Stories of Learning Trigonometry from 7^{th} graders

An Experiment of Educational Mathematics

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Abstract

This paper reports an innovative three-year mathematics teaching experiment, held in a middle school in Guangzhou, Guangdong, China. Applying a research result in educational mathematics, which rebuilds the logical structure of elementary mathematics, the teacher introduced to the 7^{th} graders the concept of the trigonometric function sinA in a new way, and used it to integrate trigonometry, geometry, and algebra. He first defined sinA as the area of a unit rhombus with an angle A, then, intuitively and rigorously, derived the fundamental properties of the function sin(A), showing the equivalence of the new definition and the traditional one. Using area computation method, he guided his students to discover the law of sine and the formula of sin(A + B). Furthermore, he introduced the concept of cosine function using the sine function, and derived the law of cosine using graphic and algebraic method. These trigonometric functions played powerful role in revealing geometric properties of triangles, polygons, and circles. The three-year experiment has greatly enhanced students' maturity in mathematics, and their ability of problem solving by combining computation, logical reasoning, and algebraic transforming. These students have performed much better than those studying using traditional textbooks in a variety of standardized tests.

1. Introduction

For more than a half century, people have paid great attention and strived hard to reform the mathematics education. For examples, the USA has experienced periods of "New Math Movement", "Go Back to the Basic", "Problem-Solving", "Constructivism", etc. However, none of these tries has achieved great successes. Zhang first proposed the idea of educational mathematics in 1989: the task of Educational Mathematics is to reorganize and recreate mathematics for the purpose of mathematics education. [Zhang1989]. He pointed out that the three books --- *Elements* (Euclid, 300BC), *Cours d'Analyse* (Cauchy, 1821), and *Elements de Mathematique* (Bourbaki, 1939-) --- are typical classics in the field of educational mathematics.

Zhang's educational mathematics research results in the areas of geometry, trigonometry, and calculus are described in [Zhang2009/2009-1/2010] and [Zhang&Peng2010]. To see more results achieved by other mathematicians in China, please refer [lin2009], [Li2010], [Zhu2009/2009-1], and [ZhangD.2011], etc. Tong and his colleagues has established the first educational mathematics graduate program in the USA.

This paper reports an innovative three-year mathematics teaching experiment, held in a middle school in Guangzhou, Guangdong, China. The second section explains the fundamental idea of the experiment. The third describes the students background and introduces the experiment design. The fourth tells the mathematics contents of key classes in each semester during the experiment. The fifth provides the assessment data, and the last section is the summarization and our prospect for the further work.

2. Fundamental Idea: Rebuild Trigonometry, Activate Whole Learning

The knowledge content of geometry, algebra, and trigonometry were formed in different eras in history and different regions in the world. They have had their own structure, terminology, and symbols. A great portion of them were not created for the purpose of teaching and learning. To include them in a unified mathematics course, we need to reorganize them.

Our strategy of the reorganization is to redefine the sine function, put it at the core, and use it as the holistic starting point to explore the whole elementary mathematics.

The word “Sine”, as well as the symbol “sin”, is a term in trigonometry and represents a trigonometric function. Hipparchus of Nicaea, a Greek astronomer in 200s BC, first introduced the term “Sine of an arc”. Given a circle and an arc, the length of the chord subtending the arc is called the sine of the arc. According to the mathematics course standards, the sine and other three trigonometric functions are introduced at the same time to the ninth graders. Given a right triangle and an acute angle, say A, the ratio of its opposite side to its adjacent side is defined the sine of the angle A, denoted sinA. Historically, this definition was first introduced in the 16^{th} century. The general definitions of trigonometric functions, created by Euler in the 18^{th} century, are introduced to high school students.

In the current mathematics curriculum, the sine is the concept with deep layers. To understand the sine of an acute angle, one must possess knowledge of similar triangles, which is why students cannot learn trigonometry if they are the 8^{th} graders or lower.

The first two years in middle school, when students first experience abstract mathematics as algebra and geometry, are crucial for the young students to develop their logical thinking habit and ability. It will be extremely beneficial to them to show them the connection and relationship of knowledge of different type and in different areas, which will stimulate their intellectual curiosity and their habit of deep thinking. The concept of trigonometric functions, especially of the sine function, is the integration of shape and numbers and the bridge linking geometry and algebra. Is it possible to to introduce the sine function to the 7^{th} graders, so that the students have opportunity to integrate geometry, algebra, and trigonometry, and to understand the power of the modern thought of functions?

To solve this problem, Zhang has discovered and published a new way of introducing the sine function, sin□, which is defined as the area of a unit rhombus with an angle □. [Zhang1980].

Elementary school students are familiar with the figure below (Figure 1), which illustrates that the area of a rectangle is equal to the product of its length and width:

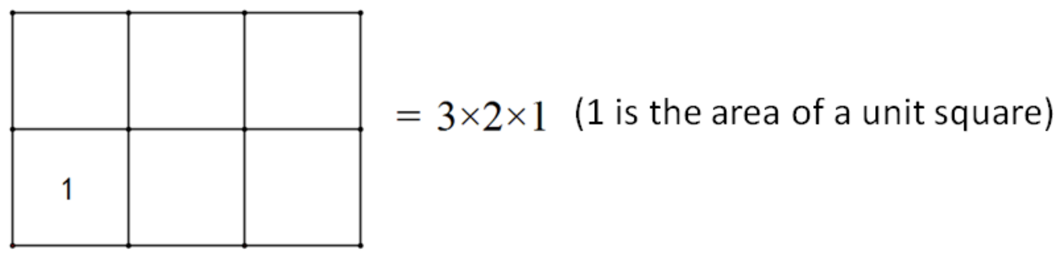

Fig.1

When the rectangle is skewed, each unit square becomes a unit rhombus, and the area formula becomes the following:

