}^{n}i^2=\frac{(2n+1)(n+1)n}{6}=2C(n,3)+3C(n,2)+C(n,1)$.

We can further find that $S(k,1)=1$, $S(k,k)=(k-1)!$, $\sum_{j=1}^{k}(-1)^jS(k,j)=0$ and

$$S(k,j)=(j-1)S(k-1,j-1)+jS(k-1,j)\quad 1\le j\le k, \qquad (4.2)$$

via which we can obtain the second Stirling triangle $S\Delta$ as follows.

$k \setminus j$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	2			
4	1	7	12	6		
5	1	15	50	60	24	
6	1	31	180	390	360	120

Table 4.2

Next we shall come up with Bernoulli coefficients $b(k, j)$ by observing Table 4.1 and Table 4.2 simultaneously:

$\frac{s(4,0)S(4,4)}{4!} = b(3,4)$, $\frac{s(3,0)S(4,3)}{3!} - \frac{s(4,1)S(4,4)}{4!} = b(3,3)$,

$\frac{s(2,0)S(4,2)}{2!} - \frac{s(3,1)S(4,3)}{3!} + \frac{s(4,2)S(4,4)}{4!} = b(3,2)$ and

$\frac{s(1,0)S(4,1)}{1!} - \frac{s(2,1)S(4,2)}{2!} + \frac{s(3,2)S(4,3)}{3!} - \frac{s(4,3)S(4,4)}{4!} = b(3,1)$.

In general, we have

$$b(k, j) = \sum_{t=0}^{k+1-j} \frac{(-1)^t s(j+t,t)S(k+1, j+t)}{(j+t)!}. \tag{4.3}$$

By substituting (4.3) into (1.1), we obtain $\sum_{i=1}^{n} i^k = \sum_{j=0}^{k} \left[\sum_{t=0}^{k+1-j} \frac{(-1)^t s(j+t,t)S(k+1, j+t)}{(j+t)!} \right] n^{j+1}$,

which is equivalent to

$$\sum_{i=1}^{n} i^k = \sum_{r=0}^{k} \sum_{j=k+1-r}^{k+1} (-1)^{j-k-1+r} \frac{1}{j} \begin{bmatrix} j \\ k+1-r \end{bmatrix} \begin{Bmatrix} k+1 \\ j \end{Bmatrix} n^{k+1-r} \tag{4.4}$$

since

$$s(k, j) = \begin{bmatrix} k \\ k-j \end{bmatrix}, \tag{4.5}$$

with the Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ being the number of ways of sorting the first n terms of $(i)_1^\infty$ into k cycles and

$$S(k, j) = (j-1)! \begin{Bmatrix} k \\ j \end{Bmatrix}, \tag{4.6}$$

with the Stirling number of the second kind $\left\{ {n \atop k} \right\}$ being the number of ways of sorting the first n terms of $(i)_1^\infty$ into k subsets (see [1]).

So (4.1) is equivalent to

$$\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right] \tag{4.7}$$

and (4.2) is equivalent to

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}. \tag{4.8}$$

In fact, we can generate $\left[{n \atop k} \right]\Delta$ diagonally based on $\left[{n \atop n} \right] = 1$, $\left[{n \atop n-1} \right] = C(n,2)$ and

$$\left[{n \atop n-2} \right] = \frac{3n-1}{4}C(n,3),\ \left[{n \atop n-3} \right] = \frac{n^2-n}{2}C(n,4),\ \left[{n \atop n-4} \right] = \frac{15n^3-30n^2+5n+2}{48}C(n,5),\ldots \tag{4.9}$$

where $1 = \sigma_1(n)2!$, $\frac{3n-1}{4} = \sigma_2(n)3!$, $\frac{n^2-n}{2} = \sigma_3(n)4!$, $\frac{15n^3-30n^2+5n+2}{48} = \sigma_4(n)5!,\ldots$ with $\sigma_r(x)$ being Stirling polynomials or via $\left[{n \atop 1} \right] = (n-1)!$, $\left[{n \atop n} \right] = 1$ and

$$\left[{n+1 \atop k+1} \right] = \sum_{j=0}^{n-k} C(k+j, j)\left[{n \atop k+j} \right], \tag{4.10}$$

which can be proved by way of $P(n,k) = \sum_{j=1}^{k}(-1)^{j-1}\left[{k \atop k-j+1} \right]n^{k-j+1}$ as follows. Taking $P(n,4) = n(n-1)(n-2)(n-3) = nP(n-1,3)$ for example, we can derive

$$\left[{5 \atop 5} \right]n^5 - \left[{5 \atop 4} \right]n^4 + \left[{5 \atop 3} \right]n^3 - \left[{5 \atop 2} \right]n^2 + \left[{5 \atop 1} \right]n$$

$$= n\left\{ \left[{4 \atop 4} \right](n-1)^4 - \left[{4 \atop 3} \right](n-1)^3 + \left[{4 \atop 2} \right](n-1)^2 - \left[{4 \atop 1} \right](n-1) \right\}$$

$$\begin{aligned} &= C(4,0)\left[{4 \atop 4} \right]n^5 - \left\{ C(4,1)\left[{4 \atop 4} \right] + C(3,0)\left[{4 \atop 3} \right] \right\}n^4 + \left\{ C(4,2)\left[{4 \atop 4} \right] + C(3,1)\left[{4 \atop 3} \right] + C(2,0)\left[{4 \atop 2} \right] \right\}n^3 \\ &\quad - \left\{ C(4,3)\left[{4 \atop 4} \right] + C(3,2)\left[{4 \atop 3} \right] + C(2,1)\left[{4 \atop 2} \right] + C(1,0)\left[{4 \atop 1} \right] \right\}n^2 \\ &\quad + \left\{ C(4,4)\left[{4 \atop 4} \right] + C(3,3)\left[{4 \atop 3} \right] + C(2,2)\left[{4 \atop 2} \right] + C(1,1)\left[{4 \atop 1} \right] \right\}n \end{aligned}$$

