Quadratic and Cubic Polynomials in Applied Problems: Finding Maximum – no Calculus, using CAS (Maple)

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Abstract

We use CAS to provide a new algebraic approach in some optimization applications where the objective function (to be minimized or maximized) is a quadratic polynomial. These problems can be solved just by knowing properties of quadratics and so give context to why we want to complete the square. Without calculus, to find the max/min of cubics, we introduce a new straightforward algebraic method (no calculus). The use of a Computer Algebra System, CAS, such as Maple easily deals with any messy algebra!

The key property of a function is that, near a maximum or minimum, the function "looks like a quadratic". This visual idea is combined with some straightforward algebra to find this local quadratic approximation of a cubic near the maximum.

Traditional "find the maximum . . . " problems are introduced to senior school or first year undergraduate students in their calculus course. With our approach, these applied problems and the Polya method of problem solving can be introduced to pre-calculus students. We use the CAS, Maple, for algebra and visualization. We use small group collaborative learning in the computer laboratory, so we parameterize the problem and recommend the use of Computer Aided Assessment (such as provided by the package MapleTA). Students engage with the visualization and algebra, are active learners with deep learning of the concept of maximum and have fun doing so.

1 Introduction

Students of secondary school mathematics courses often struggle with both visualization (and intuition) and algebra. Algebraic skills are often deficient for later mathematics. Year 10 and 11 students practise strange rituals such as "completing the square" with little understanding and lots of frustration. Graphical calculators can help visualization, and CAS can perform algebra (and provide visualization), but problems still persist.

In this paper we consider standard optimization problems which are usually studied as applications of calculus. When the function to be maximized (or minimized) is a quadratic polynomial then the optimaization problem can be solved by using properties of quadratics.

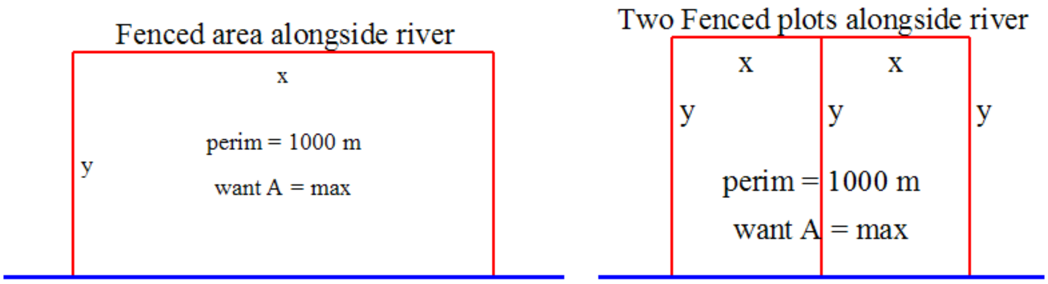

Figure 1: The farmer wants maximum area, A, with 1 plot or with 2 plots.

This can give students a clear reason for completing the square in an applied setting. Section 2 of this paper discusses problems with Quadratic objective functions.

In Section 3, we discuss how the quadaratic approximation to a cubic polynomial can be obtained by some straightforward algebra; how to use this to find the location of the maximum and visualization of the cubic "looking like" a quadratic near the maximum (or minimum). Section 4 is a brief Cconclusion.

2 Problems with Quadratic objective functions

A time honoured (and still good) place to start is the Farmer Max Problem.

2.1 The Farmer wants maximum area for fixed length of fencing

One of the early and very traditional problems is that of a farmer fencing a plot of land along a river to obtain a maximum area of land (with a given length of fence), see [1]. We call the length of fencing the perimeter (since we do not consider the side formed by the river). Here, we assume that the perimeter is 1000 m. For a diagram, see the first half of Fig. 1.

It is easy to show that the objective function (that is, the function to be maximized) is the area, $A = 2\,x\,(1000 - x)$. This is a "symmetric" quadratic with the axis of symmetry about the mid point of the domain. From the basic property of quadratics, the maximum (or minimum) is the value of the objective function at the axis of symmetry (in this case, at the midpoint of the domain). No calculus should sensibly be used here, despite this problem being the first of the optimization problems in very many calculus textbooks.

The traditional problem of a farmer fencing one plot of land along a river to obtain a maximum area of land (with a given length of fence) can be modified to want two (rectangular) plots of equal size (or, more precisely, two congruent rectangles). For a diagram, see Fig. 1.

It is easy to show that the objective function (that is, the function to be maximized with two plots) is $A = 4/3\,x\,(500 - x)$. This gives a "symmetric" quadratic with the axis of symmetry about the mid point of the domain. So the mathematics for this problem is not qualitatively different from the original, traditional problem of a farmer with one plot.

Teaching Note. The farmer problem can be further modified to have three (or more) equal sized plots. However all of these problems are qualitatively the same. Students initially meeting quadratics might solve the two or three plots problem and enjoy the redundancy. It is easy to change the length of fencing for different students (or, preferably for different student groups).

