From One to Infinity: What DGS Has or Could Have Changed in our Teaching and Learning

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Abstract: *The collaborative work enhanced by the communication tools developped in the last decade is perceived by everybody as a progress for teaching and learning. We will see in this paper that this way of working avoids to use all the knowledge developped by experts especially in the use of the technological tools. We will focus our analysis on « dynamic geometry software » and especially on the Cabri environments. We will remind the special role of DGS in a more experimental way of practising maths (researching, teaching and learning) sustained by the theoretical framework I have developped in my PhD thesis (different stages of an experimental process of discovery, different levels of techniques of investigations known as praxeologies G1, G1 informatique, G2 and G2 Informatique, the heuristic power of dynamic approach of figures…). We will give a lot of examples showing the new tools of exploration or investigation provided by DGS (such as traces, loci, animation, redefinition, macro, sliders…) to understand the difference between a paper and pencil approach or a DGS approach of a problem (that can be not necessarily a geometric problem). In these examples, we will present different techniques that must be tought in order to be used appropriately by the users of DGS. At last we will present my Youtube channel where lots of situations are provided to help teachers and students to use DGS without ignoring all the work of the experts during the last 30 years.*

1. New tools for an experimental approach of mathematics

1.1. From one to infinity already at ATCM Melbourne 2001 (Figures 1, 2 and 3)

As I was attending ATCM for the first time in 2001, I presented the representation of a sphere in parallel perspective with the help of *Cabri 2 plus* (see videos about parallel and military perspectives: [21] and [22]) and its shadow when lit by the sun (and where the direction $\overrightarrow{dd'}$ of the rays is changeable). In this example the equatorial circle and a variable meridian passing through a variable point *m* of the equatorial circle are constructed as ellipses defined with a macro (allowing us to define an ellipse with three points knowing that this ellipse is inscribed in a parallelogram). The locus of this meridian gives a first representation of the sphere (Figure 1 in the middle).

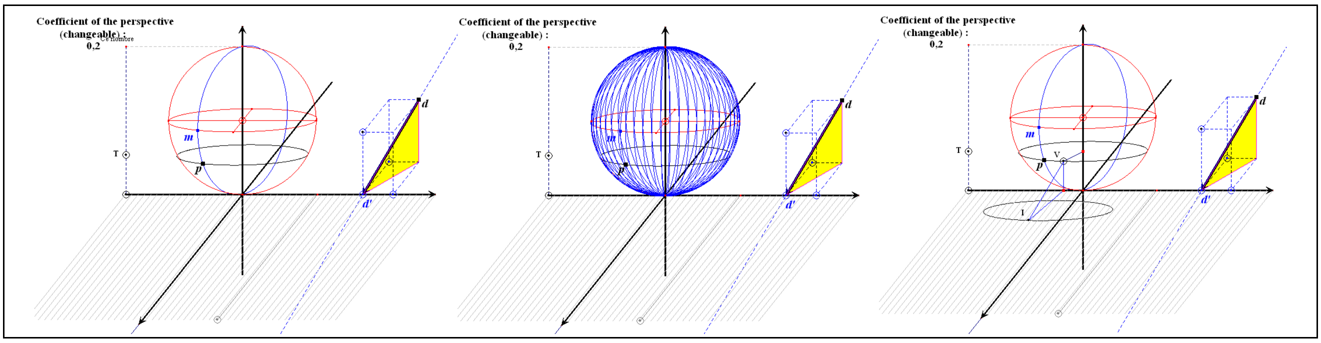

Figure 1: Representations of a sphere in parallel perspective

Another one is obtained with the locus of a variable parallel circle of the sphere (Figure 2 on the right). The shadow of a parallel or the shadow of a meridian are obtained as the locus of the shadow of one point of each (Figure 3 on the left and Figure 1 on the right) . The shadow of the sphere is obtained either as the locus of the first previous locus or the locus of the second previous locus (remark : the tool « locus of loci » was not avalable in 2001 but I managed to perform practically the same result as the one presented here) : see Figure 3 in the middle and Figure 2 on the left.

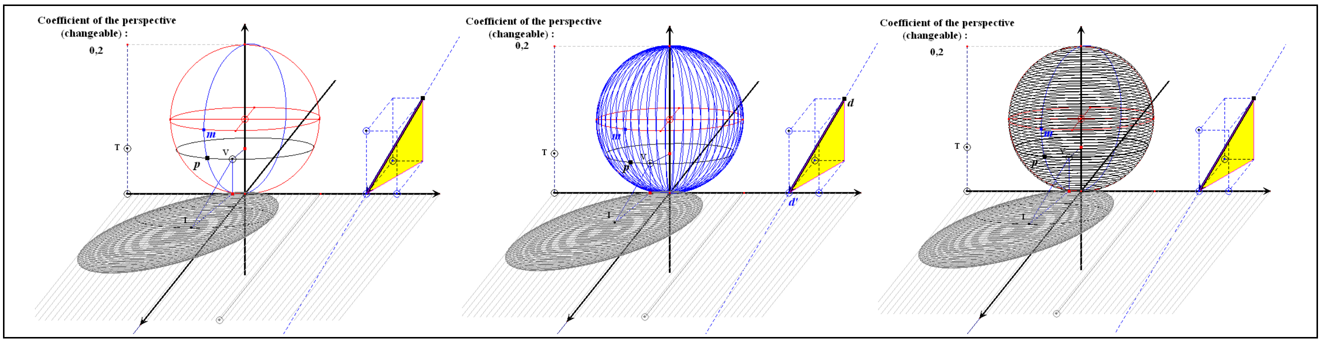

Figure 2: Shadow of a sphere in parallel perspective (first technique)

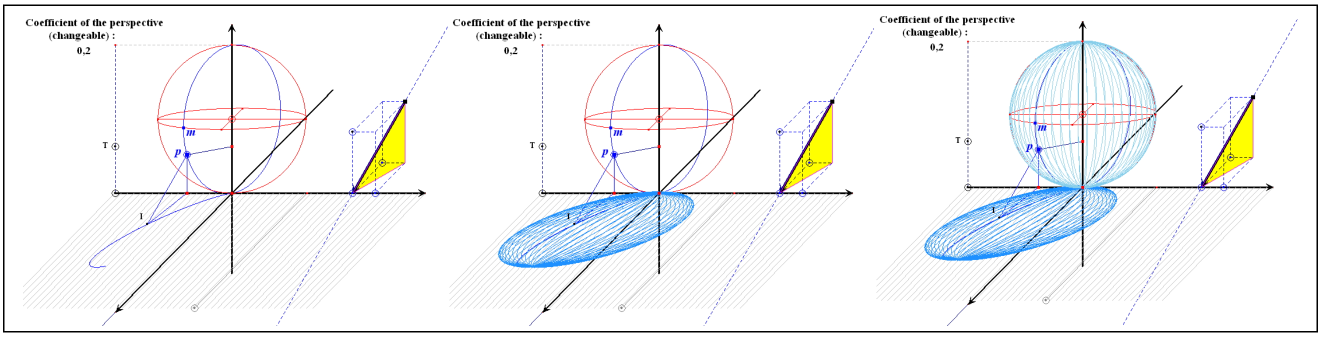

Figure 3: Shadow of a sphere in parallel perspective (second technique)

