Rhombohedra Everywhere

Jen-chung Chuan
jcchuan@gmail.com
National Tsing Hua University, Taiwan

Abstract

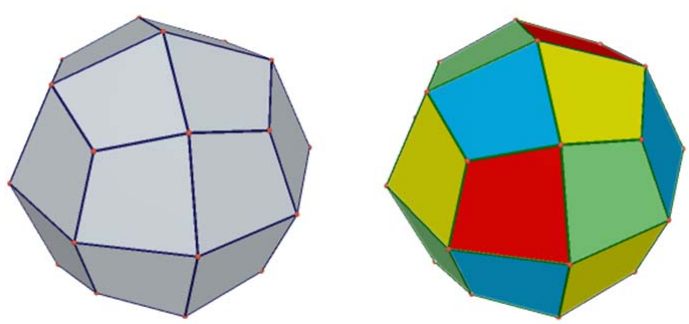
A rhombohedron is a 6-face polyhedron in which all faces are rhombi. The cube is the best-known example of the rhombohedron. We intend to show that other less-known rhombohedra are also abundant. We are to show how the rhombohedra appear in the algorithm of rhombic polyhedral dissections, in designing 3D linkages and in supplying concrete examples in mathematics amusement.

A tongue-twisting jargon: in case all six rhombic faces are congruent, the rhombohedron is known as a trigonal trapezohedron.

1. Minimum Covering for the Deltoidal Icositetrahedron

Why choosing deltoidal icositetrahedron?

It is known that every convex polyhedron is the intersection of the half spaces defined by the supporting planes. It is not straightforward to find one such 6n-face polyhedron so the supporting planes enclose n rhombohedra exactly. It turns out that the deltoidal icositetrahedron, having 24 congruent "kites" as its faces, fits such requirement. The coloring scheme below shows the possibility of painting the faces with four colors so each edge-adjacent faces shall receive distinct colors:



Figures 1(a) and (b)

It turns out that each fixed color, say the blue, occupies exactly three pairs of parallel faces. The six supporting planes enclosed the red-edge rhombohedron. This together with green-, yellow- red-edge rhombohedra cover all faces.

The rich symmetries possessed by the deltoidal icositetrahedron enables us to visualize the interplay between the dynamic geometry and combinatorial algorithm.

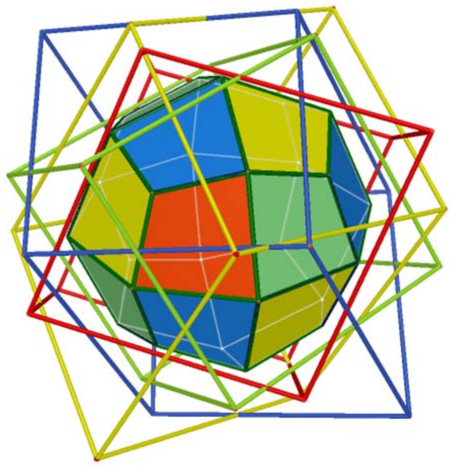
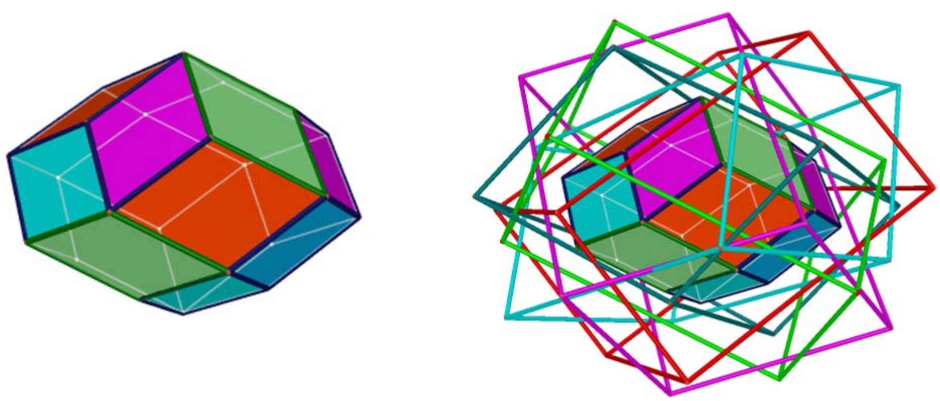


Figure 2

2. Minimum Covering for the Rhombic Icosahedron

Unlike the deltoidal icositetrahedron, no choice for any collection of six faces of the rhombic icosahedron can have non-overlapping vertices. A rhombohedron therefore can have at most two pairs of opposite faces supporting the rhombic icosahedron. Therefore, it takes a minimum of six rhombohedra to form a minimum covering.