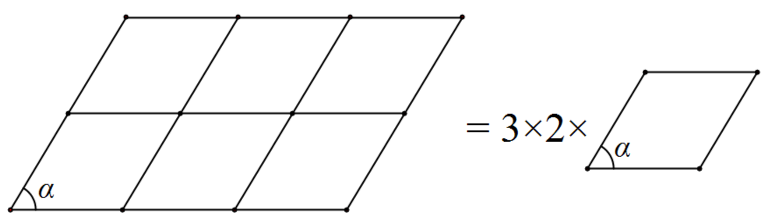

Fig.2

The right figure of the above equality represents "**the area of a unit rhombus with an angle α**", which, for simplicity, is called the "**the sine of angle α** ", denoted as sin **α**. Thus, we have a new formula of the area of a parallelogram. Its half is the well-known formula of a triangle with two sides and their included angle:

$$S_{ABC} = \frac{bc\sin A}{2} = \frac{ac\sin B}{2} = \frac{ab\sin C}{2} \tag{1}$$

This formula can yield many interesting conclusions. For example, if two angles in a triangle are equal to each other, then their opposite sides are equal to each other too. Another example: multiplying the above equality by 2 and divided by *abc*, the area formula becomes the famous law of sine. The law of sine is usually learned in high school. The high-school students have not much need to apply the law of sine. But, the 7th graders can greatly enjoy using it to solve a lot of geometric problems.

Compared to the old definition of sine, the new one is simpler, more intuitive, more rigorous (for example, $\sin 90^{o} = 1$ needs to be proved by using limit, but it is obvious using the new definition because the area of the unit square is 1), and more general (for example, the old definition is restricted to acute angles, while the new one covers all acute, right, obtuse, and straight line angle).

In 2006-2007, Zhang proposed to use unit rhombus area to define sine so that seven graders can learn trigonometry. [Zhang2006/2007]. Professor Dianzhou Zhang, a well-known scholar in the field of mathematics education in China, immediately responded enthusiastically by publishing an article titled *Let's Re-identify Trigonometry*, in which he proposed to have a teaching experiment on introducing sine to 7th graders.

Professor Dianzhou Zhang is very insightful. He wrote about this experiment in his book, titled *The Mathematics Education I personally experienced*, in 2009, "This would be a historical starting point if the trigonometry will be learned by elementary school students in the future." [ZhangD.2009].

In the book [Zhang2009], the author provided a systematical and feasible plan. This book was chosen as the source of the partial teaching material for this teaching experiment. In addition to the method of defining the sine as the unit rhombus area, this book provided an alternated way: defining the sine as the double of the area of an isosceles triangle with two unit sides. Though these two definitions are equivalent, they are of different styles. The former is more intuitive, while the later is more rigorous. We have not reached the conclusion about which one is better. Different teachers may have different preference and choices.

3. Experiment Background and Experiment Designs

Mr. Wenjun Wang was a high school teacher and a graduate student pursuing M.S. degree at the East China Normal University. His M.S. theses in 2008, titled *Initial Experiment of Teaching "Defining the Sine by Using Areas" to High School Students* [WangW2008], described the three experimental lessons (35 minutes per lesson) in the 2008 summer. In these lessons, he taught four classes of 198 freshmen and sophomores, at Furen High School in Wuxi, on using unit rhombus area to define the sine function and related topics. He analyzed the teaching effectiveness and students' comments, and he consulted more than a dozen teachers for their comments. As the summarization, he wrote in his thesis: "Overall, both students and teachers welcome the new definition. Compared to the old dry definition, the new one seems vivid and stylish. The new logic system evolves more concisely and smoothly. Therefore, it is not surprising that most students found the new definition is easier to understand."

More detailed statistical data shows that 53% of freshmen thought the new definition is easier to understand and more acceptable, 18% prefer the old definition, and 29% see no difference of difficulty in understanding both definitions. The conclusion: 82% accepted the new definition.

The author also found that 36% of sophomores prefer the new definition, 19% prefer the old definitions they learned before, and 45% see no big difference of difficulty in understanding both definitions. The conclusion: 82% accepted the new definition.

The author also saw that the sophomores were much less enthusiastic than the freshmen toward the new definition. According to the author's analysis, the sophomores are much more familiar with the traditional definitions because they had had 22 class hours more than the freshmen in applying the traditional definitions. This intimacy makes them to be less enthusiastic than the freshmen.

The author also pointed out that Ms. Caifeng Chen, a mathematics teacher at the Jiangcui High school, Taibei, Taiwan, once taught advanced students the trigonometry with defining the sine by the unit rhombus area, and her teaching was enthusiastically welcomed by her students. Unfortunately, we have not got the chance to see the related paper and/or report.

Ms. Yajing Wang, a professor of mathematics at the Qinghai Nationality College, once did the similar experiment to high school students. Her paper, titled *Study Trigonometric Functions Using the Unit Rhombus Area Formula* [WangY2008], was published on the journal *Mathematics Teaching*, No. 11, 2008.

Professor Xuefang Cui, collaborated with an experienced teacher, conducted an experiment lesson, teaching "the Sine of an Angle" to an ordinary class of 7^{th}

graders, at the end of 2007. The experiment result was summarized as a paper, titled A Teaching Exploration on Defining the Sine using Rhombus Area [Cui2008], which was also published on the journal *Mathematics Teaching*, No. 11, 2008. The author concluded, "As the experiment shows, students easily understood the concept of sine, because the new definition is more intuitive than the traditional "opposite over hypotenuse one."

