Hexagrammums

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Abstract: *In this paper we consider hexagrammums, plane geometric configurations based on six points. Pairs of given points form straight lines. Points of intersection of these lines form the new daughterly points. We get the net configuration of points and lines. The work is based on firstly, delightful drawings of Hirotaka Ebisui* [1], *and secondly, the GInMA software* [2], *which makes it easy to explore the geometric configurations and exercise their conversions. Typically the only one unusual point found by Ebisui has served the source for the study. As a result, the net has been found. In each case, the method of barycentric coordinates has been used for the formal proof. The evident solutions have been found in many cases with the use of collineations. This may introduce solutions of problems into standard courses of geometry. All the pictures in the paper are interactive. So they come to life, install on your computer GInMA software from the website* [2]. *Free basic version will allow you to get acquainted with all the materials, to conduct a study, to create your file, but not to keep it. See the video* How to convert pictures from the text in interactive drawings

1. Introduction

Straight lines and points are the simplest geometry objects. We arbitrarily take several basic points and the rule by which we find the daughterly points using straight lines. As a rule, only two lines intersect at one point when the reference points have an arbitrary position, and only two daughterly points lie on one straight line. Therefore, it is unexpected and very nice to observe the situation, when three, seven or nine straight lines intersect at one point or triples (or 4, or even 5) points belong to one straight line.

In this paper the plane configuration of 6 points is given. Points may have general position or some restrictions may be. Let the plane configuration generated by six points A, B, C, A', B', C' be given, ***m*** of which are selected arbitrarily and ***n*** satisfy a certain condition, that is, they have some connection with the first ***m*** points. We name this construction the ***m* + *n*** hexagrammum. We group the points into two triangles ABC and $A'B'C'$. For example, in the **5 + 1** hexagrammum, we have five arbitrary points A, B, C, A', B' of general position and one point C' on the straight line defined by the previous five points. For example, in the Pappus hexagram triples of points lie on two straight lines.

To facilitate our discoveries and calculations, we have used the dynamic geometry software *GInMA.*

2. Parallelogram Hexagrammum 4 + 2

Given points A, B, C and A' in general position. The points B' and C' lie on straight lines, passing through A' parallel to AB and AC, respectively. The straight line, which is parallel to AB and passing through $D = BC\cap B'C'$ intersects AC and $A'C'$ at the points H and G, respectively,

$D_0 = BB'\cap CC'$, $E_0 = AA'\cap BB'$, $F_0 = AA'\cap CC'$, $G' = BC\cap GF_0$, $H' = B'C'\cap F_0H$, $K = BC\cap GD_0$,
$K' = B'C'\cap HD_0$, $L = AB\cap GE_0$, $L' = A'B'\cap HE_0$.

In this case the points E_0, G', H', and K, K', L, L' lie on straight lines, parallel to AC.

Proof. We use the coordinate system with the origin at the point $A(0,0)$, the abscissa axis along AC and the ordinate axis along AB. We denote the points $C(x_1,0)$, $B(0, y_1)$, $B'(x_2, y_2)$, $C'(x_3, y_3)$. Then

$$E=(x_2,y_3)\frac{y_1}{y_3+y_1-y_2},\quad F=(x_2,y_3)\frac{x_1}{x_1+x_2-x_3},\quad H'=\left(x_2+(y_1-y_2)\frac{x_3}{y_3},y_1\right)\frac{y_3}{y_3+y_1-y_2},$$

$$G'=\left(\frac{(y_1-y_2)}{y_3}x_1,y_1\right)\frac{y_3}{y_3+y_1-y_2}.$$ The ordinates of the points E_0, G', H' are equal.

Similarly, we find that the ordinates of points K, K', L are L' equal.

2.1. Hexagrammum 6 + 0

Given six points A, B, C, A', B' and C' in general position. The points D, E, F, D_0, F_0, E_0 are marked. The straight line DE intersects AC and $A'C'$ at the points H and G, respectively. We denote the points $G' = BC \cap GF_0$, $H' = B'C' \cap F_0H$, $K = BC \cap GD_0$, $K' = B'C' \cap HD_0$, $L = AB \cap GE_0$, $L' = A'B' \cap HE_0$.

In this case points of the tetrad E_0, F, G', H' are collinear. Also, five points F, K, K', L, L' are collinear.