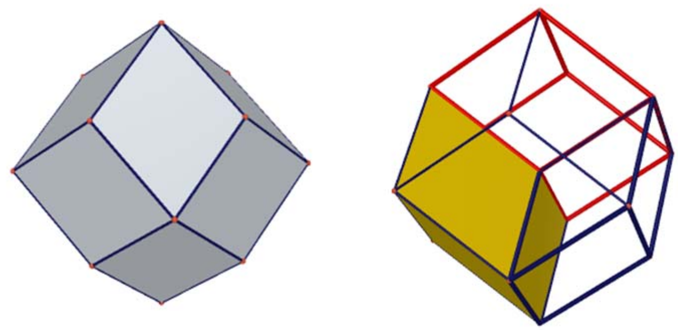
Figures 3(a) and (b)

3. 3D Jigsaw Puzzle with Rhombic Dodecahedron

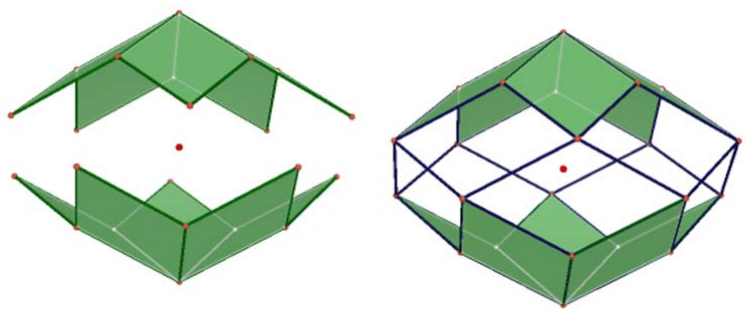
The problems of geometric dissection have been tough nuts to crack. The Dudeney Dissection [2] offers an example where the amateur mathematician can solve hard problems. Here is a much simple problem of this sort: dissect a rhombic dodecahedron into four congruent rhombohedra.

4. Assembly of Two Non-Congruent Rhombohedra to Form a Rhombic Icosahedron

The rhombic icosahedron can be derived from the rhombic triacontahedron by taking the convex hull of the top five faces and the translation by the vertical length of the bottom five faces:



Figures 4(a) and (b)



Figures 5(a) and (b)

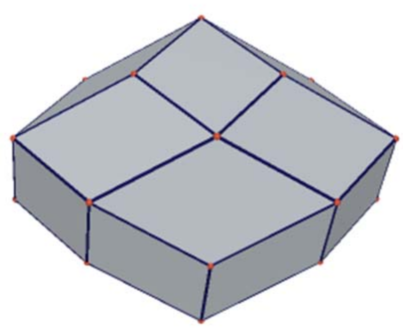


Figure 6

The rhombic icosahedron can be dissected into ten rhombohedra: among them are flat

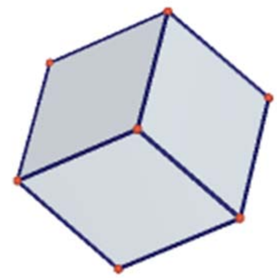


Figure 7

and the other five are rounded

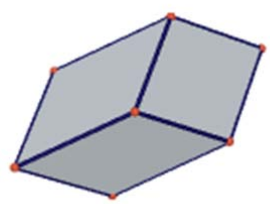
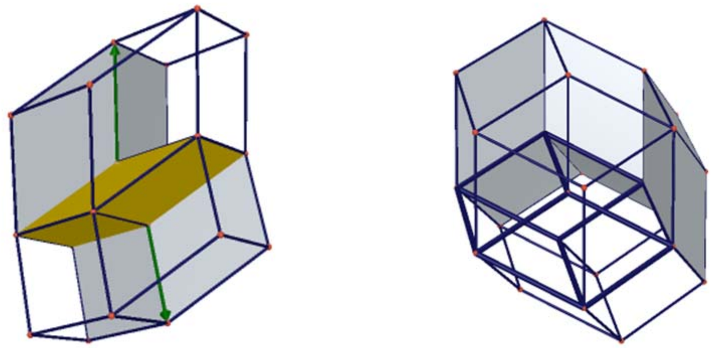


Figure 8

Borrowing from the language of cell division, we may speak of the formation process of normal growth and of abnormal growth.