In general, $P(n,k) = nP(n-1,k-1)$ gives

$$\sum_{j=0}^{k}(-1)^j\left[{k+1 \atop k+1-j} \right]n^{k+1-j} = n\sum_{j=0}^{k-1}(-1)^j\left[{k \atop k-j} \right](n-1)^{k-j}, \tag{4.11}$$

from which we can obtain (4.10) by equating the like terms.

As a matter of fact, (4.4) can be derived directly. To attain our goal, we first derive the following identity

$$C(n,k)=\sum_{j=k}^{n}(-1)^{j-k}\begin{bmatrix} j \\ k \end{bmatrix}\begin{Bmatrix} n+1 \\ j+1 \end{Bmatrix}, \tag{4.12}$$

which can be verified for $n=5$ and $k=3$ as follows.

$$C(6,3)=\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 4 \end{Bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 5 \end{Bmatrix}+\begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 6 \end{Bmatrix}-\begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ 7 \end{Bmatrix}=1\times 350-6\times 140+35\times 21-225\times 1=20,$$

$$C(5,3)=\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}+\begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}=10=\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix}-\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}+\begin{bmatrix} 4 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}-\begin{bmatrix} 5 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}-C(5,2),$$

Next we use (4.7), (4.8) and (4.11) to show the inductive step:

$$\begin{aligned}
C(6,3)&=\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}+\begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}\right)+\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}+\begin{bmatrix} 5 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}-\begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}\right)\\
&=\begin{bmatrix} 2 \\ 2 \end{bmatrix}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix}-\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}-4\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}+\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}-5\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}-\left(\begin{bmatrix} 6 \\ 3 \end{bmatrix}-6\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}\\
&=\left(4\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}+\begin{Bmatrix} 6 \\ 3 \end{Bmatrix}\right)-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\left(5\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}+\begin{Bmatrix} 6 \\ 4 \end{Bmatrix}\right)+\begin{bmatrix} 5 \\ 3 \end{bmatrix}\left(6\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}+\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}\right)-\begin{bmatrix} 6 \\ 3 \end{bmatrix}\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}=\sum_{j=3}^{6}(-1)^{j-3}\begin{bmatrix} j \\ 3 \end{bmatrix}\begin{Bmatrix} 7 \\ j+1 \end{Bmatrix}.
\end{aligned}$$

We then derive the following identity

$$(1+n)^k=\sum_{j=1}^{k+1}P(n,j-1)\begin{Bmatrix} k+1 \\ j \end{Bmatrix}. \tag{4.13}$$

Again, we only look at the case where $k=4$.

Starting from (2.1), we can use (4.12) and (4.13) to write

$$\begin{aligned}
(1+n)^4&=\sum_{j=0}^{4}C(4,j)n^{4-j}\\
&=1+\left(\sum_{t=1}^{4}(-1)^{t-1}\begin{bmatrix} t \\ 1 \end{bmatrix}\begin{Bmatrix} 5 \\ t+1 \end{Bmatrix}\right)n+\left(\sum_{t=2}^{4}(-1)^{t}\begin{bmatrix} t \\ 2 \end{bmatrix}\begin{Bmatrix} 5 \\ t+1 \end{Bmatrix}\right)n^2+\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\begin{Bmatrix} 5 \\ 4 \end{Bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}\right)n^3+n^4\\
&=1+\begin{bmatrix} 1 \\ 1 \end{bmatrix}n\begin{Bmatrix} 5 \\ 2 \end{Bmatrix}+\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}n^2-\begin{bmatrix} 2 \\ 1 \end{bmatrix}n\right)\begin{Bmatrix} 5 \\ 3 \end{Bmatrix}+\left(\sum_{t=0}^{2}(-1)^{t}\begin{bmatrix} 3 \\ 3-t \end{bmatrix}n^{3-t}\right)\begin{Bmatrix} 5 \\ 4 \end{Bmatrix}\\
&\quad+\left(\sum_{t=0}^{3}(-1)^{t}\begin{bmatrix} 4 \\ 4-t \end{bmatrix}n^{4-t}\right)\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}=\sum_{j=1}^{5}P(n,j-1)\begin{Bmatrix} 5 \\ j \end{Bmatrix}.
\end{aligned}$$

Readers are encouraged to use mathematical induction to prove

$$\sum_{i=1}^{n}i^k=\sum_{j=1}^{k+1}\frac{1}{j}P(n,j)\begin{Bmatrix} k+1 \\ j \end{Bmatrix}, \tag{4.14}$$

with (4.13) being used in the inductive step.