Proof. We use the collineation with EF as the special straight line of a collineation and reduce the problem to the previous problem.

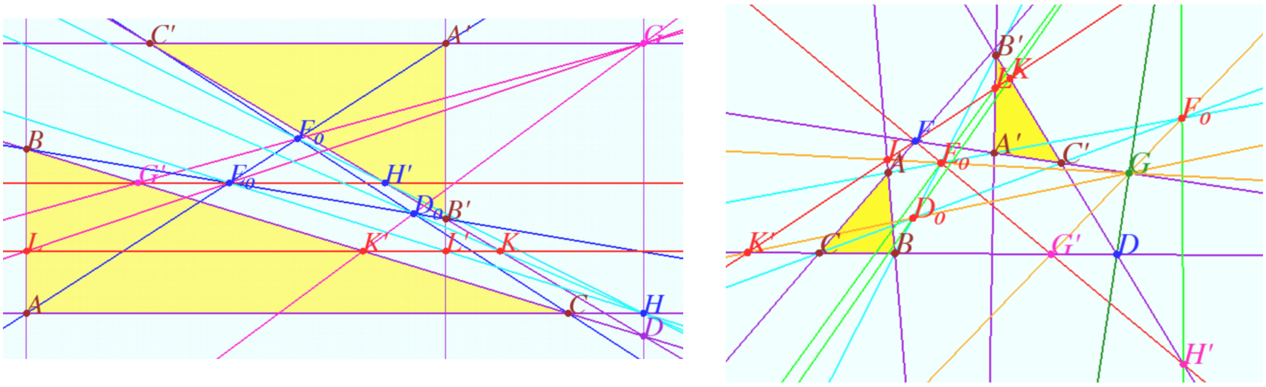

Figure 2.1 a) 4+2 parallelogram Hexagrammum, b) 6+0 Hexagrammum

2.2. Hexagrammum Ebisui 6 + 0

Given six arbitrary points A, B, C, D, E and F in general position. The straight lines AD, BE and CF intersect at the points 1, 2, 3. The straight lines $F1$ and $B2$ intersect at the point 4, $5 = C1 \cap E2$, $6 = B2 \cap D3$, $7 = A3 \cap E2$, $8 = D3 \cap F1$, $9 = A3 \cap C1$. The straight lines 45 and 89 intersect at the point $\boldsymbol{K}$, 45 and 67 intersect at the point $\boldsymbol{L}$, 67 and 89 intersect at the point $\boldsymbol{M}$.

$K' = DM \cap LE$, $K'' = AM \cap BL$, $L' = AM \cap FK$, $L'' = CK \cap DM$, $M' = BL \cap CK$, $M'' = EL \cap FK$.

In this case tetrads of points K, K', K'' and 1, L, L', L'' and 2, M, M', M'' and 3 are collinear.

If $ABCDEF$ is a quadric, these straight lines are concurrent.

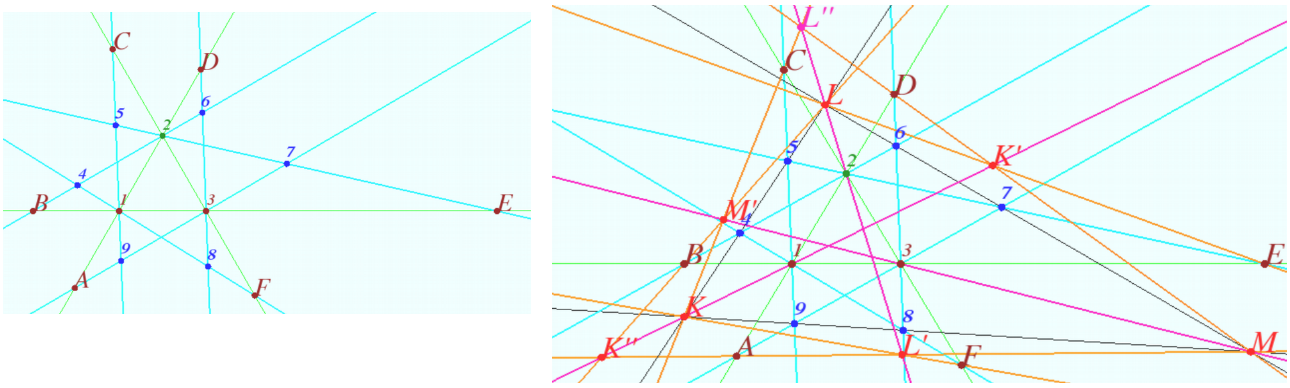

Figure 2.3 a) The base of Ebisui Hexagrammum, b) tetrads of points $KK'K''1$, $LL'L''2$, $MM'M''3$

