Using the Riemann Sums to Evaluate Areas and Volumes within DGS and CAS environments (TI-Nspire and Cabri): Enriching Dialectic Between Math Knowledge and Technical Skills

Jean-Jacques Dahan
jjdahan@wanadoo.fr
IRES of Toulouse
Paul Sabatier University, Toulouse
FRANCE

Abstract: *Creating math resources using technology is still an important issue for both teachers and experts in order to provide new tools for teaching that can enrich the approach of important concepts and enhance a motivating practice of math. But there is a huge difference between the work of a teacher and the one of an expert in such a process of creation. Most of the time, teachers use technology (calculators or software) in using the tools provided by their calculators or software to obtain immediate results (what I call push button results). So when they use technology to create resources, they do use a lot of the available tools (and often not the more appropriate ones) and suggest push button techniques to the users but not more. For example they can use the tool "integral" to evaluate the integral of a function or visualize its interpretation as an area because they know the tools allow that. But the experts have a global view of the concept to teach and know that the purpose of the use of technology "is not only to do faster what was possible to do by hand" in a push button approach "but to do it differently" (Colette Laborde), even if they have to create some microworlds not provided directly by the calculator or the software. We will see in this paper, that during the creation of these microworlds, the experts will be going back and forth between their technological skills their math knowledge: for example, in using their math knowledge to create some appropriate tools not available in their technology or in using some technological tricks to improve dynamically the approach of complicated concepts of mathematics. In this paper we will focus our attention on this dialectic during the process of creation of microworlds within Cabri and TI NSpire.*

1. Visualization of what we want to create and how to get it with technology

Before creating any resource about integration or any other subject using technology, an expert must know what technology can provide to help the process of understanding of a concept and to improve the paper and pencil approach. It is what we are going to do.

1.1. In 2D with Cabri

1.1.1. What we want to create

For a given function $\boldsymbol{f}$ defined on a given interval $[\boldsymbol{a},\boldsymbol{b}]$, we want

- to represent its curve and the Riemann rectangles (the two types shown in Figure 1) associated with this function for a given subdivision of this interval into $\boldsymbol{n}$ intervals.
- to have the possibility to modify the function, to modify the interval and to modify the number of points of the subdivision with a slider (in a domain that can be also modified).
- to be able to display the upper and lower Riemann rectangles and the sum of their areas of in the two cases shown in Figure 1.

1.1.2. How to obtain it

For example, for the blue step function (Figure 1, left), we create a formula that can allow us to represent one point of its curve and the curve as the locus of this point. Let us call $\boldsymbol{d}$ the length of the subdivision associated with the integer $\boldsymbol{n}$: $\boldsymbol{d} = \frac{\boldsymbol{b-a}}{\boldsymbol{n}}$.

If $\boldsymbol{\beta}$ is the bijection defined from $[\boldsymbol{a},\boldsymbol{b}]$ to $[0,\boldsymbol{n}]$ by the formula $\boldsymbol{\beta}(\boldsymbol{x}) = \frac{\boldsymbol{x}-\boldsymbol{a}}{\boldsymbol{d}}$, the inverse function $\boldsymbol{\beta}^{-1}$ is defined from$[0,\boldsymbol{n}]$ to $[\boldsymbol{a},\boldsymbol{b}]$ by the formula $\boldsymbol{\beta}^{-1}(X)= \boldsymbol{d}.\boldsymbol{X} + \boldsymbol{a}$.

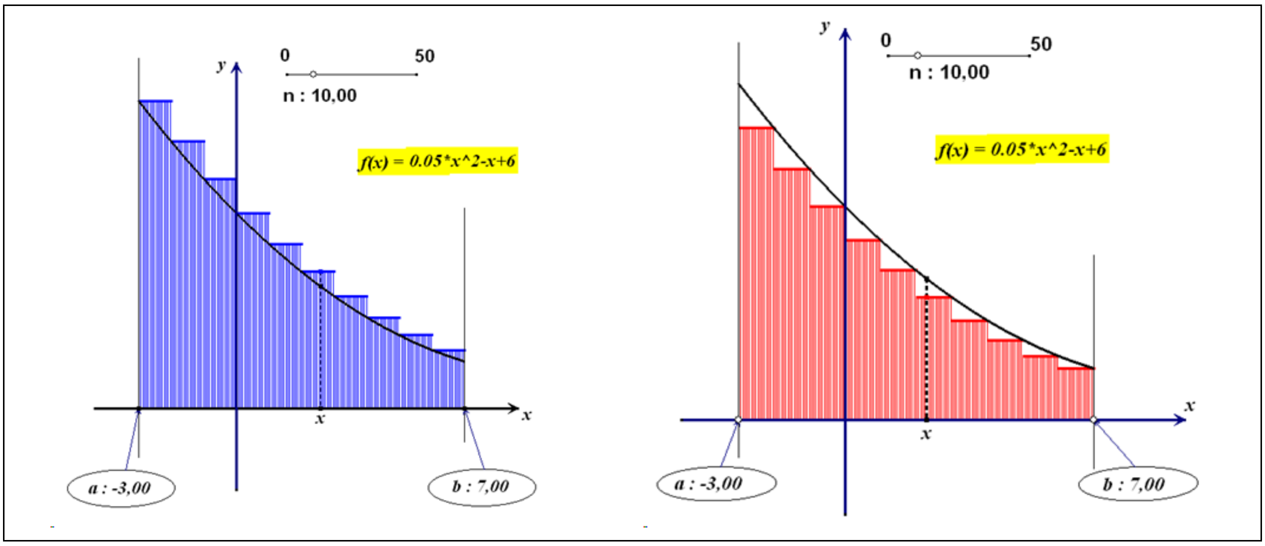


Figure 1: Riemann rectangles associated with a given function

Then the blue step function ***BS*** can be defined on $[\boldsymbol{a},\boldsymbol{b}]$ with the formula: $\boldsymbol{BS(x)} = \boldsymbol{f}(\boldsymbol{\beta}^{-1}(\boldsymbol{int}(\boldsymbol{\beta}(\boldsymbol{x}))))$. *This formula shows that it is possible to define a step function without using the "piecewise function" tool available in any CAS. We will see later that knowing this trick will allow us to solve a problem that technology cannot solve. It is also possible to visualize with Cabri 2 Plus step by step why this formula is really appropriate. It can help teachers to show the power of linear functions in relation to proportionality in the programming process.*

At last, we construct the point ($\boldsymbol{x}$, $\boldsymbol{BS(x)}$) and the blue curve is the locus of this point when $\boldsymbol{x}$ moves along segment $[\boldsymbol{a},\boldsymbol{b}]$. The blue Riemann rectangles corresponding to this curve are obtained as the locus of the dotted segment joining point $\boldsymbol{x}$ to point ($\boldsymbol{x}$, $\boldsymbol{BS(x)}$).

Verification: when $\boldsymbol{x} \in [a + k\frac{b-a}{n}, a + (k+1).\frac{b-a}{n}]$, $\boldsymbol{\beta}(\boldsymbol{x}) \in [\boldsymbol{k},\boldsymbol{k}+1]$, then $\boldsymbol{int}(\boldsymbol{\beta}(\boldsymbol{x}))=\boldsymbol{k}$ and finally, $\boldsymbol{BS(x)} = \boldsymbol{f}(\boldsymbol{\beta}^{-1}(\boldsymbol{int}(\boldsymbol{\beta}(\boldsymbol{x})))) = \boldsymbol{f}(a + k\frac{b-a}{n})$.

The same technique can be used to define similarly the red step function of Figure 1 on the right

1.2. In 3D with Cabri 2 Plus (military perspective [7])

1.2.1. What we want to create

For a given function defined on a given interval, we want

- to represent the solid of revolution obtained by rotation of the curve of the given function around the $\boldsymbol{x}$-axis (wire representation).
- to model dynamically this rotation from the initial curve to the entire representation of the solid with the help of a slider (Figure 2 on the left).
- to be able to modify the function and its domain.
- to model the rotation of the Riemann rectangles (obtained in 1.1.) and represent the Riemann cylinders (as shown in Figure 2 on the right).
- to model the rotation of the Riemann rectangles (obtained in 1.1.) and represent the Riemann cylinders (as shown in Figure 2 on the right).

