A New Model for Calculating Sphere Volume

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Abstract

Though mankind's knowledge of sphere volume had begun from the great Archimedes 2200 years ago, the proof of corresponding formula has been indirect. Even if Chinese mathematician Zu Geng (lived in the 5th century AD) and Italian mathematician Cavalieri (lived in 14-15th century AD) arrived at a useful principle (so-called Cavalieri's principle in western world and Zu Geng's principle in China) independently, their models were not direct either. This paper introduces a model of tetrahedron, whose volume equals to a sphere directly. This method may benefit high school students in understanding the sphere volume formula more easily without the preparation of calculus.

1. Zu Geng's & Cavalieri's methods

Early in 212 BC, Archimedes was able to find the volume of a sphere given the volumes of a cone and cylinder. His method borrows some notions from physics. Afterwards, in the 5th century AD, Zu Chongzhi and his son Zu Geng established a method named Zu Geng's principle to find a sphere's volume. It may be the first effort for the mankind in solving volume of sphere in pure geometry way. However Zu Geng's model is somewhat awkward. About 1100 years later, this principle was generalized by an Italian Mathematician Cavalieri to both 2-dimensional and 3-dimensional cases. Unfortunately, only Cavalieri's name is known as this principle's discoverer in western world.

So in geometry, Cavalieri's principle (or Zu Geng's principle) is as follows [2]:

- 2-dimensional case: Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.

- 3-dimensional case: Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

However, as to apply the above theory in calculating sphere volume, neither Zu Geng's nor Cavalieri's method is direct. They both calculated the volume of a sphere by linear combination of two other solids. (Fig.1 - Fig. 2.)

Figure 1　Zu Geng's method

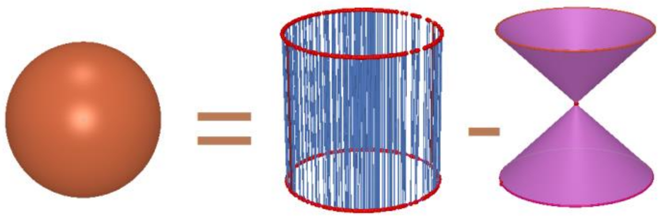

Figure 2　Cavalieri's method

2. New model for sphere volume

Now I will introduce a model different from what have mentioned above.

Following Cavalieri's principle, we only need to find another equivalent solid sharing the same area of parallel cross-section with sphere.

Given the radius of a sphere r and the distance between the intersection plane and the equatorial plane h, the radius of the cross-section circle would be$\sqrt{r^2-h^2}$（Fig. 3）. Thus, it an area of

$$S=(r^2-h^2)\pi. \tag{2.1}$$

Note that the right hand of equation （2.1）can be rewritten as $(r+h)(r-h)\pi$ which suggests an area of rectangle.

To construct a solid with more symmetric features, I would like replace π with $\sqrt{\pi}\sqrt{\pi}$ so that the length and width of rectangle equals to $(r+h)\sqrt{\pi}$ and $(r\text{-}h)\ \sqrt{\pi}$ respectively.

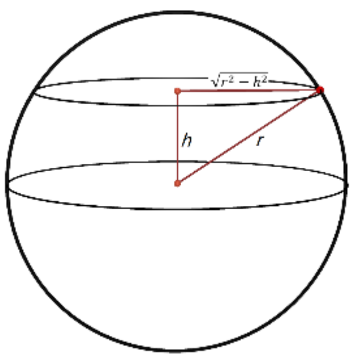

Figure 3 small circle at height *h*

Thus we have finally defined a special tetrahedron with 4 right-triangle faces: Letting A, B, C, D be 4 points in 3 dimension satisfying AB=CD=2r$\sqrt{\pi}$, BC=2r, and AB, BC，CD perpendicular to each other (Fig. 4), tetrahedron ABCD is defined.

(In ancient China, any tetrahedron having 4 right-triangular faces is named as a 鱉臑 (biē nào). however in western world they are included in a collection of Schläfli orthoscheme.

Schläfli orthoscheme is a type of simplex defined by a sequence of mutually orthogonal edges, i.e. (v_0, v_1), (v_1, v2), …, (v_{n-1},v_n). So, a Schläfli orthoscheme in 2D is any right triangle.)