This introductory example shows how, from one point constructed under constraints, the locus tool can provide a lot of points satisfying the same property to give us a good modelling of the infinite set of points verifying the same constraints. Same remark for the tool « Locus of loci » that allows, from the starting point of an object constructed under constraints, to get instanteously an accurate idea of the set of objects verifying these constraints. Dragging or animating the points commanding the constructed loci allows the experimenter to check that these loci are really the good ones.
Why the choice of this special introductory example ? There are two reasons :
The first one is because when I taught the representation in parallel perspective of a sphere (before we knew Dynamic Geometry Systems DGS) some of my preservice teachers did not believe that the representation of a sphere is an ellipse (really because they did not know the definition of the parallel perspective even though this was used in their textbooks).
The second reason is to justify the title of this paper from one to infinity. A part of the justification of the use of DGS is that it enhances the successful use of inductive reasoning.

1.2. Experimenting with DGS (the theoretical background)
1.2.1. The experimental process of discovery (Experiment, exploration, investigation)
Here are some of the concepts studied in my PhD. study ([6]) in order to get the same understanding of the concepts used in experimental mathematics ([1]).

An experiment : every process generating data the interpretation of which can lead to the discovery of a conjecture or the refutation of a conjecture.
An exploration : every process leading to a better understanding of a phenomenon.
An investigation : an exploration conducted by using specific techniques (which means existing techniques known or that must be known by the user : technological or mathematical ones)

1.2.2. The crucial role of techniques of investigation (G1 or G2 and G1 or G2 informatique)
I noticed in my PhD study that different sorts of techniques of validation are used during the different stages of an experimental process of discovery. These techniques characterize the praxeologies (recently called paradigms) under which the research work is performed ([6] and [7]):
Praxeology G1, characterized by perceptive techniques of validation in a paper an pencil environment.
Praxeology G1 informatique, characterized also by perceptive techniques of validation in technological environments which is very different from the previous one : the example of the superimposition in these two environments shows that a superimposition noted on the screen of a DGS is more reliable that the superimposition in a figure constructed by hand on a piece of paper.
Praxeology G2, characterized by deductive techniques of validation in a paper an pencil environment : most of the time, this comprises a deductive proof leading to the conclusion which is considered as true (called demonstration).
Praxeology G2 informatique, characterized by deductive techniques of validation assisted by the software : this is a verification of a conjectured result in several different cases. These verifications are based on special tools of the software such as the oracle of Cabri : this tool allows for example to check whether or not a point belongs to an object. Finally, working under G2 informatique leads to a conclusion which is considered as plausible or very plausible (in the meaning of Polya).
Working at this level can be very successful and rewarding especially when the deductive proof is out of the field of the researcher ([9], [14] and [17]).

1.2.3. The heuristic role of dynamicity (Figure 4)
We know that a dynamic representation of a figure in a paper and pencil environments (Duval : [2]) or a technological environments (Dahan : [6]), is more heuristic than the usual deductive approach. The discovery is not inevitable but the dynamic approach enriches and improves our ability to observe relationships that cannot be seen classically. Here is an example to illustrate this point :
Example to discover that two symetries give a translation ([10]**)**
Let us show first what could be a discursive approach (approach under G2) of the figure of this problem: it means that we recognize a known situation (one of the midpoint theorems). So we can deduce a result in reasoning for example like that:
As the conditions of the midpoints theorem are verified, I can use it to reach the conclusion :
(MM’’) is parallel to (C1C2) and MM’’ = 2.C1C2. Which can be completed by: the transformation mapping M onto M’’ is the translation having as a vector $2\overrightarrow{\boldsymbol{C1C2}}$.
We can imagine a more complex problem where it would be less evident to guess the appropriate theorem that can help with the appropriate sorted data. That justifies the weak heuristicity of the discursive approach compare to the next one.
Let us show now what could be an operative approach (approach under G2 informatique) of the figure of this problem: here the invariant discovered previously thanks to a deductive reasoning will appear now thanks to the dynamicity of the figure which is the core of the operative approach (so-called by Duval).

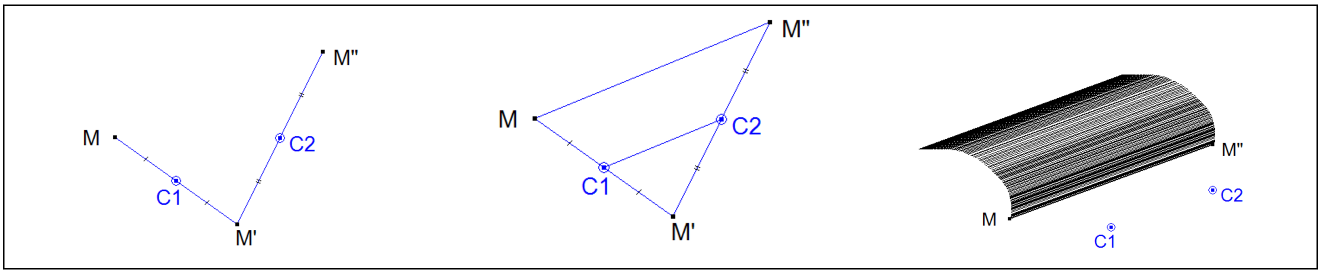

Figure 4 : Composition of two central symmetries

We can state when dragging randomly point M that the corresponding position of point M''leads the experimenter naturally to the expected conclusion « the transformation is a translation » (letting the traces of segment MM'' on during the motion helps or reinforces the conjecture). See Figure 4 on the right.

Remark : when a figure is constructed under a given DGS, dragging some points changes the aspect of the figure but keeps the properties characterizing the figure. What is very interesting here is that conversely : when points are dragged in a figure constructed under a DGS, as the original properties of the figure are kept, the way the aspect of the figure changes helps the experimenter to guess what these properties are, even if he or she is not able to discover them by using the technique of demonstration.

