Outbox Centroid Theorem:
An Episode of Dynamic Geometry Exploration

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Abstract

An outbox of a given convex quadrilateral is a rectangle such that each vertex of the quadrilateral lies on one side of the rectangle and all the vertices lie on different sides, with all the sides of the rectangle external to the quadrilateral. This paper reports on a new geometrical result concerning outboxes of convex quadrilateral – the Outbox Centroid Theorem, and gives a new proof of an existing result of M. F. Mammana. Interestingly, the investigation that leads to this new result comes from dynamic-geometry explorations.

1 Introduction

An outbox of a given convex quadrilateral $ABCD$ is a rectangle $PQRS$ such that each vertex of $ABCD$ lies on one side of $PQRS$ and all vertices lie on different sides, with all sides of $PQRS$ being external to $ABCD$. An example is shown in Figure 1 and two non-examples[1] in Figure 2.

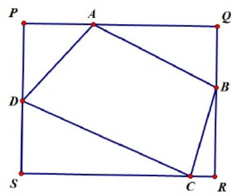

Figure 1: An example of an outbox

Coined by D. Zhao in [7], the notion of 'outbox' is not new; already in the Book IV of "Elements" Euclid recorded the following definitions:

[1]Such non-examples are sometimes termed as 'illegal' outboxes.

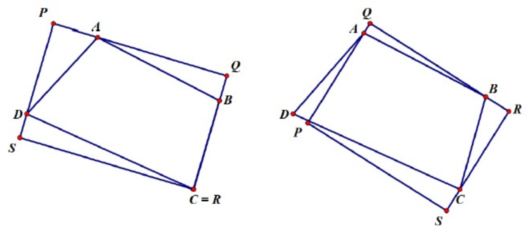

Figure 2: Two 'illegal' outboxes

Definition 1 (Inscribing rectilinear figure) *A rectilinear figure is said to be* inscribed *in a rectilinear figure when the respective (vertices) angles of the inscribed figure lie on the respective sides of the one in which it is inscribed.*

Definition 2 (Circumscribing rectilinear figure) *A rectilinear figure is said to be* circumscribed *about a rectilinear figure when the respective sides of the circumscribed figure pass through the respective (vertices) angles of the one about which it is circumscribed.*

In this ancient lingo, the 'modern' definition of an outbox can be given as follows: An outbox of a convex quadrilateral $ABCD$ is a rectangle $PQRS$ circumscribed to it. A quadrilateral is *rectangle-inscribable* (*r-inscribable*, for short) if it has an outbox. We use all the above terminologies in our ensuing discussion.

A *maximal outbox*, if it exists, is one with the largest area. D. Zhao in [7] posed the maximal outbox problem asking for the area of the maximal outbox for a given convex quadrilateral. Therein, he provided a flawed 'solution' using calculus. Firstly, Zhao assumed implicitly that every convex quadrilateral is r-inscribable. A deeper analysis carried out by Mammana in [4] already characterized r-inscribable convex quadrilateral to be those for which the sum of every pair of consecutive angles (i.e., angles adjacent to each other) is less than three right angles. Indeed this characterisation is visually compelling – it is impossible for the corner (right angle) of a rectangle to 'fit in' so as to circumscribe an isosceles trapezium with each base angle measuring $40°$ each (see Figure 3) because the angle at which the two non-parallel sides (extended) meet already exceeds $90°$. Secondly, it was never verified whether the stationary configuration

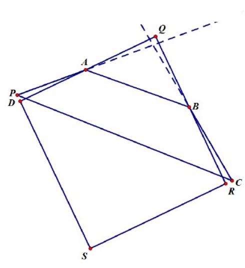

Figure 3: A non-r-inscribable convex quadrilateral

obtained by the calculus approach in [7] yields a configuration is actually attainable in a given situation. Also, the maximal outbox problem arises from a geometrical situation and so a purely geometrical solution is more natural. Note that Mammana ([4]) already gave a complete solution by characterising those convex quadrilaterals whose maximal outbox exists based on geometry and trigonometry.

This paper exploits a DGS (Dynamic Geometry System) for geometrical exploration into the geometrical properties of outboxes. Via DGS exploration, we discover and prove a novel theorem in geometry – *Outbox Centroid Theorem*. Using this, not only do we obtain the formula for the area of the maximal outbox provided it exists but also re-establish Mammana's characterisation theorem for r-inscribable convex quadrilaterals to admit maximal outboxes [4, Theorem 4]. Throughout, we adopt a sign convention for angles: counterclockwise angles are positive. To illustrate this convention, we refer to Figure 3. The angle $\angle DAP$ has an anti-clockwise sense and is defined to be positive; the implicit reference line being AP. Thus, $\angle PAD$ is negative. The background knowledge for this paper includes high-school geometry and trigonometry.

2 DGS-aided discovery

Solving the maximal outbox problem inevitably compels one to the drawing board. Traditional paper-and-pencil method in seeking any geometrical invariants is tedious. We turn to *Geometer's Sketch Pad* (GSP, for short) – a DGS. The application of DGS as an experimental approach to theoretical thinking has recently been given a thorough treatment in [1, 6]. To consider all possible outboxes of $ABCD$, if they exist at all, we construct a dynamic prototype of an outbox. The "dragging" feature of GSP allows the user to range over all the outboxes of a fixed r-inscribable convex quadrilateral and observe any geometrical invariants associated with it.

To understand the logical dependence of the elements involved in the construction of an outbox for a given r-inscribable convex quadrilateral, the reader can follow the steps in Figure 4. The segment XY controls the direction of the side PQ of the outbox by moving the point Y . Dragging Y enables us to range over all possible outboxes (including 'illegal' ones). Using the area-measuring facility of GSP, one adjusts the direction of XY to obtain the maximum area by trial-and-error.