Normal Growth: It begins with the flat “core”, the only block hidden from the view after all others have added, and then the two congruent flat ones (the grey portion) and then the four congruent rounded ones (the empty portion). Each of the six rhombohedra having a face in common with the core.

The normal growth completes when the last three, one rounded and two flat are added. Each of the last three has an edge in common with the core.



Figures 9(a) and (b)

Abnormal Growth: The abnormal growth begins with five rounded ones, after which the pocket to build the core becomes visible.

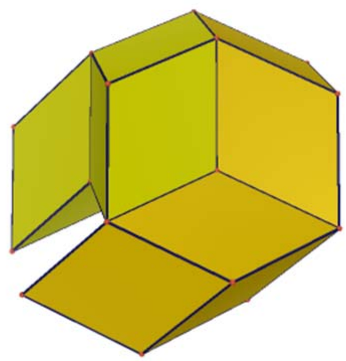


Figure 10

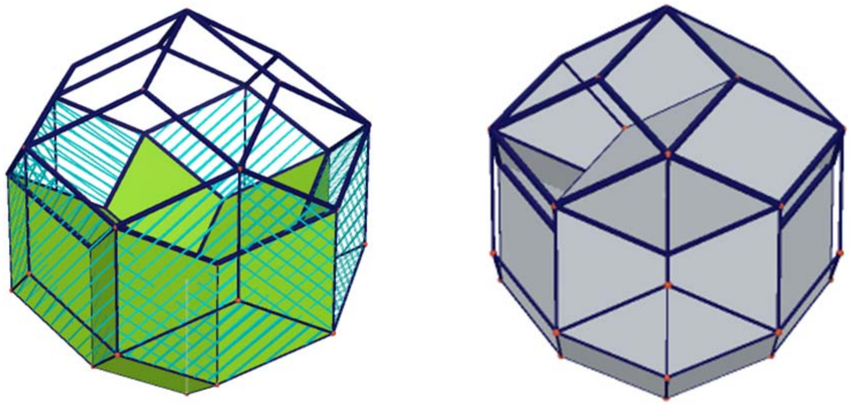
The other four now follow from the core by taking plane reflections successively.

The notion of Greedy Algorithm was never in the mind in classical geometric construction. The process of formatting the rhombic icosahedron by juxtaposition of rhombohedra appears a promising approach to solving this kind of geometric dissection problem.

5. Dissections of Rhombic Triacontahedron

Primary Dissection: Dissect the rhombic triacontahedron into two parts: the part occupied by rhombic icosahedron which is uniquely determined by the top 5 faces of the rhombic triacontahedron and the part of rhombic triacontahedron outside the rhombic icosahedron.

Secondary Dissection; The rhombic icosahedron is dissected into 10 rhombohedra as in D. The portion of rhombic triacontahedron outside the rhombic icosahedron can now be dissected into five round ones each having a face in common with rhombic icosahedron, creating five holes to fit the flat ones.



Figures 11(a) and (b)

There are altogether a total of 20 Rhombohedra in the completed dissections.

6. Dissections of Rhombic Enneacontahedron

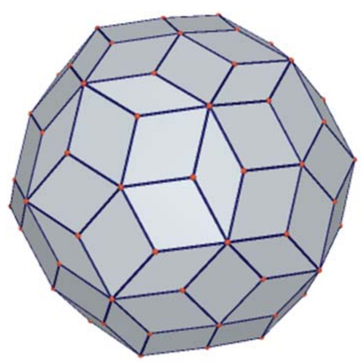


Figure 12

Instead of the Greedy Algorithm, the method of Divide and Conquer is employed here.