1.3. Very special tools for a very special use

1.3.1. Representing surfaces with loci and loci of loci ([4] and [5])

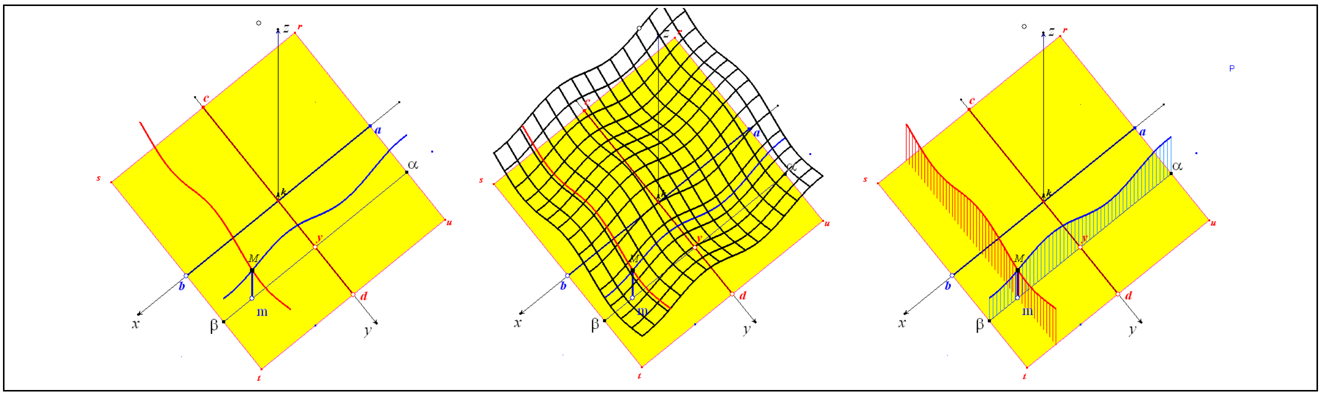

Figure 5 : Representation of a surface in military perspective

In the left image in Figure 5: we have represented in military perspective only one point of the surface of equation $z = sin(x)+cos(y)+5$, the point M above the yellow square. All the points M which vertical projection belong to the yellow square are controlled by point *y* and point *m* moving along two segments.. The blue and the red curves, *x* and *y*-sections of this surface are obtained as loci of M when *y* and *m* moves respectively on their segments. The middle of Figure 5 shows a wire representation of this surface where 15 *x*-sections are obtained as the locus of the first *x*-section and similarly for the *y*-section. The vertical sections shown in Figure 5 on the right are obtained as appropriate loci of segment [mM].

This example shows how the locus tool can help to move from one to infinity which means to get a good representation of an object defined by some specific constraints where we are able to construct only one point verifying these constraints.

1.3.2. Examples of soft loci in 2D and 3D (an appropriate and tricky use of the traces)

Example in 2D : Visualizing and conjecturing the solutions of *sin(x)=sin(y)* (([13])

This example uses a locked variable on *TI-NSpire* and a conditionnal construction in *Cabri 2 Plus*.

With TI N'Spire

PART 1: We chose the page layout with the "Graphs and Geometry" application on the leftside and the "Calculator" application on the right side (Figure 6).

On the leftside: We create a point *P* and display its coordinates. We store these coordinates in variables called vxP and vyP. We insert two sliders linked to these variables. We change the settings of these sliders in order to get -10≤ vxP ≤10 and -10 ≤vyP ≤10. Now point *P* can be dragged directly or moved with the sliders horizontally or vertically).

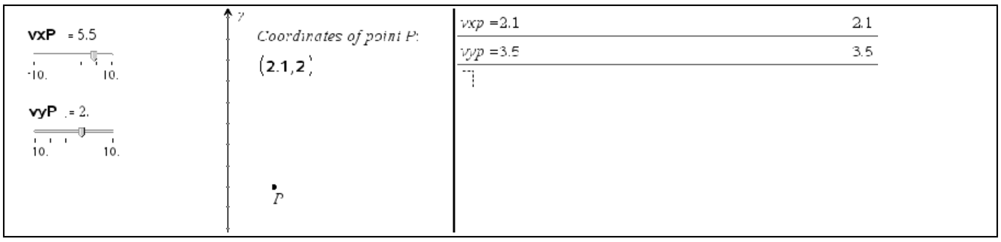

Figure 6: Graphs and a Note page of *TI-NSpire*

On the right side: We choose to give particular values to **vxP** and **vyP**. The position of point *P* changes on the left side.

PART 2: Solving with geometrical experiments the equation *sin(x)* - *sin(y)*=0 (Figure 7)

Preparation of the figure: Check that the "zoom square" option is chosen. Edit the text *sin(x)-sin(y)*; calculate this expression for *x* = **vxP** and *y* = **vyP**. Change the position of *P* until the last result *r* is very close to 0. Lock this result (with the tool "attributes"); now *P* can only reach positions where *sin*(**vxP**) - *sin*(**vyP**) = ***r*.** Activate the "geometric trace" of *P*.

Experiment 1: Let us drag point *P* everywhere we can. The trace which appears represents the locus of point *P* verifying the equation *sin*(*x*) - *sin*(*y*) = *r*. This trace is the set of data generated with this experiment which will helps us to conjecture a response to our problem. The trace we got is displayed in Figure 7 on the left. This trace is called "soft locus".

Interpretation 1: **These points seem to be located on a double set of perpendicular lines regularly organised.**

Let us create another experiment to obtain more precise interpretations.

Experiment 2 (Figure 7 in the middle): We create two lines superimposed on some points of our data. We display their equations and we obtain $y = x$ and $y = -0.99x+3.17$.

Interpretation 2: As *r* is not exactly equal to 0 we can imagine that lines that could be solutions of our equations are lines with equations: $y = x$ and $y = -x+\pi$.

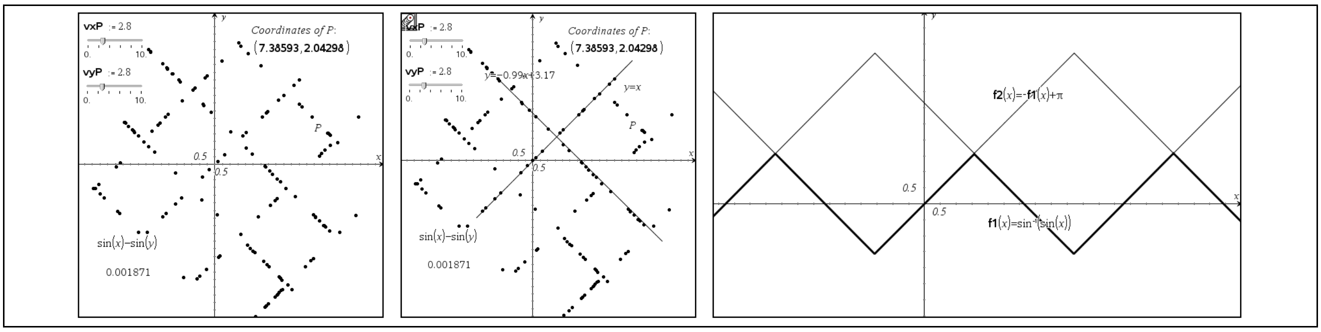

Figure 7: Solving *sin(x)=sin(y)* with locked variables and traces (*TI-NSpire*)

Final conjecture: If we group interpretation 1 and 2, we can conjecture that the solutions of the equation *sin(x)-sin(y)*= 0 are ordered pairs (*x*,*y*) so that $y = x+k.2\pi$ or $y =-x + (2k+1)\ \pi$

Verification with the software: With the "Calculator" software we get (Figure 8):

$$\text{solve}(\sin(x)-\sin(y)=0,y)$$
$$y=-\sin^{-1}(\sin(x))+2\cdot \boldsymbol{n1}\cdot\pi+\pi \text{ or } y=\sin^{-1}(\sin(x))+2\cdot \boldsymbol{n1}\cdot\pi$$

Figure 8: Solving *sin(x)=sin(y)* with the CAS of *TI-NSpire*

This result confirms our conjecture where $sin^{-1}sin(x)=x$ and ***n1***=*k*.