2.2.1. According to the conditions of the previous problem

$A' = KF \cap A3$, $A'' = BL \cap A3$, $B' = AM \cap B2$, $B'' = CK \cap B2$, $C' = DM \cap C1$, $C'' = BL \cap C1$,

$D' = CK \cap D3$, $D'' = EL \cap D3$, $E' = DM \cap E2$, $E'' = FK \cap E2$, $F' = AM \cap F1$, $F'' = EL \cap F1$.

$P = KD \cap 25$, $P' = DL \cap 57$, $P'' = AK \cap 14$, $P''' = AL \cap 24$.

In this case tetrads of points A', D', L and 1; A'', D'', K and 2; B', E', K and 3; B'', E'', M and 1; C', F', L and 3; C'', F'', M and 2; as well as five points P, P', P'', P''' and 3 are collinear.

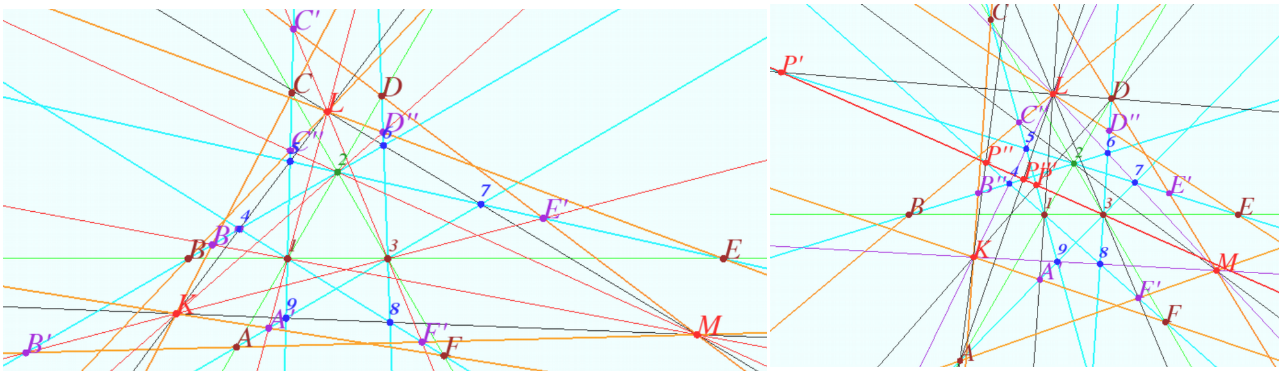

Figure 2.4 a) The tetrad points lines (red), b) the five points line (red)

Proof. We use the collineation with 1-2, 1-3 or 2-3 as the special straight lines and reduce the problem to the previous problem (see Figure 2.5, point 3 in right-left infinity, point 1 is in up and down infinity).