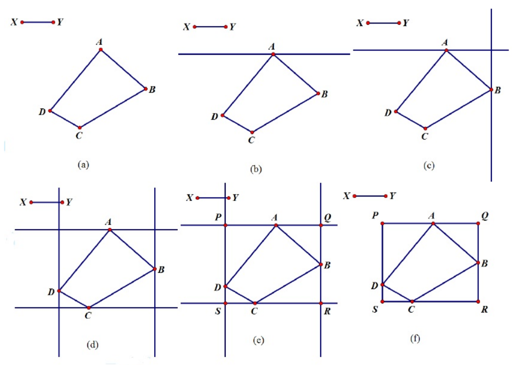

Figure 4: Step-by-step construction of a dynamic outbox

3 Outbox centroid theorem

To determine the position of the maximal outbox of a given r-inscribable convex quadrilateral $ABCD$, one may wish to 'parameterize' the position of an outbox by the motion of a single point. The special point we choose is the centroid of an outbox (the intersection of the medians, or equivalently, the diagonals). But we have two questions:

A. Can we determine if a point is the centroid of some outbox?

B. Given that a point is the centroid of some outbox, can the position (and dimensions) of this outbox be determined?

To answer these, the locus of the centroid of an outbox for a fixed r-inscribable convex quadrilateral must be fully determined. By dragging Y and tracing the centroid K, one pleasantly discovers that K traces out what seems to be circular arc:

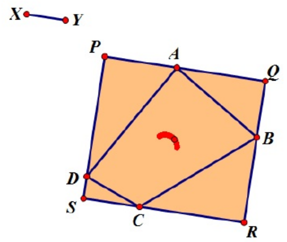

Figure 5: Locus of the centroid of an outbox

To develop the proof for this observation, we modify a geometrical setup due to Mammana [4, p.84–86]. Given a convex quadrilateral $ABCD$, where $\angle BAD = \alpha$, $\angle CBA = \beta$, $\angle DCB = \gamma$ and $\angle ADC = \delta$, assume without loss of generality that $\alpha = \max\{\alpha, \beta, \gamma, \delta\}$.

Clearly, $\alpha = \frac{\pi}{2}$ or α is obtuse; otherwise, $\alpha < \frac{\pi}{2}$, together with $\alpha \geqslant \beta$, γ, δ, implies that $\alpha + \beta + \gamma + \delta < 2\pi$ – a contradiction. If $\alpha = \frac{\pi}{2}$, then it could not be the case that one of the angles β, γ and δ is acute (i.e., strictly less than α); otherwise, $\alpha + \beta + \gamma + \delta < 2\pi$, another contradiction. So, if $\alpha = \frac{\pi}{2}$, it must be that $\beta = \gamma = \delta = \frac{\pi}{2}$ or equivalently that $ABCD$ is a rectangle. Since a rectangle clearly has an outbox, we restrict to the case where α is obtuse.

Because $\alpha \geqslant \beta$, γ, δ, it follows that $\alpha + \beta > \beta + \gamma$ and $\alpha + \delta > \gamma + \delta$. Thus, the maximum sum of the adjacent pairs of interior angles must be equal to $\max\{\alpha + \beta, \alpha + \delta\}$. Here, we deviate from the labelling convention of [4] in that $\alpha + \beta$ need not be the largest amongst the sum of adjacent interior angles.

Suppose $PQRS$ is an outbox of $ABCD$ as shown in the Figure 6. Since $\angle PQR = \frac{\pi}{2}$, the locus of Q is a subset of the open semicircular arc Γ_{AB} whose diameter is AB and which lies external to $ABCD$. As to which subset this is, we shall subsequently determine it.

Since PQ is a straight line segment, $\angle DAP + \angle BAD = \angle BQA + \angle ABQ = \frac{\pi}{2} + (\pi - \angle CBA - \angle RBC)$ and thus, $\alpha + \beta = \angle BAD + \angle CBA = \frac{3\pi}{2} - \angle DAP - \angle RBC$. Since AQ (respectively, QR) is external to the quadrilateral $ABCD$, we have $\angle DAP > 0$ and $\angle RBC > 0$ and so,

$$\alpha + \beta < \frac{3\pi}{2}. \tag{1}$$

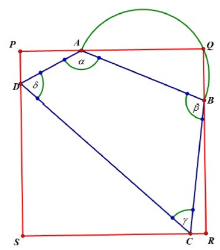

Figure 6: A particular outbox of an r-inscribable quadrilateral

Analyzing each remaining side of the quadrilateral similarly, we have:

$$\beta + \gamma < \frac{3\pi}{2},\ \gamma + \delta < \frac{3\pi}{2} \ \&\ \delta + \alpha < \frac{3\pi}{2}. \tag{2}$$

Thus, it is necessary that the sum of any adjacent pair of interior angles is less than $\frac{3\pi}{2}$.

This condition turns out to be sufficient for $ABCD$ to be r-inscribable. To see this, note that Q is the vertex of an outbox of $ABCD$ with $\angle BQA = \frac{\pi}{2}$ if and only if

(1) the line QA (respectively, QB) extended is external to $ABCD$, and

(2) the perpendicular to QA (respectively, QB) that passes through D (respectively, C) is external to $ABCD$.

In (1), the straight line QA extended is external to $ABCD$ if and only if $\angle QAB < \pi - \alpha$. Denoting by A_1' the point of intersection of DA produced with the semicircle Γ_{AB} (see either of the diagrams in Figure 7), the upper bound described by the preceding inequality corresponds to one extreme position where $\angle A_1'AB = \pi - \alpha$.

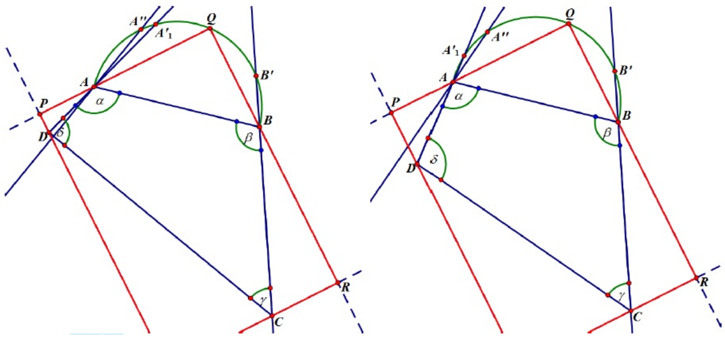

Figure 7: $A \neq A''$: $\angle A_1'AB < \angle A"AB$ (left), $\angle A_1'AB \geqslant \angle A"AB$ (right)

For the necessary and sufficient condition for the straight line QB extended to be external to $ABCD$, one considers two mutually exclusive cases.

- $\beta = \frac{\pi}{2}$ or obtuse. Then the line segment BC is tangential to Γ_{AB} or meets Γ_{AB} at some point $B' \neq B$. This extreme position B' gives rise to a lower bound for $\angle QAB$, i.e., $\angle QAB > \angle B'AB = \beta - \frac{\pi}{2}$. (See Figure 8 (Case 1).)