Geometrical verification: In Figure 7 on the right, we have displayed two particular solutions given by the software in another page of the "Graphs and geometry" application. This experiment validates our conjecture as well as the result given by the CAS.

With Cabri 2 Plus

Figure 9 shows the use of *Cabri 2 Plus*; instead of locking a variable, we use conditionnal constructions and traces of a blue and a read point superimposed with *M*(*x*,*y*) (a movable point) respectively when *sin(x)-sin(y)>0* and *sin(x)-sin(y)<0*. This technique was demonstrated for the Pythagorean theorem at ATCM Melacca in 2002 (([3]).

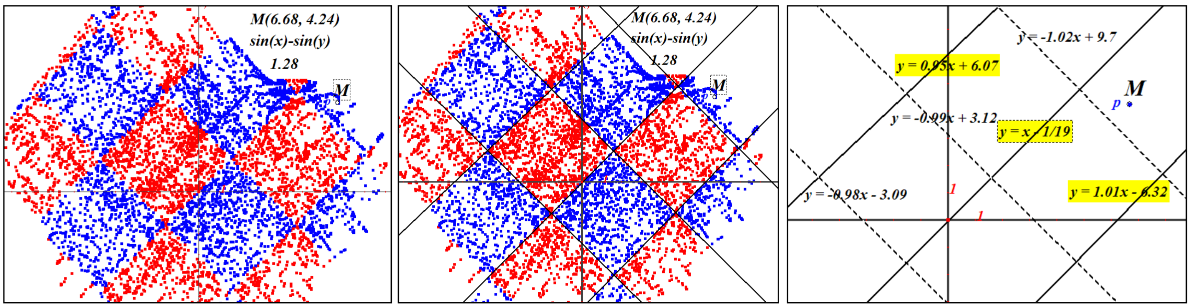

Figure 9 : Solving *sin(x)=sin(y)* with conditional constructions and soft loci (*Cabri 2 Plus*)

Example in 3D : Conjecturing that *ax+by+cz+d=0* is the equation of a plane

The following file was created for a French teacher teaching at the French scientific baccalaureat level (French highschool Lyautey of Casablanca Morocco). The teacher proposed to the students an experiment to visualize on a *Cabri 3D* page a maximum of points *M(x,y,z)* such that *x-y-z+2=0* (Figure 10). She constructs a moving point m(*x,y,0*) on the horizontal plane (*z=0*). She uses the calculator to evaluate *z=x-y+2* which is the third coordinate of a point in space lying in the same vertical line as *m* and verifying the previous equation. She activates the trace of point *m*. Then she randomly drags point *m* and obtains the traces of points *M* verifying the equation. In the center of Figure 10 we can see these traces, but a static visualization does not generate any conjecture. Then, the teacher changes slowly the point of view and everybody can see that all these points seem to belong to the same plane ([23]).

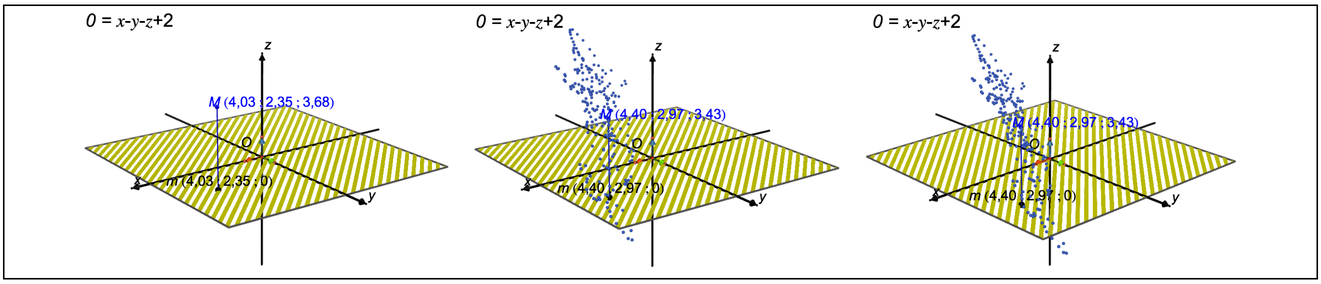

Figure 10 : Conjecturing that *ax+by+cz+d =0* is the equation of a plane

In Figure 11 on the left we have represented the plane defined by this equation and in Figure 11 on the right we have extanded this plane to suggest that all the points obtained with these traces lie on this plane.

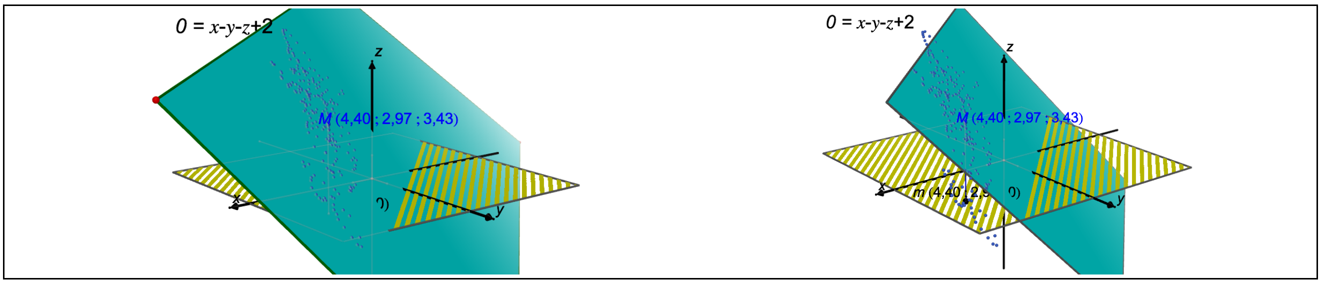

Figure 11 : Checking the previous conjecture

Hidden behind this example is a technique of investigation allowing an experimenter to visualize the points of a surface *f(x,y,z)=0* when it is possible to write it as *z=g(x,y)*. We can imagine the richness of such a technique especially to investigate surfaces depending on several parameters.