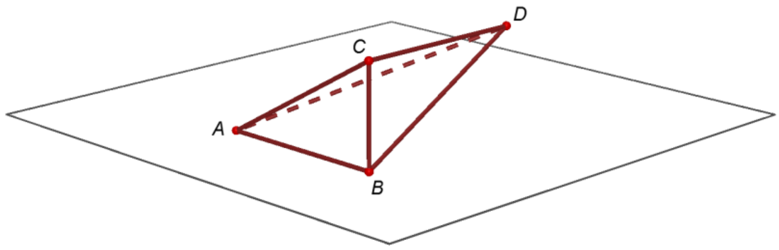

Figure 4 A 3D Schläfli orthoscheme in special gesture

It is easy to verify that the tetrahedron we have just defined has a volume equals to that of a sphere, since

$$V_{ABCD} = \frac{1}{3} \cdot S_{\Delta BCD} \cdot AB = \frac{1}{6} \cdot BC \cdot CD \cdot AB = \frac{4}{3}\pi r^3.$$

Now we can put both solids in between two parallel planes shown in Fig 5. Note that the tetrahedron is set in an "unbalanced" gesture: it has two edges lie on upper and lower planes and none of its 4 faces is in the two planes (Fig. 6).

So we just need to prove that the cross-sections of this tetrahedron from a third parallel plane at height h has area of$(r^2 - h^2)\pi$.

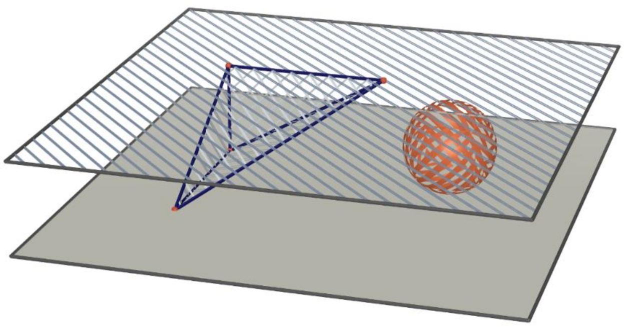

Figure 5 A sphere and its equavalent tetrahedron in between two parallel planes

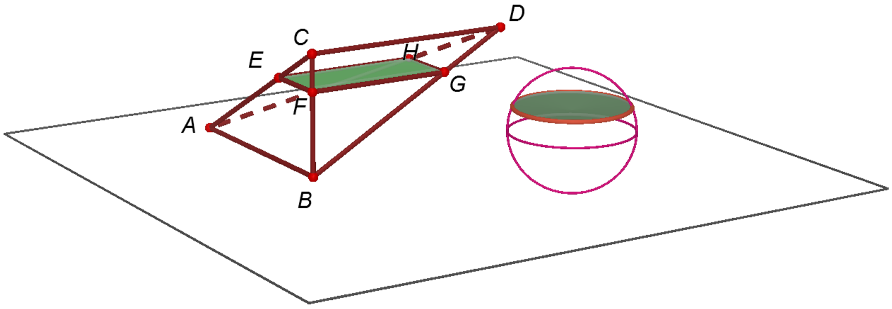

Figure 6 both solids have same area of intersection

Since $EF \parallel AB$, $FG \parallel CD$, yet $AB \perp CD$, So $\angle EFG = 90°$. Similarly, $\angle FGH = \angle GHE = \angle HEF = 90°$. It is obviously that the quadrilateral *EFGH* at the height h is a rectangle.

Furthermore, since

$EF{:}AB = CF{:}CB = (r-h){:}2r$, $EF = AB \times (r-h)/2r = (r-h)\sqrt{\pi}$; similarly we can also get $FG{:}CD = BF{:}BC = (r+h){:}2r$, $FG = CD \times (r+h)/2r = (r+h)\sqrt{\pi}$, Thus $S_{EFGH} = EF \cdot FG = (r^2 - h^2)\pi$.

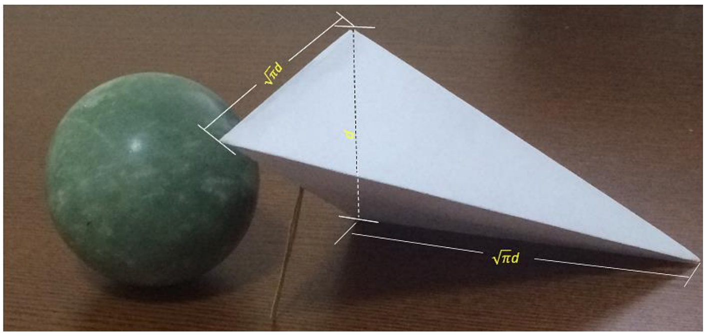

Figure 7 picture of a real model

Such a tetrahedron can be easily made from a sheet of paper. Here is a template for this construction in Figure 8. (Note: Dash lines is for valley folds, solid lines for mountain folds. Extra tab is for gluing.)