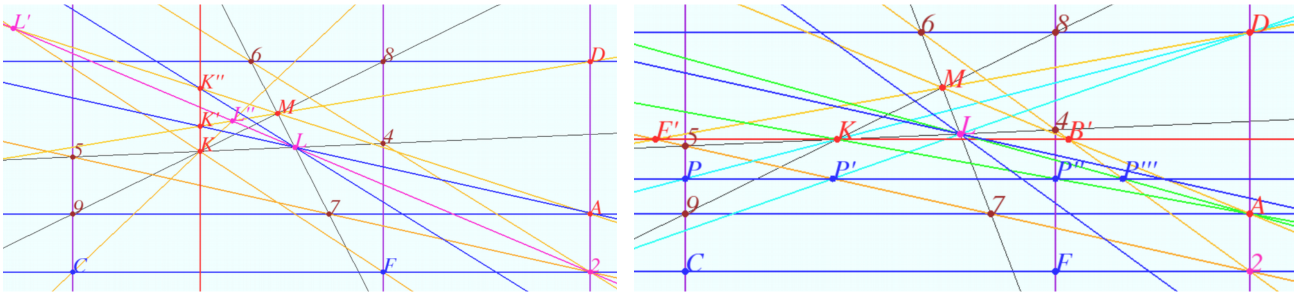

Figure 2.5 The images of the lines of Figure 2.4

3. Hexagrammum Ebisui 4 + 2

Given four arbitrary points *A, B, C,* and *B′* in general position. The points *A′* and *C′* lie on the straight line *AC*. Let *M* (*M′*) be a midpoint of *AC* (*A′C′*). The straight line, which is parallel to *AC* and passing through *E* intersects *B′M′, BM* and *B′C′* at the points *K, L* and *G,* respectively. The straight line, which is parallel to *AC* and passing through *D* intersects *BM, B′M′, AB* and *A′B′* at the points *K′, L′, P′* and *P″*, respectively. We denote the points as $K'' = C'E \cap AP$, $L'' = A'D \cap CP''$.
In this case the tetrad of points *L, L′, L″, F″*, and the tetrad of points *K, K′, K″* and *F′* are collinear.

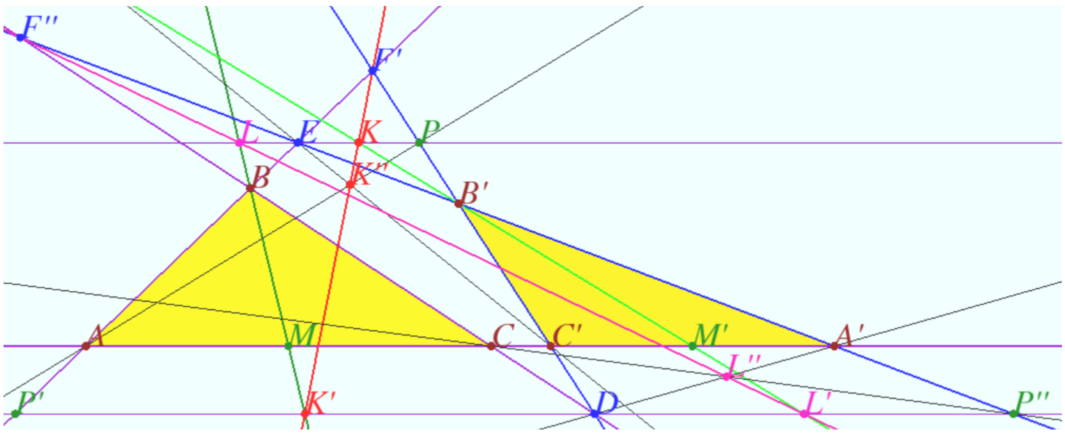

Figure 3.1 Ebisui Hexagrammum 4 + 2

Proof. The triangles *A′B′C′* and *EB′P* are similar, hence *B′M′* is a median of *EB′P*, the point *K* is a midpoint of *EP*. The triangles *ABC* and *BDP′* are similar, hence *BM* is a median of *BDP′*, the point *K′* is a midpoint of *DP′*. In the trapezium *EPDP′* the straight line *KK′* connects the midpoints of bases and contains points of intersection of lateral sides *K″* and diagonals *F′*.

Similarly we construct the proof for the straight line *LL′L″F″*.

3. 1. Hexagrammum 4 + 2, seven straight lines are concurrent

Given four arbitrary points A, B, C, and B' in general position. The points A' and C' lie on the straight line AC. We denote the points as $D = BC \cap B'C'$, $E = AB \cap A'B'$, $F' = AB \cap B'C'$, $F'' = BC \cap A'B'$, $F_1 = AB \cap B'C$, $F^1 = BC \cap AB'$, $F_3 = A'B \cap B'C'$, $F^3 = BC' \cap A'B'$, $F_2 = A'B \cap B'C$, $F^2 = AB' \cap BC'$, $E^0 = AB' \cap A'B$, $D^0 = BC' \cap B'C$.

In this case seven straight lines DD^0, EE^0, F_1F^1, F_2F'', F^2F', F_3F^3 and BB' are concurrent (at O).