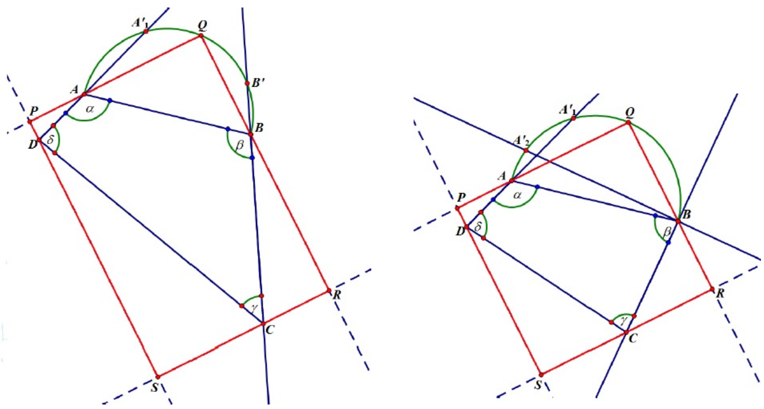

Figure 8: Case 1: $\beta = \frac{\pi}{2}$ or obtuse; Case 2: β is acute

- β is acute. Then the perpendicular to BC through B meets Γ at some point A'_2. This extreme position A'_2 gives rise to an upper bound for $\angle QAB$, i.e., $\angle QAB < \angle A'_2AB = \beta$. (See Figure 8 (Case 2).)

In summary, when $\frac{\pi}{2} \leqslant \beta < \pi$, the condition (1) holds if and only if Q lies on the open arc $\widehat{A'_1B'}$; and when $0 < \beta < \frac{\pi}{2}$, the condition (1) holds if and only if Q lies on the open arc $\widehat{A'_1B} \cap \widehat{A'_2B}$.

As for the condition (2), one must consider the perpendicular to CD through A (respectively, through B), and label A'' (respectively, B'') the point of intersection of this perpendicular with the semicircular arc Γ. Note that AB is parallel to CD if and only if $A = A''$ and $B = B''$. Taking logical negations, this would mean that AB is not parallel to CD if and only if $A \neq A''$ or $B \neq B''$. In this case, the first possibility of $A \neq A''$ entails that $B = B''$; (see Figure 7) and by symmetry of the situation, the second possibility of $B \neq B''$ entails that $A = A''$ (see Figure 9). Note that Figures 7 and 9 illustrate only the case where $\frac{\pi}{2} \leqslant \beta < \pi$.

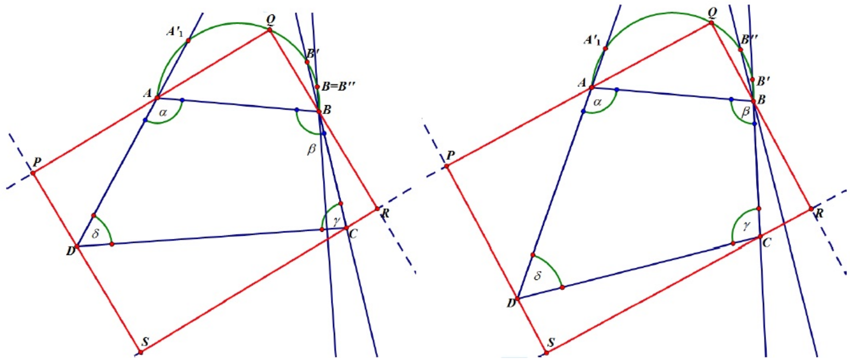

Figure 9: $B \neq B''$: $\angle B'AB > \angle B''AB$ (left), $\angle B'AB \leqslant B''AB$ (right)

So, (2) is equivalent to Q lying on the open arc $\widehat{A''B''}$ of Γ_{AB}. Three notations:
(i) $x \dot{-} y := \begin{cases} x - y, & x > y; \\ 0, & x \leqslant y. \end{cases}$ (ii) $\hat{\beta} := \frac{\pi}{2} - \left(\frac{\pi}{2} \dot{-} \beta\right)$. (iii)$\phi := \begin{cases} \frac{3\pi}{2} - (\alpha + \delta), & \alpha + \delta > \pi \\ \frac{\pi}{2}, & \alpha + \delta \leqslant \pi. \end{cases}$
Thus, we have:

Proposition 3 *The following are equivalent for a given convex quadrilateral $ABCD$ and a point Q on Γ_{AB}:*

1. *Q is the vertex of an outbox for $ABCD$ with $\angle BQA = \frac{\pi}{2}$.*

2. *Q lies on the non-empty intersection of the open arcs $\widehat{A_1'B'}$, $\widehat{A_1'B}$, $\widehat{A_2'B}$ and $\widehat{A''B''}$ of Γ_{AB}.*

3. *The acute angle $\angle QAB$ satisfies the inequalities:*

$$\max\{\angle B'AB, \angle B''AB\} < \angle QAB < \min\{\angle A_1'AB, \angle A_2'AB, \angle A''AB\}.$$

4. *The acute angle $\angle QAB$ satisfies the inequalities:*

$$\max\{\beta \dot{-} \frac{\pi}{2}, \beta + \gamma \dot{-} \pi\} < \angle QAB < \min\{\pi - \alpha, \hat{\beta}, \phi\}.$$

In particular, when $\alpha + \beta$ is the largest amongst possible sums of adjacent interior angles, the condition $\alpha + \beta < \frac{3\pi}{2}$ is equivalent to (4), and hence (1), i.e., $ABCD$ has an outbox.

Note that $ABCD$ has an outbox if and only if (2) holds, i.e., the intersection of the open arcs $\widehat{A_1'B'}$, $\widehat{A_1'B}$, $\widehat{A_2'B}$ and $\widehat{A''B''}$ is non-empty. Denote this non-empty intersection of these open arcs by the open arc $\widehat{A'''B'''}$, where A''' ie either A_1', A_2' or A'', depending on the comparison between $\angle A_1'AB$, $\angle A_2'AB$ and $\angle A''AB$, and B'' is either B' or B'' depending on the comparison between $\angle B'AB$ and $\angle B''AB$.