Professor Cui reflected the teaching and said, defining sine using rhombus area can "lower the prerequisite, and make it easier for students to comprehend the new concept"; it "overcomes the drawback of from-abstract-to-abstract in the traditional way of teaching sine functions"; "Logic in teaching and learning evolves more smoothly. Drills of expression transforming are much less difficult. Students are greatly interested in whole process, have strong desire for the continuing learning, and are highly inspired"; "This innovative new logic structure helped students to integrate numbers and shapes and to expend their thinking space greatly in learning subsequent courses"; it "builds a thinking net connecting trigonometry, geometry, and algebra." [Cui2008]

Later, she organized a two-year experiment involving seven classes from four middle schools in Ningbo, Zhejiang. These four schools represent four different categories of students: very good, good, average, and below average in their academic aptitudes. The experiment results, teachers' comments, and experts' analysis showed that (1) The 7th graders can understand and accept the Sine definition using unit rhombus area; (2) The trigonometry introduced by area method can help middle school students to construct intuitive models for trigonometric functions, to develop a variety of mathematics learning methods, and to grasp the multi-facets of the essence of mathematics; and (3) The logic of "Rebuilding Trigonometry" can greatly help students in learning mathematics. Cui has published a paper, titled The Teaching Experiments on Defining Sine Using "Rhombus Area" [Cui2011], summarizing the two-year experiment in detail. She proposed to write the new definition in official textbooks and to do more experiments.

The key assessment for any teaching reform experiment is the students' performance in the standardized tests. With the support of the Science Association of Guangzhou, the Haizu Experimental Middle School launched a three-year teaching experiment of "Rebuilding Trigonometry" in 2012.

In June, 2012, this school chose 105 students to form two classes, named *classes for innovative experiment in mathematics*. The chosen students were fairly good in Chinese and English, but relatively weak in mathematics. The average mathematics entrance test scores were 62.5 for Class 1, and 64 for Class 2. In Class 1, four students were diagnosed as learning disability and ten were regarded as poor-performed students.

Ms. Dongfang Zhang, a young teacher, was appointed as their mathematics teacher. The experiment reorganized and integrated the mathematics material from the book *Integrated Elementary Mathematics* [Zhang2009] and the standard textbook published by the People Educational Publisher, forming teaching material with a new systematical structure. There were totally 358 mathematics lessons, 45 minutes per lesson, in three years of middle school. The contents of 96 lessons were designed by using [Zhang2009], and the contents of the remaining 262 lessons were designed by using the standard textbook.

Three out of the 96 lessons were taught in the first semester (fall semester, 7th grade). These lessons are warm-up, guiding students to discuss and solve a few interesting problems, and stimulating their interest in thinking and exploration. Twenty-three lessons were taught in the second semester (spring semester, 7th grade), defining sine, deriving the law of sine, and introducing its applications in geodetic survey and exploration of triangle properties. Twenty-four lessons were taught in the third semester (fall semester, 8th grade), deriving the sine sum angle formula and some values of the sine of certain specific angles, and proving the Pythagorean Theorem (GeGu Theorem). Fifteen lessons were taught in the fourth semester (spring semester, 8th grade), reviewing the above knowledge and exploring the properties of quadrilaterals. Thirty-one lessons were taught in the fifth semester (fall semester, 9th grade), fifteen of which were used to define cosine, to derive the law of cosine, and to introduce its applications, and sixteen of which were used to explore properties of circles and regular polygons, to introduce tangent and cotangent functions, and to explore the relationships and applications of the four trigonometric functions.

During the three years of experiment, the teacher used the dynamical geometry software "Super-Painter" to support the teaching, and organized a variety of extra programs, which raised students' interest and motivation.

4. The Examples of Key Lesson in Each Semester

Now let us see some examples of key lessons in each semester.

【Lesson example 1】Warm up lesson in 1st semester: Numbers on Monthly calendar.

Teachers guide students to observe some interesting relationship between the digitals on calendar, along with discussions to explore the truth. As shown in Figure 3 and Figure 4.