However for higher level students, the multiple plots (of land) could be used as an extra way of "individualizing" problems for the small groups of students working collaboratively.

2.2 Completing the square

Completing the square is usually introduced to school students in Year 10 as part of algebra: for example, in Chapter 5–Quadratic equations of the textbook ICE–EM Mathematics Secondary 4A, see [6]. This textbook is one of a series and has an eminent team of authors. However we wait until Chapter 7–The parabola for sketching of parabolas, translations of parabolas and applications to finding the minimum or maximum (including the familiar Farmer Max problem). In schools in Victoria, students have had, for a couple of decades, at least a graphical calculator, CAS calculator or laptop with CAS software. We recommend the use of visualization to motivate completion of the square. Plotting several monic quadratic polynomials (that is, quadratics where the coefficient of x^2 is 1) and their translations can provide good intuition and motivation for completing the square and connect the algebra with the geometry of the vertex and axis of symmetry. Generalization to the general quadratic is then simple.

2.2.1 Norman Window problem

For our First Year undergraduate students, we use the Norman Window problem as an assignment, see [1]. For middle school students, say Year 10 (or Year 9 enrichment), this would be done as an example. We find that the objective function is quadratic that is not symmetric on the domain, so the max is at the vertex which is not at the midpoint of the domain. In a pre-calculus course, we can easily find where the max is by completing the square.

It is easy to show that the objective function (that is, the function to be maximized) is

$$A = r\,perim - 2\,r^2 - \frac{1}{2}\pi r^2 \ .$$

We define the perimeter to be the average student number, gpN, of the small number of students in each group in our collaborative teaching and learning class. We insist the exact rational number gpN be used in the problem (that is, no floating point numbers are allowed to be used in the algebra). For a diagram of the Norman Window, see Fig. 2.

Teaching Note. The Norman Window problem, in a pre-calculus course, requires completion of the square. However, since π appears as part of the coefficient of r^2, the algebra for the completion of the square requires manipulation involving the symbol π. For algebraic beginners, we recommend styarting with the modified problem where the semi-circular top is replaced by a triangular top, see below.