From this point till the end of Section 5, we assume that the convex quadrilateral $ABCD$ is not a parallelogram. The case of a parallogram is an easy exercise for the reader. Let W be the intersection of the diagonals AC and BD of the given r-inscribable convex quadrilateral. In addition to that labelling convention of $\angle DAB := \alpha$ being the largest interior angles, from this point onwards we also adopt the orientation of $\angle AWB := \vartheta$ is either $\frac{\pi}{2}$ or obtuse. Our labelling convention coincides with [4, Section 3].

Lemma 4 *Let $ABCD$ be an r-inscribable convex quadrilateral.*

1. *For each point Q of Γ_{AB} (as described above), the centroid K_Q of the rectangle $\mathcal{R}_Q := PQRS$, formed by the extensions of QA and QB, and the perpendiculars to QA through D and to QB through C, lies on the circle with diameter LN.*

2. *The assignment $Q \mapsto K_Q$ in (1) defines an injective function $K : \Gamma_{AB} \longrightarrow \Lambda$.*

Proof. (1) Given Q on Γ_{AB}, let P, R and S be the rest of the vertices of the rectangle $\mathcal{R}_Q$ formed by the extensions of QA and QB, and the perpendiculars to QA through D and to QB through C. Note that $\mathcal{R}_Q := PQRS$ may not be an outbox of $ABCD$ since it might well be the case that Q lies outside the open arc $\widehat{A'''B'''}$. For the ensuing argument, refer to Figure 10. Let U be the midpoint of PQ, V that of RS, X that of PS and Y that of QR. It is clear that the line segments XY and UV are respectively parallel to PQ and QR. Moreover, XY is the perpendicular bisector of PQ and SR, while XY is the perpendicular bisector of PS and QR. Clearly, the centroid K_Q of the rectangle $\mathcal{R}_Q := PQRS$ is the intersection of UV and XY.

We first show that XY meets AC at L the midpoint of AC. To this end, construct the perpendicular AA' to SR that passes through A. Since XY bisects PS and QR, it must also bisect AA' at E. Since LE is parallel to CA', it follows that the triangle LAE is similar to the triangle CAA'. Moreover, because E bisects AA', one has that L bisects AC as desired.

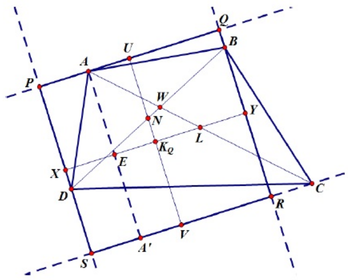

Figure 10: A construction used in the proof of (1)

Similarly, one can conclude that UV meets BD at N, the midpoint of BD. Since XY and UV are perpendicular, together with L lying on XY and N on UV , it follows that $\angle NK_QL$ is a right angle. Because Q is an arbitrary point on Γ_{AB}, it then follows from Thale's theorem that the locus of K_Q is a subset of the circle Λ with diameter LN.

(2) Since every rectangle $\mathcal{R}_Q$ possesses only one centroid K_Q and by (1) $K_Q \in \Lambda$, it follows that the assignment $K : Q \mapsto K_Q$ is a function from Γ_{AB} to Λ. It remains to show that it is injective. Let l_A be the perpendicular to AB through L and l_B be the parallel to AB through L (see Figure 11). Denote by K_A (respectively, K_B) the intersection of l_A (respectively, l_B)

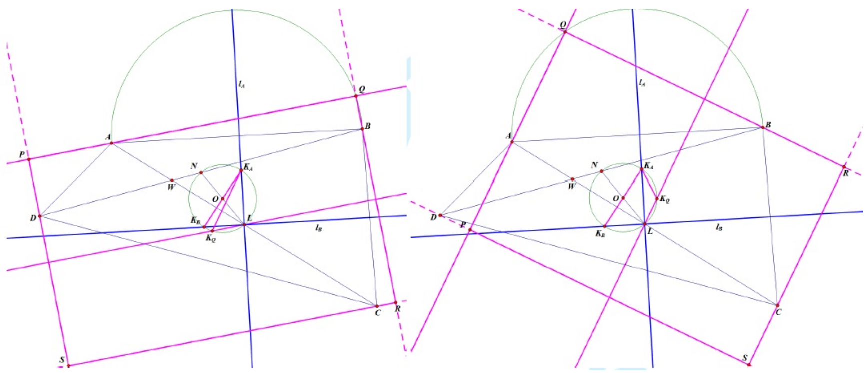

Figure 11: K_A and K_B in the proof of (1)

with the circle Λ (other than the point L whenever there are two points of intersection). Since l_A and l_B are perpendicular by construction, it follows that $\angle K_BLK_A = \frac{\pi}{2}$ so that by Thales theorem K_A and K_B are diametrically opposite.

We claim that $\angle QAB - \angle K_QK_AK_B$ for all $Q \in \Gamma_{AB}$. There are two possible cases to prove:

- K_A and L are on the same side with respect to the chord K_BK_Q (see Figure 11 (top)). By the inscribed angle theorem, $\angle K_QK_AK_B = \angle K_QLK_B$. Notice that the latter angle is that made between the lines l_B and K_QL produced. Since these two lines are respectively parallel to AB and AQ, it follows by the virtue of corresponding angles that $\angle QAB = \angle K_QLK_B$ and hence $\angle QAB = \angle K_QK_AK_B$.

- K_A and L are on opposite sides of the chord K_BK_A (see Figure 11 (bottom)). By the cyclic quadrilateral theorem, $\angle K_QK_AK_B = \pi - \angle K_BLK_Q$. But the latter angle is equal to the acute angle between the line K_QL and l_B, and in turn this equal to $\angle QAB$ due to corresponding angles. Hence $\angle QAB = \angle K_QK_AK_B$.

Suppose that $K_Q = K'Q$ for some $Q, Q' \in \Gamma_{AB}$. Then, by the preceding result, $\angle QAB = \angle K_Q K_A K_B = \angle K'_Q K_A K_B = \angle Q'AB$. Since Q and $Q' \in \Gamma_{AB}$, we must have $Q = Q'$. This then completes the proof that the function $K : Q \mapsto K_Q$ is injective. ■

We now turn to the problem of determining the locus of the centroid KQ of an outbox $PQRS$ of $ABCD$, where $Q \in \Gamma_{AB}$. This is equivalent to locating all the possible positions of K_Q on the circle Λ as Q moves on the open arc $\widehat{A'''B'''}$ as described earlier. In what follows, we continue to denote by $\mathcal{R}_Q$ the rectangle formed by the extensions of QA and QB, and the perpendiculars to QA through D and to QB through C (where Q is a point on the open semi-circular arc Γ_{AB}). We also use the notations K_A and K_B as described above.