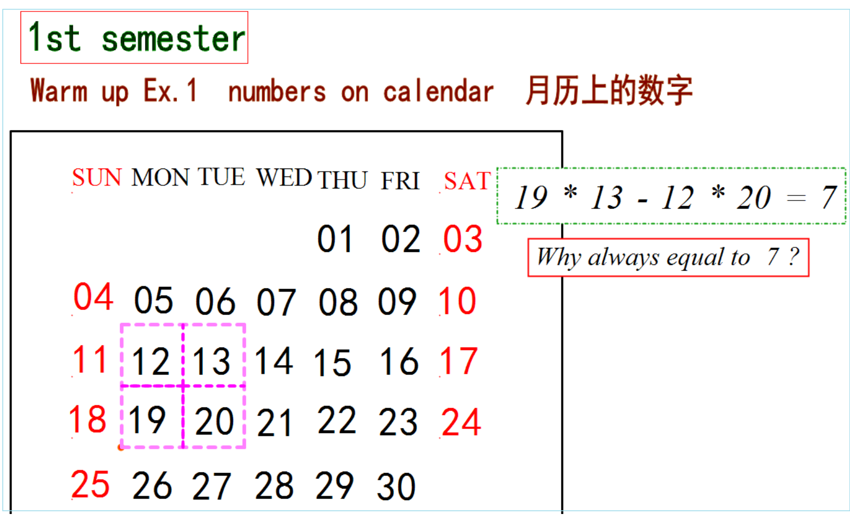

Fig. 3

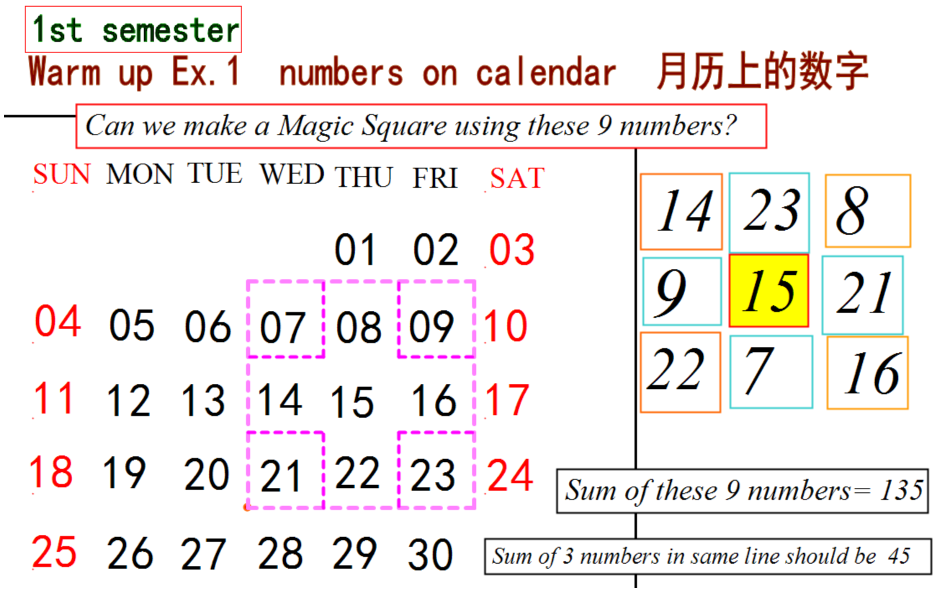

Fig.4

【Lesson example 2】Warm up lesson in 1st semester：The new discovery in multiplication table.

Teachers guide students to observe the relationship between digital in multiplication table, found some interesting rules and explore the truth. As shown in Figures 5 and 6.

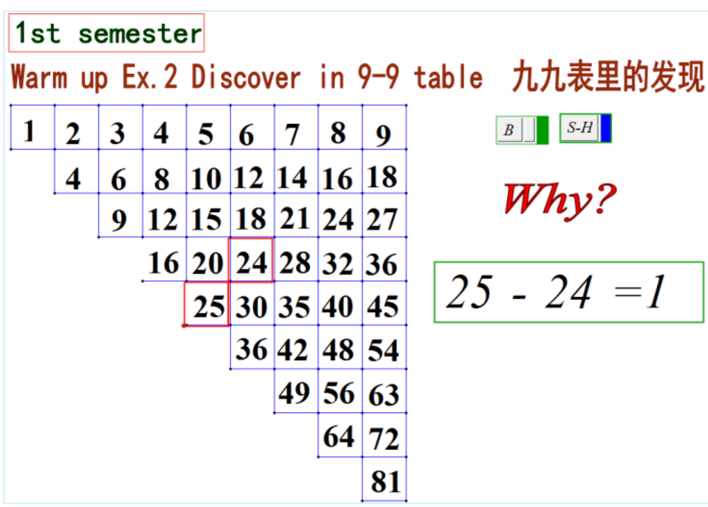

Fig.5

Fig.6

【Lesson Example 3】 2nd semester: Introduction of sine and a new formula for the triangle area.

Analogy calculate area of a rectangle, to explore how to find the parallelogram area starting from the data of two sides and the angle between the two sides, thereby introducing an alternative method defined sine. Then obtained the new formula to calculate the area of a triangle. As shown in Fig. 7 ~9.

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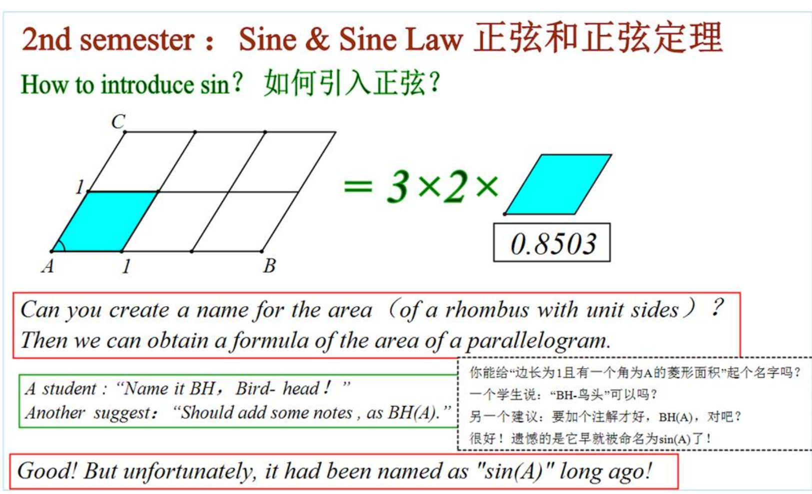