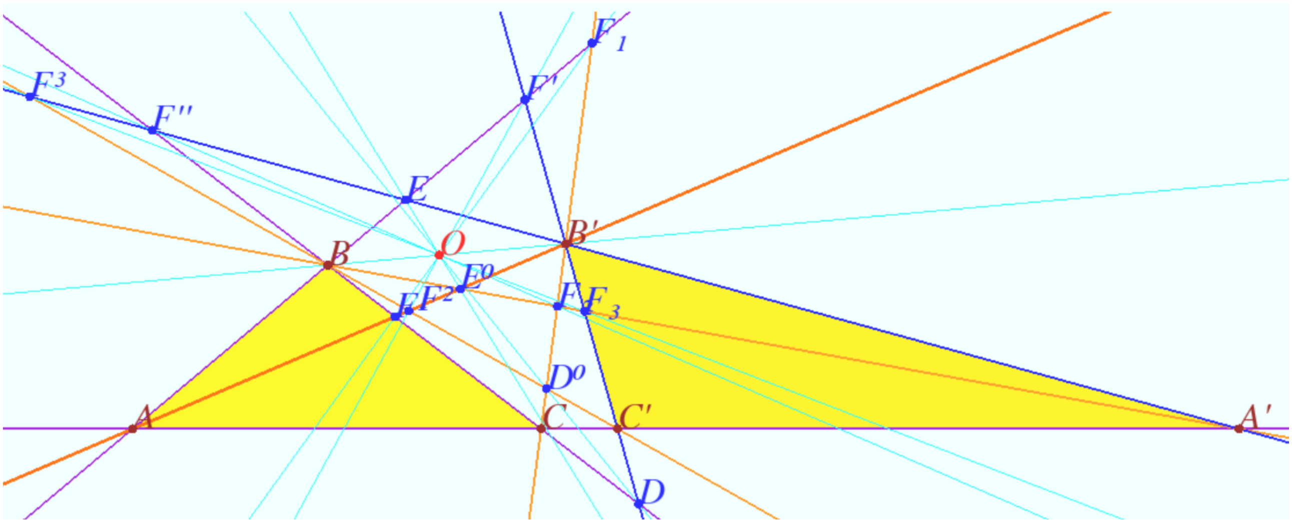

Figure 3.2 Hexagrammum 4 + 2, seven concurrent lines

Proof. We use the collineation with the special straight line containing the point of intersection of the lines AA' and BB'. These seven lines become parallel. Trapeziums arise with the bases on these lines, for example, the trapezium $ABB'C'$. The point O is mapped into the midpoint of the common base BB' of such trapeziums. For example, the trapezium $ABB'C'$ generates the straight line $F'F^2$, which passes through the point F' of intersection of lateral sides AB and $B'C'$, through the point F^2 of intersection of diagonals AB' and BC', and through the midpoint of BB'.

4. Hexagrammum Pappus 4 + 1 + 1

Given four arbitrary points A, B, A', and B' in general position. The points C and C' lies on straight lines $A'B$ and AB', respectively. We denote the points as $D = BC \cap B'C'$, $E = AB \cap A'B'$, $F = AC \cap A'C'$, $D' = AC \cap A'B'$, $D'' = AB \cap A'C'$, $E_5 = AB \cap CC'$, $E^5 = A'B' \cap CC'$, $K = BC \cap AE^5$, $K' = B'C' \cap A'E_5$, $L = AC \cap A'E_5$, $L' = A'C' \cap AE^5$.

In this case five straight lines BB', CC', $D'D''$, KK', LL' are concurrent (at the point O).

Six straight lines AA', DF, BL, $B'L'$, $D'K$, $D''K'$ are concurrent (at the point O').