Define the open semi-circular arc $\Lambda_{K_A K_B}$ of Λ to be that with diameter $K_A K_B$ such that for every point M on it, $0 < \angle MK_A K_B < \frac{\pi}{2}$. Note that by our sign convention of angles, $\angle MK_A K_B$ has a counterclockwise sense and so the open semi-circular arc $\Lambda_{K_A K_B}$ is uniquely determined.

Proposition 5 *Let $ABCD$ be an r-inscribable convex quadrilateral. Then, the co-restriction of the function K on the open semi-circular arc $\Lambda_{K_A K_B}$, i.e., $K : \Gamma_{AB} \longrightarrow \Lambda_{K_A K_B}$, $Q \mapsto K_Q$ is a bijection between Γ_{AB} and $\Gamma_{K_A K_B}$. Indeed, the locus of K_Q as Q varies on the open semi-circular arc Γ_{AB} is exactly the open semi-circular arc $\Lambda_{K_A K_B}$.*

Proof. By Lemma 4, it suffices to show that this co-restriction of K on $\Lambda_{K_A K_B}$ is surjective. To this end, let M be any point on $\Lambda_{K_A K_B}$. Construct a line k parallel to LM through A. Since $\angle MK_A K_B$ is equal to the acute angle between $K_B L$ and LM, it follows that the angle between k and AB which is equal to $\angle MK_A K_B$ is acute. Thus, k meets Γ_{AB} non-emptily at some point QM. We now show that $K_{Q_M} = M$. First construct the rectangle $\mathcal{R}_Q := PQRS$ which is formed by the lines AQ and QB produced and the perpendiculars to AQ and QB through D and C respectively. It follows that PQ is parallel to LM and to SR. Since L is the midpoint of AC, it follows from arguments involving similar triangles that LM extended bisects QR at a point Y , and PS at a point X so that XY is a median of the rectangle $\mathcal{R}_Q$. Similarly, the line MN extended bisects PQ at a point U and SR at a point V since N is the midpoint of BD. So, UV is the other median of the rectangle $\mathcal{R}_Q$. Since M is the intersection of the medians XY and UV by construction, it follows that M is the centroid of the rectangle $\mathcal{R}_Q$. This proves that $K_{Q_M} = M$, and thus the co-restriction $K : \Gamma_{AB} \longrightarrow \Lambda_{K_A K_B}$ is surjective, as desired. ■

Since $Q \in \Gamma_{AB}$ is completely determined by $\angle QAB$ and $K_Q \in \Lambda_{K_A K_B}$ by $\angle K_Q K_A K_B$, the equality $\angle QAB = \angle K_Q K_A K_B$ then allows one to perceive the bijection K as the identity map on the open interval $\left(0, \frac{\pi}{2}\right)$. Proposition 3 asserts that $Q \in \Gamma_{AB}$ is the vertex of an outbox $PQRS$ (with $\angle AQB = \frac{\pi}{2}$) if and only if $\max\{\angle B'AB, \angle B''AB\} < \angle QAB < \min\{\angle A'_1 AB, \angle A'_2 AB, \angle A''AB\}$. Applying the bijection K, it follows that

$$\max\{\angle K_{B'} K_A K_B, \angle K_{B''} K_A K_B\} < \angle K_Q K_A K_B < \min\{\angle K_{A'_1} K_A K_B, \angle K_{A'_2} K_A K_B, \angle K_{A''} K_A K_B\}$$

or equivalently, $\angle K_{B'''} K_A K_B < \angle K_Q K_A K_B < \angle K_{A'''} K_A K_B$, i.e., $\angle B'''AB < \angle K_Q K_A K_B < \angle A'''AB$. Thus, by Proposition 5, we have established the main theorem of this paper:

Theorem 6 (Outbox Centroid Theorem) *Let $ABCD$ be an r-inscribable convex quadrilateral. The locus of the centroid K_Q of an outbox $PQRS$ of $ABCD$ is an open arc $\widehat{K_{A'''} K_{B'''}}$ of the semi-circular arc $\Lambda_{K_A K_B}$ such that $\angle K_{B'''} K_A K_B < \angle K_Q K_A K_B < \angle K_{A'''} K_A K_B$.*

Remark 7 *1. Every parallelogram has an outbox since the sum of adjacent angles is always π (which is less than $\frac{3\pi}{2}$). In the case where $ABCD$ is a parallelogram, the points L and N coincide with K. Thus, the locus of the centroid of an outbox reduces to a point (i.e., the radius of the circle Λ is zero). Thus, the case of a parallelogram can be seen as a limiting case of what we are considering in this section.*

2. Theorem 6 answers both questions (A) and (B) raised at the beginning of this section.

3. Our DGS-aided discovery made in the preceding theorem exploits the wandering dragging *approach – a method described in [1] as "moving the basic point(s) on the screen randomly, without plan, in order to discover interesting configurations or regularities in the figures". In our case, the basic point is Y and the regularity is the locus of the centroid of the outbox. For the use of dragging in dynamic geometry environment, the reader is referred to [3].*

4 Characteristic triangles

Amongst all the possible outboxes of a given r-inscribable convex quadrilateral $ABCD$, which, if it exists, is the one with the maximum area? Further experimentation using GSP reveals more. Let I (respectively, J) be the foot of the perpendicular from L (respectively, N) to the diagonal BD (respectively, AC). See Figure 12 (left). Recall also that $\angle DAB := \alpha$ is the largest of the interior angles of $ABCD$ and $\angle AWB := \vartheta$ is either $\frac{\pi}{2}$ or obtuse.