Fig.7

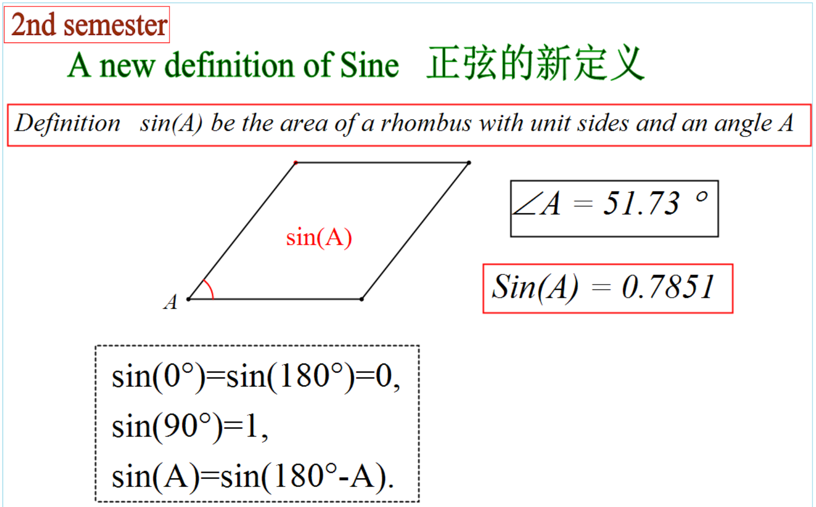


Fig.8

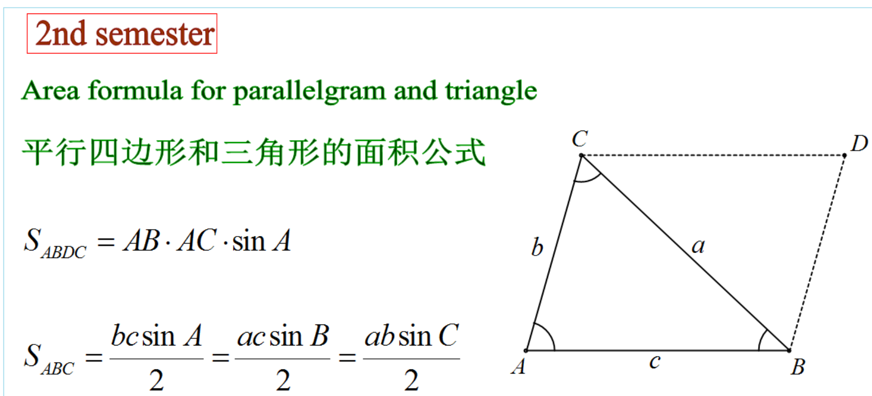


Fig. 9

【Lesson example 4】 2nd semester: Introduction of sine law.

By the new formula for triangle area

$$S_{ABC} = \frac{bc\sin A}{2} = \frac{ac\sin B}{2} = \frac{ab\sin C}{2}, \qquad (1)$$

When $\angle C$=90°，We get the following equalities which as same as in traditional definition:

$$\sin A = \frac{a}{c}, \ \sin B = \frac{b}{c}. \qquad (2)$$

Take equality (1) times 2 and divide by *abc*, then have the law of sine:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{3}$$

Then, we can introduce some applications and useful inferences. As shown in Fig.10~13.