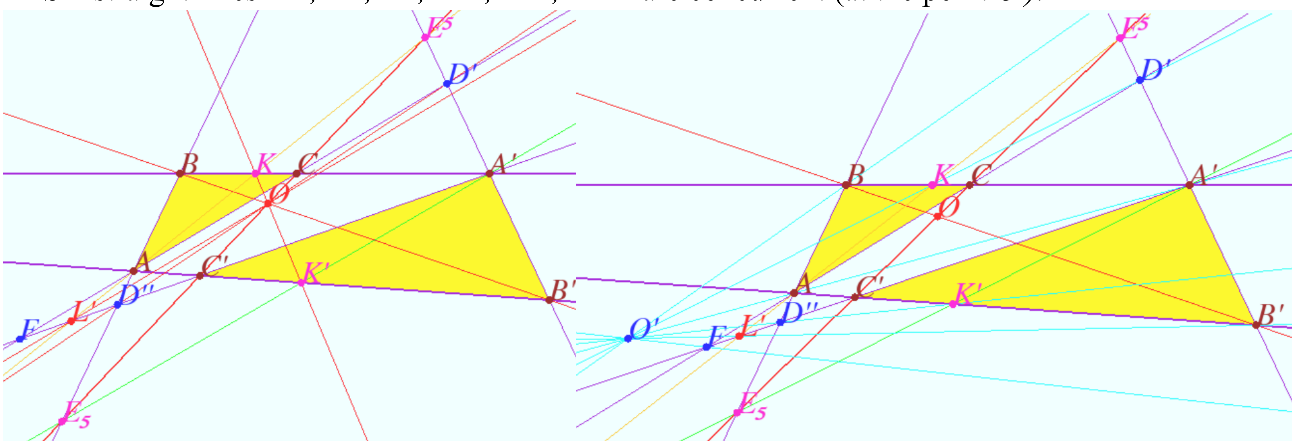

Figure 4.1 a) Hexagrammum 4 + 1+1, five concurrent lines, b) six concurrent lines

Proof. We use the collineation with the special straight line CC' in which the images of straight lines AB' and $A'B$ are perpendicular. The problem takes the following form.

Given points A and A', which define the rectangle $AFA'D$ with the horizontal side. We assume that C is the point at infinity on the horizontal, C' on the vertical line. Hence the point B lies on the horizontal $A'C$, the point B' lies on the vertical line AC'. The point F^5 lies on CC', hence $AKL'||A'B'D'$ (yellow lines).

The point F_5 lies on CC', hence $A'K'L||ABD''$ (green lines), $LD'||KD$ (blue lines contain C), $DB'||D''A'$ (violet lines contain the point C').

$KAD'A'$ is a parallelogram with the center O', then $KO' = D'O'$. Similarly, $K'O' = D''O'$. Hence, $KK'D'D''$ is a parallelogram and $D'D''||KK'$. Similarly, red lines $BB'||D'D''||KK'||LL'$ are parallel. Corresponding straight lines of the preimage intersect at the point O on the special line. The second diagonals of the parallelograms with the vertices A and O' intersect at the point O'.

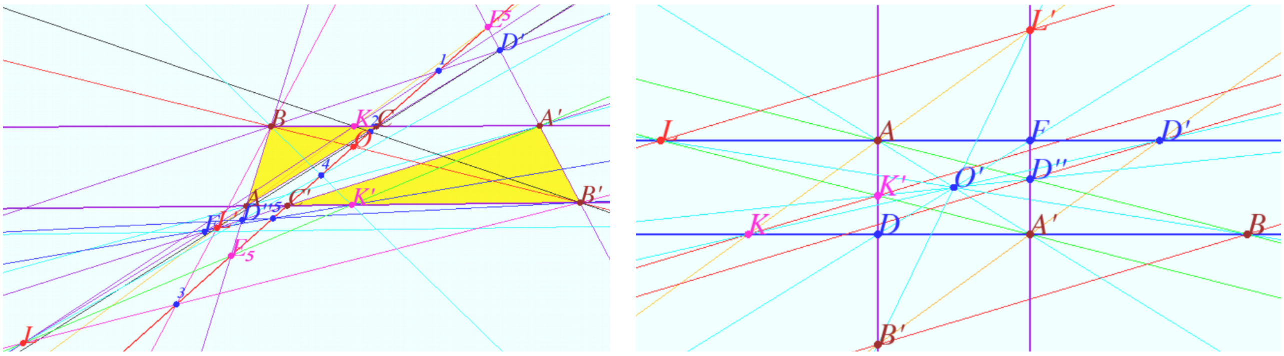

Figure 4.2 a) Hexagrammum 4 + 1+1, six concurrent lines, b) result of collineation

4. 1 Hexagrammum Pappus 4 + 1 + 1

According to the conditions of the previous problem:

1. Five pairs of straight lines BL' and $B'L$, BD' and KL, $B'D''$ and $K'L'$, KB' and $D'L'$, BK' and $D''L$ are intersecting on the straight line CC'. The images of these lines are evidently parallel in the plane of projection.