By construction $\angle LIB = \angle LIN = \frac{\pi}{2}$ and LN is the diameter of the circle Λ (as defined in the preceding section), it follows that I coincides with the point of intersection of the diagonal BD with the circle Λ. Likewise, J coincides with the point of intersection of the diagonal AC with the circle Λ. DGS experiments allow us to observe something very special about $\triangle IKJ$ that corresponds to a given outbox $PQRS$ (we call this the *characteristic triangle* of $PQRS$). Here K is the centroid of the outbox $PQRS$. Whenever the area of $PQRS$ (denoted by $[PQRS]$) collapses to 0 (in which case this is an illegal' outbox), the area of IKJ (denoted by $[IKJ]$) is 0. This leads us to conjecture that the ratio of the area of an outbox to that of its characteristic triangle is a *constant* – which is further reinforced by compelling evidence via DGS (see Figure 12). Our observations made in Figure 12 using DGS show clearly that as K moves along the circle Λ, the angle IKJ is constant by virtue of the inscribed angle theorem. This indicates that the lengths of IK and JK are the only measurements which completely determine the area of the triangle IKJ by virtue of the sine rule. This train of thought leads us to the following lemma.

Lemma 8 *Let $ABCD$ be a fixed r-inscribable convex quadrilateral as shown in Figure 13. The point U (respectively, V) is the midpoint PQ (respectively, SR) while the point X (respectively, Y) is the midpoint of PS (respectively, QR). Then for any outbox $PQRS$ of $ABCD$,*

$$\frac{IK}{XK} = \frac{LN}{DN} \;\&\; \frac{JK}{VK} = \frac{NL}{CL},$$

where IKJ denotes the characteristic triangle of $PQRS$. In particular, these ratios are invariants over all possible outboxes $PQRS$ of $ABCD$.

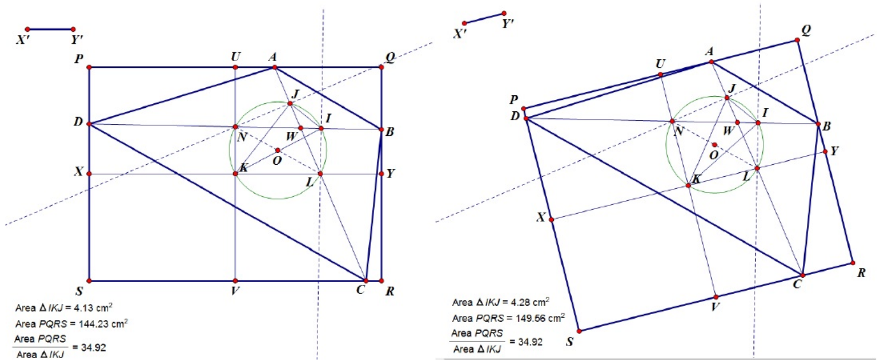

Figure 12: Two characteristic triangles

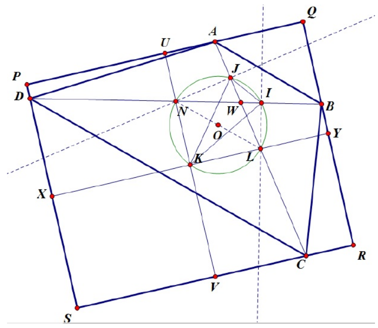

Figure 13: Characteristic triangle of $PQRS$

Proof. Since the segment IL subtends both $\angle IKL$ and $\angle INL$, by the inscribed angle theorem it follows that $\angle IKL = \angle INL$. So, $\angle XKI = \angle DNL$. Next we show that $DXLI$ is a cyclic quadrilateral. To this end, note that $\angle LXD = \frac{\pi}{2}$ by construction. Also, $\angle LIN = \frac{\pi}{2}$ because I is the foot of perpendicular to BD from L. Since D, N and I (irrespective of the order) are collinear, it follows that $\angle LID = \angle LIN = \frac{\pi}{2}$. By the cyclic quadrilateral theorem, $DXLI$ is a cyclic quadrilateral and so, by the inscribed angle theorem, $\angle LDI = \angle LXI$. So, $\angle LDN = \angle KXI$. Hence $\triangle IKX$ is similar to $\triangle LND$. Consequently, $\frac{IK}{XK} = \frac{LN}{DN}$. Since the points D, L and N are fixed for a given r-inscribable convex quadrilateral $ABCD$, it follows that the ratio $\frac{IK}{XK}$ is invariant over all outboxes $PQRS$ of $ABCD$. Similarly, one can show that $\frac{JK}{VK} = \frac{NL}{CL}$ is also invariant over all possible outboxes $PQRS$ of $ABCD$. ■

Theorem 9 *Let $ABCD$ be a fixed r-inscribable convex quadrilateral. Then the ratio of the area of any outbox $PQRS$ to that of its characteristics triangle, i.e., $[PQRS] : [IKJ]$, is an invariant over all possible outboxes $PQRS$.*

Proof. Suppose $PQRS$ is an outbox of $ABCD$ and K is the centroid of $PQRS$. By the sine rule, the area of $\triangle IKJ$ is $\frac{1}{2} IM \cdot JM \sin \angle IKJ$. Because IJ is a fixed chord of the circle Λ, $\angle IKJ$ and hence $\sin \angle IKJ$ is a constant over all possible outboxed by the inscribed angle theorem. Applying Lemma 8, it follows that

$$[IKJ] = \frac{1}{2}\left(XM \cdot \frac{LN}{DN}\right)\left(VM \cdot \frac{NL}{CL}\right)\sin \angle IKJ = 2\left(\frac{LN^2}{CL \cdot DN}\right)\sin \angle IKJ \cdot [PQRS].$$

Since $k = 2\left(\frac{LN^2}{CL\cdot DN}\right)\sin\angle IKJ$ is an invariant over all outboxes $PQRS$, we are done. ■

Corollary 10 *Let $ABCD$ be a fixed r-inscribable convex quadrilateral, and the points I and J are defined as above. Then the maximal outbox of $ABCD$, if it exists, is achieved when its centroid K is at the point M on Λ which is furthest away from the chord IJ.*

Proof. Assume the existence of some maximal outbox of $ABCD$. By Theorem 9, a characteristic triangle with the maximum area yields a maximal outbox. In turn, a characteristic triangle (with a fixed base IJ) attains maximum area when the vertex K is the point on Λ which is furthest away from IJ. ■

