Solve First – Ask Questions Later: discovering geometry using Symbolic Geometry and CAS

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Abstract
Using symbolic geometry and CAS facilitates a new kind of geometric discovery. Rather than relying on geometric insight to solve a problem, it can be expressed algebraically using the symbolic geometry system then solved in the CAS. The solution can then be examined geometrically in the symbolic geometry system and insight attained after the fact. We illustrate this approach with a number of examples from students' work over the last decade.

1. Introduction

The thing which makes geometry so appealing is that sometimes cleverness can yield simple solutions to complicated problems. This can also make geometry frustrating on the occasions when the cleverness eludes us.
Symbolic geometry software, such as Geometry Expressions facilitates the creation of mathematical models and the expression of the geometry as algebra (or trigonometry). The algebraic form may be solved using standard approaches or copied into a CAS and simple operations applied to work to a solution. The solution may in turn be copied back into the symbolic geometry system and analysed. Examination of the solution obtained in this manner can yield geometric insight which can lead to more succinct, simpler proofs. This yields a fundamentally new approach to solving geometry problems, which allows the insight to come after the problem is solved. In this way, progress can be made, even without the insight.
In this talk we present a number of geometry problems whose analysis yielded geometric insight in just this way. Many of these problems arose during summer mathematics projects undertaken by students at the Portland Saturday Academy over the course of the last 10 years.
The more engaging student projects involved the analysis of real-world problems, the formulation of tractable mathematical models, and solution of the mathematical problem posed by the model. We give examples of mathematics motivated by application domains from design of solar cookers to telescope aberration. We highlight the formulation of mathematical problems from application domains as a highly creative mathematical activity and one to which the student is traditionally not exposed.

Saturday Academy

Most of the projects described below were undertaken by students in the Portland Saturday Academy Apprenticeship in Science and Engineering (ASE) program [1]. Under this program, high school students (typically at the end of year 10 or 11) work during the summer for 8 weeks as interns at local engineering or scientific labs. The students are typically of the highest caliber, as the main motivation for participating is to boost their chances of being admitted to exclusive universities. Saltire Software has participated in the program since 2006.

A major component of the work done by Saltire's interns has been a substantial mathematical research project. The student is given technology: Geometry Expressions for modeling, Maple and Mathematica for solving and graphing. They select an open ended research topic, and with guidance from their mentor (this paper's author) develop the questions they wish to address, create mathematical models and attain some sort of answers to their questions. Typically these research projects account for about half of their time spent as an intern, approximately 160 hours of work.

2. An Inscribable Circumscribable Pentagon

A problem which nicely illustrates the style of geometrical discovery described in this paper was posed to the author at the 2006 ACTM conference.

> *Find a pentagon, other than the regular pentagon which is both inscribable and circumscribable.*

That is, find a pentagon whose vertices lie on a circle, and whose edges are tangential to another circle.

$$\Rightarrow \left| \frac{-\left(9+12\cdot b-12\cdot b^2+4\cdot b\cdot\sqrt{9-6\cdot b}+\sqrt{3+4\cdot b-4\cdot b^2}\cdot\left(-3\cdot\sqrt{3}+2\cdot\sqrt{3}\cdot b-3\cdot\sqrt{6+4\cdot b-2\cdot\sqrt{9+12\cdot b-12\cdot b^2}}\right)\right)}{2\cdot\left(-3\cdot\sqrt{3}+2\cdot\sqrt{3}\cdot b+4\cdot b\cdot\sqrt{3-2\cdot b}+3\cdot\sqrt{3+4\cdot b-4\cdot b^2}-3\cdot\sqrt{6+4\cdot b-2\cdot\sqrt{9+12\cdot b-12\cdot b^2}}+2\cdot b\cdot\sqrt{6+4\cdot b-2\cdot\sqrt{9+12\cdot b-12\cdot b^2}}\right)} \right|$$

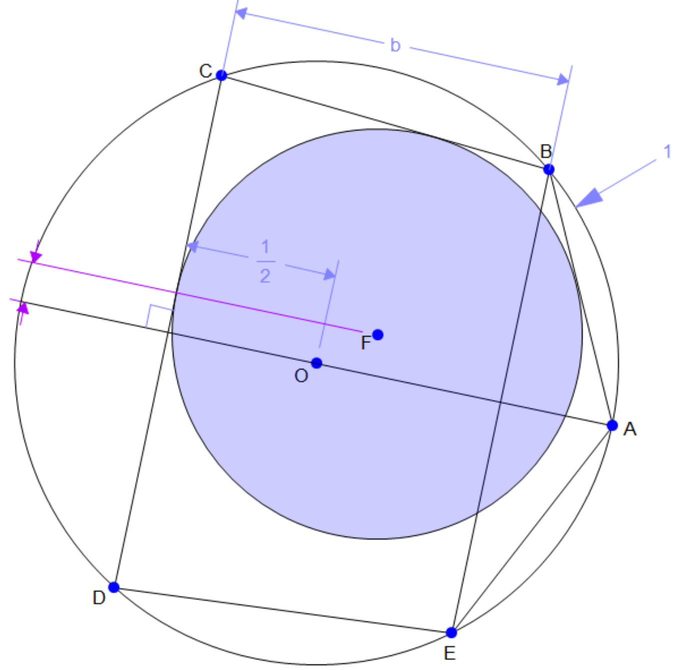


Figure 1: An axisymmetric circumscribable pentagon, b is the parallel distance between BD and EF. The inner circle is tangent to EF, BF and BC. Displayed is the distance between the circle's centre and the axis of symmetry.

There are, of course, many such pentagons and we are asked to find only one, so we can make choices to simplify the problem. We choose to look for an axisymmetric pentagon where the edge perpendicular to the axis is half way between the centre and the circumference. Let A be a point on the circumference of a circle centred at O of unit radius. Let CD be perpendicular to AO at a distance ½ from O, let BE be parallel to CD and at a distance b (figure 1). We now create a circle tangent to AB, BC and CD. For the pentagon to be inscribable, this circle must also be tangential to AE and DE, and hence by the symmetry of the drawing, its centre F must lie on OA.