1.3.3. Transformations and animations (([16] and [19])
The floating flag presented in Figure 12 on the left was originally one of the files created to teach techniques of animation to middle school students during annual *Cabri 3D* workshops in a French highschool in Casablanca Morocco. On Figure 12 on the right, we can see the constructions using transformations (translations, symmetries and reflections). The trick is the animation of point *V* of arc *AVB* along segment [*MN*]. Figure 12 in the middle shows such a modelling performed with *TI-NSpire* (which is more difficult and less realistic than using *Cabri 3D* because the floating impression is obtained here by the morphing of only one arc instead of three with *Cabri 3D*).
After students had performed the *Cabri 3D* modelling, one of them came and told me that my modelling was false because the length of the flag was not constant : this student was absolutely right. So his remark was the starting point of some research in which I modelled the flag in respecting this constraint with *TI-NSpire*. To do this, I used the CAS and the connection between the CAS and the geometry application of the sofware. At the end of the research, the result I obtained was correct but perceptively not very realistic… because the result was displayed in a 2D

DGS in military perspective which is a parallel perspective as the result performed with *Cabri 3D* (which is a three-dimensional DGS) was modelled in central perspective ([15] and [16]).

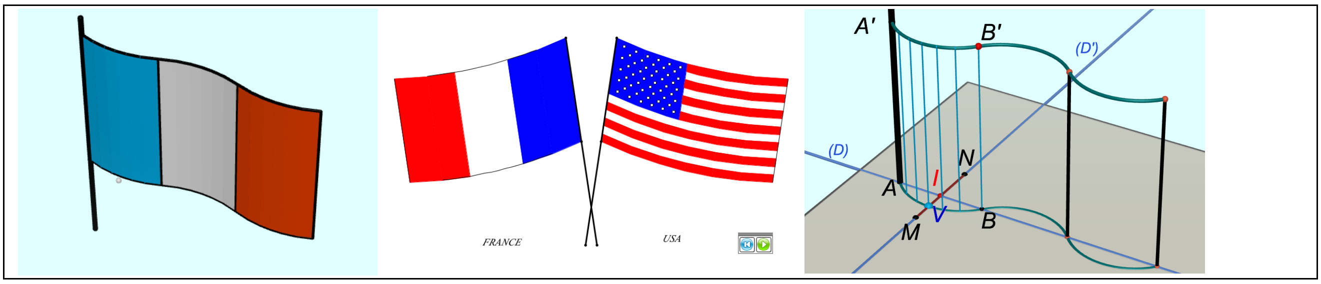

Figure 12 : Modelling a floating flag

The workshops we conducted with a French teacher during three years to train middle school students in the creation of animations in 3D with *Cabri 3D* let us conclude that modelling objects of the real world with an appropriate use of techniques based on transformations, animating them with other appropriate techniques enhances a better entry into the mathematical world of 3D. The previous example is only one of those we have used to train these students. Lots of these training examples and lots of creations of the students can be found in my YouTube channel « jjdahan »

1.3.4. Introduction to Euler's method with the help of a macro (([8]))

Construction of a circle with a macro point after point

Figure 13 on the left shows how to use the geometric definition of a circle to get its representation with a pen on a paper by dragging a compass in a rotating motion.
Figure 13 in the middle shows the construction of a point *M'* on the tangent line to a given circle at point *M*, when distance *MM'* is a given number that can be changed.
Figure 13 on the right shows that *M'* can be considered as a good approximation of a point of this circle close to *M* when the given distance is a number close to 0, constructed clockwise.
This suggests the idea of creating a macro with initial objects, the center of the circle, one point of this circle (*M*), number 1 (distance *MM'*) and number 2 (90°) and with final object point *M'* constructed with the tools of *Cabri* knowing the initial objects.

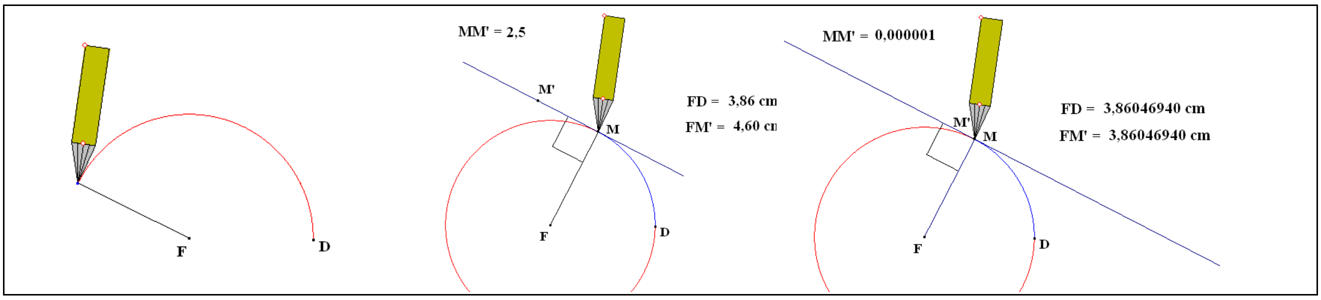

Figure 13 : A circle and a construction on one of its tangent lines

Figure 14 on the left shows all the points obtained by applying the previous macro an important number of times. As distance *MM'* is controlled with a slider, we can see that we do not obtain points of the expected circle (in reality we have used an intermediate macro constructing ten points after M in iterating the previous one).

But in Figure 14 in the middle, if *MM'=p* is close to 0, the points obtained with these constructions are almost superimposed on the circle.
In Figure 14 on the right, when p = 0.02, we can see that the construction of the circle point by point with these macros is very accurate for half a circle

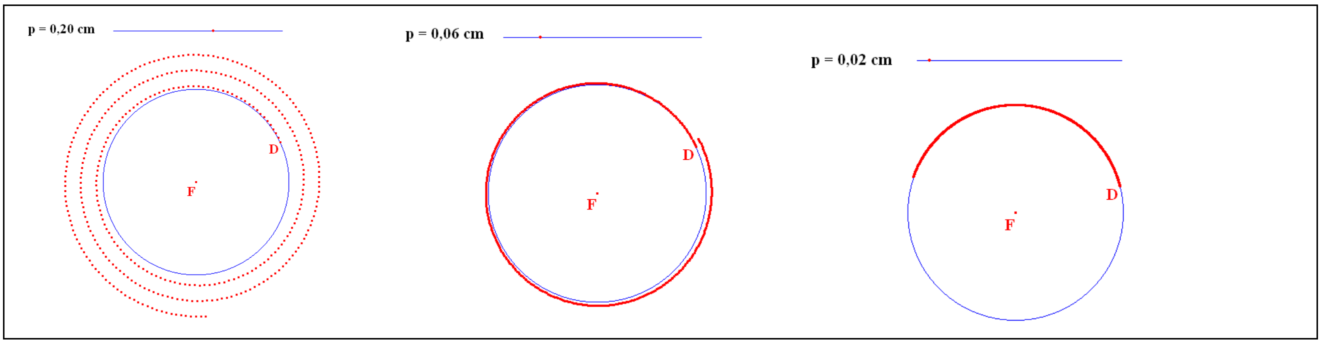

Figure 14 : Construction of a circle with the Euler's method performed with macros of *Cabri 2 Plus*

This example shows the power of macros to represent any curve knowing one of its points and a property of the tangent line at each point of the curve (for example the slope in a system of axis). This work has been published several times ([8]). The technique was used to solve graphically ordinary differential equations of the first and second degree. The particular case of the graphic construction of the curves of an antiderivative function was presented in different congresses (T3 Calgary 2002, TIME Montreal 2004 and TIME Dresden 2006).