Assuming for the moment the given r-inscribable convex quadrilateral has a maximal outbox, we derive the formula for its area. Denote the point of intersection of the diagonals of the given r-inscribable convex quadrilateral by W. To analyze the position of the centroid of the maximal outbox (assuming it exists), we zoom into the circle Λ and a characteristic triangle IJK, where K lies on the major arc subtended by the chord IJ . Since $\angle AWB := \vartheta$ is $\frac{\pi}{2}$ or obtuse, either (1) I and N are on the same side with respect to W along the diagonal BD, or (2) I and N are on opposite side with respect to W along the diagonal BD. Assume first that I and N are on the same side with respect to W, as shown in the two situations of Figure 14. In first situation as shown in Figure 14 (left), I and N lie on the same side of W along the diagonal BD. By the inscribed angle theorem, $\angle IKJ = \angle INJ$. So $\angle IKJ = \angle INJ = \angle JWI - \angle WJN = \vartheta - \frac{\pi}{2}$. In the second situation as shown in Figure 14 (right), I and N lie on opposite sides of W along the diagonal BD. So, $\angle IKJ = \pi - \angle INJ = \pi - (\angle NWJ + \angle NJW) = \pi - \left(\pi - \angle AWB + \frac{\pi}{2}\right) = \vartheta - \frac{\pi}{2}$. Thus, we have:

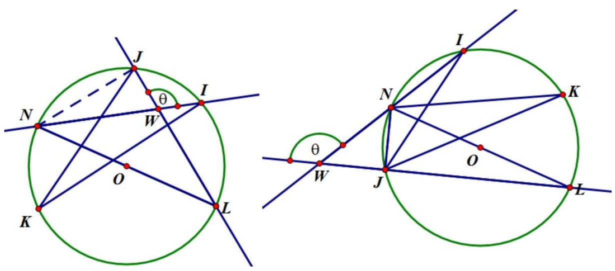

Figure 14: Zoom-in

Lemma 11 *Let $ABCD$ be a given r-inscribable convex quadrilateral and the points W, I and J as defined above. Then for any outbox $PQRS$ with centroid K, we have $\angle JKI = \angle AWB - \frac{\pi}{2}$.*

Theorem 12 *Let $ABCD$ be an r-inscribable convex quadrilateral whose diagonals AC and BD are of length d_1 and d_2 respectively, and make an angle of $\angle AWB := \vartheta$ (where $\frac{\pi}{2} < \vartheta < \pi$). Then, the maximal outbox of $ABCD$, if it exists, has area $\frac{1}{2}d_1 d_2(1+\sin\vartheta)$.*

Proof. From Lemma 8, we have $\frac{JK}{VK} = \frac{LN}{LC}$ and $\frac{IK}{XK} = \frac{LN}{DN}$. Because $LC = \frac{1}{2}d_1$ and $DN = \frac{1}{2}d_2$, we have $VK = \frac{JK}{LN}\cdot\frac{1}{2}d_1$ and $XK = \frac{IK}{LN}\cdot\frac{1}{2}d_2$. Denoting the radius of the circle Λ by r , we have $LN = 2r = 2\cdot OK$. When K represents the centroid of the maximal outbox, K is the furthest point on the circle Λ away from IJ by Corollary 10. Thus, by Lemma 8, $\angle OKJ = \angle IKO = \frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$. So, at this position where K is the centroid of the maximal outbox, $VK = \frac{1}{2}d_1 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$ and $XK = \frac{1}{2}d_2 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$. Finally, by the sine rule and the double angle formula, the area of the maximal outbox, if it exists, is given by $4 \cdot \frac{1}{2}d_1 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right) \cdot \frac{1}{2}d_2 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right) = \frac{1}{2}d_1 d_2(1 + \sin\vartheta)$. ■

Remark 13 *The formula for the area of the maximal outbox derived in [7] by D. Zhao was* $d_1 d_2 \left|\cos\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)\sin\left(\frac{\pi}{4} + \frac{\vartheta}{2}\right)\right|$, *which is equivalent to ours via the factor formula.*

5 Existence of maximal outbox

We now turn to characterizing those r-inscribable convex quadrilaterals that admit maximal outboxes. By Proposition 3 and Corollary 10, it suffices to find the necessary and sufficient condition for the inequalities $\max\{\beta \dot{-} \frac{\pi}{2}, \beta + \gamma \dot{-} \pi\} < \angle QAB < \min\{\pi - \alpha, \hat{\beta}, \phi\}$ to hold when K_Q is the furthest point on Λ from the chord IJ . For this purpose, it is important to

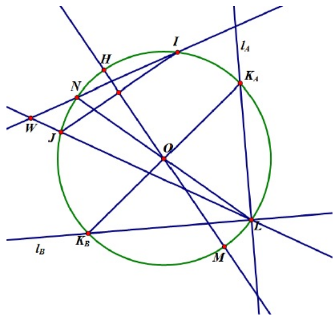

Figure 15: Analysis of the size of $\angle K_Q K_A K_B$

relate the size of $\angle K_Q K_A K_B$ (which is equal to $\angle QAB$ in size) with the geometrical structure of $ABCD$. Let H be the diametrically opposite of M with respect to O (see Figure 5). Since H and M are diametrically opposites (respectively, K_A and K_B), it follows that the chords HK_A and MK_B are of the same length. Thus, the angles subtended by these chords are equal in size, i.e., $\angle MK_AK_B = \angle HMK_A$. But we have $\angle HMK_A = \angle HMI + \angle IMK_A$. Now, $\angle HMI = \frac{1}{2}\angle JMI$, and $\angle JMI = \angle JLI = \vartheta - \frac{\pi}{2}$. Also, $\angle IMK_A = \angle ILK_A = \angle DBA$. Thus, $\angle HMKA = \frac{1}{2}(\vartheta - \frac{\pi}{2}) + \angle DBA = \frac{1}{2}(\vartheta - \frac{\pi}{2}) + \pi - \angle BAC - \vartheta = \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC$.

Hence $ABCD$ has a maximal outbox if and only if each of the six inequalities are satisfied for the acute angle $\angle HMK_A$:

$$\max\{\beta \dot{-} \frac{\pi}{2}, \beta + \gamma \dot{-} \pi\} < \angle QAB = \angle HMK_A < \min\{\pi - \alpha, \hat{\beta}, \phi\}.$$

These inequalities can be presented as follows:

1. $0 < \angle HMK_A < \frac{\pi}{2}$. This is equivalent to $0 < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{\pi}{2} \iff \frac{\pi}{4} - \frac{1}{2}\vartheta < \angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$. But $\vartheta \geqslant \frac{\pi}{2}$ so that $\frac{1}{2}\vartheta \geqslant \frac{\pi}{4}$. Since $\angle BAC > 0$, it follows that $\angle BAC > \frac{\pi}{4} - \frac{1}{2}\vartheta$ is already satisfied. Thus, $\angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$.