Let us now define the first Euler numbers $e(k,j)$ by $e(k,1)=e(k,k)=1$ and

$e(k,j)=\sum_{t=0}^{j-1}(-1)^t C(k+1-j+t,t)S(k+1,j-t)$ such as

$e(3,2)=C(2,0)S(4,2)-C(3,1)S(4,1)=4$, $e(4,2)=C(3,0)S(5,2)-C(3,1)S(5,1)=11$,

$e(4,3)=C(2,0)S(5,3)-C(3,1)S(5,2)+C(4,2)S(5,1)=11$,

$e(5,2) = C(4,0)S(6,2) - C(5,1)S(6,1) = 26$,
$e(5,3) = C(3,0)S(6,3) - C(4,1)S(6,2) + C(5,2)S(6,1) = 66$,
$e(5,4) = C(2,0)S(6,4) - C(3,1)S(6,3) + C(4,2)S(6,2) - C(5,3)S(6,1) = 26$.

In addition to $e(k, j) = e(k, k+1-j)$, we can further observe

$e(3,2) = (3+1-2)e(2,1) + 2e(2,2)$, $e(4,2) = (4+1-2)e(3,1) + 2e(3,2)$,
$e(4,3) = (4+1-3)e(3,2) + 3e(3,3)$, $e(5,2) = (5+1-2)e(4,1) + 2e(4,2)$,
$e(5,3) = (5+1-3)e(4,2) + 3e(4,3)$, $e(5,4) = (5+1-4)e(4,3) + 4e(4,4)$

to come up with

$$e(k, j) = (k+1-j)e(k-1, j-1) + je(k-1, j) \tag{4.15}$$

and the first Euler triangle $e\Delta$.

$k \setminus j$	1	2	3	4	5
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

Table 4.3

Since $(n+1)^1 = C(n+1,1)$, we can also obtain $e(k, j)$ via

$$(n+1)^k = \sum_{j=1}^{k} e(k, j)C(n+j, k):$$

$(n+1)^2 = C(n+1,2) + C(n+2,2)$, $(n+1)^3 = C(n+1,3) + 4C(n+2,3) + C(n+3,3)$,
$(n+1)^4 = C(n+1,4) + 11C(n+2,4) + 11C(n+3,4) + C(n+4,4), \ldots$

By virtue of (4.15), we can use mathematical induction to establish

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k} e(k, j)C(n+j, k+1): \tag{4.16}$$

$$\sum_{i=1}^{n+1} i^k = \sum_{j=1}^{k} e(k, j)C(n+j, k+1) + \sum_{j=1}^{k} e(k, j)C(n+j, k) = \sum_{j=1}^{k} e(k, j)C(n++1+j, k+1).$$

For $k = 3$, we can write

$$\sum_{i=1}^{n} i^3 = e(3,1)C(n+1,4) + e(3,2)C(n+2,4) + e(3,3)C(n+3,4)$$
$$= e(3,1)[C(1,1)C(n,3) + C(1,0)C(n,4)]$$
$$+ e(3,2)[C(2,2)C(n,2) + C(2,1)C(n,3) + C(2,0)C(n,4)]$$
$$+ e(3,3)[C(3,3)C(n,1) + C(3,2)C(n,2) + C(3,1)C(n,3) + C(3,0)C(n,4)]$$
$$= \sum_{t=1}^{3} e(3,t)C(t,0) \sum_{j+0}^{3} \frac{(-1)^j s(4, j)n^{4-j}}{4!} + \sum_{t=1}^{3} e(3,t)C(t,1) \sum_{j=0}^{2} \frac{(-1)^j s(3, j)n^{3-j}}{3!}$$

$$+\sum_{t=2}^{3} e(3,t)C(t,2)\sum_{j+0}^{1}\frac{(-1)^j s(2,j)n^{2-j}}{2!} + e(3,3)C(3,3)\frac{s(1,0)}{1!}.$$

Hence we have

$$b(3,4)=\sum_{t=1}^{3} e(3,t)C(t,0)\frac{s(4,0)}{4!}=\frac{1}{4},$$

$$b(3,3)=-\sum_{t=1}^{3} e(3,t)C(t,0)\frac{s(4,1)}{4!}+\sum_{t=1}^{3} e(3,t)C(t,1)\frac{s(3,0)}{3!}=\frac{1}{2},$$

$$b(3,2)=\sum_{t=1}^{3} e(3,t)C(t,0)\frac{s(4,2)}{4!}-\sum_{t=1}^{3} e(3,t)C(t,1)\frac{s(3,1)}{3!}+\sum_{t=2}^{3} e(3,t)C(t,2)\frac{s(2,0)}{2!}=\frac{1}{4},$$

$$b(3,1)=-\sum_{t=1}^{3} e(3,t)C(t,0)\frac{s(4,3)}{4!}+\sum_{t=1}^{3} e(3,t)C(t,1)\frac{s(3,2)}{3!}-\sum_{t=2}^{3} e(3,t)C(t,2)\frac{s(2,1)}{2!}$$

$$+e(3,3)C(3,3)\frac{s(1,0)}{1!}=0.$$

Since $S(k,j)=\sum_{t=k-j}^{k-1} e(k-1,t)C(t,k-j)$ and $S(k,k)=\sum_{t=1}^{k-1} e(k-1,t)$, we can write (4.16) as

$$\sum_{i=1}^{n} i^k = \sum_{j=1}^{k}\frac{e(k,j)}{(k+1)!}\left\{\sum_{t=0}^{k}\left[\sum_{r=0}^{t}(-1)^r s(k,r)C(k+1-r,t-r)i^{t-r}\right]n^{k+1-t}\right\}.$$