In Figure 1, we see the algebraic output from Geometry Expressions for the distance between F and the axis of symmetry OA. Copying this result into Maple we can find a value of b which makes it identically zero (figure 2).

```
> solve(%=0,b);
```

$$\frac{3}{2}-\left(-\frac{(-4\sqrt{3}+4I)^{(1/3)}}{4}-\frac{1}{(-4\sqrt{3}+4I)^{(1/3)}}-\frac{1}{2}I\sqrt{3}\left(\frac{(-4\sqrt{3}+4I)^{(1/3)}}{2}-\frac{2}{(-4\sqrt{3}+4I)^{(1/3)}}\right)\right)^2/2$$

```
> simplify(evalc(%));
```

$$\frac{1}{2}+\frac{1}{2}\sqrt{3}\sin\left(\frac{4\pi}{9}\right)-\frac{1}{2}\cos\left(\frac{4\pi}{9}\right)$$

Figure 2: Maple solution of the equation from figure 1.

Having found a solution, we notice that the angle $\frac{4\pi}{9}$ appears in the equation. This suggests that perhaps the solution is related to the regular nonagon. In fact, we find that the solution can be constructed from edges and diagonals of the regular nonagon as shown in figure 3. At this point, we can use elementary geometry to prove that the pentagon so described is inscribable [2].

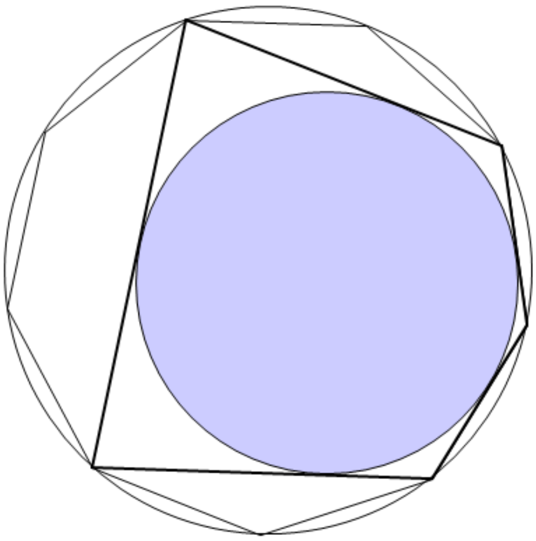


Figure 3: Inscribable pentagon inside a regular nonagon

While this example was not done by one of our Saturday Academy students it admirably illustrates the solve first – enlightenment later approach. A rather ugly solution using Geometry Expressions and Maple led to a geometric characterization in terms of the regular nonagon. At this point an elegant geometric solution was attainable.

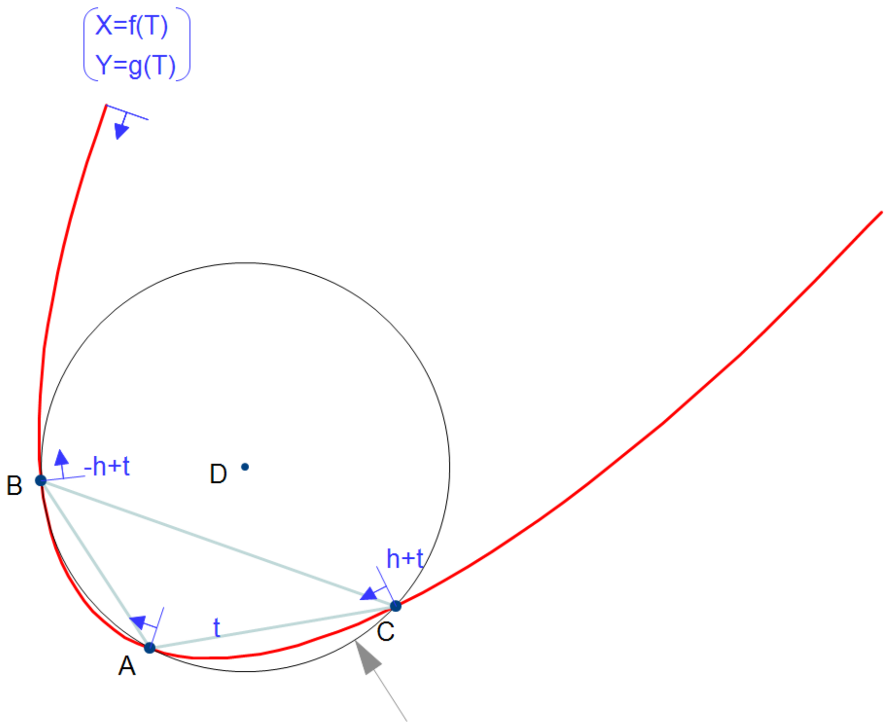
3. Limiting forms of Triangle Defined Circles

The problem which our first ASE student selected to address was this:

What happens, in the limit, to the circumcircle of a triangle when the three points coalesce?

By numerical experimentation with Geometry Expressions, the student convinced himself that the answer to the question depended on the path along which the points coalesced.

If the triangle had vertices ABC, he reasoned that one could apply transformations to keep A fixed, then the paths of B and C would form two arms of a curve which meet at A. He was able to prove that the circumcircle tended to 0 unless that curve was sufficiently smooth at A. In the case where it was smooth, he used Maple to find the radius of the limit circle. He was able to confirm that this was the radius of curvature of the curve (figure 4).



$$\Rightarrow \frac{\sqrt{(f(t)-f(h+t))^2+(g(t)-g(h+t))^2}\cdot\sqrt{(f(t)-f(-h+t))^2+(g(t)-g(-h+t))^2}\cdot\sqrt{(-f(h+t)+f(-h+t))^2+(-g(h+t)+g(-h+t))^2}}{2\cdot(-f(h+t)\cdot g(t)+f(-h+t)\cdot g(t)+f(t)\cdot g(h+t)-f(-h+t)\cdot g(h+t)-f(t)\cdot g(-h+t)+f(h+t)\cdot g(-h+t))}$$

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> limit(%,h=0,right);
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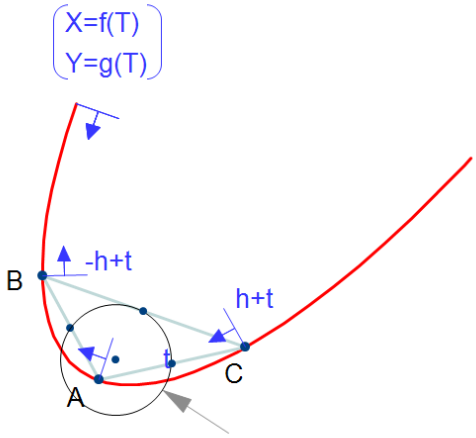
$$-\frac{\left(D(f)(t)^2+D(g)(t)^2\right)^{(3/2)}}{-D^{(2)}(g)(t)\,D(f)(t)+D(g)(t)\,D^{(2)}(f)(t)}$$

Figure 4 (a) Circumcircle of points at parametric locations t-h, t, t+h on curve (f(t),g(t)) along with expression for the radius. (b) Limit of the radius as h→0 computed in Maple.