Of course CAS can be used as scaffolding [7] which allows students to make progress with higher level work even if all the lower level skills are not yet fully mastered. Maple has a command in the student package, which is loaded by using the with( ) command. Here,

```
[> with(Student[Precalculus]):
[> CompleteSquare(A,r);
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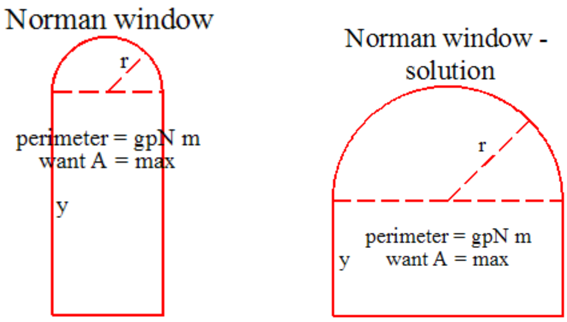

Figure 2: The Norman Window problem (with a given perimeter) where the maximum area, A, is wanted. The shape of the solution, where $y = r$, is also displayed.

The Maple answer is $(-2-\frac{\pi}{2})\left(r+\frac{1}{2}\frac{perim}{-2-\frac{\pi}{2}}\right)^2-\frac{1}{4}\frac{perim^2}{-2-\frac{\pi}{2}}$ which looks more frightening than it is: remember that each student group would have a given value for $perim$, and the coefficient of r^2 just appears in several places! Thus the axis of symmetry is at $r+\frac{1}{2}\frac{perim}{-2-\frac{\pi}{2}}=0$, that is, $r=\frac{perim}{4+\pi}$. Since the coefficient of r in A is negative, A has a maximum there.

Note. An alternative provided by CAS

An alternative (to the above "completing the square") for a quadratic objective function is to use the solve( ) command from the CAS to find the two zeros since we know that the maximum (or minimum) will be where the variable equals the average value of the zeros. Here:
[> solve(A=0,r): gives the two zeros: 0 and $r=\frac{2\,perim}{4+\pi}$. Although these solutions are not physically achievable, the average of these zeros still gives the radius for the maximum area.

Students can be asked whether they can suggest this alternative to the standard completing the square approach. There is a slight "cheat" here (since the completion of the square leads to the derivation of the formula to find the zeros of a quadratic). There is valuable reinforcement of the fundamental symmetry property of quadratics, as well as an opportunity for (another) discussion about the fundamental connection between the completion of the square and the "quadratic formula"! A better option here is to use factorization to obtain the zeros (see the discussion for the Window with Equilateral Triangular Top).

2.2.2 Norman Window problem with different transmission coefficients

A Norman Windows might use stained (or leadlight) glass for the top part of the window and clear glass in the bottom part (that is, the rectangular part). Now we could ask: what shape gives the maximum amount of light transmitted through the window?

The amount of light passing though the glass per unit area is given by the **transmission coefficient**. In this case, we only need the relative transmission coefficient as a ratio between the top glass and the bottom glass: we denote this by tc. The amount of light (relative to all glass being clear) is $2\,r\,y+\frac{1}{2}\,tc\,\pi\,r^2$, where $tc=\frac{1}{2}$, say. Now the objective function (depending

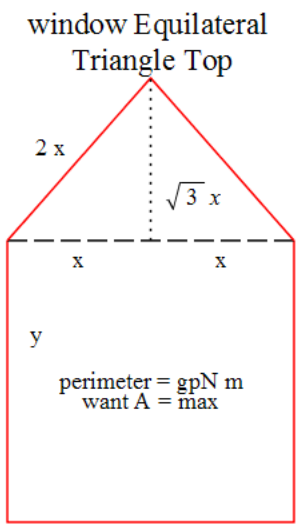

Figure 3: The Window with Equilateral Triangular Top problem (with a given perimeter) where the maximum area, A, is wanted.

on r only) to be maximized, is the modified area (from the original Norman Window):

$$A = r\,perim - 2\,r^2 - \pi r^2 + \frac{1}{2}\,tc\,\pi r^2\ .$$

2.2.3 Window with Triangular Top problem

As mentioned above, for middle school students the Norman Window problem could be done as an example, or follow the Triangular Top problem discussed here.

Equilateral Triangle Top

The Window with Triangular Top problem is the Norman Window problem but with the top semi-circle replaced by an equilateral triangle, see Fig. 3. The objective function, A, is:

$$A = 2\,x\,y + \frac{1}{2}\,2\,x\,\sqrt{3}\,x = 2\,x\,y + \sqrt{3}\,x^2\ .$$

Since we also know that the perimeter (which has a known value) is $perim = 2\,y + 6\,x$, we can write y in terms of x as $\frac{1}{2}\,perim - 3\,x$. Thus y can be eliminated from the formula for A to give

$$A = \sqrt{3}\,x^2 - 6\,x^2 + x\,perim\ .$$

This is a quadratic where the coefficient of x^2 is $\sqrt{3} - 6$ which is negative, so this quadratic has a maximum.

Before solving for A to be a maximum, we ask: what are the allowed values for the ONE independent variable, x? From the diagram, $x > 0$ and $y > 0$ (in theory, 0 values might be allowed but are clearly not practical!). Now the endpoint $y = 0$ corresponds to an x value of

$$x_{y0} = \frac{1}{6}\,perim$$

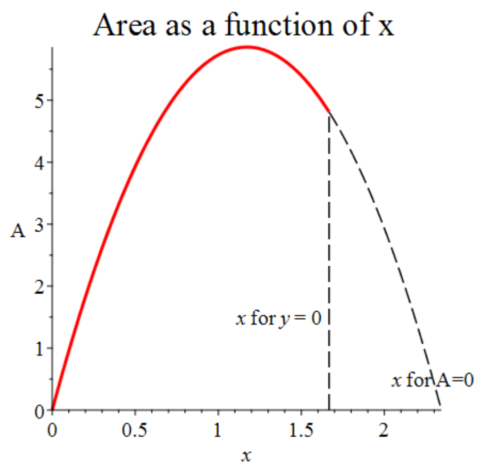

Figure 4: The Area, as a function of x, of the Window with Equilateral Triangular Top problem. The dashed part is not physically allowed.