2. Three straight lines BC', $B'C$ and EF are concurrent (at the point D^0).

Proof. The statement is true because the images of the triangles $D''FE$ and BD^0E are similar.

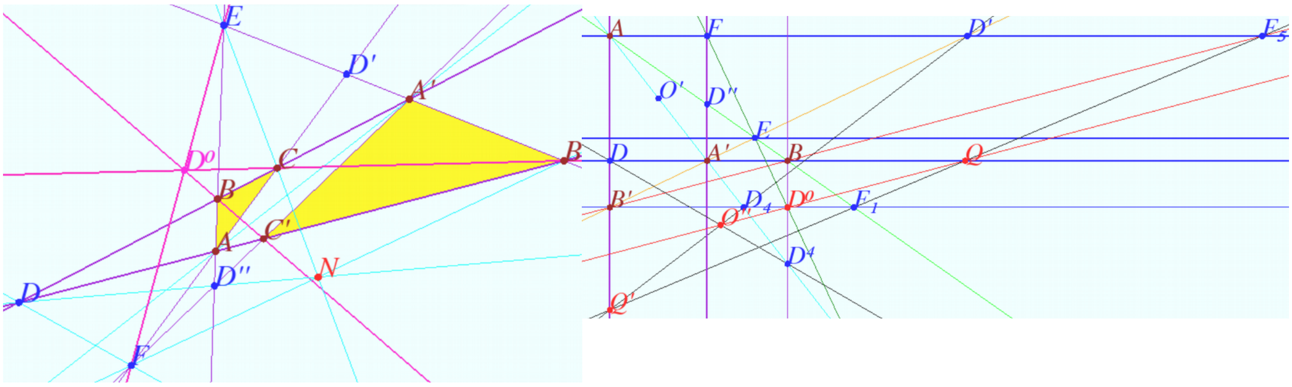

Figure 4.3 a) Three concurrent lines at D^0, b) result of collineation for the points D^0, Q, Q', O''

3. Three straight lines DD'', $B'F$ and CE are concurrent (at the point N).

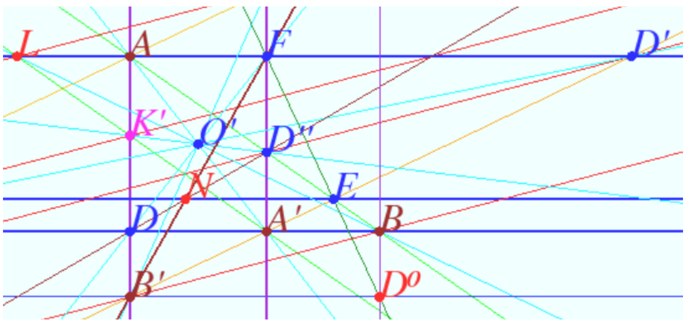

Figure 4.3 The result of collineation at the point N

Proof. The point N of the intersection of DD'' and $B'F$ divides $B'F$ in the same ratio

$$D''F : BD^0 = D''F : DB',$$

in which the point E divides FD_0, that is in the plane of images EN is parallel to $B'D^0$.

4. We denote $D_4 = AA' \cap B'C$, $D^4 = AA' \cap BC'$, $F_5 = BB' \cap AC$, $F_1 = B'C \cap AB$.
Three straight lines DD^4, $D'D_4$ and OD^0 are concurrent (at the point O'').

5. Three straight lines F_1F_5, $B'C$ and OD^0 are concurrent (at the point Q).

6. Three straight lines $D'D_4$, $B'C'$ and F_1F_5 are concurrent (at the point Q').

5. *Star Hexagrammum 3 + 3* (*8 concurrent straight lines*)

Given three arbitrary points A, B, and C in general position. The points K, L and M lie on straight lines BC, AB and AC, respectively.