2. $\beta \dot{-} \frac{\pi}{2} < \angle HMK_A$.

   (i) If $\beta \geqslant \frac{\pi}{2}$, we have $\beta - \frac{\pi}{2} < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC$ if and only if

$$\angle BAC + \beta - \pi + \frac{1}{2}\vartheta < \frac{\pi}{4} \iff \frac{1}{2}\vartheta - \angle ACB < \frac{\pi}{4} \iff \angle ACB \geqslant \frac{1}{2}\vartheta - \frac{\pi}{4}.$$

(ii) If $\beta \leqslant \frac{\pi}{2}$, we have $0 < \frac{3\pi}{4} - \frac{1}{2}\theta - \angle BAC \iff \angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to (1). Notice also that if this condition holds, then one also has $\angle ACB > \pi - \beta - \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle ACB > \frac{1}{2}\vartheta - \frac{\pi}{4} + \left(\frac{\pi}{2} - \beta\right) \geqslant \frac{1}{2}\vartheta - \frac{\pi}{4}$ so that the inequality in (2)(i) is also true.

3. $\beta + \gamma \dot{-} \pi < \angle HMK_A$.

(i) If $\beta + \gamma > \pi$, we have $\beta + \gamma - \pi < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC$ if and only if

$$\angle BAC + \beta - \pi + \gamma < \frac{3\pi}{4} - \frac{1}{2}\vartheta \iff \angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta.$$

(ii) If $\beta + \gamma \leqslant \pi$, we have $0 < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC \iff \angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to (1). Furthermore, this condition also implies that

$$\pi + \angle DCA - \gamma - \beta < \frac{3\pi}{4} - \frac{1}{2}\vartheta \iff \angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta - (\pi - \gamma - \beta)$$

which implies that $\angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, i.e., the inequality in (3)(i) also holds.

4. $\angle HMK_A < \pi - \alpha$. We have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \pi - \alpha$ if and only if

$$\alpha - \angle BAC < \frac{\pi}{4} + \frac{1}{2}\vartheta \iff \angle DAC < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

Since $\angle DAC + \angle ADB = \vartheta$, the preceding inequality is equivalent to $\angle ADB > \frac{1}{2}\vartheta - \frac{\pi}{4}$.

5. $\angle HMK_A < \hat{\beta}$.

(i) If $\beta < \frac{\pi}{2}$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \beta$ if and only if

$$\frac{3\pi}{4} - \frac{1}{2}\vartheta < \pi - \angle ACB \iff \angle ACB < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

(ii) If $\beta \geqslant \frac{\pi}{4}$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{\pi}{2} \iff \angle BAC > \frac{\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to 1(i). Moreover, this inequality also implies 5(i) because

$$\angle ACB < \pi - \beta - \frac{\pi}{4} + \frac{1}{2}\vartheta \implies \angle ACB < \frac{\pi}{4} + \frac{1}{2}\vartheta - \left(\beta - \frac{\pi}{2}\right) < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

6. $\angle HMK_A < \phi$.

(i) If $\alpha + \delta > \pi$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{3\pi}{2} - \alpha - \delta$ if and only if

$$\alpha + \delta - \angle BAC < \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle DAC + \delta < \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle BDC < \frac{3\pi}{4} - \frac{1}{2}\vartheta.$$

(ii) If $\alpha + \delta \leqslant \pi$, then $\frac{\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{\pi}{2} \iff \angle BAC > \frac{\pi}{4} - \frac{1}{2}\vartheta$, which is just (1). Also, one has $\angle BDC < \pi - \left(\frac{\pi}{4} - \frac{1}{2}\vartheta\right) - \angle DAC - \angle ADB \implies \angle BDC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$ since $\angle DAC + \angle ADB = \pi - \vartheta$. So, the inequality in 6(i) holds.

Note that since $\vartheta \geqslant \frac{\pi}{2}$ holds and $\angle DCB > 0$, it holds that $\angle DCB > 0 > \frac{\pi}{4} - \frac{1}{2}\vartheta$. So, $\angle DCB + \angle DCA + \vartheta = \pi$ then guarantees that $\angle DCA = \pi - \vartheta - \angle DCB$, and thus $\angle DCA < \pi - \vartheta - \frac{\pi}{4} + \frac{1}{2}\vartheta$ if and only if $\angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta$. Similarly, since $\angle DCA > 0$, it holds that $\angle DCA > 0 > \frac{\pi}{4} - \frac{1}{2}\vartheta$. Hence $\angle DCB + \angle DCA + \vartheta = \pi$ so that $\angle DCB < \pi - \vartheta - \frac{\pi}{4} + \frac{1}{2}\vartheta$ if and only if $\angle DCB < \frac{3\pi}{4} - \frac{1}{2}\vartheta$. Thus, the configuration that $\vartheta \geqslant \frac{\pi}{2}$ we assume guarantees that the inequalities in 3(i) and 6(i) to hold automatically. All in all, we have a new proof for:

Theorem 14 *([4, Theorem 4]) An r-inscribable quadrilateral ABCD has a maximal outbox if and only if $\vartheta = \frac{\pi}{2}$ or obtuse and the following inequalities are simultaneously satisfied:*

$$\angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta,\ \angle ACB > \frac{1}{2}\vartheta - \frac{\pi}{4},\ \angle ADB > \frac{1}{2}\vartheta - \frac{\pi}{4},\ \&\ \angle ACB < \frac{1}{2}\vartheta + \frac{\pi}{4} \tag{3}$$

6 Conclusion

The maximal outbox problem can be seen as a generalization of the maximal 'out-triangle' problem. The older out-triangle problem was proposed, studied and solved completely in [5], and again independently in [2]. For any given triangle $\mathcal{T}$ the set F of equilateral triangles circumscribed to $\mathcal{T}$ is non-empty. Furthermore, if A, B and C are vertices of the triangle $\mathcal{T}$, such that $AB \geqslant AC \geqslant BC$, among the triangles of the set F there exists one of maximum area (i.e., a maximal 'out-triangle') if and only if the median of the side AB with the side BC forms an angle smaller than $\frac{5\pi}{6}$. It is natural to guess that a similar kind of centroid theorem exists for the case of triangles (or even more generally any convex polygon), and can thus yield an alternative proof of the aforementioned result.

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