Thus the physically allowable values are $0 < x < x_{y0}$ so now a plot can be made for $A(x)$ for the physically allowed values of x, see Fig. 4 for a graph when $perim = 10$.

Now we find the x for which A is a maximum, and we assume the given value of the perimeter is 10 (length is assummed to be in metres). To find the maximum of the A, WITHOUT CALCULUS, the "standard" approach is to complete the square and so find the location of the axis of symmetry (and hence the x value for the max). So we find, by competion of the square, that

$$A = \sqrt{3}\,x^2 - 6\,x^2 + 10\,x = (\sqrt{3} - 6)\left(x + \frac{5}{\sqrt{3} - 6}\right)^2 - \frac{25}{\sqrt{3} - 6}$$

and it is easy to see thst this is correct. Thus the axis of symmetry is at $x + \dfrac{5}{\sqrt{3} - 6} = 0$, that is, $x = \dfrac{5}{\sqrt{3} - 6}$. Since the coefficient of x in A is negative, A has a maximum there.

Note. An alternative with or without CAS

An alternative to "completinging the square" for a quadratic objective function is to find the location of the zeros of the quadratic. The average of these zeros is the x value that gives the maximum A. In the Norman window problem, above, we used the solve( ) command from the CAS to find the two zeros. However for this example (and the Norman window problem) the constant term in the quadratic is zero. This means that factorization is trivial and so the zeros are very easy to write down.

Here, A factorizes as $A = x\,((\sqrt{3} - 6)\,x + 10)$ giving the two zeros by setting each factor zero: $x = 0$ and $(\sqrt{3} - 6)\,x + 10 = 0$. Although these solutions are not physically achievable, the average of these zeros still gives the radius for the maximum area, see Fig. 4.

Teaching Note. In addition to the Teaching Note for the Norman Window problem, there are three further comments. Firstly, there is an opportunity to dicsuss physical and unphysical values of variables and the fact that unphysical values may be useful in the solution method.

Secondly, there are two pre-calculus solution methods discussed here. It is important for

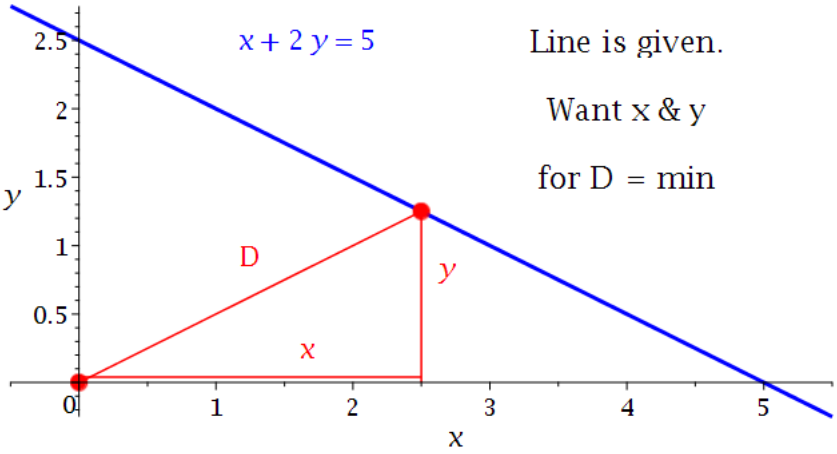

Figure 5: The diagram for the Minimum Distance between a Point and a Line problem.

students to realize that multiple solution methods are often available and to consider some of these. In practice, mathematicians try what appears to be the "best" approach, where "best" might mean the easiest, or the most elegant, or the easiest to use with a CAS.

Thirdly, completion of the square for the Norman Window problem, above, needed algebraic manipulation with the symbnol π. Similarly the Window with Triangular Top problem, here, needed (simple) algebraic manipulation with a surd. The Triangular Top problem can easily be modified to avoid irrational numbers by using a Triangular Top where half of the triangle has integer ratios (that is, Pythagorian triples) such as 3:4:5. 4:3:5, 12:5:13 etc.

2.2.4 Minimum Distance between Point and Line problem

The original problem is "Find the point on the straight line $x + 2\,y = 5$ which is closest to the origin." See Fig. 5 for a diagram. Note that parameterization for different student groups is simple here since this line is a particular case of $a\,x + b\,y = c$.

This is a very attractive problem which often appears as an optimization problem in a calculus course. However is can be solved by multiple methods, including

- Minimize Distance – using calculus
- Minimize Distance – algebraically, without calculus
- Minimize Distance – geometrically using intersecting lines
- Minimize Distance – geometrically using right angle triangles
- Minimize Distance – visually with multiple representations and animations

see [3] for a detailed discussion. Here, we only consider the second method: Minimize Distance – algebraically, without calculus. However we strongly recommend that other methods be introduced to students (preferably where other approachs are suggested by students).

Let D be the distance from the given point, the origin, to the line $x+2\,y = 5$. The problem is to minimise the distance, $D = \sqrt{x^2 + y^2}$. Since the distance must be positive it is easier if we avoid the square root by solving the equivalent problem: minimise $D^2 = x^2 + y^2$. It is easy to show that minimizing D is equivalent to minimizing D^2 by considering that if $D_M < D$ for $D_M, D > 0$ then multiply both sides of the inequality by D_M to show that $D_M^2 < D\,D_M$. Using the original inequality again in the right hand side gives $D_M^2 < D\,D$ and hence the result.)