Having turned the original question into a more tractable mathematical formulation, and successfully analyzed this using symbolic geometry and CAS, the student proceeded to apply the same technique to a number of other circles defined by triangles which he encountered on the web [3]. For example, his analysis of the limit of the 9-point circle is shown in Figure 5. Comparison of this limit with that of figure 1 reveals that the limiting radius of the 9 point circle is half the radius of curvature of the curve.



$$\Rightarrow \frac{\sqrt{\left(\frac{-f(t)}{2}+\frac{f(h+t)}{2}\right)^2+\left(\frac{-g(t)}{2}+\frac{g(h+t)}{2}\right)^2}\cdot\sqrt{\left(\frac{-f(t)}{2}+\frac{f(-h+t)}{2}\right)^2+\left(\frac{-g(t)}{2}+\frac{g(-h+t)}{2}\right)^2}\cdot\sqrt{\left(\frac{-f(h+t)}{2}+\frac{f(-h+t)}{2}\right)^2+\left(\frac{-g(h+t)}{2}+\frac{g(-h+t)}{2}\right)^2}}{\frac{-f(h+t)\cdot g(t)}{2}+\frac{f(-h+t)\cdot g(t)}{2}+\frac{f(t)\cdot g(h+t)}{2}-\frac{f(-h+t)\cdot g(h+t)}{2}-\frac{f(t)\cdot g(-h+t)}{2}+\frac{f(h+t)\cdot g(-h+t)}{2}}$$

```
> limit(%,h=0,right);
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$$-\frac{\left(D(f)(t)^2+D(g)(t)^2\right)^{(3/2)}}{-2\,D^{(2)}(g)(t)\,D(f)(t)+2\,D(g)(t)\,D^{(2)}(f)(t)}$$

Figure 5: (a) Geometry Expressions Model of the 9 point Circle radius. (b) limit of this radius as h→0 computed in Maple.

The student found definitions of many more triangle circles online [3], and expanded his investigation to 50 such circles. Of these 50 he found 20 tended to zero radius and 2 to infinite radius, while 22 tended to a radius which is a constant multiple of the radius of curvature, and 6 tended to some other radius (figure 6).

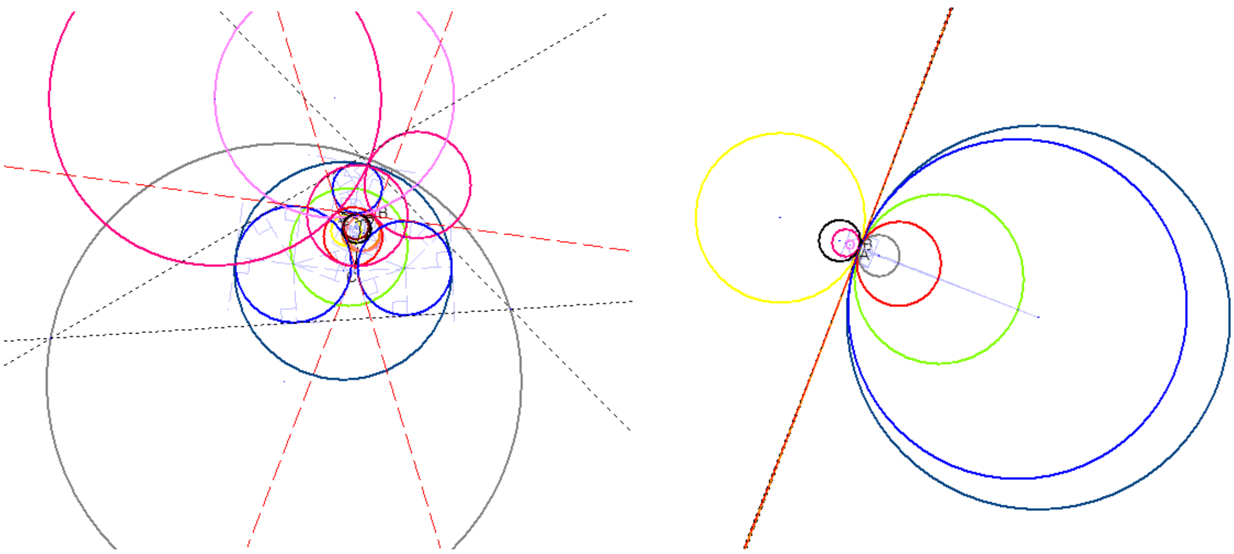


Figure 6: A number of different triangle defined circles and their limiting forms

4. Telescope Aberration

An article in Mathematics Teacher on telescope optics [4] was the starting point of the second year's investigation. The article described how incoming light parallel to the axis of a parabolic reflector would all converge once reflected on the focus of the parabola. Incoming light at an angle to the axis was discussed and in relation to this, two terms were used without definition: "optical focus" of the reflected light, and the "focal surface" of the parabola.

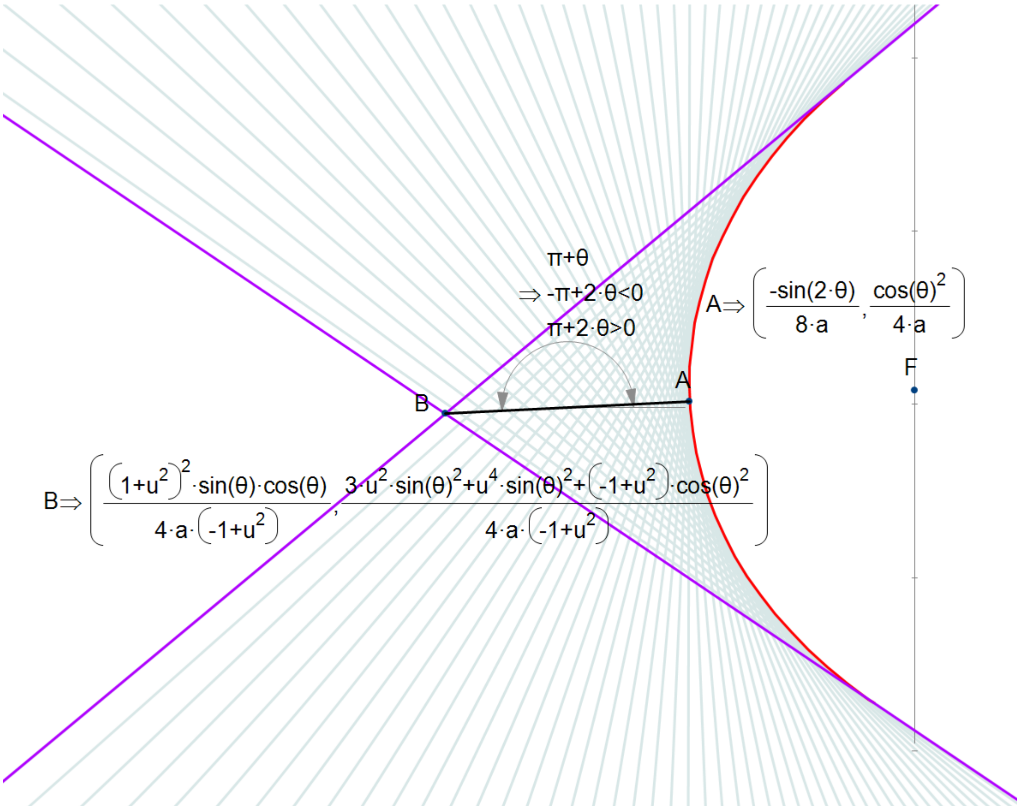


Figure 7: Model of the "optical focus" or region of concentration of reflected light for incident rays at angle θ to the axis.

The optical focus defined the location where parallel rays at a specific angle to the axis converge. As they do not in general all intersect at a point, there is a need to come up with a definition of the point of convergence.