- To display the volume of the solid of revolution and the sum of the volumes of the Riemann cylinders in the two cases corresponding to the two sorts of Riemann rectangles.

1.2.2. How do we obtain it

To obtain representations in military perspective (which is a parallel perspective where the objects included in horizontal planes are represented in true dimensions and where the vanishing line is the vertical axis *Oz*), we rotate the system of axis ***xOy*** of **Cabri 2 Plus** as shown in Figure 2 and we add the third axis *Oz*. To simplify our work, we chose 1 as the coefficient of the perspective.

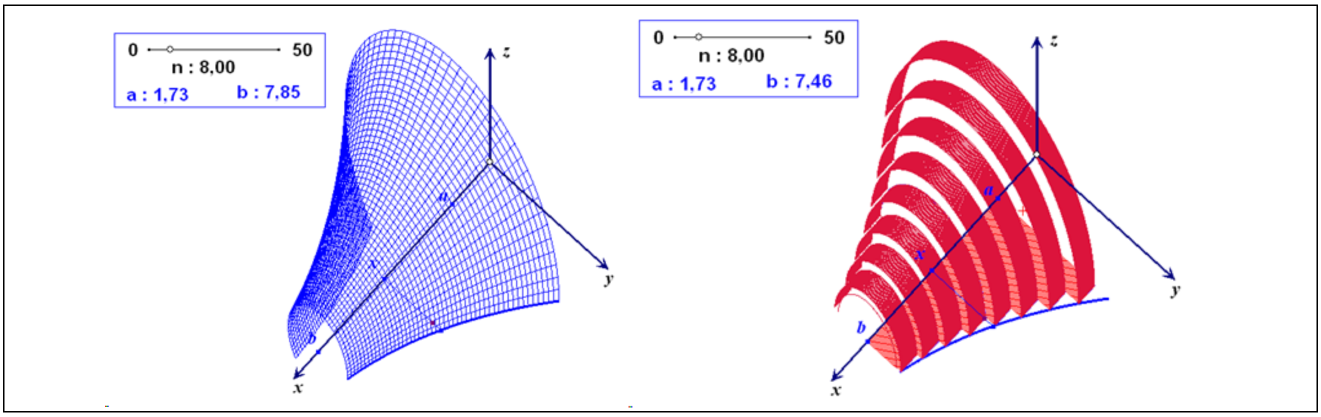


Figure 2: Solid of revolution and Riemann cylinders associated with a given function

1.2.2.1. How to obtain the wire representation of a solid of revolution

In the system ***xOy***, we represent the curve of a given function ***f*** defined on a given interval [***a***,***b***] of the ***x***-axis as the locus of point *m*(***x***, ***f*** (***x***)) when ***x*** moves along [***a***,***b***].

As shown in Figure 3, we construct the red circle centred in ***x*** having ***xm*** as a radius.

A point ***p*** is constructed on this circle and point ***M*** is constructed with the condition ***hp*** = ***hM***.

Point ***M*** is the representation in military perspective of the point obtained with respect to the rotation around the ***x***-axis and which angle is the angle ∠***mxp***. Therefore, the locus of ***M*** when ***p*** moves along the red circle is a circle represented by the blue ellipse in Figure 3 on the left.

The locus of this ellipse when ***x*** moves along [***a***,***b***] gives an idea of the solid of revolution obtained by rotation of the curve of ***f*** around the ***x***-axis (Figure 3 on the right)

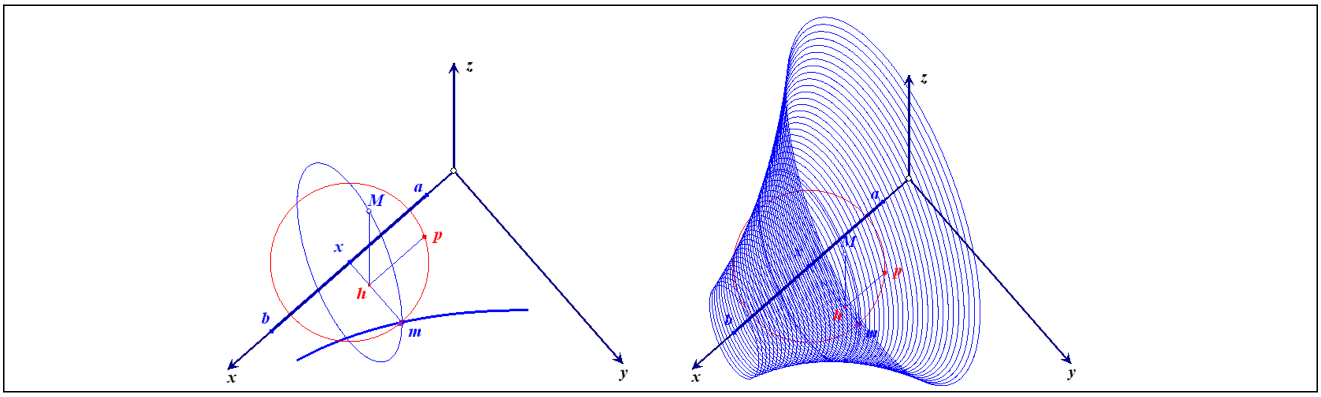


Figure 3: First representation of a solid of revolution

The locus of ***M*** when ***x*** moves along [***a***,***b***] is the rotated curve of the initial one with the previous rotation (Figure 4 on the left).

The locus of this curve when ***p*** moves along the red circle gives an idea of the solid of revolution obtained by rotation of the curve of ***f*** around the ***x***-axis (Figure 4 on the right).
If we superimpose all these constructions, we obtained the wire representation of the surface shown in Figure 2 on the left. In reality only a part thanks to the slider controlling the rotation.

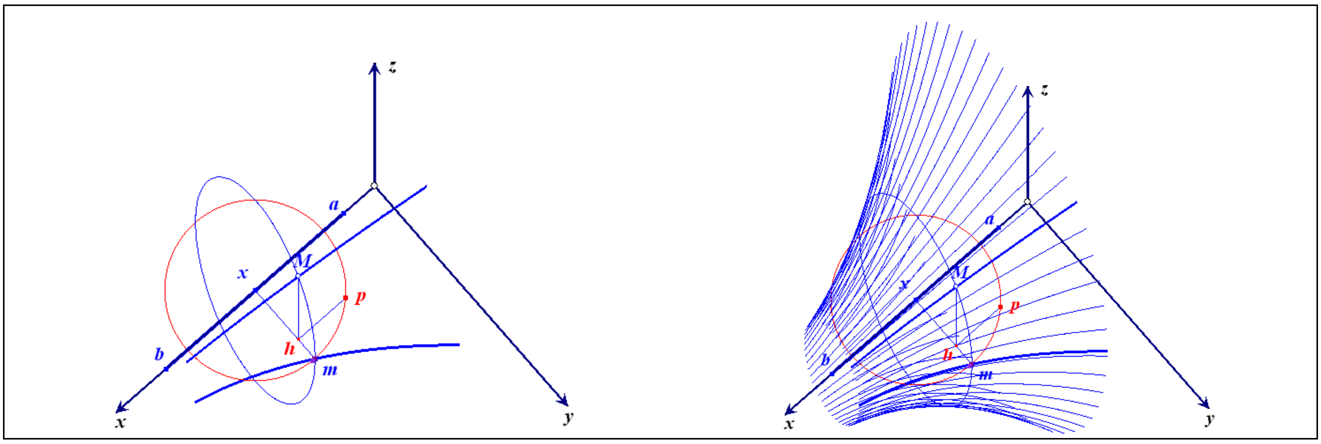


Figure 4: Second representation of a solid of revolution

1.2.2.2. How to approximate the wire representation of a solid of revolution (starting with a step function)
Here we will explain how to obtain the representation shown in Figure 2 on the right.
We use the technique presented in 1.1.2 to represent in the ***xOy*** plane the step function associated with the given function ***f*** on the interval [***a***,***b***] and associated with number ***n*** to obtain the result shown in Figure 5 on the left.
To rotate this curve around the ***x***-axis, we use now the same technique as the one described in 1.2.2. (Figure 5 on the right)
Finally we use also the same techniques to give two representations of the solid obtained in rotating the step function around the ***x***-axis (Figure 6 left and right).