In Maple, D is a 'reserved word' used to denote the differentiation operator. Also, with CAS, it is not permitted to assign to a variable squared. To avoid any difficulties, we denote D^2 by DSq. To write the expression for D^2 in terms of one variable only, we choose to eliminate y (since students are usually more comfortable with x as the independent variable). Thus

$$DSq = \frac{5}{4}\,x^2 - \frac{5}{2}\,x + \frac{25}{4} = \frac{5}{4}\,(x^2 - 2\,x + 5)\ .$$

In this case, the quadratic does not have any real zeros, but completion of the square is very simple and leads to the solution $DSq = 5$ as the minimum distance squared when $x = 1$. Hence the solution to the original problem $D = \sqrt{5}$ as the minimum distance when $x = 1$.

3 Problems with Cubic objective functions

Leigh-Lancaster, an experienced mathematics teacher and manager of school mathematics in Victoria (Australia) wrote "Why Cubic Polynomials?" see [8]. His first sentence reads

> The study of cubic polynomials of a single real variable is typically introduced in Year 11 as a generalization of work on linear functions and quadratic functions across Years 7 10, and then extended to include calculus.

To provide more detail, we extract a small part of the official Study Design for Mathematical Methods Unit 1, see [9], (normally studied in the first half of Year 11):

> Functions and graphs
> ...The behaviour of functions and their graphs is explored in a variety of modelling contexts and theoretical investigations. This area of study includes [some items omitted in the following]:
> • ...examples of relations that are not functions and their graphs such as $x = k$, $x = a\,y^2$ and circles in the form $(x-h)^2 + (y-k)^2 = r^2$
> • graphs of power functions $f(x) = x^n$ for $n \in \mathbb{N}$ and $n \in \{-2, -1, \frac{1}{3}, \frac{1}{2}\}$, and transformations of these graphs to the form $y = a\,(x+b)^n + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$
> • graphs of polynomial functions to degree 4 and other polynomials of higher degree such as $g(x) = (x+2)^2\,(x-1)^3 + 10$

Thus cubic polynomials and applications play an important role in the school mathenmatics curriculum by Year 11. We present a new algebraic approach (pre-calculus) to cubic polynomials in several applied optimization problems. As a preliminary we illustrate the key idea of locating the maximum and minimum of an example.

3.1 Finding max or min of a cubic: an example

One of the examples from [8] is the cubic polynomial

$$p_3 = 2\,x^3 + 4\,x^2 - 6\,x + 7$$

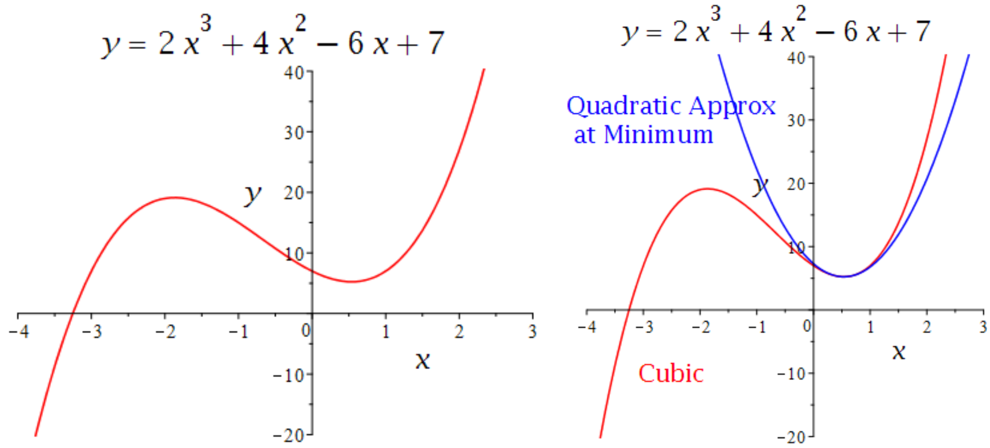

Figure 6: The graph of the cubic polynomial example on the left, with the cubic and its Quadratic Approximation at the Minimum on the right.

which is graphed in Fig. 6. Most of focus on cubics is on finding the zeros which was one of the great mathematical problems historically. A general formula to find the zeros of a cubic was found, then a general formula for the quartic (a polynomial with terms up to power 4) but eventually it was proven that no general formula to solve quintic polynomials exists.

For this case, there is only one real zero (at about -3.25). From the theory of complex variables and the Fundamental Theorem of Algebra, we know there is also a pair of complex conjugate zeros. Unfortunately, even for this "easy" looking problem, the solution is very messy with terms of cubic roots of surds: the real zero is

$$-\frac{1}{6}\,(658+6\sqrt{8121})^{1/3} - \frac{26}{3\,(658+6\sqrt{8121})^{1/3}} - 2/3 \approx -3.252983649\,.$$

Derivation and use of the general formula for the zeros of a cubic is not taught in school or undergraduate university mathematics. The messy formula can be evaluated using CAS to any required accuracy for applications (or we can use a numerical computation method). As noted in [8], it is easy to construct cubics which have "nice" behaviour such as an integer real solution: this can be found by using the remainder theorem, then long division of polynomials to find the quadratic factor of the cubic. Then use the quadratic formula to find any other real solutions (or indeed the complex conjugate solutions). Students practise these problems at school and first year university and can be excused for thinking that this is a useful strategy in general (it is not - consider the real zero above!).

However there are interesting applied optimization problems (for which the maximum or minimum is sought) for which the objective function is a cubic polynomial. The study of these problems is usually delayed to be part of the applications of calculus. However they could be introduced to pre-calculus students. **The key observation is that the cubic "looks like" a quadratic near the min or max!** Students notice this, but this observation is usually ignored. We formalize this (and use arguements of "in the small" that are so important in higher level mathematics, especially in applied mathematics and computational mathematics).