The optical surface is then the locus of these convergence zones for a range of incoming angles. We asked the additional question: can we define, geometrically, the aberration: loosely the width of the convergence zone. Aberration should be zero for light entering parallel to the axis and, one would assume, increase as the angle of the incoming light increases.

Figure 7 shows a Geometry Expressions model of the region of concentration of incident rays at angle θ to the axis of the parabola. It is bounded on the left by the reflections of light which impinges on the edge of the mirror and on the right by the envelope of the reflected rays. This envelope may be thought of as the locus of the intersections of the reflected images of "neighboring" rays, where "neighboring" is to be interpreted as a limit of ray pairs with finite separation. The edges of the mirror are defined to lie at parametric location u and -u on the parabola. (Assuming the mirror has its center at the origin, and its axis aligned with the y axis, and that its focal length is f, then the coordinates of the edge points are $\left(2fu, fu^2\right)$).

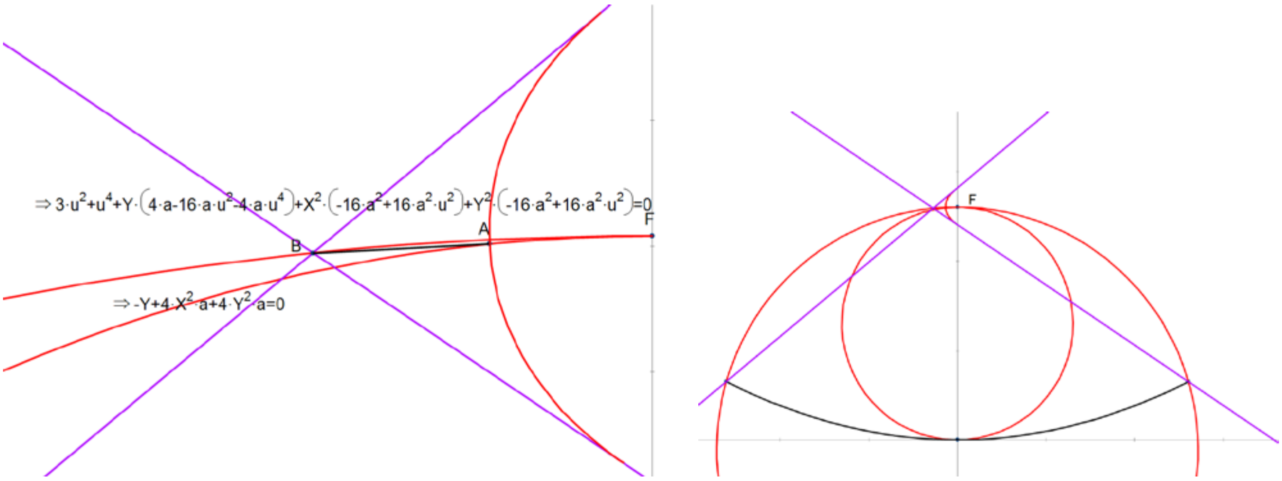


Figure 8: (a) Loci of points A and B are circles passing through the focus F. (b) Both circles pass through the focus. One is tangent to the parabola's vertex, the other passes through the edges of the parabola.

Point B on the diagram is the intersection of the reflected rays from the edge of the mirror, while point A is the point on the envelope curve whose tangent is the ray reflected from the center of the mirror. Alternatively, this can be thought of as the intersection of the reflection of "neighboring" rays which hit the center of the mirror. The length of the line joining A and B gives a measure of the width of the area of concentration of the light and thus defines, in some sense, the aberration. We can observe that AB is angle θ below the horizontal and that AB extended passes through the focus.

The locus of A and B can be computed by Geometry Expressions and their implicit equations calculated (figure 8). Inspection shows these to be circles, one being the circle through the focus which is tangent to the parabola at its vertex, the other being the circle through the focus and the two edges of the mirror.