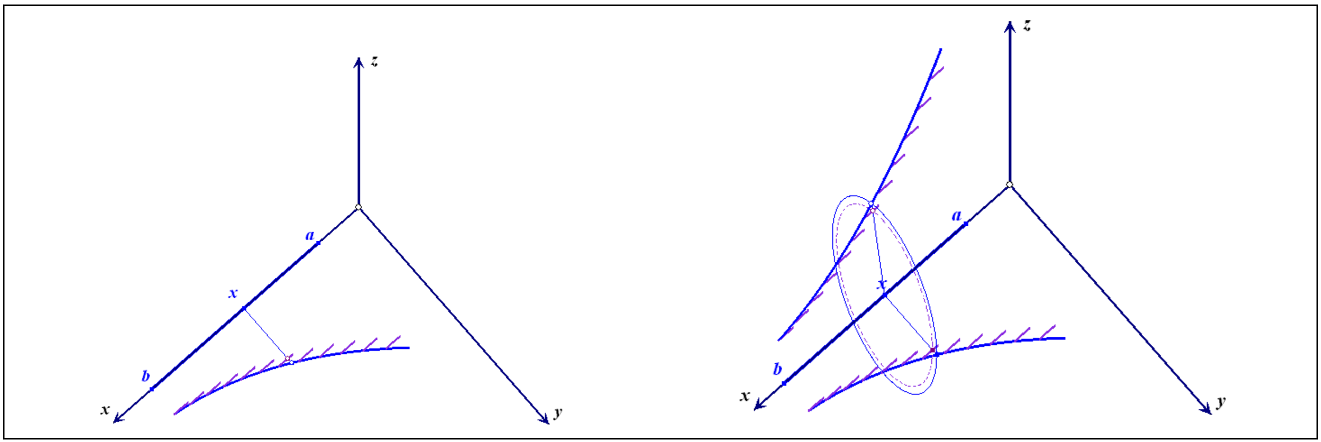


Figure 5: representations of a step function curve and its rotated one

If we superimpose all these constructions, we obtained the wire representation of the surface shown in Figure 2 on the right.

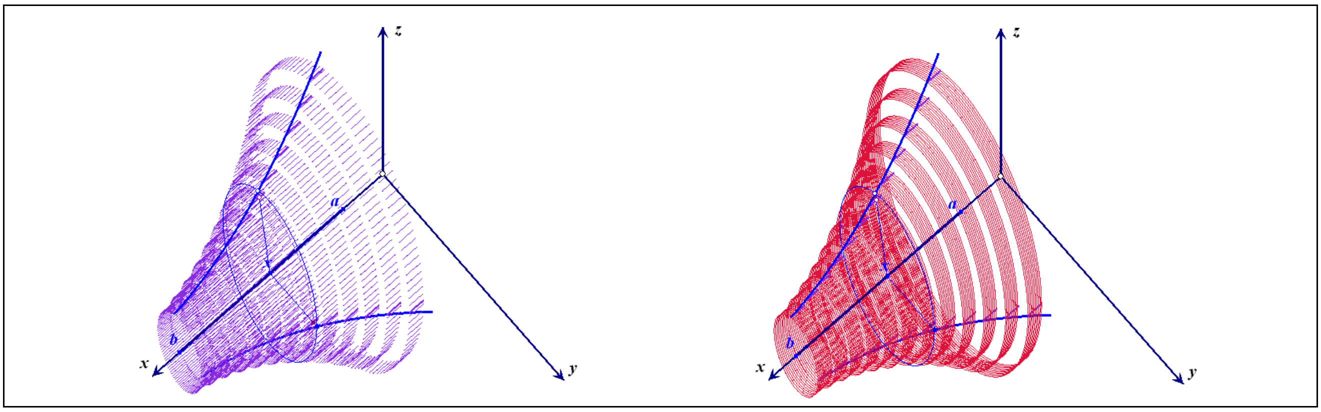


Figure 6: representations of the rotation of step function curves

2. Pedagogical utility of such files

2.1. Microworlds to improve the teaching of the concept of integral

The files displayed in Figure 1 contain sliders controlling the value of ***n*** between 0 and 50 (this value can be modified and especially increased if necessary). Increasing the value of ***n*** can lead the observer (Figure 7) to the conjecture stating that the blue area below the blue step function and the red area below the red step function approach the area below the given curve.

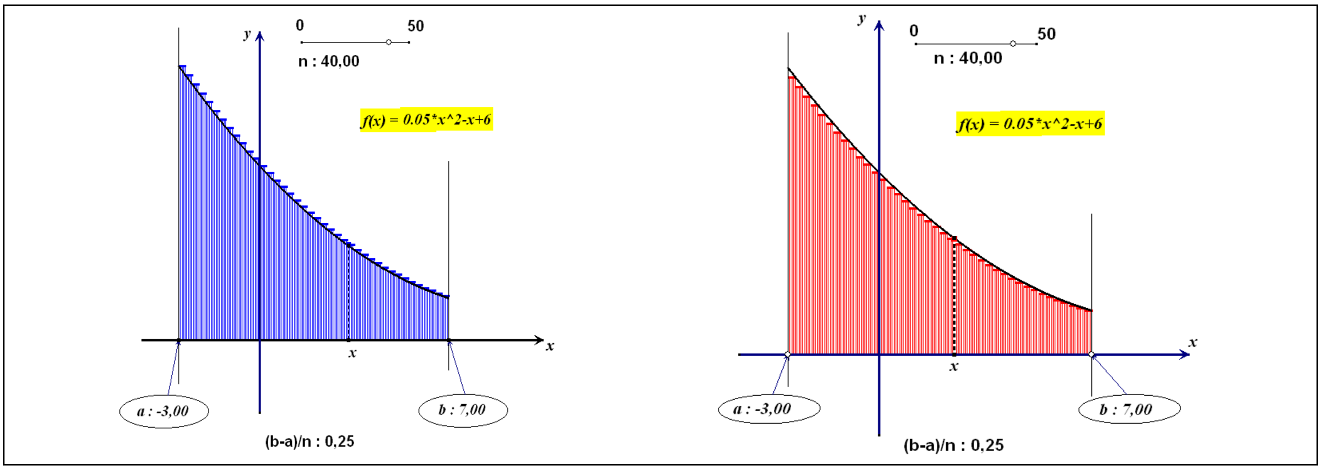


Figure 7: How to approach the area below the curve of a function

Remark 1: Within **the Cabri 2 Plus** environment, it is not possible to display interactively the previous areas, but this is possible within the TI-Nspire environment. We will show how below.
Remark 2: We must never forget that a perceptive approach of a figure ([3]) even on a screen of a computer can be misleading. Here is an example suggested by Helmut Heugl illustrating this remark. We have constructed in Figure 8 a yellow square with side measured as 10 and therefore (via a deductive approach) the diagonal (represented as a part of the curve associated with the function $\boldsymbol{f(x)=-x+10}$) is measured as $10\sqrt{2}$. But the visualization of this diagonal and the polygonal line representing a stair above it (a stair with ***n*** steps) leads us to the conjecture that the length of the diagonal is approached better and better with the length of this polygonal line when increasing the value of ***n***. However the length of this polygonal line is constant and equal to 20 because the sum of

the lengths of the horizontal steps is equal to the side of the square and same thing for the sum of the lengths of the vertical steps. That is a paradox. We must keep such an example in our mind to teach our students to be careful before stating that: “what I see is what is true!”.