Next, let us observe Table 4.2 diagonally. In addition to $\frac{S(n,n-1)}{(n-2)!}=C(n,2)$, we can obtain $\frac{S(n,n-2)}{(n-3)!}=C(n+1,4)+2C(n,4)$, $\frac{S(n,n-3)}{(n-4)!}=C(n+2,6)+8C(n+1,6)+6C(n,6)$, $\frac{S(n,n-4)}{(n-5)!}=C(n+3,8)+22C(n+2,8)+58C(n+1,8)+24C(n,8),\ldots$

In such manner, we can define the second Euler number $E(k,j)$ by way of $S(n,n-k)=(n-k-1)!\sum_{j=1}^{k} E(k,j)C(n+k-j,2k)$ and come up with $E(k,1)=E(k-1,1)$, $E(k,k)=kE(k-1,k-1)$ and in general

$$E(k,j)=(2k-j)E(k-1,j-1)+kF(k-1,j), \qquad (4.17)$$

via which we can generate the second Euler triangle $E\Delta$.

$k \setminus j$	1	2	3	4	5
1	1				
2	1	2			
3	1	8	6		
4	1	22	58	24	
5	1	52	328	444	120

Table 4.4

On the other hand, we can also obtain

$$s(n,k)=\sum_{j=1}^{k}E(k,j)C(n+j-1,2k)\,. \tag{4.18}$$

Since (4.18) is true for $k=1$: $s(n,1)=\sum_{j=1}^{1}E(1,j)C(n+1-1,2\cdot 1)$, we need only show that

$$s(n,m+1)=\sum_{j=1}^{m+1}E(m+1,j)C(n+j-1,2m+2) \tag{4.19}$$

by assuming (4.19) for $k=m$. Prior to proving (4.19) by mathematical induction, let us do it in the case of $m=4$. Using (4.1) and (4.17), we can write

$$\begin{aligned}
&s(n,5)=(n-1))s(n-1,4)+s(n-1,5)\\
&=(n-1)[E(4,1)C(n-1,8)+E(4,2)C(n,8)+E(4,3)C(n+1,8)+E(4,4)C(n+2,8)]\\
&\quad+E(5,1)C(n-1,10)+E(5,2)C(n,10)+E(5,3)C(n+1,10)+E(5,4)C(n+2,10)+E(5,5)C(n+3,10)\\
&=\sum_{j=1}^{4}[(n-1)C(n+j-2,8)+jC(n+j-2,10)+(9-j)C(n+j-1,10)]E(4,j)\,;
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{5}E(5,j)C(n+j-1,10)\\
&=E(4,1)C(n,10)+[8E(4,1)+2E(4,2)]C(n+1,10)+[7E(4,2)+3E(4,3)]C(n+2,10)\\
&\quad+[6E(4,3)+4E(4,4)]C(n+3,10)+5E(4,4)C(n+4,10)\\
&=\sum_{j=1}^{4}[jC(n+j-1,10)+(9-j)C(n+j,10)]E(4,j)\,.
\end{aligned}$$

The coefficients of the like term $E(4,j)$ are equal, since

$$\begin{aligned}
&[(n-1)C(n+j-2,8)+jC(n+j-2,10)+(9-j)C(n+j-1,10)]\\
&-[jC(n+j-1,10)+(9-j)C(n+j,10)]\\
&=(n+j-1)C(n+j-2,8)-[jC(n+j-2,8)+jC(n+j-2,9)+(9-j)C(n+j-1,9)]\\
&=9C(n+j-1,9)-[jC(n+j-2,8)+jC(n+j-2,9)+(9-j)C(n+j-1,9)]=0\,.
\end{aligned}$$

Thus we have proved that $s(n,5)=\sum_{j=1}^{5}E(5,j)C(n+j-1,10)$. In general, we can write

$$\begin{aligned}
&s(n,m+1)-\sum_{j=1}^{m+1}E(m+1,j)C(n+j-1,2m+2)\\
&=\sum_{j=1}^{m}[(n-1)C(n+j-2,2m)+jC(n+j-2,2m+2)+(2m+1-j)C(n+j-1,2m+2)]E(m,j)\\
&\quad-\sum_{j=1}^{m}[jC(n+j-1,2m+2)+(9-j)C(n+j,2m+2)]E(m,j)\,.
\end{aligned}$$

The coefficients of the like term $E(m,j)$ are equal, since

$$\begin{aligned}
&[(n-1)C(n+j-2,2m)+jC(n+j-2,2m+2)+(2m+1-j)C(n+j-1,2m+2)]\\
&\quad-[jC(n+j-1,2m+2)+(2m+1-j)C(n+j,2m+2)]\\
&=(n+j-1)C(n+j-2,2m)
\end{aligned}$$

$$-[jC(n+j-2,2m)+jC(n+j-2,2m+2)+(2m+1-j)C(n+j-1,2m+1)]$$
$$=(2m+1)C(n+j-1,2m+1)$$
$$-[jC(n+j-2,2m)+jC(n+j-2,2m+1)+(2m+1-j)C(n+j-1,2m+1)]=0.$$

We have completed the proof of (4.14) by mathematical induction. Therefore, by virtue of (4.3), (4.10) and (4.11), $b(k,j)$ can be expressed in terms of the second Euler numbers.