We assume that the min or max value xM of x that we want is NOT "known", but we will find the value(s). If x is nearly xM, then we consider values that only vary a small amount from xM. Define $x = xM + \delta$. As we vary δ by small values, the graph "looks like" a quadratic: we derive this quadratic (and plot it in Fig. 6). Now, substitute $x = xM + \delta$ in the formula for p_3 to give

$$\begin{aligned} p3_{delta} &= 2\,(xM+\delta)^3 + 4\,(xM+\delta)^2 - 6\,xM - 6\,\delta + 7 \\ &= 2\,\delta^3 + 6\,xM\,\delta^2 + 6\,xM^2\,\delta + 2\,xM^3 + 4\,\delta^2 + 8\;xM\delta + 4\,xM^2 - 6\,\delta - 6\,xM + 7\;. \end{aligned}$$

Since we are only interested in small values of δ, then terms in δ^2 are much smaller than terms in δ. Similarly, terms in δ^3 are much smaller than terms in δ^2, so we ignore any terms with a power higher than 2. The function is renamed (to indicate that this is now a quadratic approximation) and the terms are collected (in the powers of δ):

$$p3_{delta}2 = (6\,xM + 4)\,\delta^2 + (6\,xM^2 + 8\,xM - 6)\,\delta + 2\,xM^3 + 4\,xM^2 - 6\,xM + 7\;.$$

This is the local (near xM) quadratic approximation. (We could provide an animation of this as hM varies.) For xM to give a max or min, this must have the δ term zero,.Thus we solve

$$solve(6\,xM^2 + 8\,xM - 6 = 0,\; xM)$$

using the CAS solve( ) command, or just use the quadratic formula, to give

$$xM = -\frac{2}{3} + \frac{1}{3}\sqrt{13}\,,\; -\frac{2}{3} - \frac{1}{3}\sqrt{13}\; \approx 0.5351837583\,, -1.868517092\;.$$

There are two solutions, as we expected from the plot. We can determine which is a max or min by looking at the plot, or by whether the coefficient of the power two term is negative or positive. Thus we evaluate $6\,xM + 4$ at each of our solutions for xM to obtain $2\sqrt{13}\,, -2\sqrt{13}$ (respectively). Thus the first solution (the positive one) is a min and the second solution (the negative one) is a max, as expected from the plot.

The quadratic approximation at the minimum, $p3_{delta}2Min$, is given by substitution of the second solution into the general local quadratic aproximation. This looks messy but can be simplified (using CAS makes this easy!) to give

$$p3_{delta}2Min = 2\sqrt{13}\,\delta^2 + 329/27 - \frac{52}{27}\sqrt{13}\;.$$

Finally we plot the quadratic approximation at the minimum on the original plot of the cubic: to do this we first have to rewrite the quadratic in terms of the original variable $x = xM + \delta$. Thus the substitution $\delta = x - xM$ is used so that the cubic and its quadratic approximation can be plotted together, see the Fig. 6.

3.2 Maximum Volume of Open Box

A well known calculus problem, from senior school or first year university is the Open Box problem, see [2]. A rectangular piece of cardboard (or sheet metal, say) where each corner has a square cut away and the resulting object has the sides folded up to form the Open Box, see Fig. 7. Find x such that the box has the maximum volume.

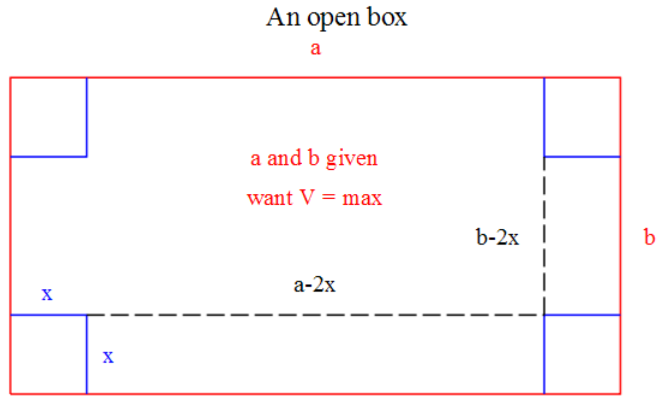

Figure 7: The diagram of the rectangular piece of cardboard, with sides of lengths a and b, from which the Open Box is constructed.

The original rectangle has sides of lengths a and b with the side of each corner square that is cut out, has side of length x. Many texts choose $a = 8$ and $b = 5$ since the solution gives $x = 1$. The objective function here is the Volume of the box

$$V = (a - 2\,x)\,(b - 2\,x)\,x$$

which is a cubic. For a detailed discussion of the pre-calculus solution and the teaching and learning of of this problem, see [2].

The original problem has its maximum for $x = 1$, which is simplisitic. We favour providing the solution of problems for the small student group to follow (and learn). This is followed by a closely related problem (perhaps the same problem but with different parameters) to be solved and submitted for assessment. Extensive experience shows that students regard problems with different parameters as essentially different problems, The small group collaborative learning we favour is best conducted with "different" problems for each group to discourage the laziness of just copying the work of others! In [2] we discuss this issue and the list of 24 different rectangular shapes that can be used (and scaled as well) to provide "different" problems which all have rational lengths for the original sides and for the cut out length x.