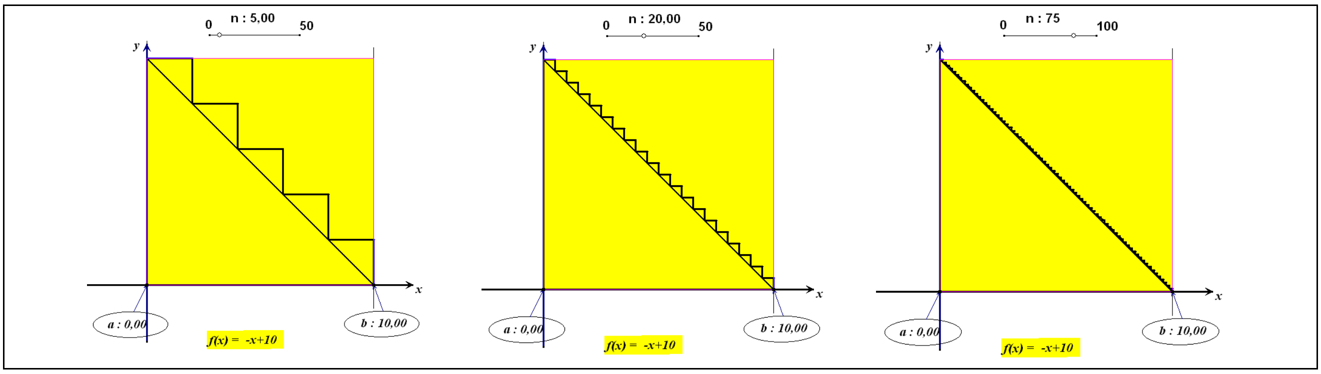


Figure 8: Illustration of a perceptive paradox

2.2. A possible introduction of the concept of integral within TI-Nspire
2.2.1. Linear functions and integral tools

Here is a possible scenario (tested in 2016) and improved (in 2017) with a math teacher of the scientific baccalaureate level of the French curriculum ([6]).

The student can use a double page of **TI-Nspire** containing a **Graphs&Geometry** page and a **Note** page below. A linear function ***f1*** is provided with two sliders controlling ***m*** and ***p***. Two other sliders are available to change the positions of ***a*** and ***b*** on the ***x***-axis. The ‘integral’ tools can be used as black boxes:

- In the **Graphs&Geometry** page, the integral of ***f1*** between ***a*** and ***b*** displays a number (here 16.5) as well as a grey trapezoid (Figure 9 on the left).

- In the **Note** page, the computation of the integral of ***f1*** between ***a*** and ***b*** displays the formula $\int_a^b f1(x)dx$ and the same number 16.5. We can check that this equality is verified even if we change the values of ***m***, ***p***, ***a*** or ***b*** but only if ***f1***(***x***) is positive between ***a*** and ***b***.

The conjecture stating that this result could be the area of the grey trapezoid can be validated in the **Note** page in evaluating the area of this trapezoid with its geometric formula (Figure 9 on the right)

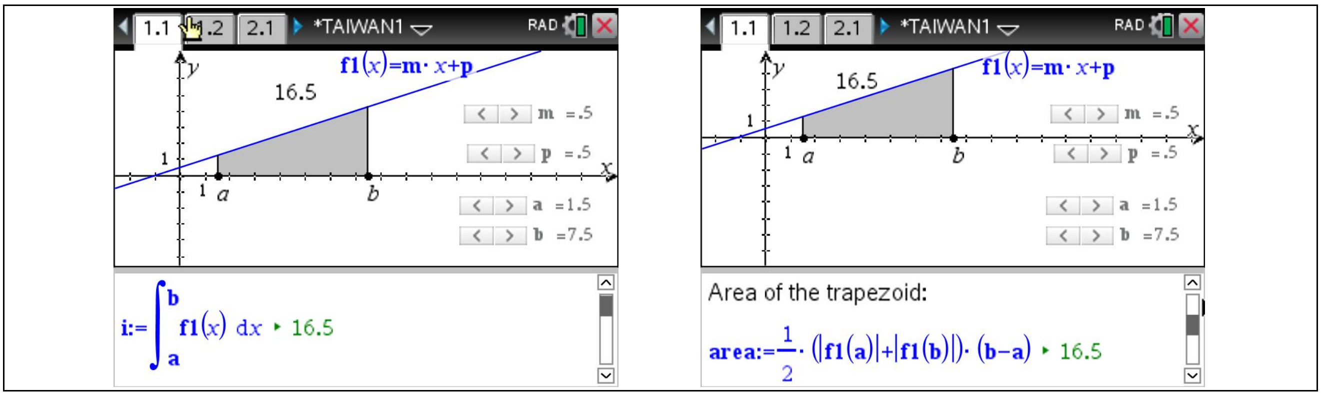


Figure 9: Integral of a linear function and area of a trapezoid

If we investigate figures like Figure 10, we can notice that the value obtained for the integral (-9 here) does not correspond with the number given by the previous formula which is not unexpected.

But, if we delete the absolute value in the previous formula, we can check that we obtain again the value given for the integral (Figure 10 on the right). We call this integral the algebraic area defined by ***f1***, ***a*** and ***b***.

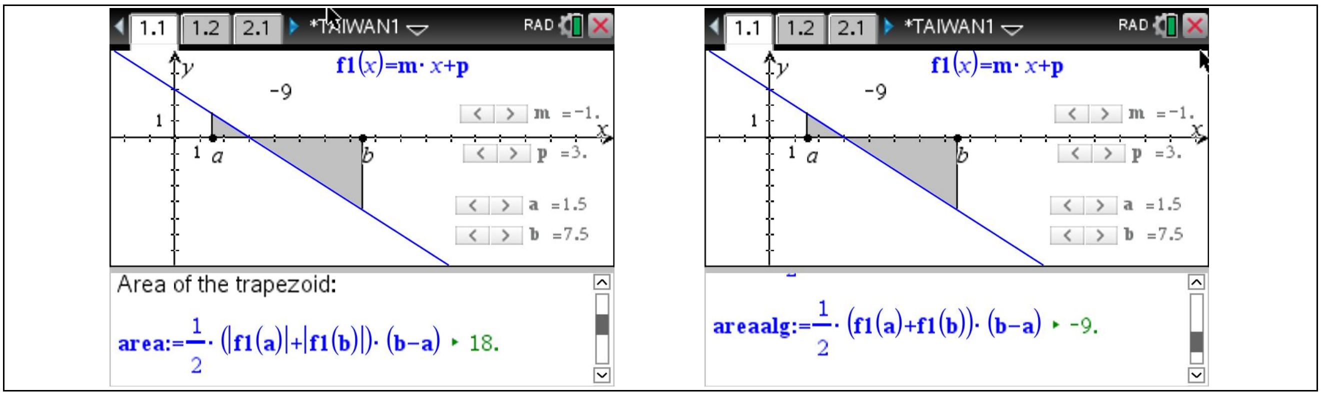


Figure 10: Integral of a linear function and algebraic area of a trapezoid

Another file can be provided to students to understand the link between the value of the integral of a linear function and areas defined by the curve of a linear function, line x=***a***, line x= ***b*** and the x-axis. The students can experiment when the sign of ***f1***(x) is not constant on [***a***,***b***]. They can display the two triangles defining the previous trapezoid, a blue and a red with their areas. Therefore it is easy to conjecture that the value of the integral is equal to the difference of the blue area and the red one (Figure 11 on the left). When ***f1***(x) is negative along [***a***,***b***], we can check that the integral is the opposite of the area of the trapezoid (coloured in yellow) which is coherent with the previous experiment (Figure 11 on the right).