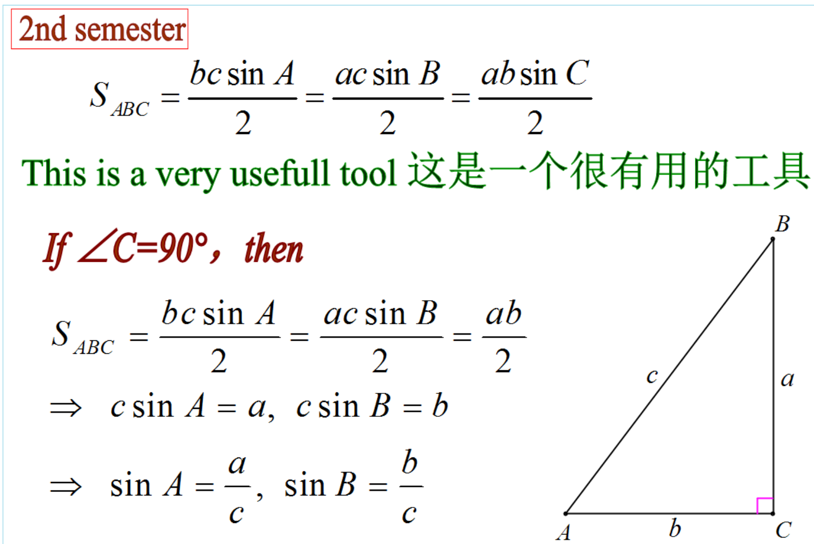


Fig.10

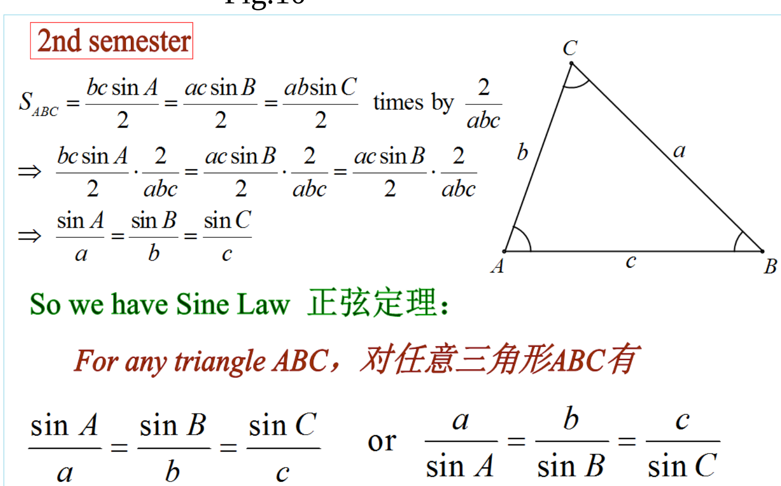


Fig.11

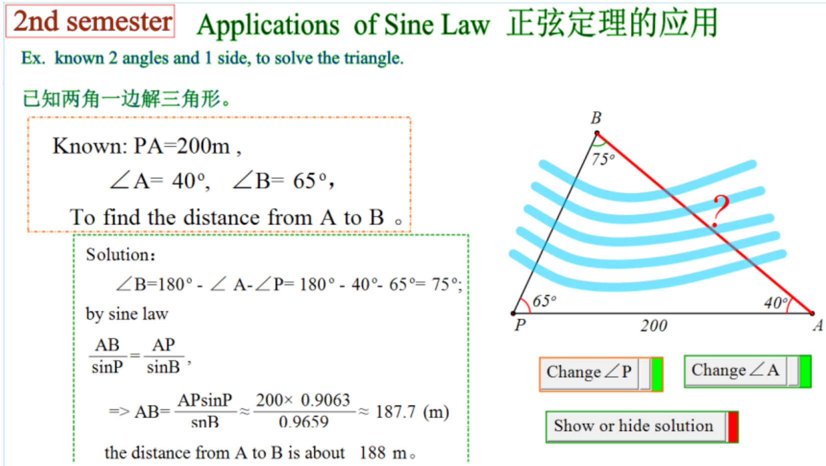


Fig.12

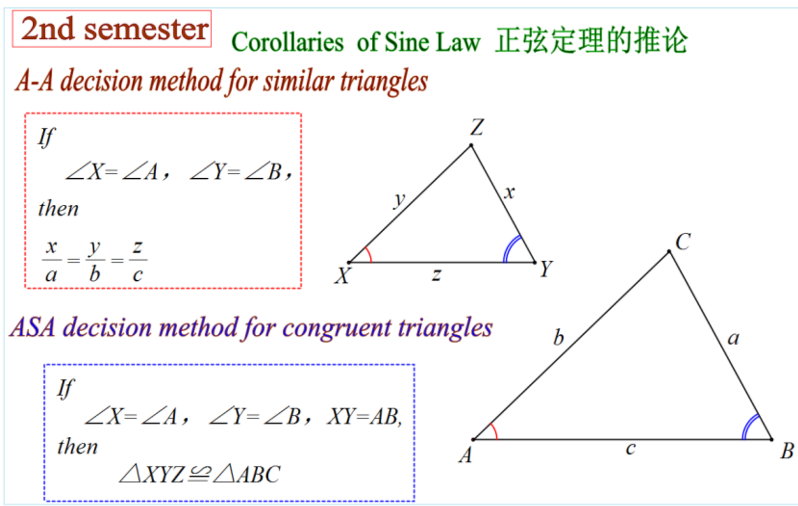


Fig.13

【Lesson example 5】 2nd semester: increasing or decreasing of sine.

Using area formula (1), we can explore the increasing or decreasing of the value of sine, as shown in Fig.14. Combined with Sine law, we can get "in any triangle, the larger angle facing larger side, and the larger side facing large angle" , and other useful properties.

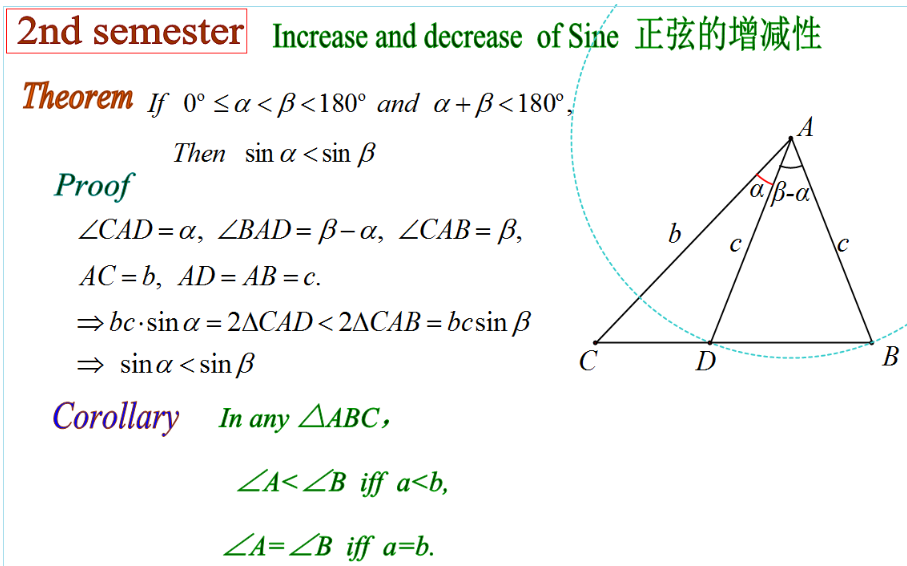


Fig.14

【Lesson example 6】 3rd semester: Sine addition theorem.

High line of a triangle divide it into two parts. With the area formula (1), were calculated throughout the area of a triangle and the two parts, then a equation are listed . To sort out it, we will have the sine addition theorem：

$$\sin(\alpha + \beta) = \sin\alpha \cdot \sin(90^\circ - \beta) + \sin\beta \cdot \sin(90^\circ - \alpha). \tag{4}$$

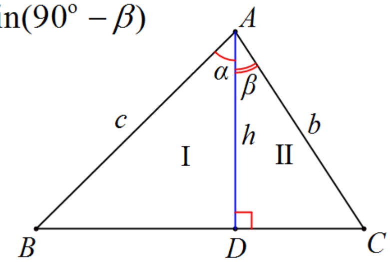
Take its special case; we got the values of sine for some special angles and Pythagorean Theorem. The way leads to the secondary radical and quadratic equations and so on. As shown in Fig.15~20.

3rd semester

Sine addition theorem 正弦加法定理

$$\sin(\alpha+\beta)=\sin\alpha\cdot\sin(90^\circ-\beta)+\sin\alpha\cdot\sin(90^\circ-\beta)$$

Proof



As the figure, $\Delta ABC=\Delta\mathrm{I}+\Delta\mathrm{II}$

$$\Rightarrow\ \frac{b\cdot c\cdot\sin(\alpha+\beta)}{2}=\frac{h\cdot c\cdot\sin\alpha}{2}+\frac{b\cdot h\cdot\sin\beta}{2}$$

$$\Rightarrow\ \sin(\alpha+\beta)=\frac{h}{b}\cdot\sin\alpha+\frac{h}{c}\cdot\sin\beta=\sin\alpha\cdot\sin(90^\circ-\beta)+\sin\beta\cdot\sin(90^\circ-\alpha)$$

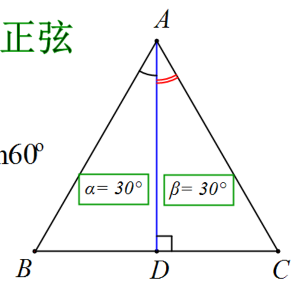
Fig. 15

3rd semester

Applications of Sine addtion theorem 正弦加法定理的应用

$$\sin(\alpha+\beta)=\sin\alpha\cdot\sin(90^\circ-\beta)+\sin\alpha\cdot\sin(90^\circ-\beta)$$