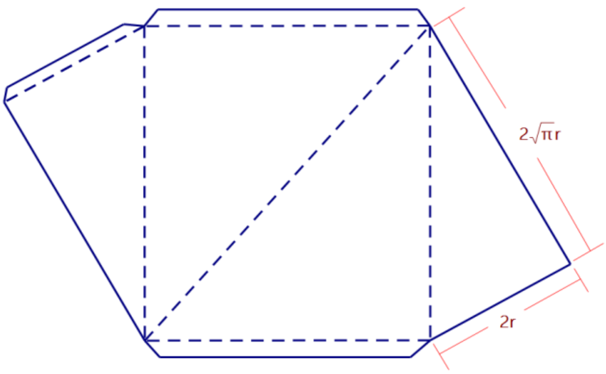

Figure 8 template for a paper model

3. Some variations

The above model is not the only tetrahedron having equal cross-section area with a sphere. Actually there are infinite such tetrahedra. The reason is by Cavalieri's principle, there are infinite tetrahedra equivalent to each other as their cross-section area are concerned.

First variation is to make a more symmetric tetrahedron: Each of the new tetrahedron face will be congruent isosceles triangle. So it will no longer be a Schläfli Orthoscheme, but the distance between upper and lower edge remains to be *2r*. (See Fig. 9.)

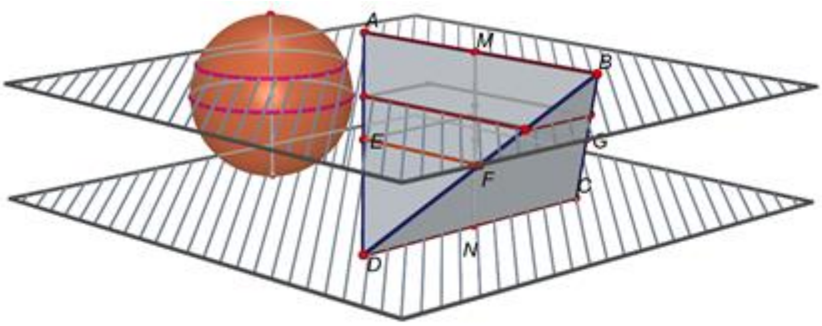

Figure 9 the tetrahedron model can be more symmetry

Tracing Zu Geng's method and comparing with mine, another modified model can be found. i.e., combine 4 uniform orthoschemes into a ring of tetrahedral. Since the whole volume should be the original sphere, every single tetrahedron should have a narrowed upper and lower edges like illustrated in Fig. 10.

By origami, the crease pattern shown will bring a structure much like Zu Geng's model i.e., a cube with two pyramids removed (Fig. 11). To make such a structure, one need first stick the left and right edge to form a hollow prism with sleeve turned up. Then a twist need to make from the upper and lower crease of the sleeve. Finally, the two ends meet at the center part and make a seam.

Note that the elements of each tetrahedron equal to 1/4 of last tetrahedron model in Fig. 9.