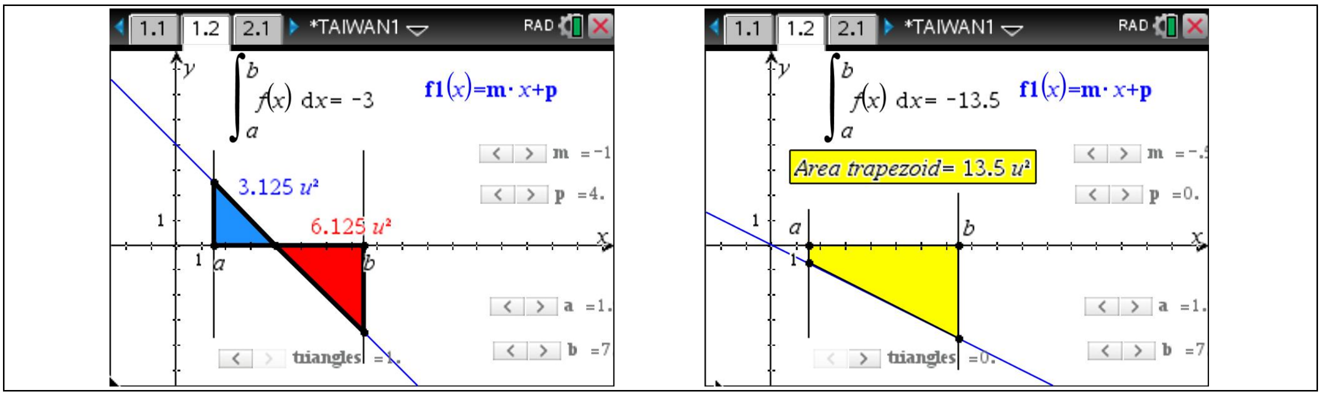


Figure 11: Integral and areas of trapezoids or triangles in the general case

2.2.2. Other functions and integrals

In the next file available in Figure 12, the linear function is changed for a quadratic one which is positive on the interval [***a***,***b***]. The value of the integral is displayed (Figure 12 on the left).

Next, the students are asked to create a polygon with area as close as possible to the area defined by the curve of the function, the x-axis and the two vertical lines x=***a*** and x=***b***. They can create a polygon like the red one of Figure 12 (on the right) and state that, again, this area seems to be given by the value of the integral. They can validate this conjecture by changing ***f1***, ***a***, ***b*** in taking care of the construction of the red polygon.

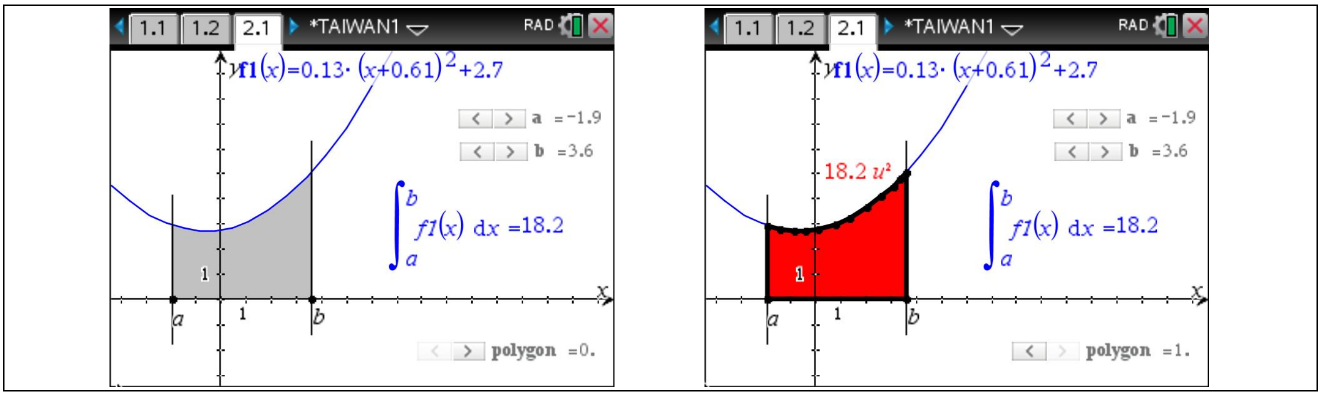


Figure 12: Integral and area

2.3. Conclusion of such work

At the end of such investigations enhanced by the interactive files provided to the students, they can understand that, evaluating an integral is only evaluating an area, the opposite of an area or a difference of areas or… In order to simplify the purpose of my paper, we will use only positive functions on a domain [***a***,***b***].

3. Approximations of integrals with Riemann rectangles (TI-Nspire)

3.1. Approximations with lists and scatterplots

In the **Note** page shown below (Figure 13 on the left), we have defined ***f1***, ***a***, ***b*** evaluated $\boldsymbol{\frac{b-a}{n}}$ and stored it in ***d***. In the **Graphs** page (Figure 13 on the right), we have displayed the curve of function ***f1*** and two scatterplots: **s1** defined by **list10** and **list11** (blue points between ***a*** and ***b***) and then **s2** defined by **list10** and **list1** (blue points on the curve of ***f1*** between ***a*** and ***b***) where **list10** is defined by **seq**(***x***, ***x***, ***a***, ***b***-***d***, ***d***) and **list11** by **seq**(**0**, ***x***, **1**, ***d1***, **1**).

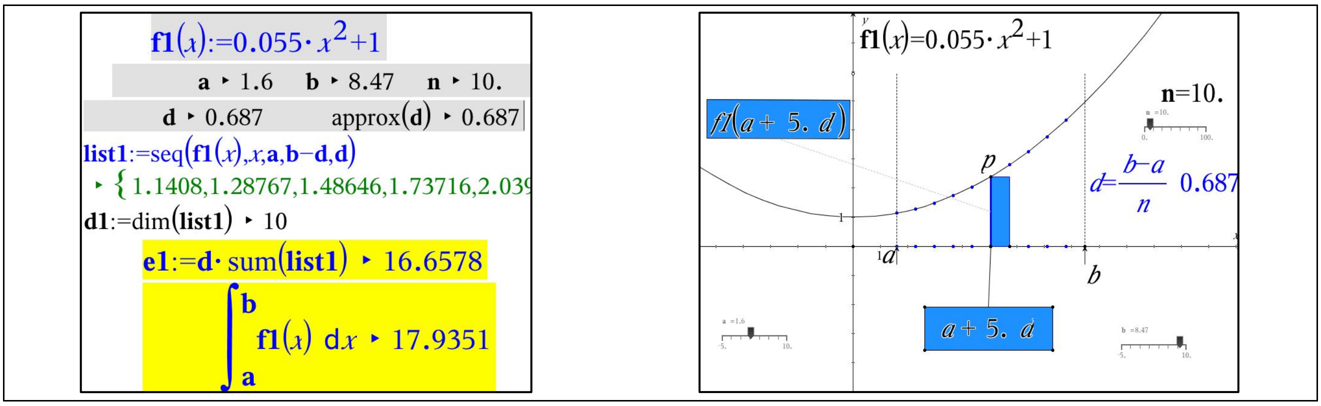


Figure 13: Riemann rectangles and their areas with lists

In yellow we can see the value of **e1**, which is the sum of the areas of the blue Riemann rectangles below ***f1*** and the value of the integral of ***f1*** along [***a***,***b***]. When increasing the value of ***n*** we can state that the value of **e1** increases and approaches closer and closer to the value of the integral of ***f1*** (Figure 14 on the left).

We can do the same work for to define the Riemann rectangles above ***f1*** and state that the value of **e2** decreases and approaches closer and closer to the value of the integral of ***f1*** (Figure 14 on the left)

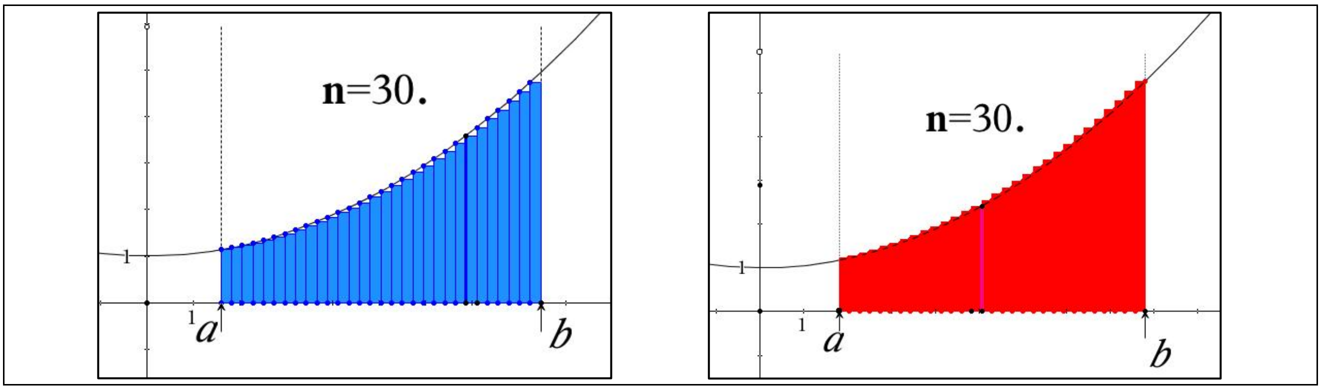


Figure 14: Improving accuracy of areas through the use of Riemann rectangles

When $\int_a^b f1(x)dx$ = 17.7032 we can obtain on the **Note** pages the following results

n	e1	e2
30	17.2874	18.1251
100	17.5779	17.8292
1000	17.6907	17.7158

3.2. Approximations with piecewise functions

3.2.1. First approximations

Each Riemann step function is defined by **ff**(k,x) in the screenshot of the **Note** page below (Figure 15 on the left) and the Riemann step function approximating the given function **f** is the sum defined by **g**($\boldsymbol{x}$). Let us remark that this function is equal to 0 outside the interval [***a***,***b***]. Functions **f** and **g** are represented in a **Graph&Geometry** page by **f1** and **g1** (Figure 15 on the right), **f1** in black and **f2** in blue.