Sine value of some special angles 特殊角的正弦



(1) *Take* $\alpha=\beta=30^\circ$,

$$\Rightarrow\ \sin60^\circ=\sin30^\circ\cdot\sin60^\circ+\sin30^\circ\cdot\sin60^\circ=2\sin30^\circ\cdot\sin60^\circ$$

$$\Rightarrow\ 1=2\sin30^\circ\Rightarrow\ \sin30^\circ=\frac{1}{2}.$$

Fig.16

(2) *Take* $\alpha=\beta=45^\circ$,

$$\Rightarrow\ \sin90^\circ=\sin45^\circ\cdot\sin45^\circ+\sin45^\circ\cdot\sin45^\circ=2(\sin45^\circ)^2$$

$$\Rightarrow\ 1=2(\sin45^\circ)^2\Rightarrow\ \sin45^\circ=\frac{\sqrt{2}}{2}.$$

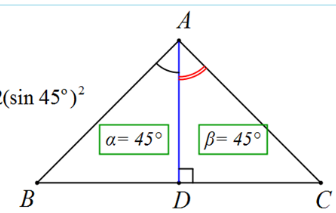


Fig.17

(3) *Take* $\alpha=30^\circ,\beta=60^\circ$,

$$\Rightarrow\ \sin90^\circ=(\sin30^\circ)^2+(\sin60^\circ)^2$$

$$\Rightarrow\ 1=\frac{1}{4}+(\sin60^\circ)^2\Rightarrow\ \sin60^\circ=\frac{\sqrt{3}}{2}.$$

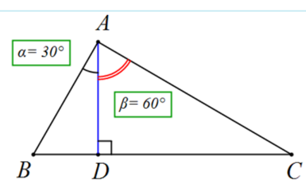


Fig.18

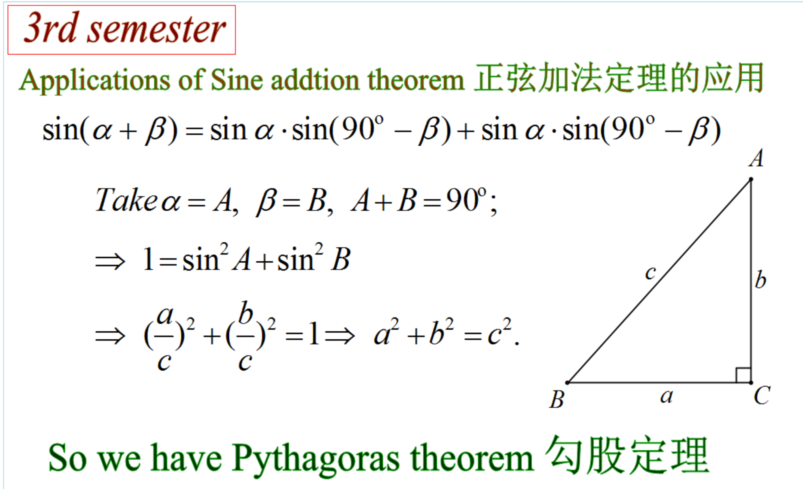


Fig.19

【Lesson example 7】 5th semester: cosine and cosine law.

The sine of the complementary angle to be called the cosine , that be the introduction of the definition:

$$\cos A=\begin{cases}\sin(90^\circ-A) & (0^\circ\le A\le 90^\circ)\\ -\sin(A-90^\circ) & (90^\circ< A\le 180^\circ)\end{cases} \tag{5}$$

Then introduce the cosine law with ternary equation. As shown in Fig.21 and 22.