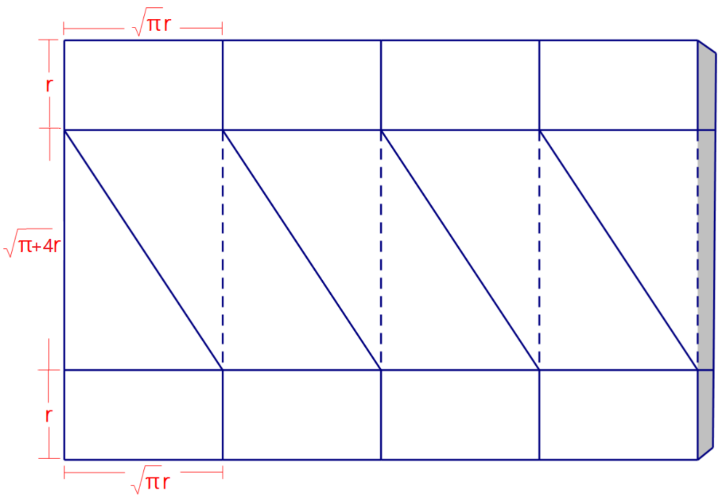

Figure 10 another template of a complex model

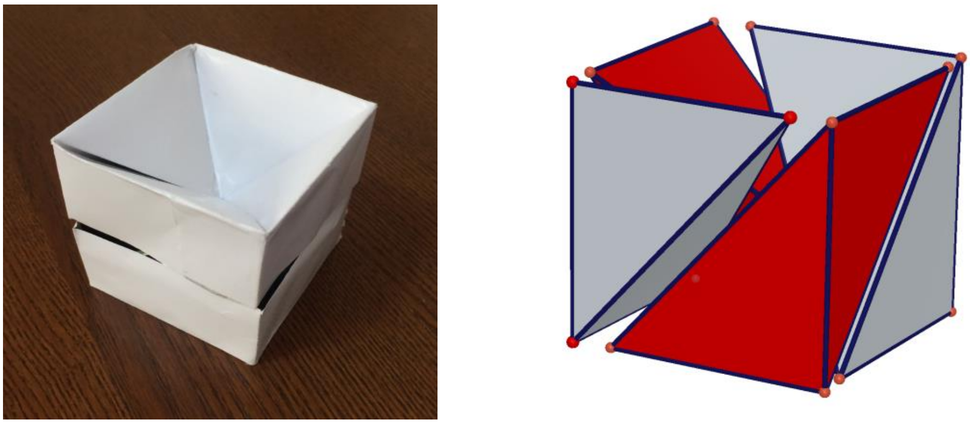

Figure 11 a real complex model and its inner parts

Another interesting variation is to identify a sphere to a regular square pyramid with base edge of 2r and height of πr (Fig. 12). When using this model it should be placed in between the two parallel planes laterally, with its base square perpendicular to them. The proof is similar to that of the tetrahedron model.

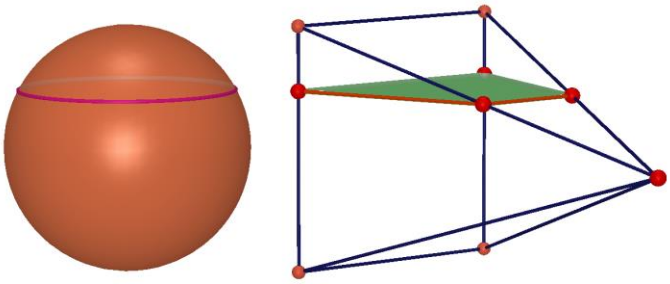

Figure 12 sphere and its regular pyramid equavalent

4. Conclusion

Archimedes himself once deducted that a disk should have an area of a right triangle whose two legs equal to radius and circumference respectively (See Fig.13) [5]. This amazing result now get generalize to 3D and has a lot of different forms. When taking disk as a 2D ball, right triangle as a 2D Orthoscheme we can parallel the result to 3D: disk corresponding to sphere and right triangle to biē nào.

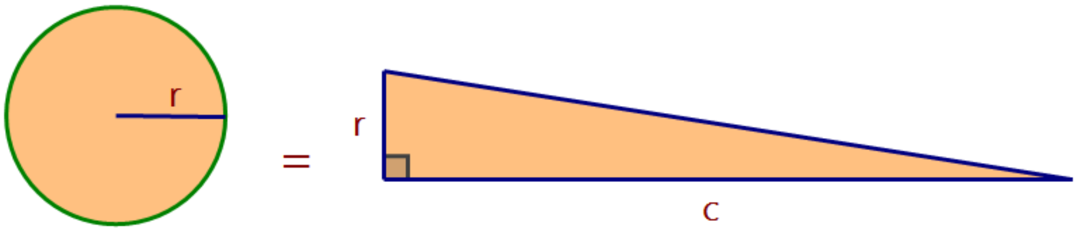

Figure 13 Archimedes' circle area model

A sphere can also transformed into other convex identities like a regular square pyramid. These convex polyhedra always stand on their edges rather on base faces when investigating by Cavalieri's principle. It remind one the Fubini's Theorem in integral that confirms two repeated integrals of a function of two variables are equal. On one hand, one repeated integral equals to volume of a sphere following Cavalieri's principle, on the other hand, another repeated integral equals to a pyramid with is easy to calculate by simple formula.

I wish this is heuristic for mathematics teachers while they teach sphere volume in the future.

References

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[3] https://en.wikipedia.org/wiki/Bonaventura_Cavalieri

[4] https://en.wikipedia.org/wiki/Schl%C3%A4fli_orthoscheme

[5] https://www.youtube.com/watch?v=whYqhpc6S6g

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