Primary Dissection:

1. Two rhombic icosahedrons

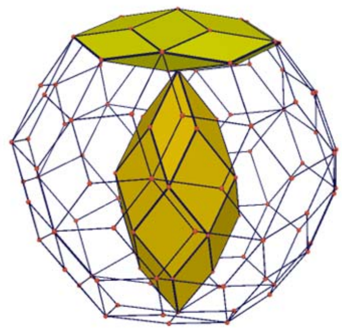


Figure 13

2. Twenty rhombic dodecahedrons

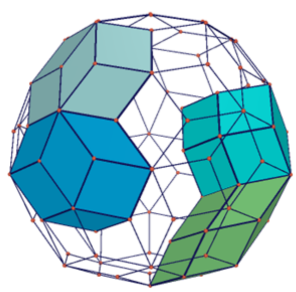


Figure 14

3. Twenty rhombohedra

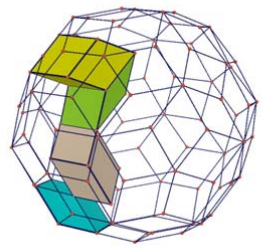


Figure 15

Secondary dissections:

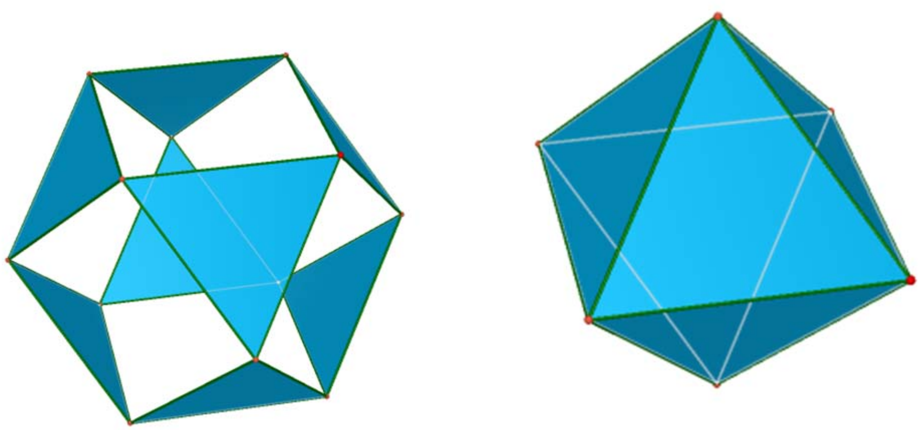
1. Each of the rhombic icosahedrons can be further dissected into ten rhombohedra.

2. Each of the rhombic dodecahedrons can be further dissected into four rhombohedra.

There are altogether a total of 120 rhombohedra in the completed dissections. The illustration of all 120 rhombohedra is not feasible on the printed media.

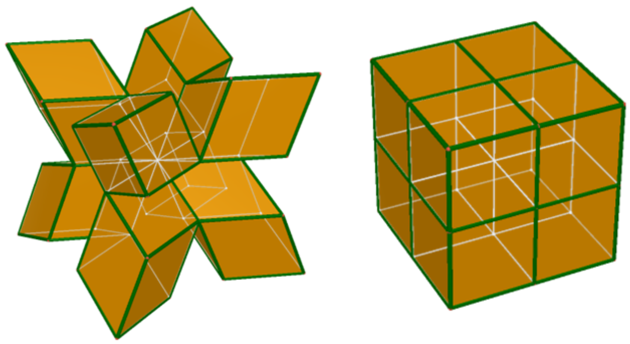
7. Variation of the Jitterbug

An animation of a conformal polyhedral linkage known as the Buckminster Fuller's Jitterbug can be found in YouTube: https://www.youtube.com/watch?v=FfViCWntbDQ



Figures 16 (a) and (b)