The aberration can thus be defined geometrically as follows (figure 9). Given a parabolic mirror with focal point F, vertex V and edges C and D, draw the circumcircle of C, D and F and the circle with diameter FV. Draw a line through F parallel to the incident rays and reflect it in a line through F at 45 degrees to the axis FV. The intersections between this line and the two circles mark the boundaries A,B of the concentration of the reflected rays and the length |AB| represents the aberration.

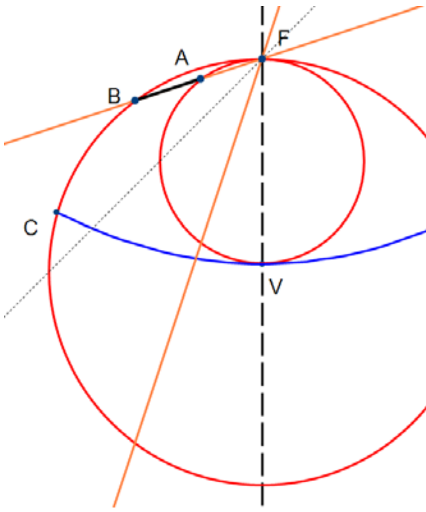


Figure 9: Geometrical method of determining the aberration in a parabolic mirror.

The student reasoned that any point on the segment AB could be used as the "optical focus" and any circle between the two circles could represent the "focal surface".
The student extended his investigation to consider a non-parabolic mirror. In generalizing the results he faced the problem that the two circles used in describing the parabolic mirror both incorporate the parabola focus in their definition, but the general curve does not have such a focus. He also recognized that his inner curve is the limiting form of the outer curve as the points C and D converge on V. His outer curve does not depend on the fact that the mirror is a parabola, but merely on the direction of the tangents to the mirror at C and D.
To analyze the general case, he took a triangle ABC and a pair of parallel rays touching AB at A and BC at C. He reflected those rays in AB and BC. He determined and proved, using elementary geometry, that if D is the circumcenter of ABC, then the locus of the intersection of the reflected rays is the circumcircle of ACD (figure 10).

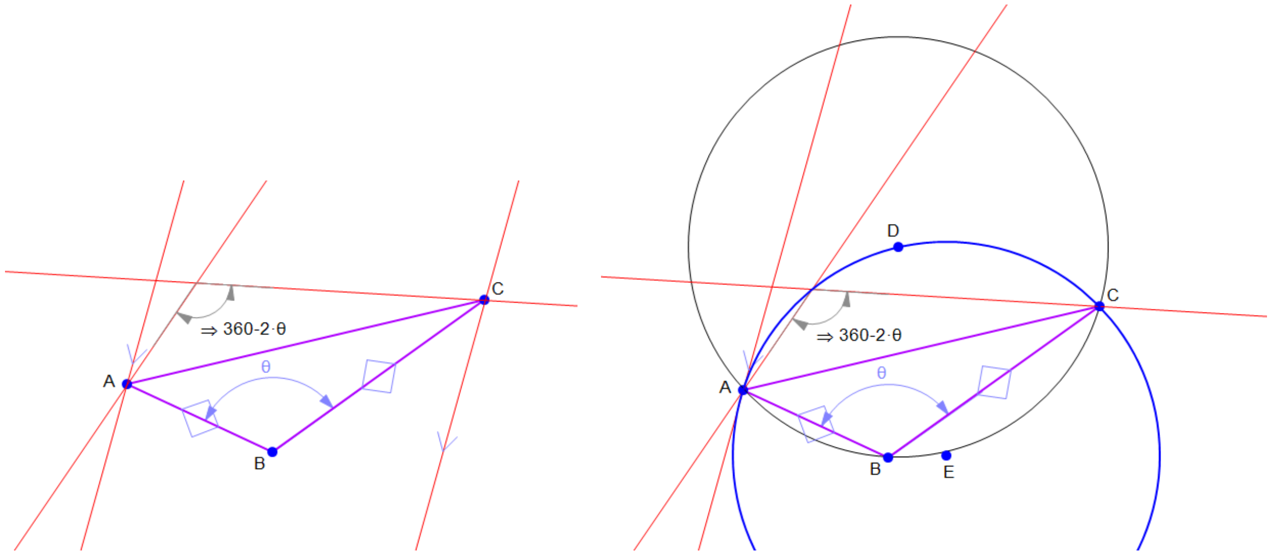
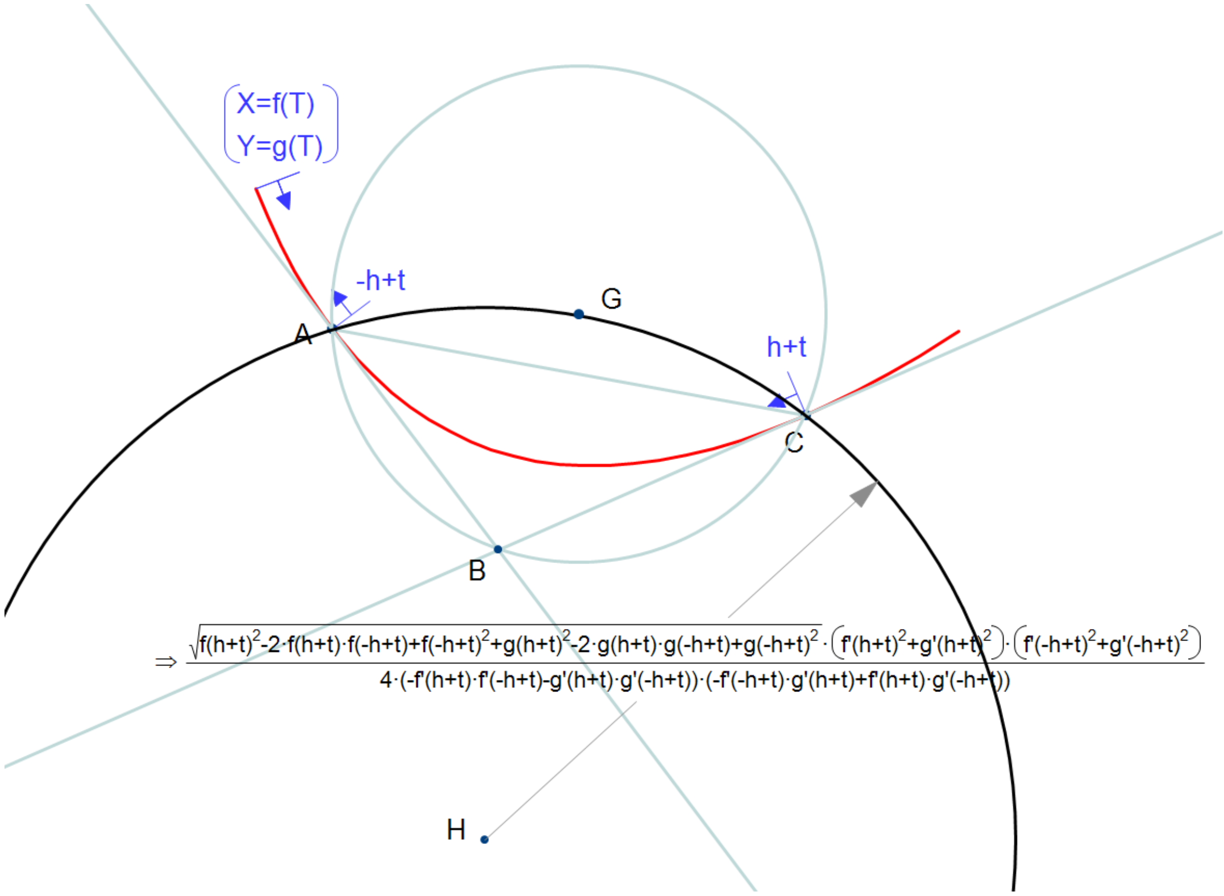


Figure 10 (a) Images of two parallel rays reflected in AB and CB meet at a constant angle. (b) The locus of the intersection points is the circumcircle of A, C and D the center of the circumcircle of ABC.

Finally, and surprisingly, he ended up using the approach of the previous year's student to determine the limit of this circle as A and C tend to B. (figure 11). He determined the limit circle has a radius one quarter of the radius of curvature.
To confirm that the circles for the general curve corresponded to the circles through the focus of the parabola, the student considered the situation where A and B in figure 10(a) lie on a parabola, and where AB and CB are tangents to the parabola. He then considered the case where the rays incident to A and C are parallel to the axis of the parabola. In this case, the focal property of the parabola demands that the reflected rays intersect at the parabola's focus. Hence the parabola's focus does indeed lie on the circumcircle of A C and D in figure 10b.