Teaching Note.
Many different assignments to mark unfortunately greatly increases the teacher's work load. Some form of Computer Aided Assessment, CAA, is highly recommended. In our case we require students to use Maple and to submit their Maple file (via the internet), so we write our own CAA procedures. Otherwise a good CAA can be used: the best commercial system is MapleTA (where Maple is used for the mathematics, but the student does not need to know anything about Maple). A brief discussion of CAA for this problem can be found in [2].

With the use of CAA, careful design of the question and the parameters choosen is required to avoid pitfalls such as no solution or singularities for some parameters. However it may be important to ask for intermediate results. For example, if a particular solution method is wanted, then just asking for the solution is not sufficient: some intermediiate results, particularly if they are specific to the required method, should be asked for. For this problem, if the pre-calculus method (introduced here) is required, then asking for the quadratic approximation at the maximum would diasallow some student use of calculus and just writing down the solution.

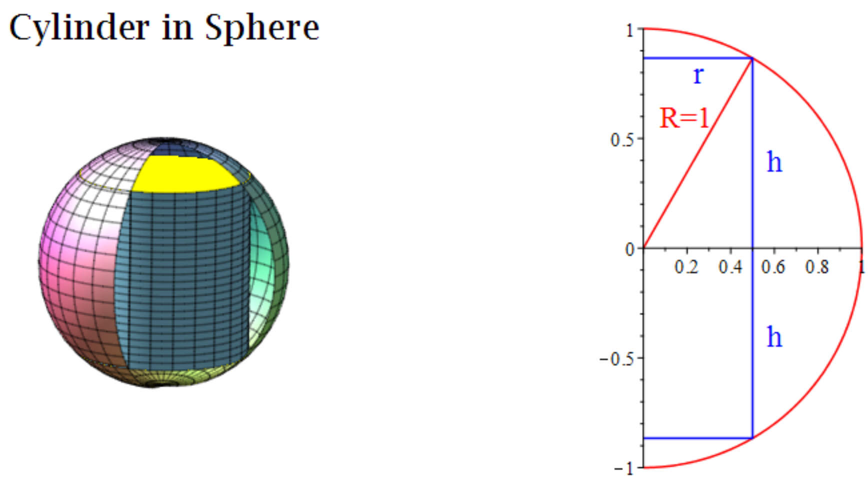

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Figure 8: The plot of a cylinder inscribed in a sphere, radius $R = 1$, with a cut out of the sphere to enable visualization of the cylinder; and diagram of a cross section.

Further, even students who meet this problem in a calculus course do so in the senior school or the first semester of first year university: since they study Taylor series later (usually in the second semester of first year university), this would still be an effective strategy.

3.3 Maximum volume of a Cylinder inscribed in a Sphere

The Problem: For a cylinder inscribed in a sphere of radius 1, find the *exact* radius of the cylinder which has the maximum possible volume.

Fig. 8 provides a plot of the cylinder in a unit sphere, radius $R = 1$, and a diagram of a cross section where the radius of the cylinder is r and the height is $2\,h$. Clearly, r and h are related: from the right angled triangle, $r^2 + h^2 = R^2$ so $h^2 = R^2 - r^2$ and h can be eliminated. The objective function is the Volume of the cylinder:

$$V = \pi\, r^2\, 2\, h = 2\,\pi\, r^2\, \sqrt{R^2 - r^2} = 2\,\pi\, r^2\, \sqrt{1 - r^2}$$

where the unit sphere has $R = 1$. This is a classic problem that appears in many textbooks. For R arbitary, the problem is simply scaled, so it is common to set $R = 1$.

For this problem, visualization of the unit sphere with an inscribed cylinder is wanted, but students in pre-calculus and early calculus courses have little experience with plots in 3D. We recommend that the 3D plot is provided in the worksheet given to students.

For detailed discussion about various solution methods for this problem, see [4], where the focus is on an experimental mathematics methodology. In particular, there is a brief introduction to the experimental mathematics methodology and visualization with zooming-in to get a sufficiently accurate solution to identify a (possible) exact solution which is then proven correct by the type of algebraic method used here.

However the algebraic method discussed here can be used to find the localation of the maximum or minimum as well as prove its correctness. The way the problem is asked, most students immediately start to solve for r directly, as above. The square root causes some difficulty, but it can be handled (as mentioned in [4]) by using the Binomial Theorem for power $\frac{1}{2}$, which is easy to prove by some simple algebra.

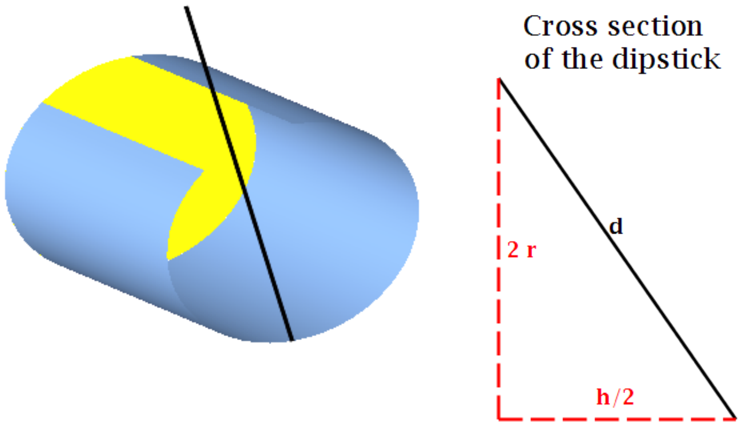

Figure 9: The Wine Barrel problem. A plot of a horizontal cylinder, radius r and length h, with a cut out to enable visualization of the dipstick; and diagram of a length-wise "section".

There is an easier way to simplify this problem by squaring both sides to eliminate the square root (similar to the Minimum Distance between a Point and a Line problem, above). Volume Squared of the cylinder, VSq is:

$$VSq = 4\,\pi^2\,r^4\,(1 - r^2)\ .$$

This problem's objective function is a degree 6 polynomial as a function of r, but can be solved directly by our algebraic method. This is easier because the square root has been eliminated.

We notice that VSq depends only on r^2, so we can rewrite in terms of $rSq = r^2$:

$$VSq = 4\,\pi^2\,rSq^2\,(1 - rSq)\ .$$

Thus we have a cubic objective function which is easy to solve.