Note that $e(n,k-1)=\left\langle {n \atop k} \right\rangle$, where the first-order Eulerian number $\left\langle {n \atop k} \right\rangle$ being the number of permutations $p_1p_2...p_n$ of the set $\{1,2,...n\}$ that have k ascents, i.e. k places where $p_j < p_{j+1}$ and $E(n,k-1)=\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$, where the second-order Eulerian number $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ being the number of permutations $p_1p_2...p_n$ of the multiset $\{1,1,2,2,...n,n\}$ that have k ascents, i.e. k places where $p_j < p_{j+1}$, provided that all numbers between the two occurrences of m are greater than m for $1\le m\le n$ (see [1]).

5. Generalization

For an arithmetically progressive sequence $(a+(i-1)d)_1^\infty$, the Stirling triangle of the first kind $\left[{n \atop k} \right]_{a;d}$ can be constructed via

$$\left[{n \atop k} \right]_{a;d}=[a+(n-2)d]\left[{n-1 \atop k} \right]_{a;d}+\left[{n-1 \atop k} \right]_{a;d} \tag{5.1}$$

with $\left[{n \atop n} \right]_{a;d}=1$ and the Stirling triangle of the second kind $\left\{ {n \atop k} \right\}_{a;d}$ can be constructed via

$$\left\{ {n \atop k} \right\}_{a;d}=\left\{ {n-1 \atop k-1} \right\}_{a;d}+[a+(k-1)d]\left\{ {n-1 \atop k} \right\}_{a;d} \tag{5.2}$$

with $\left\{ {n \atop n} \right\}_{a;d}=1$**.** When $a=d=1$, (5.1) and (5.2) become (4.1) and (4.2) due to (4.5) and (4.6).

On the other hand, $\left\{ {n \atop k} \right\}=\sum_{j=1}^{n-k}(-1)^{j-1}\left[{k+j \atop k} \right]\left\{ {n \atop k+j} \right\}$ (see [1]) can be generalized to

$$\left[{n \atop k} \right]_{a;d}=[a+(n-2)d]\left[{n-1 \atop k} \right]_{a;d}+\left[{n-1 \atop k} \right]_{a;d} \tag{5.3}$$

To generalize $\left\langle {n \atop k} \right\rangle$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$, first derive $S_{a;d}^{(1)}(n)=(d-a)C(n,2)+aC(n+1,2)$ and $S_{a;d}^{(2)}(n)=(d-a)^2C(n,3)+(-2a^2+2ad+d^2)C(n+1,3)+a^2C(n+2,3)$, whereas

$$S_{a;d}^{(3)}(n)=(d-a)^3C(n,4)+(3a^3-6a^2d+4d^3)C(n+1,4)$$

$$+(-3a^3+3a^2d+3ad^2+d^3)C(n+2,4)+a^3C(n+3,4) \qquad (5.4)$$

will be more conveniently converted from (3.1) as follows. First of all,

$$n^4 = C(n,4)+11C(n+1,4)+11C(n+2,4)+C(n+3),$$
$$n^3 = -C(n,4)-3C(n+1,4)+3C(n+2,4)+C(n+3),$$
$$n^2 = C(n,4)-C(n+1,4)-C(n+2,4)+C(n+3),$$
$$n = -C(n,4)+3C(n+1,4)-3C(n+2,4)+C(n+3)$$

can be obtained from the following layout.

	n^4	n^3	n^2	n
$C(n,4)$	1/24	6/24	11/24	6/24
$C(n+1,4)$	1/24	-2/24	-1/24	2/24
$C(n+2,4)$	1/24	2/24	-1/24	-2/24
$C(n+3,4)$	1/24	6/24	11/24	6/24

Table 5.1

Therefore, we can derive from (3.1) that

$$\sum_{i=1}^{n}[a+(i-1)d]^3 = \frac{d^3}{4}n^4 + d^2\left(a-\frac{d}{2}\right)n^3 + \frac{3d}{2}\left(a^2-ad+\frac{d^2}{6}\right)n^2 + a(a-d)\left(a-\frac{d}{2}\right)n$$

$$= \frac{d^3}{4}[C(n,4)+11C(n+1,4)+11C(n+2,4)+C(n+3)],$$

$$+d^2\left(a-\frac{d}{2}\right)[-C(n,4)-3C(n+1,4)+3C(n+2,4)+C(n+3)],$$

$$+\frac{3d}{2}\left(a^2-ad+\frac{d^2}{6}\right)[C(n,4)-C(n+1,4)-C(n+2,4)+C(n+3)],$$

$$+a(a-d)\left(a-\frac{d}{2}\right)[-C(n,4)+3C(n+1,4)-3C(n+2,4)+C(n+3)].$$

from which we can obtain (5.3).