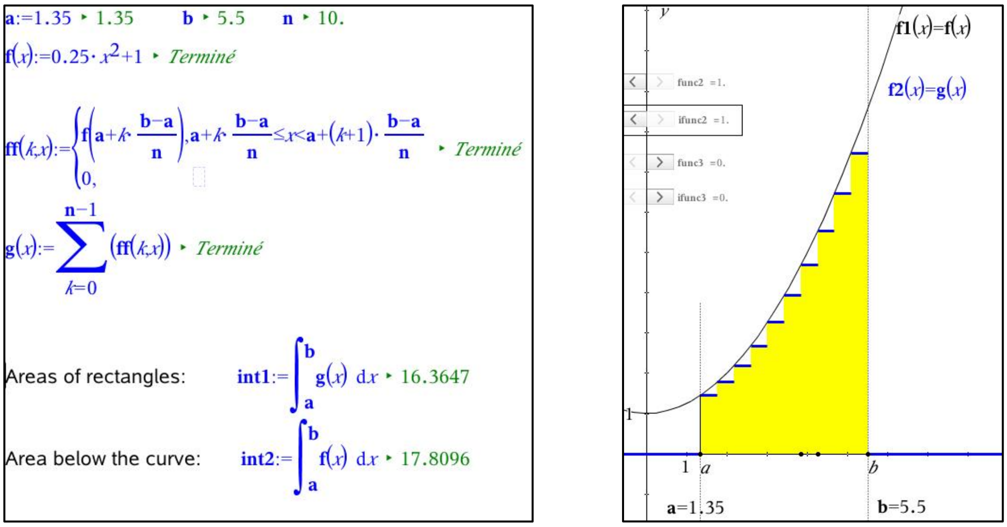


Figure 15: Integral and piecewise functions 1

We have evaluated the area of the yellow Riemann rectangles in evaluating **int1**, which is the integral of the blue Riemann step function: here 16.3647 for a subdivision of ***n*** = 10 intervals. We have also evaluated **int2**, which is the integral of the given function: here 17.8096, which is more than the previous number.

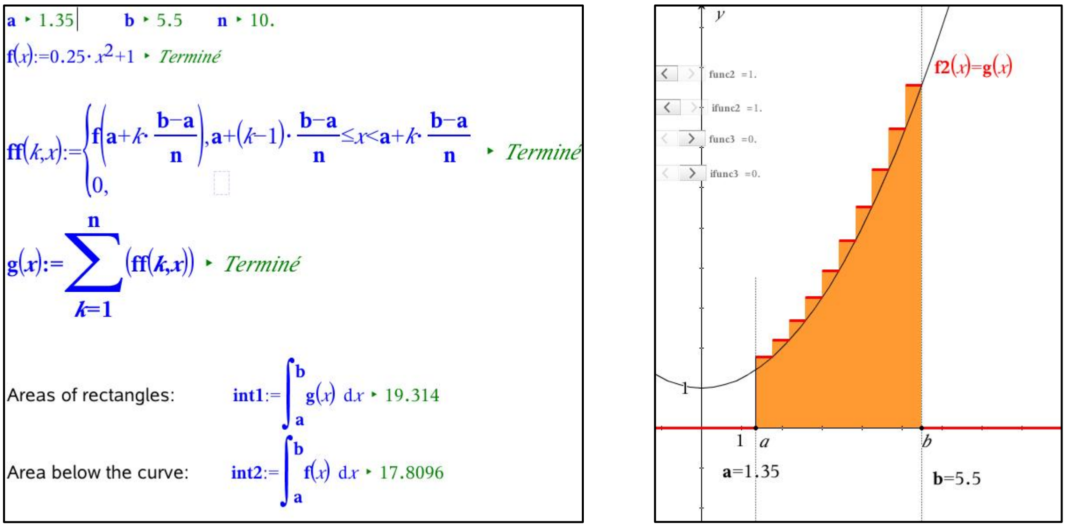


Figure 16: Integral and piecewise functions 2

3.2.2. Second approximation

We can do a similar work as that one shown in Figure 16: here the area of the rectangles is given by 19.314 which is more than the integral of ***f***.

3.2.3. Experimenting to approximate with two areas the integral of the given function

We increase the value of ***n*** as we did in 3.1. and we can observe the same phenomenon: The areas below ***f*** and the areas above ***f*** approximate increasingly well the value of the integral of the given function

4. Solids of revolution: dynamic visualizations and volumes

4.1. Dynamic visualization of solids of revolution in TI-Nspire

To represent a solid of revolution, we use the **3D graphing tool** of TI-Nspire and mostly parametric equations. To begin, we create the given function ***f1*** anywhere in a page of a **Problem**. Then, in a 3D page we display its curve in a domain which is an interval [***a***,***b***] in creating these parametric equations

xP20(t,u) = t, **yP20**(t,u) = *f1(t),* **zP20**(t,u) = 0 and settings: $t \in [-5,5]$ (Figure 17 in the middle).

For technical reasons, I have improved this representation in giving a thickness to the curve in modifying the previous equations of a curve onto the equations of a surface like this

xP20(t,u) = t, **yP20**(t,u) = *f1(t),* **zP20**(t,u) = u and settings: t ∈ [-5,5] and $u \in [0,0.05]$ (Figure 17 in the middle too). To see the trick, we can chose $u \in [0,1]$ (Figure 17 in the left).

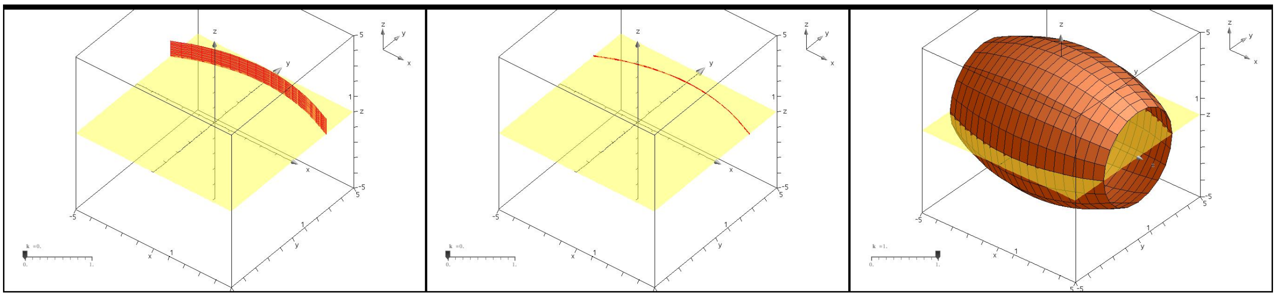


Figure 17: Solid of revolution

The solid of revolution obtained by rotation of this curve around the ***x***-axis is defined by the equations: **xP21**(t,u) = **xP20**(t,u), **yP20**(t,u) = **yP20**(t,u).cos(u), **zP20**(t,u) = **yP20**(t,u).sin(u) with the following settings $t \in [-5,5]$ and $u \in [0,2\text{pi}]$ (Figure 17 on the right)
In order to model the rotation of the initial curve and the progressive visualization of the solid, we will use a trick suggested by Steve Phelps during the T^3 congress of Orlando 2016 ([4]).
We create a slider controlling ***k*** between 0 and 1 and we change the equations for these ones:
xP21(t,u) = **xP20**(t,u), **yP21**(t,u) = **yP20**(t,u).cos(***k***.u), **zP21**(t,u) = **yP20**(t,u).sin(***k***.u) (Figure 18).