```
> limit(%,h=0,right);
```

$$\frac{\left(D(f)(t)^2 + D(g)(t)^2\right)^{(3/2)}}{4\, D(f)(t)\, D^{(2)}(g)(t) - 4\, D^{(2)}(f)(t)\, D(g)(t)}$$

Figure 11: Limit of the locus circle

5. Solar Cookers

In the third summer, we stayed on the topic of reflectors, but in the context of solar cookers. The project we identified was to characterize the sensitivity of different shapes of reflectors to errors in alignment with incoming light. While irrelevant to computer targeted reflectors in solar power stations, such sensitivity would be important to hand deployed solar cookers in the third world.

A parabolic reflector aligned perfectly with incoming light concentrates all the light on the focal point and thus is able to achieve an infinite solar concentration ratio. Conversely, if it is not aligned exactly with incoming light, all reflected rays miss the focus. In practice, however, a finite sized target is positioned at the focus. This yields a finite solar concentration ratio (the cross sectional area of the reflector divided by the area of the target) and a finite sensitivity to misalignment.

A survey of solar cooker designs on the web revealed that parabolas of a variety of different focal lengths are in use. Of course, the focal length does not affect the parabola's capability of focussing when perfectly aligned with the incoming light. A question we asked was this:

For a given cross section, what focal length is least sensitive to misalignment.

To model such a cooker, the student created a parabola whose vertex lay at the origin and whose focus was on the y axis. She constrained the distance between the focus and the vertex to be f. She centred a circular target of radius r at the focus and considered a finite portion of the parabola bounded by vertical lines distance d/2 from the y axis. With d and r considered as constants, the solar concentration ratio when adequately aligned was $\left(\frac{d}{2r}\right)^2$. She decided that her measure of sensitivity to misalignment would be the angle at which light reflected from the edge of the mirror started to miss the target. To find this angle, she created a line through the edge of the mirror and tangent to the target circle. She then reflected this line in the tangent to the parabola at the mirror's edge and measured the angle of the reflection to the vertical (figure 12).

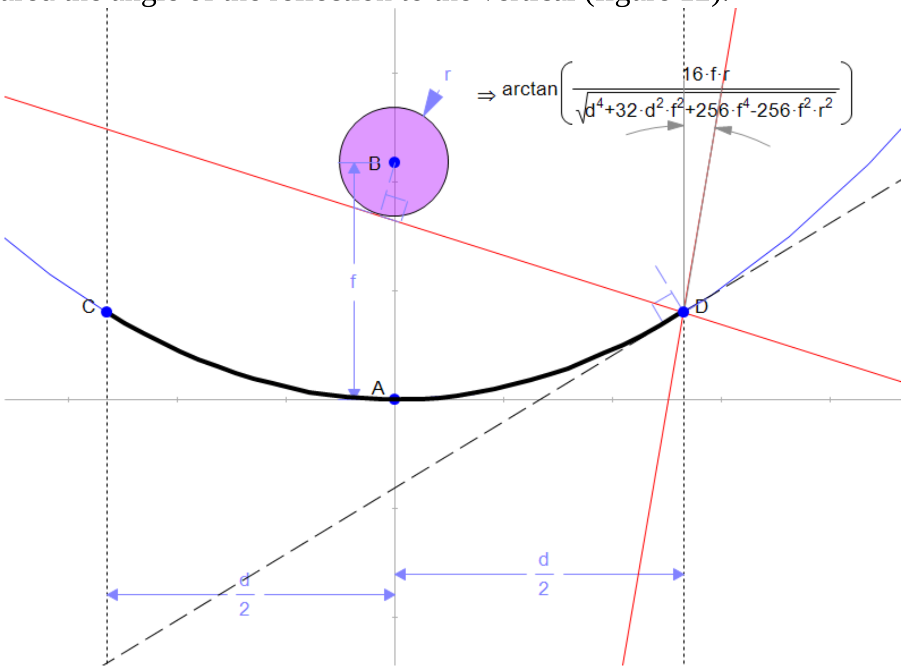


Figure 12: Solar cooker model with target of radius r and reflector of cross sectional diameter d. Focal length is f.

The least sensitive cooker would be the one whose focal length f maximizes this angle. The student copied the expression into Maple, differentiated and solved to get the result shown in figure 13.

$$\arctan\left(\frac{16\, r\, f}{\sqrt{d^4+32\, f^2\, d^2+256\, f^4-256\, r^2\, f^2}}\right)$$