Now, we are ready to define $\left\langle {n \atop k} \right\rangle_{a;d}$ according to (5.1), (5.2) and (5.3) analogous to $\left\langle {n \atop k} \right\rangle$, By virtue of (5.1) , (5.2) and (5.3), we define $\left\langle {1 \atop -1} \right\rangle_{a;d} = d-a$, $\left\langle {1 \atop 0} \right\rangle_{a;d} = a$, $\left\langle {2 \atop -1} \right\rangle_{a;d} = (d-a)^2$, $\left\langle {2 \atop 0} \right\rangle_{a;d} = -2a^2+2ad+d^2$, $\left\langle {2 \atop 1} \right\rangle_{a;d} = a^2$, $\left\langle {3 \atop -1} \right\rangle_{a;d} = (d-a)^3$, $\left\langle {3 \atop 0} \right\rangle_{a;d} = 3a^3-6a^2d+4d^3$, $\left\langle {3 \atop 1} \right\rangle_{a;d} = -3a^3+3a^2d+3ad^2+d^3$ and $\left\langle {3 \atop 2} \right\rangle_{a;d} = a^3$.

To generalize (4.15), we define $\left\langle {0 \atop -1} \right\rangle_{a;d} = 0$. Then we write $\left\langle {2 \atop -1} \right\rangle_{a;d} = (-a+d)\left\langle {1 \atop -1} \right\rangle_{a;d}$, $\left\langle {2 \atop 0} \right\rangle_{a;d} = (a+d)\left\langle {1 \atop -1} \right\rangle_{a;d} + (-a+2d)\left\langle {1 \atop 0} \right\rangle_{a;d}$, $\left\langle {2 \atop 1} \right\rangle_{a;d} = a\left\langle {1 \atop 0} \right\rangle_{a;d}$, $\left\langle {3 \atop -1} \right\rangle_{a;d} = (-a+d)\left\langle {2 \atop -1} \right\rangle_{a;d}$, $\left\langle {3 \atop 0} \right\rangle_{a;d} = (a+2d)\left\langle {2 \atop -1} \right\rangle_{a;d} + (-a+2d)\left\langle {2 \atop 0} \right\rangle_{a;d}$, $\left\langle {3 \atop 1} \right\rangle_{a;d} = (a+d)\left\langle {2 \atop 0} \right\rangle_{a;d} + (-a+3d)\left\langle {2 \atop 1} \right\rangle_{a;d}$, $\left\langle {3 \atop 2} \right\rangle_{a;d} = a\left\langle {2 \atop 1} \right\rangle_{a;d}$ and in general $\left\langle {n \atop k} \right\rangle_{a;d} = [a+(n-k-1)d]\left\langle {n-1 \atop k-1} \right\rangle_{a;d} + [-a+(k+2)d]\left\langle {n-1 \atop k} \right\rangle_{a;d}$, which is the generalization of (4.15).

Next, we write $S_{a;d}^{(1)}(n) = (d-a)C(n-1,1) + aC(n,1)$,

$S_{a;d}^{(2)}(n) = (d-a)^2 C(n-1,2) + (-2a^2+2ad+d^2)C(n,2) + a^2 C(n+1,2)$ and

$$S_{a;d}^{(3)}(n) = (d-a)^3 C(n-1,3) + (3a^3 - 6a^2 d + 4d^3)C(n,3)$$
$$+ (-3a^3 + 3a^2 d + 3ad^2 + d^3)C(n+1,3) + a^3 C(n+2,3).$$

In general, $[a+(n-1)d]^k = \sum_{j=-1}^{k-1} \left\langle {k \atop j} \right\rangle_{a;d} C(n+j,k)$, which is the generalization of (4.13) since $\left\langle {n \atop k} \right\rangle = e(n,k-1)$. Therefore, the proof of (4.15) can be generalized to prove

$$\sum_{i=1}^{n} [a+(i-1)d]^k = \sum_{j=-1}^{k-1} \left\langle {k \atop j} \right\rangle_{a;d} C(n+j+1,k+1). \tag{5.5}$$

By recalling $\left[{n \atop k} \right] = s(n,n-k)$ and $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = E(n,k-1)$, we shall generalize (4.15) as follows. Since $\left[{n \atop n-1} \right]_{a;d} = \sum_{i=1}^{n-1} [a+(i-1)d] = (d-a)C(n-1,2) + aC(n,2)$, we assume

$\left[{n \atop n-2} \right]_{a;d} = xC(n-1,4) + yC(n,4) + zC(n+1,4)$. Then we have $z = \left[{3 \atop 1} \right]_{a;d} = a^2 + ad$,

$y + 5z = \left[{4 \atop 2} \right]_{a;d} = 3a^2 + 6ad + 2d^2$, $x + 5y + 15z = \left[{5 \atop 3} \right]_{a;d} = 6a^2 + 18ad + 11d^2$ so that $x = (d-a)^2$, $y = -2a^2 + ad + 2d^2$ and $z = a(a+d)$. Thus, we have arrived at

$$\left[{n \atop n-2} \right]_{a;d} = (d-a)^2 C(n-1,4) + (-2a^2 + ad + 2d^2)C(n,4) + a((a+d)C(n+1,4).$$