However a moments reflection (before starting any calculations to find the maximum V) indicates an easier approach. At the initial modelling stage we wrote $V = \pi\,r^2\,2\,h$ and chose to eliminate h since we were asked for the r which maximizes V. A better choice would be to eliminate r to obtain

$$V = 2\,\pi\,(1 - h^2)\,h$$

which is a cubic objective function that is easy to solve for h, and then calculate the corresponding value of r.

3.4 Kepler's Wine Barrel problem

The Wine Barrel problem is famous; for historical and mathematical details, see [5]. Before calculus, the volume of wine filling a horizontal cylindrical barrel was measured by using a dipstick. The barrel had a plug hole at the midpoint of the top of the barrel: the dipstick was inserted into the plug hole down to the bottom and an end of the barrel, see Fig. 9.

A mathematician, Kepler (in the time before calculus), realized that the dipstick measurement did not properly measure the volume and so solved the problem: What is the shape of the cylinder that maximizes the volume for a given dipstick length measurement?

From Fig. 9, it is esay to see that $r^2 = (d^2 - h^2/4)/4$, so the volume of the (horizontal) cylinder is

$$V = \pi r^2 \, h = \pi \frac{(d^2 - h^2/4)}{4} \, h = \frac{\pi}{4} \, (d^2 \, h - \frac{1}{4} \, h^3) \, .$$

The objective function, V is a cubic in h. It is simple to leave d as a symbolic constant and solve algebraically (as above) for h by hand or by CAS. The quadratic approximation about the maximum is

$$V_{delta}2 := \frac{\pi}{3\sqrt{3}} \, d^3 - \frac{\sqrt{3}\,\pi}{8} \, d \, \delta^2$$

Teaching Note.
To plot the quadratic approximation with the Volume, d needs to assigned a value, say $d = 1$. A fun activity for middle school students is for student groups to solve the problem for different given values of d. They can be asked to report their result to the class and to notice that the ratio of h/r is aprroximately $2\sqrt{2}$. They can then be asked to solve the problem again with the value of d symbolic and hence prove that the shape (for maximum volume) is $h/r = 2\sqrt{2}$.

4 Conclusion

CAS such as Maple enable innovative approaches to curriculum, pedagogy and assessment at school and university. Standard problems solved in the standard way using CAS risk being another activity (such as lectures and computer sessions) that students find boring. Well designed teaching and assessment is effective, efficient and even fun. The development of these CAS based materials requires a lot of staff time, but they give high returns with staff satisfaction, student performance and attitude. The reduced marking load using automatic marking more than repays the development cost. We recommend more collaborative learning by students and more collaborative development of CAS materials by academics and teachers.

This paper shows that elementary optimization problems (such as "find the maximum volume of an open box") can introduce applications (of quadratic and cubic polynomials) and be a rich source of mathematics if the usual (efficient) calculus method is not used. We show visualization to build intuition; and several different methods using algebra to obtain the quadratic approximation of the objective function about its maximum: hence finding the location of the maximum and proving its correctness. This approach introduces an important topic in mathematics: approximation in the small. It also provides strong support for (and visualization of) the intuitive notion that the function "looks like a parabola" near its maximum.

This approach can also be used for objective functions with terms of power of $\pm\frac{1}{2}, -1$, which often occur in interesting applications, as well as appearing in the list of functions to be studied in early Year 11, see the preamble in Section 3, or the reference [9]. For an indication of the treatment of a problem with a sqrt root term, see [4]. We will provide an overview, and some details, of our algebraic approach of several of these applied problems in a future paper.

The Maple activities described here are all web mediated, so they could be part of an online or blended learning) course: students have different learning styles and some take advantage of the flexible learning that the approach here supports (since attendance in the computer lab is recommended but not required). Students enjoy the collaborative learning using Maple and learn (surprisingly quickly) to do the calculations; use visualization; and use automatic marking. Students are engaged, active and collaborative learners with these Maple sessions.

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