```
> simplify(diff(%,f));
```

$$\frac{16\,(d^2-16\,f^2)\,r}{(d^2+16\,f^2)\sqrt{d^4+32\,f^2\,d^2+256\,f^4-256\,r^2\,f^2}}$$

```
> solve(%=0,f);
```

$$\frac{d}{4}, -\frac{d}{4}$$

Figure 13: f value which maximizes the critical angle

This seemed remarkably simple, and when entered into the Geometry Expressions model the algebraic simplicity corresponded to geometric simplicity (figure 14). The focal length was such that the focus was aligned horizontally with the edges of the reflector.

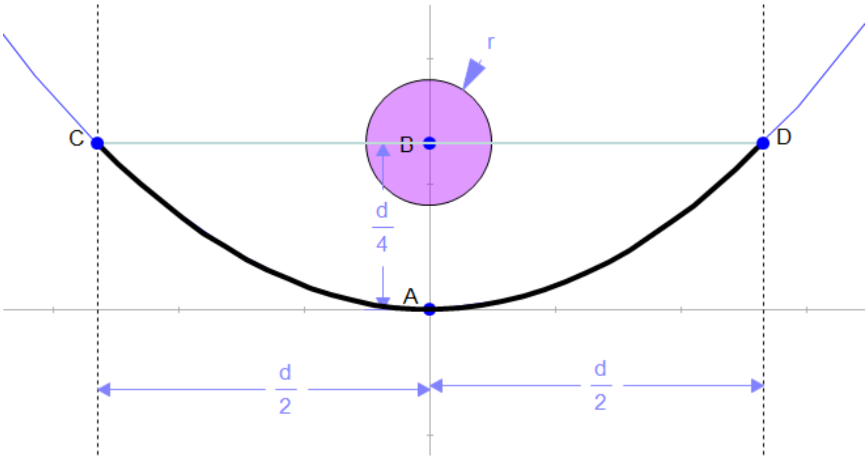


Figure 14 Geometry of the configuration which maximizes the critical angle.

The geometry of the solution, found by calculus led to a much more succinct solution of the original problem. If D is the edge of the reflecting surface, we draw lines from D tangential to the target and reflect these in the parabola at D (figure 15). The angle between the reflected rays is the critical angle which defines the tolerance of the cooker to misalignment. Clearly this is the same as the angle between the two tangents from D to the target. However, this last angle is just the apparent size of the target viewed from D. Clearly the apparent size is maximal when D is as close as possible to the target. As D can be located anywhere on the vertical line, it is closest to the target when D is horizontally aligned with B.

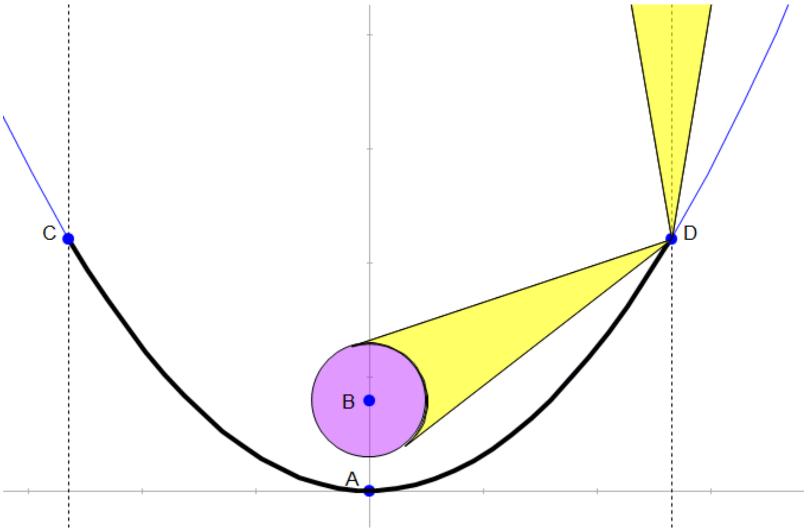


Figure15: The critical angle is the same as the apparent (angular) size of the target viewed from D. The angular size of the target is biggest when D is as close as possible.

5. Cusps on Circle Caustics

The circle caustic is the curve formed by the concentration of light reflected in a circle (figure 16). This is frequently referred to as the "coffee cup caustic". When the light source is at infinity, the shape of the caustic is a nephroid, and when the light source is at the circumference of the circle, it is a cardioid. We are interested in the situation where the light source is inside the cylinder, and in particular we want to determine the location of the cusps [6].

Figure 16: Caustic curve formed by light reflected in a perspex cylinder.

Mathematically, the caustic is the envelope of the reflected rays. The envelope may conveniently be modelled in Geometry Expressions (fig 17), and the equation of the curve derived.

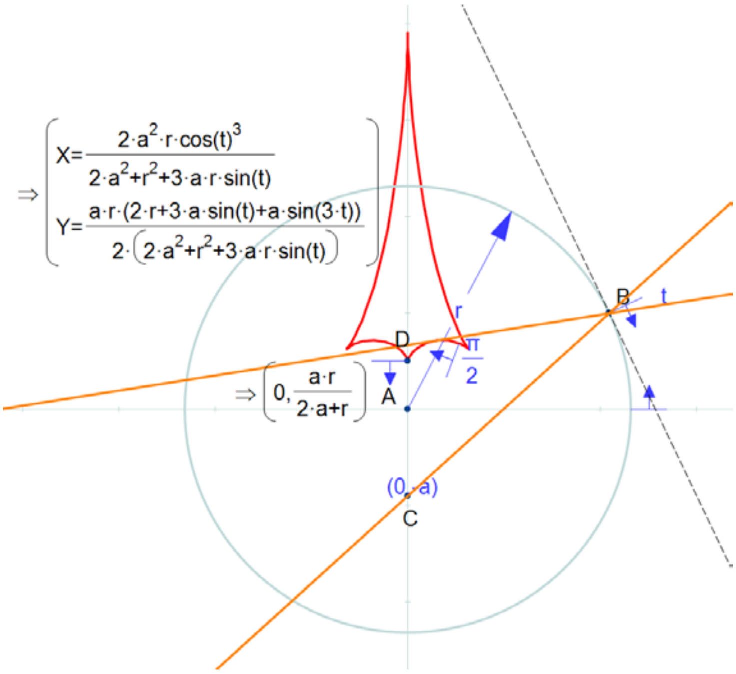


Figure 17: Geometry Expressions model of the circle caustic. Line CB is reflected in the tangent to the circle at point B. The envelope of the reflected lines as B traverses the circle is displayed. Point D is placed at parametric location $\frac{\pi}{2}$ on the envelope.

In general there are four cusps. Two are on the axis of symmetry, corresponding to B at parametric locations $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ on the circle. Their location can readily be found by placing a point on the envelope at those parametric locations. The cusp at parametric location $\frac{\pi}{2}$ is shown in figure 17. To locate the off-axis cusps, we look for points where the tangent to the curve is not well defined. Such a point is where both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vanish. Figure 18 shows this computation in Maple.

$$x := \frac{2\cos(t)^3\, r\, a^2}{2\, a^2 + r^2 + 3\sin(t)\, r\, a}$$

$$y := \frac{(2\, r + 3\sin(t)\, a + \sin(3\, t)\, a)\, r\, a}{2\,(2\, a^2 + r^2 + 3\sin(t)\, r\, a)}$$