Similarly, we can obtain

$$\left[\begin{matrix} n \\ n-3 \end{matrix}\right]_{a;d} = (d-a)^3 C(n-1,6) + (3a^3 - 3a^2 d - 7ad^2 + 8d^3) C(n,6)$$

$$+(-3a^3 - 3a^2 d + 8ad^2 + 6d^3) C(n+1,6) + a((a+d)(a+2d) C(n+2,6).$$

Defining $\left\langle\!\left\langle \begin{matrix} 0 \\ -1 \end{matrix} \right\rangle\!\right\rangle_{a;d} = 1$, $\left\langle\!\left\langle \begin{matrix} 1 \\ -1 \end{matrix} \right\rangle\!\right\rangle_{a;d} = d-a$, $\left\langle\!\left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle\!\right\rangle_{a;d} = a$, we can tabulate $\left\langle\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\right\rangle_{a;d}$ via

$$\left\langle\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\right\rangle_{a;d} = [a + (2n-2-k)d] \left\langle\!\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle\!\right\rangle_{a;d} + [-a + (k+2)d] \left\langle\!\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle\!\right\rangle_{a;d} : \qquad (5.6)$$

$n \backslash k$	-1	0	1	2
0	1			
1	$d-a$	a		
2	$(d-a)^2$	$-2a^2 + ad + 2d^2$	$a(a+d)$	
3	$(d-a)^3$	$3a^3 - 3a^2 d - 7ad^2 + 8d^3$	$-3a^3 - 3a^2 d + 8ad^2 + 6d^3$	$a(a+d)(a+2d)$

Table 5.2

Accordingly, we can derive $\left[\begin{matrix} n \\ n-k \end{matrix}\right]_{a;d} = \sum_{i=-1}^{n-1} \left\langle\!\left\langle \begin{matrix} k \\ j \end{matrix} \right\rangle\!\right\rangle_{a;d} C(n+j, 2k)$ so that $\left\langle\!\left\langle \begin{matrix} k \\ j \end{matrix} \right\rangle\!\right\rangle_{a;d}$ is the second-order Eulerian number for $(a + (i-1)d)_0^\infty$.

We shall further rewrite (5.1), (5.2), (5.5), (5.6) with $a_i = a + (i-1)d$ as

$$\left[\begin{matrix} n \\ k \end{matrix}\right]_{(a_i)_1^\infty} = [a + (n-2)d] \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} + \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} = a_{n-1} \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty} + \left[\begin{matrix} n-1 \\ k \end{matrix}\right]_{(a_i)_1^\infty}, \qquad (5.7)$$

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}_{(a_i)_1^\infty} = \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}_{(a_i)_1^\infty} + [a + (k-1)d] \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}_{(a_i)_1^\infty} = \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}_{(a_i)_1^\infty} + a_k \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}_{(a_i)_1^\infty}, \qquad (5.8)$$

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(a_i)_0^\infty} = a_{n-k} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_{(a_i)_0^\infty} + [(n+k)a_n - (n+1+k)a_{n-1}] \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_{(a_i)_0^\infty}, \qquad (5.9)$$

$$\left\langle\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\right\rangle_{(a_i)_0^\infty} = [(n-k)a_n - (n-1-k)a_{n-1}] \left\langle\!\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle\!\right\rangle_{(a_i)_0^\infty}$$

$$+ [(n+k)a_n - (n+1+k)a_{n-1}] \left\langle\!\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle\!\right\rangle_{(a_i)_0^\infty}. \qquad (5.10)$$

More generally, a triangular array $T^{r,s}{}_{(a_i)_0^\infty}$ for $(a_i)_0^\infty$ can be defined as $T^{r,s}{}_{(a_i)_0^\infty}=0$ for $k\le -2$ and $k\ge n$, $T^{r,s}{}_{(a_i)_0^\infty}(1,-1)=-a_0$, $T^{r,s}{}_{(a_i)_0^\infty}(1,0)=a_1$ and

$$T^{r,s}{}_{(a_i)_0^\infty}(n,k)=M(r)T^{r,s}{}_{(a_i)_0^\infty}(n-1,k-1)+M(s)T^{r,s}{}_{(a_i)_0^\infty}(n-1,k), \qquad (5.11)$$

where $M(r)$ and $M(s)$ can be taken the following model list.

$$M(1)=1, M(2)=a_{n-1}, M(3)=a_{k+1}, M(4)=a_{n-k}, M(5)=(n+k)a_n-(n+1+k)a_{n-1},$$
$$M(6)=(n-k)a_n-(n-1-k)a_{n-1},\ldots \qquad (5.12)$$

Then (5.7), 5.8), (5.9) and (5.10) become $\begin{bmatrix} n \\ k \end{bmatrix}_{(a_i)_0^\infty}=T^{1,2}{}_{(a_i)_0^\infty}(n,k-1)$,

$\begin{Bmatrix} n \\ k \end{Bmatrix}_{(a_i)_0^\infty}=T^{1,3}{}_{(a_i)_0^\infty}(n,k-1)$, $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{(a_i)_0^\infty}=T^{4,5}{}_{(a_i)_0^\infty}(n,k)$ and $\left\langle\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\right\rangle_{(a_i)_0^\infty}=T^{6,5}{}_{(a_i)_0^\infty}(n,k)$.

In conclusion, we first derive from (5.7) for $\left(q^{i-1}\right)_1^\infty$, based on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty}=0$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\left(q^{i-1}\right)_1^\infty}=1$, that $\begin{bmatrix} n \\ k \end{bmatrix}_{(q^{i-1})_1^\infty}=q^{C(n-k,2)}\prod_{i=1}^{k-1}\frac{1-q^{n-k+i}}{1-q^i}=q^{C(n-k,2)}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$; and use (5.8) for $\left(q^{i-1}\right)_1^\infty$ to obtain $\begin{Bmatrix} n \\ k \end{Bmatrix}_{(q^{i-1})_1^\infty}=\prod_{i=1}^{k-1}\frac{1-q^{n-k+i}}{1-q^i}=\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$, where $\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ is a q - Gaussian coefficient.