We denote the points: $L_1 = LM \cap AK$, $K_1 = KM \cap CL$, $M_1 = CL \cap AK$,
$L_2 = LM \cap AK_1$, $M_2 = MB \cap AK_1$, $D_1 = BC \cap AK_1$, $K_2 = MK \cap CL_1$, $M'_2 = BM \cap CL_1$, $E_1 = AB \cap CL_1$,
$L_0 = LM \cap E_1M_1$, $K_0 = KM \cap D_1M_1$, $E_0 = LM \cap EM_2$, $D_0 = KM \cap EM_2$,
$L_3 = LM \cap AK_2$, $M_3 = MB \cap AK_2$, $D = CL \cap AK_2$, $D' = CL \cap E_0D_0$, $D_2 = BC \cap AK_2$,
$K_3 = KM \cap CL_2$, $M'_3 = AK_2 \cap CL_2$, $E = AK \cap CL_2$, $E' = AK \cap DM_2$, $E_2 = AB \cap CL_2$,
$L_4 = LM \cap AK_3$, $M_4 = MB \cap AK_3$, $D_3 = BC \cap AK_3$, $K_4 = KM \cap CL_3$, $M'_4 = MB \cap CL_3$, etc.

In this case:

– the points M_1 and M'_1, M_2 and M'_2, M_3 and M'_3,..., D and D', E and E', are coincident.
– eight straight lines BM_3, KL_2, M_1M_2, E_2D_2, LK_2, L_0D_0, K_0E_0 and L_1K_1 are concurrent (at the point Q),
– the points M_1, D_1 and E_1 are collinear,
– the points E_3, E, E_0, M_2, D_0, D, and D_3 are collinear,
– the points M_1, M_3, M_5 ,… are collinear,
– the points M_2, M_4, M_6,… are collinear at the line MB.

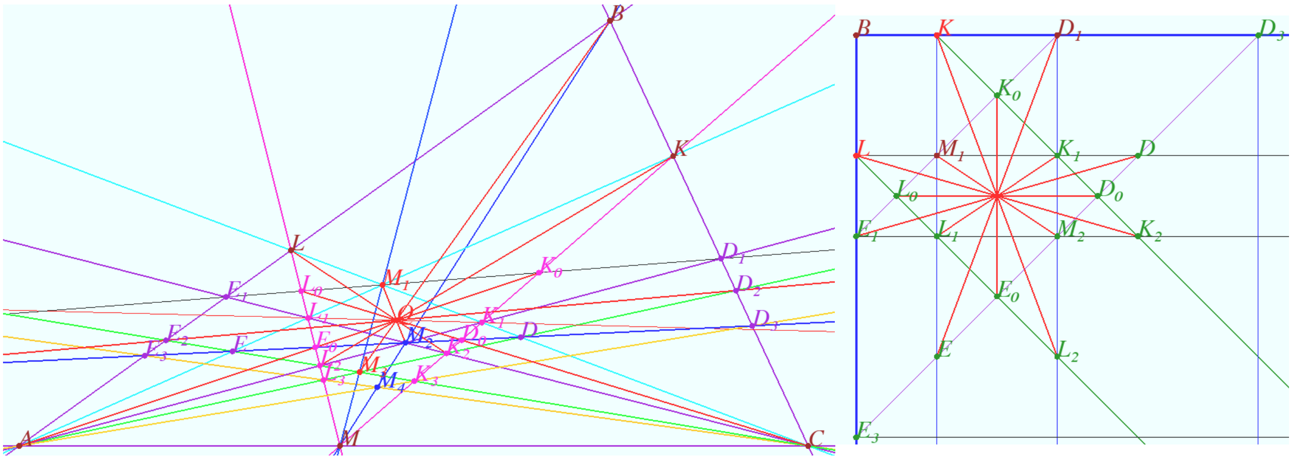

Figure 5.1 a) The star hexagrammum, b) the image of the star hexagrammum

Proof. We perform the collineation with the special straight line AC (transforming the quadrilateral $KD_1K_1M_1$ to the square). All straight lines coming out of C become horizontal. All straight lines passing through A become vertical. All straight lines passing through M are parallel to the diagonal KK_1 of the square, i.e. are angled at 45° downward to the right. Let M' be the point of intersection of AC and D_1M_1. Then all straight lines containing M' are parallel to the diagonal D_1M_1 of the square, i.e. are angled at 45 °downward to the left. All statements become trivial.

6. Conclusions. Probably, this research is not find practical application, but can deliver a certain aesthetic pleasure to the reader.

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References

[1] Hirotaka Ebisui. Collinear Note (2015).
[2] D. Shelomovskii. GInMA, 2016. http://deoma-cmd.ru/en/Products/Geometry/