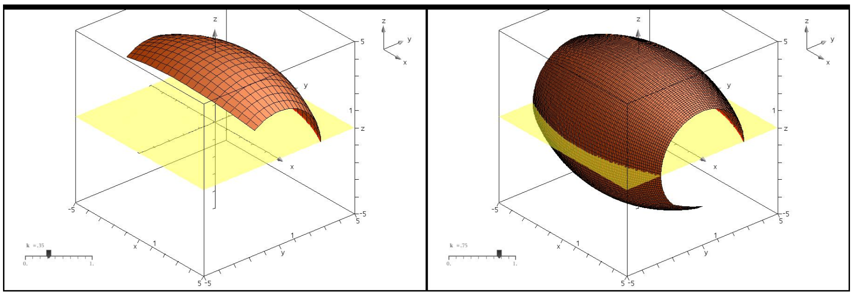


Figure 18: Dynamic representation of a solid of revolution

4.2. Representation of Riemann cylinders (beginner level)
We focus our attention on the interval [0,5] and its subdivision into 5 intervals each with width of 1 (Figure 19 on the left). We will show how to represent the cylinder obtained in rotating the segment defined by $t \in [2,3]$ and **xP5**(t,u) = t, **yP5**(t,u) = ***f1***(3) and **zP5**(t,u) = 0 around the ***x***-axis. As we did before, we represent the dynamic cylinder with similar formulas:
xP6(t,u) = **xP5**(t,u), **yP6**(t,u) = **yP5**(t,u).cos(***k***.u), **zP6**(t,u) = **yP5**(t,u).sin(***k***.u) where the settings are $t \in [2,3]$ and $u \in [0,2\text{pi}]$ (in light blue, Figure 19 on the right) and the slider ***k*** controlling numbers between 0 and 1. We can do the same work for each of the other intervals [0,1], [1,2], [3,4] and [4,5] (in blue, Figure 19 on the right).
We evaluate now the volume of these cylinders:
vapp = $\pi.(\boldsymbol{f1}(\mathbf{1})^2+\boldsymbol{f1}(\mathbf{2})^2+\boldsymbol{f1}(\mathbf{3})^2+\boldsymbol{f1}(\mathbf{4})^2+\boldsymbol{f1}(\mathbf{5})^2)$ which is an approximation of half of the volume of the solid of revolution. The value we got for ***f1***(**x**) = $-0.05x^2+4$ is **189.901.** As the exact value of this volume given by the integral **vex** = $\int_0^5 \pi\left(f1(x)\right)^2 dx$ is 203.876, the error is less than 7%.

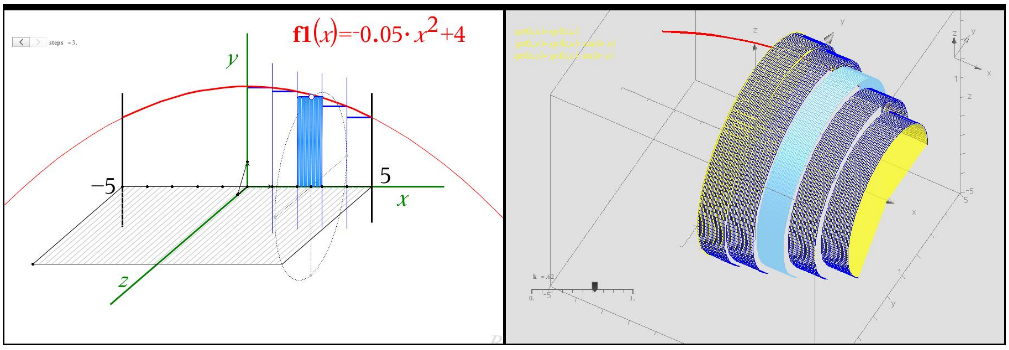


Figure 19: Dynamic Riemann cylinders

4.3. Representation of Riemann cylinders (expert level)

It would have been easy to apply the technique presented in 4.1. to the piecewise functions already used in 3.2. but the 3D application of TI-Nspire does not accept piecewise functions. So I had to circumvent this difficulty. Here is the reasoning leading to the formula which was accepted by the 3D application of TI-Nspire. $\boldsymbol{f}$ is the given function, $[\boldsymbol{a},\boldsymbol{b}]$ its domain and $\boldsymbol{n}$ the integer defining the subdivision of this domain. The points of this subdivision are: $s(k) = a+k.d$ where $d = \frac{b-a}{n}$. We use auxiliary functions:

- The first one is defined by $\mathbf{aux1}(k,x) = (x\text{-}\mathbf{s}(k)).(\mathbf{s}(k+1)\text{-}x)$ which is a quadratic function positive in $]s(k), s(k+1)[$ and negative outside.
- The second one defined by $\mathbf{aux2}(k,x) = \frac{\mathbf{aux1}(k,x)+|\mathbf{aux1}(k,x)|}{10^{-4}+\mathbf{aux1}(k,x)+|\mathbf{aux1}(k,x)|}$ which value is very close to 1 when $\boldsymbol{x}$ belong to $]s(k), s(k+1)[$ and equal to 0 outside.
- The third one is: $g(k,x) = f(\mathbf{s}(k)$. $\mathbf{aux2}(k,x)$ which value is very close to $f(\mathbf{s}(k)$ when $\boldsymbol{x}$ belong to $]s(k), s(k+1)[$ and equal to 0 outside.

And eventually function $h(x) = \sum_{k=0}^{n-1} g(k,x)$ which is the blue step function defined in 3.2.

This process is described step by step in Figure 20 on the left (in a **Graphs&Geometry** page of **TI-Nspire**). Another issue was the fact that, while $\boldsymbol{g}$ was accepted by **TI-Nspire**, $\boldsymbol{h}$ was not. So, in order to apply to $\boldsymbol{h}$ the technique shown in 4.1., I had to replace the previous formula of $\boldsymbol{h}$ by:

g(0,t)+g(1,t)+g(2,t)+g(3,t)+g(4,t)+g(5,t)+g(6,t)+g(7,t)+g(8,t)+g(9,t) (called ***ri10***(*x*)) when $\boldsymbol{n} = 10$.

With this formula, it was possible to represent the 10 dynamic cylinders associated with our given function (Figure 20 on the right).