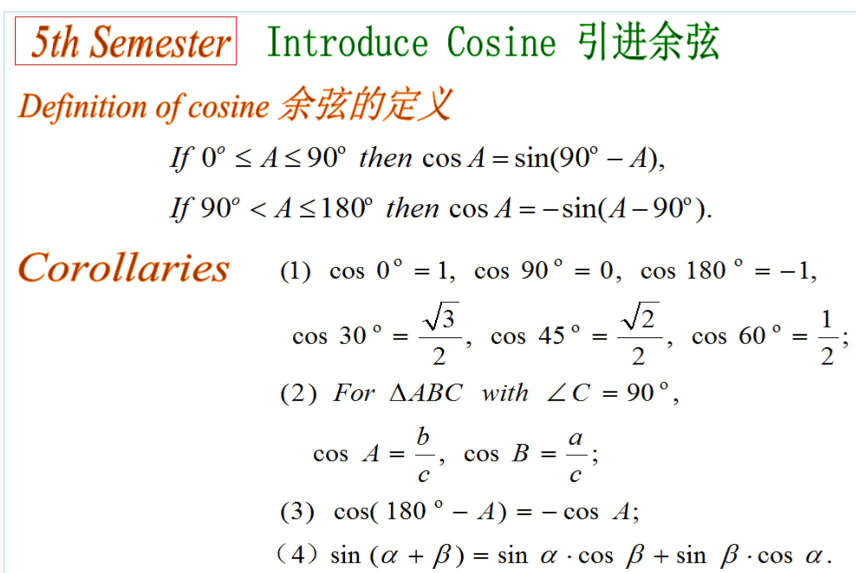


Fig. 21

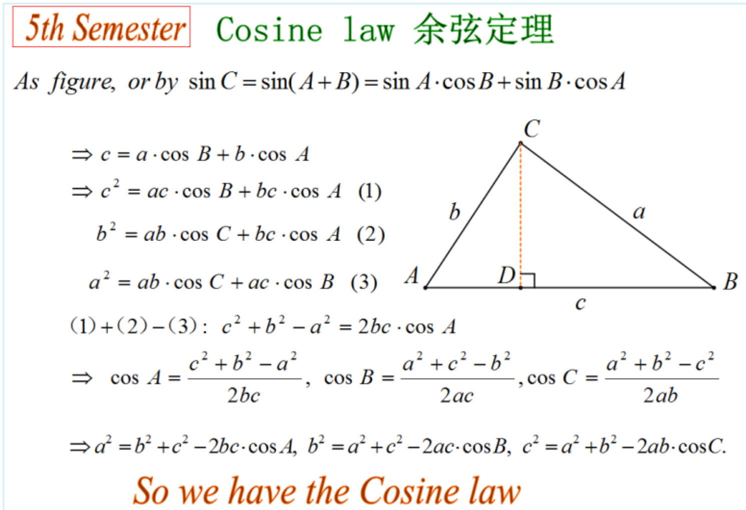


Fig. 22

【Lesson example 8】 5th semester: applications and corollaries of Cosine law.

Cosine law can be used to solve triangle, and has other important applications. As shown in Figure 23.

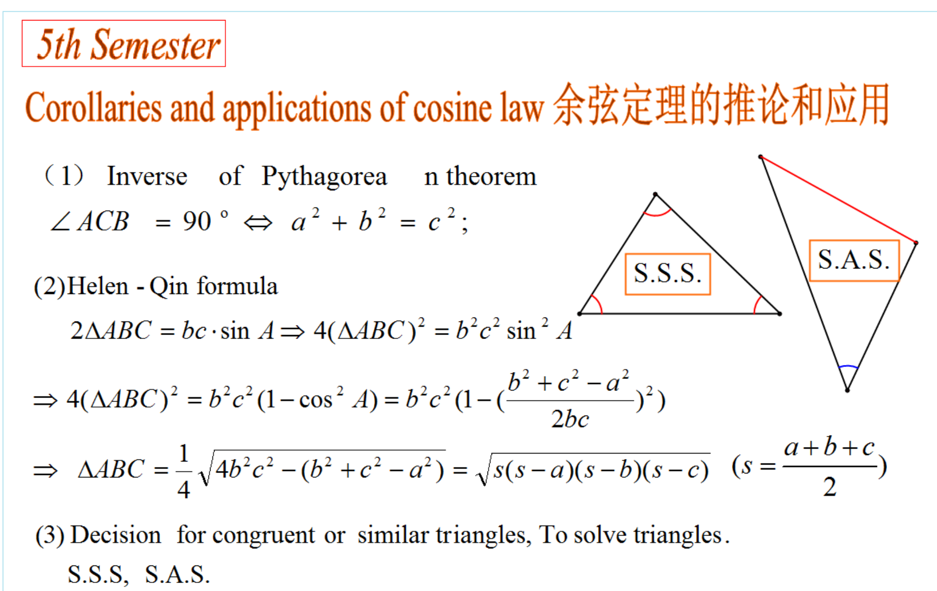


Fig. 23

5. Assessments

This three-year experiment convincingly demonstrated the power of "Rebuilding Trigonometry". With the sine function introduced using the rhombus area in 7th grade, algebra and geometry are closely connected, students' abilities in logical thinking, analysis and syntheses, problem-solving, and number-shape imagination are greatly and rapidly enhanced. This enhancement has been seen clearly from the students' performance in the regional standard mathematics tests in Haizu, Guangzhou.

In June 2013, Haizu school district held an annual standardized mathematics test. All eighty classes of 7th graders from in this district participated the test, with average scores 91 (out of 150). Of course, the students in the experiment participated the standardized test. Class 1 won the championship with average scores 140, and Class 2 rated Number 8 with average scores 138.

At the end of the fall semester of 8th grade, Haizu school district held another standardized mathematics test, with average scores 87.76 (out of 150). Class 1 won the championship again with average scores 136, and Class 2 rated Number 5 with average scores 133.

At the end of the 8th grade, Haizu school district held another standardized mathematics test, with average scores 96.83 (out of 150). Class 1 won the championship again with average scores 145, and Class 2 rated Number 3 with average scores 141.

At the end of the fall semester of 9th grade, Haizu school district held another standardized mathematics test, with average scores 93 (out of 150). Class 1 won the championship again with average scores 137.5, and Class 2 rated Number 5 with average scores 129.5.

In June of 2015, these students graduated and participated the entrance test for high schools. The average mathematics scores of the two classes are 131.47 and 131.11, and all students are rated as excellence in mathematics, while 66.91% of students in Haizu district were rated as excellence in mathematics. Their excellence in mathematics had positive impact to their performances in other courses. Their overall average in the test are 733.96 and 730.25, much higher than 532.50, the average scores of all students in Haizu district.

Ms. Dongfang Zhang said, the teaching experiment greatly enhanced and strengthened students' problem-solving ability, especially their ability in solving comprehensive challenging problems. At one Haizu District standardized test, fifteen students correctly solved the most difficult problem in the test, and twelve out of the fifteen students were from her experiment classes.

6. Prospect and Expectation for the Future

Inspired with the success of the Haizu's experiment, we plan to do the following in the near future:

(1) Reflect on the key reasons for the Haizu experiment's success, and figure out the deeper laws of educational mathematics;

(2) Conduct similar experiments in more schools and classes to collect more data and to see if it is of repeatability. In addition to Haizu Experimental Middle School, fourteen more schools in Guangzhou area has started the experiment, and more schools in other cities will also join this experiment;

(3) Follow the students in Ms. Zhang's experiment classes to see their academic performance in high schools, and compare their performance with the ones who were not in the experiment;

(4) Draft new mathematics textbooks integrating trigonometry, geometry, and algebra in the preparation for the future large scale experiments.

Haizu's experiment has promoted the research and practice of educational mathematics into a new stage and period, and has provided a feasible and promising plan for the reform of mathematics teaching. It will have profound impact on the research ideas and methods of mathematics education reform.

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