2. The process of discovery when mediated with DGS

2.1. The different stages of an experimental process of discovery

2.1.1. The micro-stages : Thanks to the research work conducted with black boxes, this process is modelled by a sequence of experiments. The word « experiment » is used to mean including preparation, protocol (the way the experiment must be conducted), exploration and interpretation (P.P.E.I.). Finally this process is modeled with a sequence of such P.P.E.I. called links of a chain. These quaternary links of a chain are the micro-stages of this process.

2.1.2. The macro-stages: We know that the sequence of micro-stages can be cut out in blocks called macro-stages, the ones before conjecture and the other ones after conjecture. This result has been discovered especially when the research is conducted in a Cabri environment. These macro-stages are summarized below in two tables (Figure 15 for the macro-stages before conjecture and Figure 16 for the macro-stages after conjecture).

ERRATIC RESEARCH	1st break	SORTED RESEARCH	2nd break	ACCELERATION OF RESEARCH	3rd break CONJECTURE

Figure 15 : Macro-stages before conjecture

3rd break CONJECTURE	EXPERIMENTAL ANALYSIS AND VALIDATION	4th break	THEORETICAL ANALYSIS AND QUASI-THEORICAL VALIDATION	5th break	CRITICAL ANALYSIS

Figure 16 : Macro-stages after conjecture

Here is a quick description of these macro-stages:
Erratic research involves a sequence of explorations without any successful interpretation or with different interpretations that gives the impression to the researcher to have made no progress.
Sorted research occurs when the researcher focuses his or her explorations around a precise topic and is successful in generating data that can be interpreted inductively.
Acceleration of research is a stage where the researcher gets the impression to grab the solution and when he or she tries in an unmethodical way to validate very quickly this solution.
The conjecture: it is the moment when the researcher thinks that he or she has found really the solution and that the plausibility of this result is very strong.
Experimental analysis is an experimental stage where the aim of the experiments is to validate the conjecture, here perceptively on the screen of the computer (stage conducted under G1 informatique).
Theoretical analysis and quasi-theoretical validation is also an experimental stage where the aim of experiments is to validate the conjecture but here with the use of the verification tools of the software (stage conducted under G2 informatique).
Critical analysis is a stage where the experimenter tries to validate or invalidate the conjecture previously “proven” experimentally.
This last stage is a crucial step in the scientific process in experimental sciences. It must be in mathematics especially when experiments are conducted using models of geometric objects of Euclidean geometry provided by the Cabri microworld.

2.2. Problem solving and DGS

2.2.1. Solving a problem by a professional mathematician: an experiment conducted during a congress in Liege (proceedings quoted in [6]) with specialists of both mathematics and DGS highlighted the fact that even professional mathematicians can traverse all the stages before making a conjecture depending on the distance between the problem proposed to them and the level of their corresponding knowledge (in this case the problem was a black box problem).

2.2.2. Solving a problem by a teacher : knowing that, a teacher must be aware that his or her way of presenting the solution of a problem sustained by DGS is crucial for the development of the problem solving skills of his or her students. For example, he or she must know that at the beginning of a very difficult problem, he or she must show that the erratic research stage is a stage where you try to use the deep knowledge that seems to be related to the problem. When some interesting ideas arise, he or she must know that it is time to show how to use some techniques that can be heuristic in order to enhance the power of investigation of the students. At last, when a very plausible conjecture arises, he or she must show how a deductive reasoning can provide logical consequences that can be validated or not experimentally.

2.2.3. Solving a problem by a learner: most of the time, what can be observed in the classrooms is the poverty of the process of research of students when research can be mediated by DGS : the principal reason is an ignorance by the teacher of the previous parameters that can be chosen on purpose if known. It is a very difficult task to require from students to work at a special stage of a research if the given problem does not allow them to reach this stage or to progress successfuly during another stage if techniques of investigations are not known.

2.2.4. The particular case of some black box problems as instruments of initiation of the experimental process of research where the experimenter can decide the way of experimenting:

Following the work of Bernard Capponi who had imagined the possible use of hidden transformations (hidden with macros), I have really used this idea in a lot of experiments (some of which are described in my PhD study). Use of these could be a perfect entry for the process of investigation. I have created for the Association of Teachers of Mathematics (in the UK) a pack of 20 of these black boxes to be used by english teachers. It would be highly pertinent to use these tools in order to help students and teachers to understand the possible stages of a process of investigation mediated by technology and especially by DGS.

2.3. The example of the technique of morphing using proportionnality ([18])

2.3.1. Morphing a graph onto a cube is a good way to convince students or teachers of the power of dynamicity provided by DGS. Figure 17 can be an introduction to the technique of morphing using proportionnality.

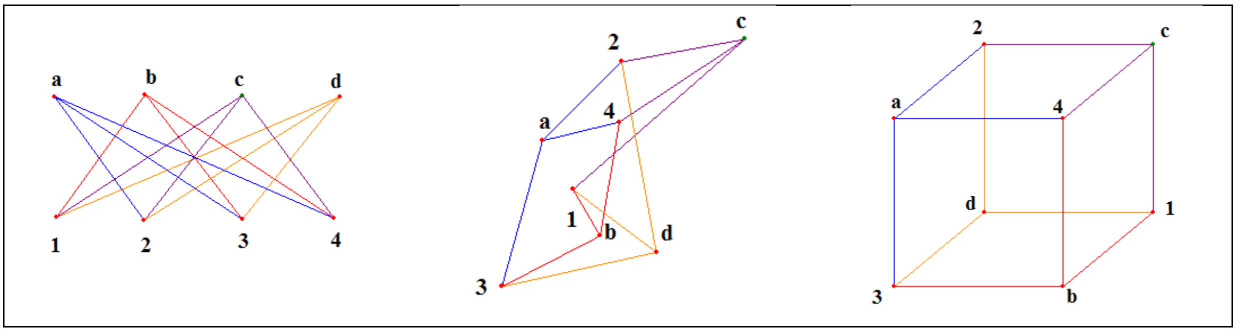

Figure 17 : Morphing a graph onto a cube

2.3.2. Morphing a triangle into an heptagon (ATCM Hong Kong 2006, non published): to understand what is really a figure with its properties (see technique in [20]) : Figure 18 on the left shows how the locus of point M on segment [AB] generates an intermediate curve between a triangle and an heptagon. The animation of point M along this segment models the morphing of the triangle into the heptagon.