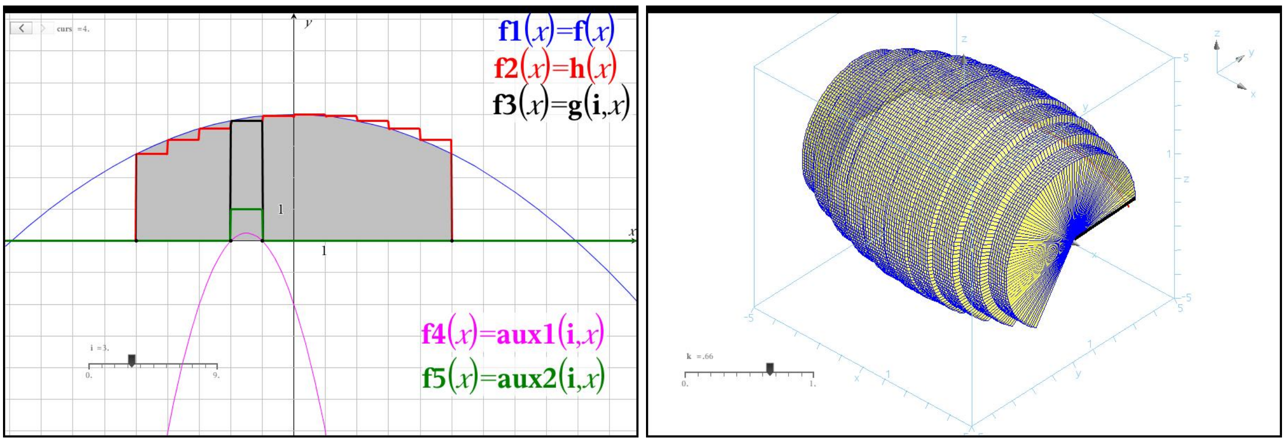


Figure 20: Solid of revolution with the 3D graphing tool of **TI-Nspire**

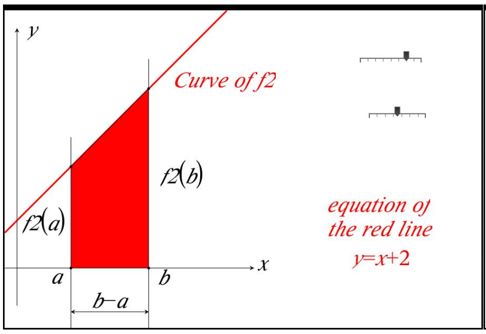
For $f(x)= -0.05x^2+4$ on the domain [-5,5] and for $n = 10$ we have obtained:
The volume of the ten Riemann cylinders is: $\frac{b-a}{n}\sum_{k=0}^{n-1}\pi(f(s(k))^2$ with the value of 406.31 displayed. The exact value of the volume of the solid of revolution is given by $\int_{-5}^{5}\pi\left(f(x)\right)^2 dx$ for which the displayed value is 407.753.

5. Integrals and antiderivative functions

The research work (quoted in **2.2.1**.) conducted with a math teacher led to the creation of the first activity about integrals ([6]). This activity was proposed to students of the level of baccalaureat during two years. The improvements we performed during the second year were very successful. The beginning of this activity looks like what is shown in **2.**. The students had to download a ***tns*** file (**TI-Nspire** file) we have prepared for them and follow the instructions page after page. Their task was:

- Each student choses a function $f1$ and a function $f2$; for each function $f1$ and each function $f2$ chosen he or she had to evaluate the areas $A1$ and $A2$ of the trapezoids defined by the curve of the function, the x-axis and the verticals $x = a$ and $x = b$.
- Then they had to find two functions $F1$ and $F2$ verifying the conditions $A1 = F1(b)-F1(a)$ and $A2 = F2(b)-F2(a)$.
- They had to conjecture a relation between functions $f1$ and $F1$ and $f2$ and $F2$.
- Eventually, they had to fill the table of Figure 21on the right.

For example: for the student who has chosen $f2$(x)= x+2 (first of the 5 students: *J.A*), the area $A2$ must be $\frac{(b-a)(a+2+b+2)}{2}$. In order to find function $F2$, he had to expand this expression (by hand or with the CAS of the calculator) and obtain $-\frac{a^2}{2}-2a+\frac{b^2}{2}+2b$. Finally he had to write it like: $(\frac{b^2}{2}+2b)-(\frac{a^2}{2}-2a)$ to recognize $F2(x) = \frac{x^2}{2}+2x$. The final conjecture for each student had to state that $F2'(x) = f2(x)$



Your results

	$f1(x)$	$F1(x)$	$f2(x)$	$F2(x)$	Relation
J.A	2	$2x$	$x+2$	$\frac{x^2}{2}+2x$	$F1'=f1$ $F2'=f2$
M	1.5	$1.5x$	$2x$	x^2	same
N	0.5	$0.5x$	$2x-2$	x^2-2x	same
Ti	3	$3x$	$x+3$	$\frac{x^2}{2}+3x$	same
TO	2.5	$2.5x$	x	$\frac{x^2}{2}$	same

Figure 21: Conjecturing a relation between integrals and antiderivative functions

The last work proposed to the students was to write down the conjecture stating that the area defined by the curve of a positive function f the x-axis and the vertical lines $x = a$ and $x = b$ is given by the formula $F(b)-F(a)$ where F is a function satisfying $F'(x)=f(x)$ (called the antiderivative function of function f). As this formula had been validated for linear functions, we suggested that they guessed what could be the area when $f(x)=x^2$ and $a = 0$ and $b = 1$. They had to find $F(x)= \frac{x^3}{3}$ and conjecture that the area must be equal to $\frac{1}{3}$ of unit which is 0.333….

To validate this conjecture they had to manipulate the last **Graphs&Geometry** page (Figure 22 on the left). They could check that the slider ***red*** controls the Riemann rectangles below the red step function and their area (here 0.285 in Figure 22 on the left). They could also check that the slider ***blue*** controls the Riemann rectangles below the blue step function and their area (here 0.385 in Figure 22 on centre). They could also check that slider ***d*** changes the number of intervals of the subdivision.

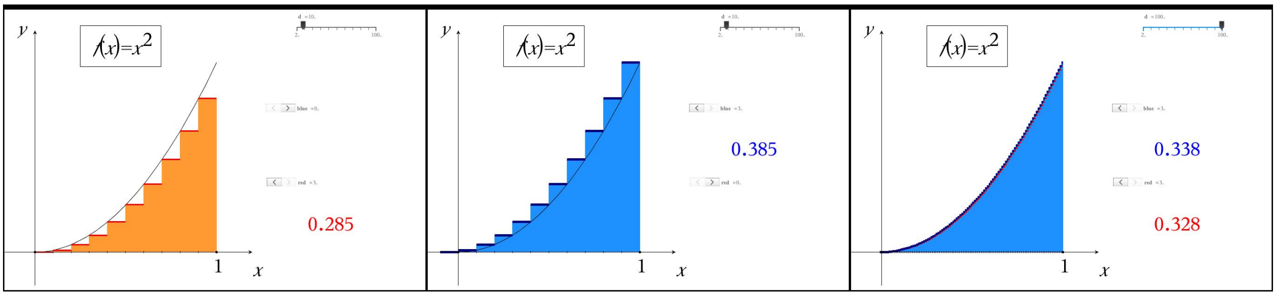


Figure 22: Validation of the conjecture about $\int_0^1 \mathrm{f(x)}dx$

In order to validate the last conjecture, they had to increase the value of ***d*** and observe the values of the two areas of the Riemann rectangles approximating the area below ***f*** and by the way the integral of ***f*** between 0 and 1. If ***d*** =1000, we obtain 0.338 and 0.328 and we get a validation in the meaning of Popper. The value conjectured for the integral cannot be refuted.

5. Conclusion

In this paper, we have illustrated the fact that math knowledge is compulsory if we want to use technology in a transparent way. Math knowledge allows the expert to create microworlds in which students can experiment without worrying about their skills with the technology used. We have seen that technology is never perfect to respond to the desires of the teachers but tricks based on the expert knowledge of the used couple math-software illustrate this dialectic, this back and forth between math and technology. Most of the time, it is the lack of technological tools to model an important math phenomenon that enhances the math and technological creativity and nourishes this dialectic. The mathematical and technological power of proportionality and linear functions shows a direction for teachers to use them with their students: in doing so they will illustrate a dialectic of modelling between reality (the technological environment) and abstraction (pure mathematics). I am convinced after all the research works conducted with teachers, after all the activities created during these researches and tested with students that our brain filled with deep math knowledge is the core of a powerful use of technology to improve math teaching.

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https://1.cdn.edl.io/8TbpwZIXVs8Trq3N6FwigBOW6G0I7gau4ZiPxzBYrBgk77Lt.pdf

[5] YouTube playlist of a previous version of this presentation:
https://www.youtube.com/watch?v=K2xifUK1Tmc&list=PLOIs4xavv0zGG7i4dgb1HYHQ8ZEyYltOE

[6] Youtube video: « Introduction de la notion d'intégrale en terminale » at
https://youtu.be/RIxdTTExIgA

[7] YouTube videos about military perspective
"T3_Chicago_MP" at http://youtu.be/vYStmrDNT8E
"T3_Chicago_1" at http://youtu.be/yXDhiOD1fvw
"T3_Chicago_2" at http://youtu.be/3rJZZMztbng

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Cabri 2 Plus and *Cabri 3D* by Cabrilog at http://www.cabri.com
TI-Nspire by Texas Instruments at http://education.ti.com/en/us/home