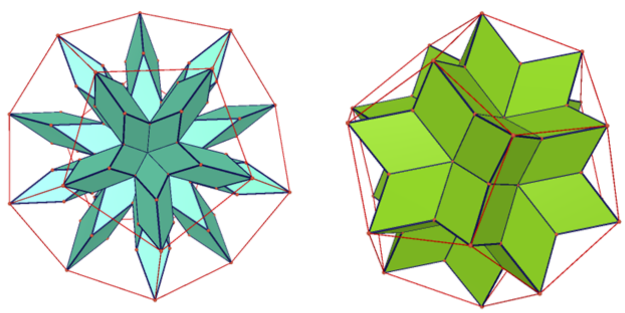
In the late 19th Century, the study of linkages [3], such as the designing of a mechanical device that transfers the circular motion to follow a straight line, occupied the attention of the best mathematical minds. Thanks to dynamic geometry, the explorations of the 3D linkage have never been made easier. The skeletal rhombohedra would serve as the spatial counterpart of the traditional 4-bar linkage in the shape of a rhombus. The motion of the Jitterbug inspires the designing of the spatial linkage consisting of eight skeletal rhombohedra as basic components.



Figures 17 (a) and (b)

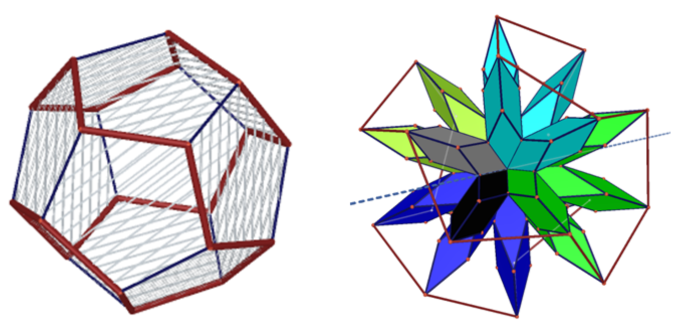
8. Connecting Two Entirely Different 20 Rhombohedral Complexes

There are two interesting 3D models formed by 20 rhombohedra, the 120-face crystal and the rhombic hexecontahedron:



Figures 18(a) and (b)

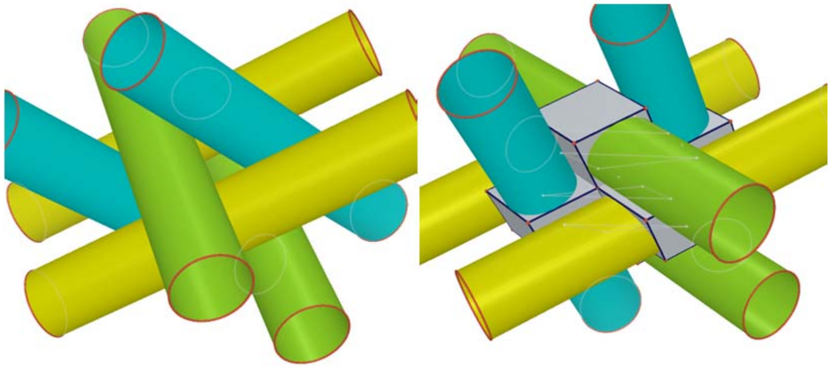
The two static models could be “connected” with 20 skeletal rhombohedral links following a Hamiltonian cycle on the 20 vertices of the dodecahedron.



Figures 19(a) and (b)

The idea behind the designing of such Hamilton-cycle linkage is taken from the mathematical game “icosian” invented in 1857 by William Rowan Hamilton. The study of polyhedrons marks the beginning of Graph Theory, which in turn, offers stimulating ideas in dynamic geometry.

I. Invisible Framework to Build Six Kissing Tubes [1]



Figures 20(a) and (b)

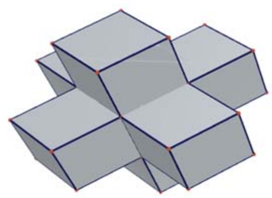


Figure 21

You are cordially invited to attend the ATCM 2016 workshop "An Animation of Six Identical Cylinders Each Touching Exactly Four Others" to find out how the above static figures become movable.

References:

[1] Stewart T. Coffin, The Puzzling World of Polyhedral Dissections, http://www.johnrausch.com/PuzzlingWorld/

[2] Henry Dudley, The Canterbury Puzzles and Other Curious Problems, 1907, https://archive.org/details/117770747

[3] A. B. Kampe, How to Draw a Straight Line; a lecture on linkages, 1922, https://archive.org/details/howtodrawstraigh00kemprich