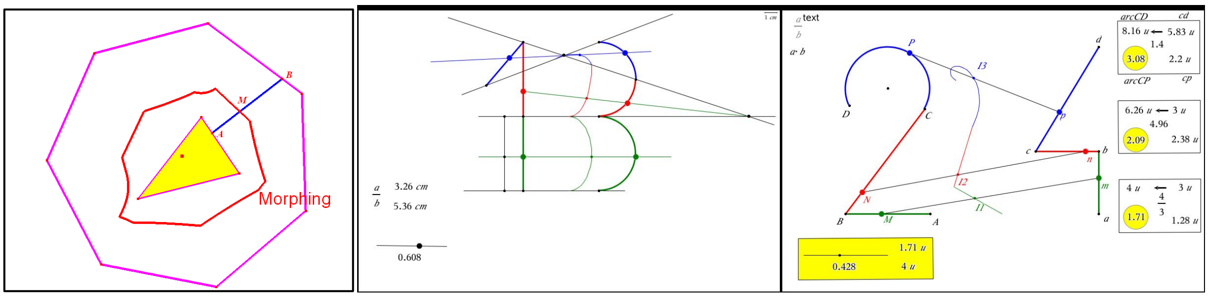

Figure 18 : Examples of morphings

2.3.3. Morphing a number onto another : to show the richness of the concept of proportionnality to create a motion modelling the « morphing process » ([15]) ; an illustration is given in Figure 18 in the middle and on the right.

2.3.4. Morphing one surface into another one (ATCM Hong Kong 2006, non published)
As presented in 1.3.1., in Figure 19 in the middle we construct point *I* and *S* which are respectively a point of surfaces $z = -2+sin(x)+cos(y)$ and $z = \sqrt{25 - x^2 - y^2}$ on the same vertical. As done in Figure 18 on the left, we create point *M* on segment [*SI*] which two loci will provide intermediate loci (one *x*-section and one *y*-section of the intermediate surface) between the loci of *S* (Figure 19 on the left) and *I* (Figure 19 on the right). Finally, to get the wire representation of the intermediate surface between the two previous ones, it is sufficient to do for *M* the job we did to represent a surface in 1.3.1. The intermediate surface is shown in Figure 19 in the middle.

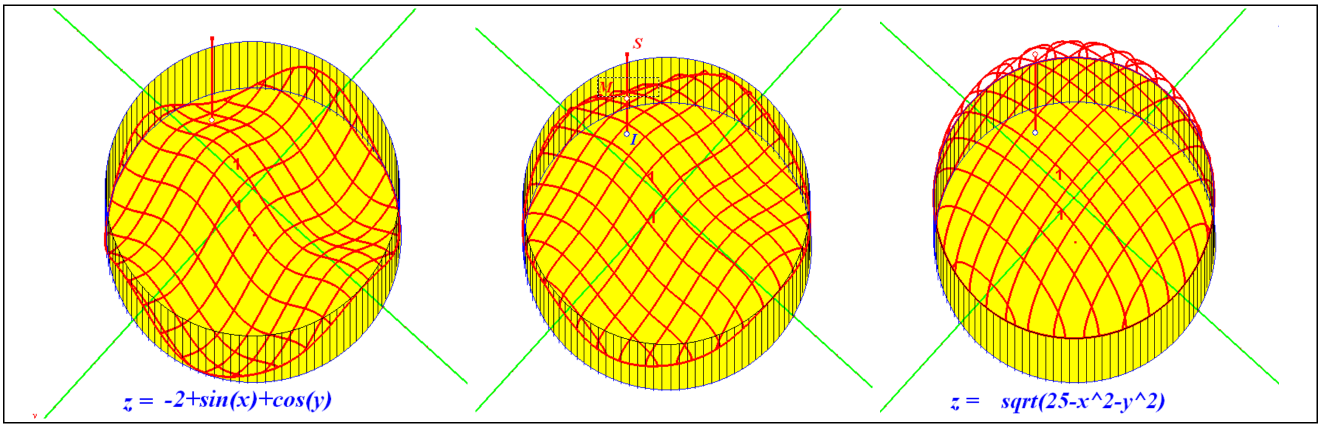

Figure 19 : Morphing a surface into another one

2.4. Powerful macros and differential equations ([8])
Using the Euler's method and some tricky macros giving a part of a solution curve of a given differential equation passing through a given point, I have been able to perform the very beautiful set of solution curves of the equation ***y'+0.5y=sin(x)*** (we have used the locus tool of each polygon modelling a solution passing through a point that can be movable). See Figure 20.

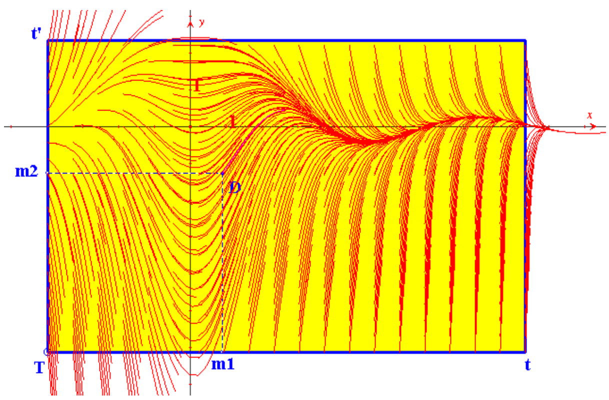

Figure 20 : Set of solutions of the differential equation ***y'+0.5y=sin(x)***

2.5. Transformations and modelling process in 3D (with *Cabri 3D*)

2.5.1. Modelling the real world with transformations in 3D : in Figure 21 are presented three examples I have created in order to show students the techniques using 3D transformations which allow them to model 3D static or dynamic objects of the real world : the double stair of the castle of Chambord in the Loire valley in France (Figure 21 on the left) and the Opera Garnier of Paris (Figure 21 on the right) are the static examples and the helicopter (Figure 21 in the middle) is the dynamic example. Cabri 3D workshops led by a mathematics teacher in the French highschool of Casablanca during three years have demonstrated the power of an initiation to the creation of animated objects of the real world with Cabri 3D to enhance the math 3D skills of middle school students. The modelling of the Opera Garnier was performed with a sixteen year old student (level : one year before french baccalaureat) to illustrate an essay she had to present (the subject was the Haussmanian architecture in Paris in the XIXth century). See [24], [25] and [26].