Next, we can use (5.7) for $\left(C(i+1,2)\right)_1^\infty$ to tabulate $\begin{bmatrix} n \\ k \end{bmatrix}_{(C(i+1,2))_1^\infty}$ as follows.

$n\setminus k$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	3	4	1					
4	18	27	10	1				
5	180	288	127	20	1			
6	2700	4500	2193	427	35	1		
7	56700	97200	50553	11160	1162	56	1	
8	1587600	2778300	1512684	363033	43696	2730	84	1
9	57153600	101606400	57234924	14581872	1936089	141976	5754	120
10	2571912000	4629441600	359163558	713419164	101705877	8325009	400906	11154

Table 5.3

6. Differentiation

The expression $\sum_{j=0}^{4} c_{4-j}^{4} n^{4-j}$ for $\sum_{i=1}^{n} i^3$ can also be obtained via the method of common differences by solving $\sum_{t=0}^{3} c_{4-t}^{4} = 1, \sum_{t=0}^{3} 2^{4-t} c_{4-t}^{4} = 9$, $\sum_{t=0}^{3} 3^{4-t} c_{4-t}^{4} = 36$, $\sum_{t=0}^{3} 4^{4-t} c_{4-t}^{4} = 100$ and $\sum_{t=0}^{3} 5^{4-t} c_{4-t}^{4} = 225$. The process of successive reductions is as shown.

$c_4^4 + c_3^4 + c_2^4 + c_1^4 + c_0^4 = a_1$, $\Delta n^4 c_4^4 + \Delta n^3 c_3^4 + \Delta n^2 c_2^4 + \Delta n c_1^4 = \Delta a$,

$\Delta^2 n^4 c_4^4 + \Delta^2 n^3 c_3^4 + \Delta^2 n^2 c_2^4 = \Delta^2 a$, $\Delta^3 n^4 c_4^4 + \Delta^3 n^3 c_3^4 = \Delta^3 a$ and $\Delta^4 n^4 c_4^4 = \Delta^4 a$, where

$\Delta n^4 = 15$, $\Delta^2 n^4 = 50$, $\Delta^3 n^4 = 60$, $\Delta^4 n^4 = 24$, $\Delta n^3 = 7$, $\Delta^2 n^3 = 12$, $\Delta^3 n^3 = 6$, $\Delta n^2 = 3$, $\Delta^2 n^2 = 2$, $\Delta n = 1$, $a_1 = 1$, $\Delta a = 8$, $\Delta^2 a = 19$, $\Delta^3 a = 18$ and $\Delta^4 a = 6$.

Thus $c_4^4 = \frac{\Delta^4 a}{\Delta^4 n^4} = \frac{1}{4}$, $c_3^4 = \frac{\Delta^3 a - \Delta^3 n^4 c_4^4}{\Delta^3 n^3} = \frac{1}{2}$, $c_2^4 = \frac{\Delta^2 a - \Delta^2 n^4 c_4^4 - \Delta^2 n^3 c_3^4}{\Delta^2 n^2} = \frac{1}{4}$
$c_1^4 = \frac{\Delta a - \Delta n^4 c_4^4 - \Delta n^3 c_3^4 - \Delta n^2 c_2^4}{\Delta n} = 0$ and $c_0^4 = 1 - \sum_{t=0}^{3} c_{4-t}^4 = 0$, which agree with Table 1.3.

In general, $c_k^k = \frac{\Delta^k a}{\Delta^k n^k}$, $c_{k-j}^k = \frac{\Delta^{k-j} a - \sum_{t=0}^{j-1} \Delta^{k-j} n^{k-t} c_{k-t}^k}{\Delta^k n^k}$, $1 \le j \le k-1$ and $c_0^k = a_1 - \sum_{t=0}^{k-1} c_{k-t}^k$. The polynomial expressions for $\sum_{i=1}^{n} i^l$, $l \ge 5$, can be obtained in the similar fashion. Note that each Bernoulli number c_1^k can now be calculated independently.

As we can check in Table 5.3, the common difference for $\begin{bmatrix} n \\ n-2 \end{bmatrix}_{\left(\binom{i+1}{2}\right)_1^\infty}$ is of degree 6.

So by assuming $\begin{bmatrix} n \\ n-2 \end{bmatrix}_{(C(i+1,2))_1^\infty} = \sum_{j=0}^{6} c_{6-j}^6 n^j$ and taking $n = 3,4,5,6,7,8,9$ respectively, we can solve the simultaneous equations to come up with $\begin{bmatrix} n \\ n-2 \end{bmatrix}_{(C(i+1,2))_1^\infty} = \frac{5n^2 + n - 3}{15} C(n+1,4)$.

Readers are encouraged to implement (5.11) and (5.12) so that triangular arrays such as Table 5.3 be displayed and the polynomial expressions for diagonal sequences be derived and the graph of which be drawn.

Reference

[1] Ronald L.Graham, Donald E. Knuth and Oren Patashnik, *Concrete Mathematics* (2nd edn.), (*Addison-Wesle*) 1994.