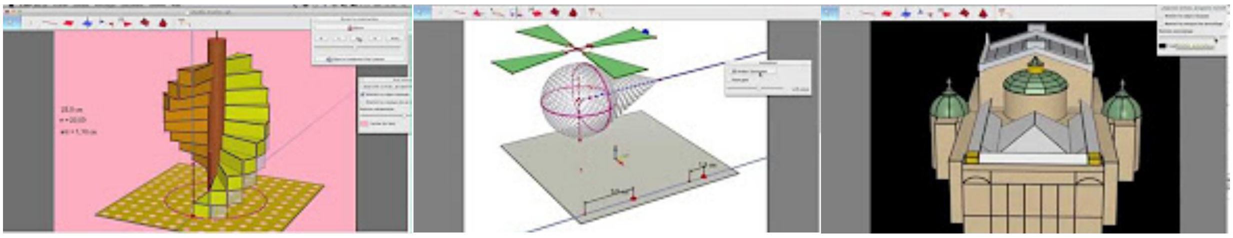

Figure 21 : Modelling the real world with Cabri 3D

Six videos of my YouTube channel (jjdahan) which title begins by APM TOULOUSE had been recorded by six of the students of the workshop to present their skills in 3D DGS.

2.5.2. Tricky constructions and unusual links between calculus ans geometry
- **Modelling the basic steps of the Cha Cha dance** was a research work started unsuccessfuly with CAS and performed successfuly with tricky geometric constructions avoiding to use complicated piecewise functions. I have used this special technique of modelling to train mathematics teachers to use *Cabri 3D* during an annual ATM conference in England (Figure 22, the left one for the thechnique of modelling). See ([11] and ([12]))

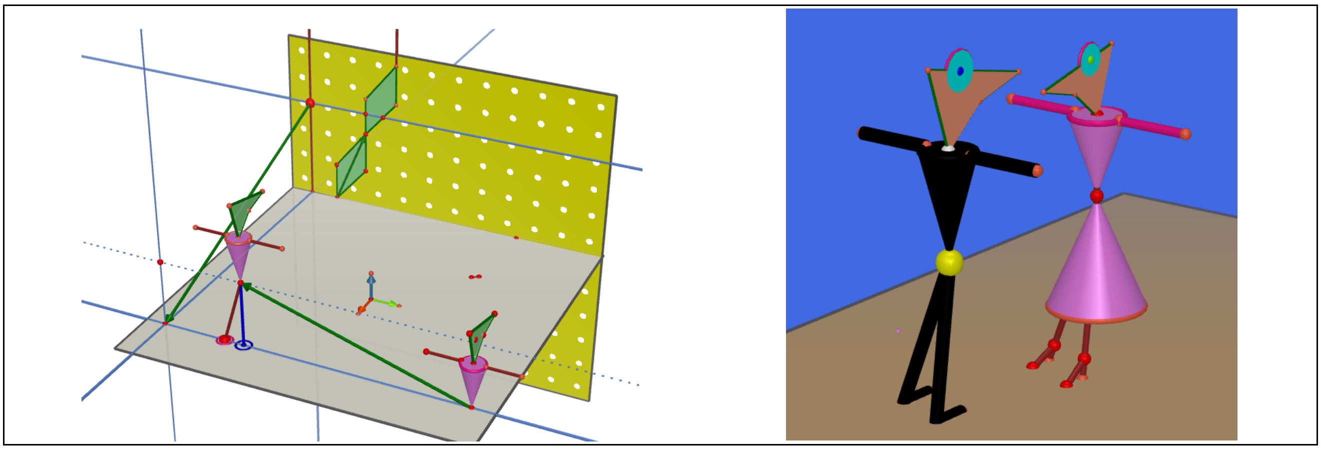

Figure 22 : Modelling Cha Cha dance

- **Back to the representation in military perspective of the surface** $z = f(x,y)$, if we represente the two surfaces $z_1 = f(x,y) + \sqrt{f(x,y)} - \sqrt{f(x,y)}$ and $z_2 = f(x,y) + \sqrt{-f(x,y)} - \sqrt{-f(x,y)}$, we get for the first one the positive part of the surface in blue and the negative in red (Figure 23 in the left). If we hide the negative one, we can see the intersection between the surface and the plane $z = 0$. We can state that this intersection in Figure 23 on the right is a set of perpendicular lines giving the solution of the equation $f(x,y)=0$ which is here $sin(x)=sin(y)$ ([13])

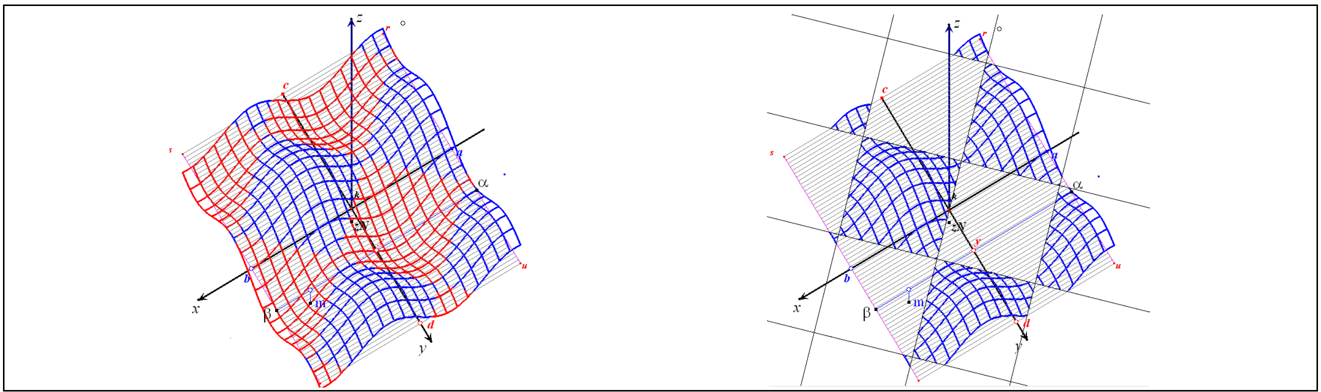

Figure 23 : Representations of the positive and negative parts of a surface $z = f(x,y)$

3. Conclusion

As stated during my presentation "Modelling Cha Cha dance with *TI-NSpire* in South Africa,*"The modelling process shows the immediate "utility" of a math activity but helps strongly the modeller to reach the real "purpose" of such a work: a deep understanding of some focused math knowledge"*. We can add that knowing the techniques of experts enhances the power of modelling and by the way the math creativity especially in 3D within the Cabri 3D environment. Knowing all the possible stages of an experimental process of research leading to discovery must help the teacher to chose appropriate problems before proposing them to students in order to expect them to work in an expected stage. Knowing the techniques of validation used during such a process must also help teachers or students to know if they will reach the truth or the plausibility of the property they have pointed during their work

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[21] Video: Parallel_perspective-cavaliere_english http://youtu.be/wjApHcBcki4
[22] Video:, Parallel_perspective_military_english http://youtu.be/LcOlX_l1TLk
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Software : *Cabri 2 Plus* and *Cabri 3D* by Cabrilog at http://www.cabri.com
TI-Nspire by Texas Instruments at http://education.ti.com/en/us/home