```
> solve({diff(x,t)=0,diff(y,t)=0},t);
```

$$\left\{ t = -\frac{1}{2}\pi \right\}, \left\{ t = \frac{1}{2}\pi \right\}, \left\{ t = \arctan\left(-\frac{a}{r}, \frac{\mathrm{RootOf}(a^2 - r^2 + \_Z^2, label = \_L2)}{r} \right) \right\}$$

```
> allvalues(arctan(-a/r, RootOf(a^2-r^2+_Z^2, label = _L2)/r));
```

$$\arctan\left(-\frac{a}{r}, \frac{\sqrt{-a^2 + r^2}}{r} \right), \arctan\left(-\frac{a}{r}, -\frac{\sqrt{-a^2 + r^2}}{r} \right)$$

Figure 18: Maple computation for cusp location

The results can be copied back into Geometry Expressions (Figure 19). Let point E lie at parametric location $-\arctan(\frac{a}{\sqrt{r^2 - a^2}})$ on the circle, and point F at the same parametric location on the envelope curve. Point F lies at the cusp. We can observe and confirm geometrically that:

1. CE is horizontal
2. F is the image of C under reflection in AE.

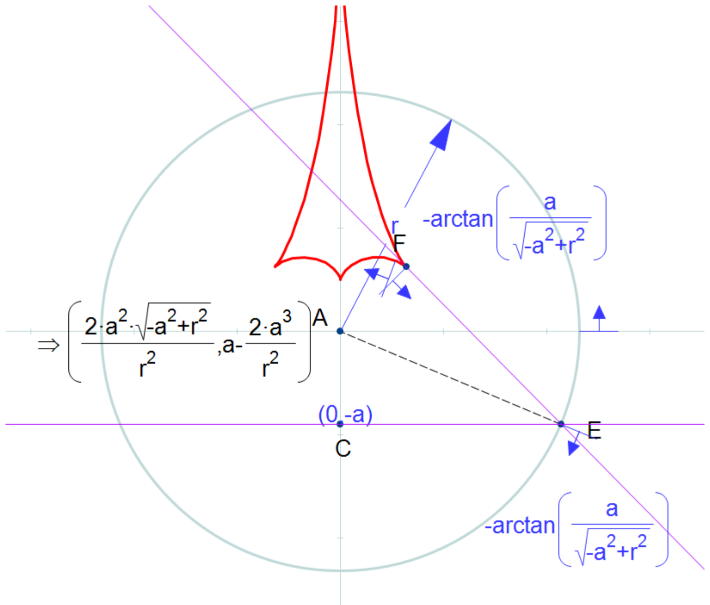


Figure 19: Location of the off-axis cusp and corresponding incident and reflected rays

Again, our symbolic solution yields simple geometry. An intuitive geometric argument for this location of the cusp considers the reflection C' of C in the line AG where G lies at general location on the curve (figure 20). The locus of C' is a circle through C centred at A. C also lies on the reflection of CG in the circle, though not necessarily on the envelope. In fact the caustic is inside the locus of C' part of the time and outside part of the time, with the division between inside and outside happening at the cusp, just when GC' is tangent to the locus of C'. (Another way of thinking of this is that this is the instant when the reflected ray is stationary, as C' is moving along the direction of the ray). By symmetry, if GC' is tangent to the circle centred at A through C then GC is also tangent, and angle ACG is right.

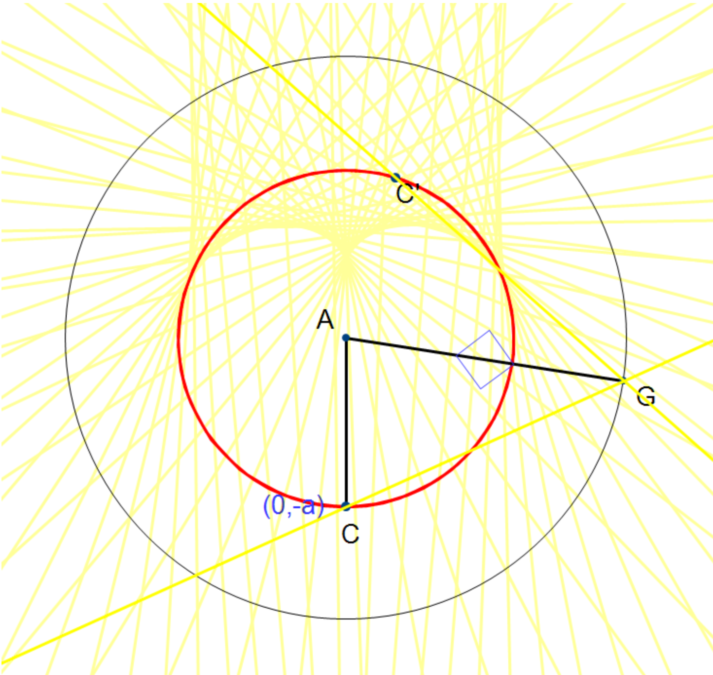


Figure 20: C' is the reflection of C in AG. It lies on the circle centered at A through C and on the reflection of the line CG.

6. Conclusion

Creating a tractable mathematical model of a real life problem is difficult. The mathematical model is an idealization of the real life problem such that it can be analysed and solved with the tools at hand. Giving a student a sophisticated mathematics environment such as Maple or Mathematica greatly extends the class of mathematical models which can successfully be analysed, and hence makes the modelling task easier. For the modelling itself to be done by students a symbolic geometry system such as Geometry Expressions, which converts a graphically expressed model into explicit mathematics has proven important. Solvability of the mathematical model frequently is dependent on both its geometrical expression and the parametrization used. The capability of reliably converting a geometrical expression of a problem into symbolic mathematics allows the student confidently to create multiple models. The automation of solution through a CAS can lead to a rapid cycle round the model / solve loop.

The general approach used with the students was to acquire a mathematical model using Geometry Expressions, to extract some measurement from the Geometry Expressions mode, expressed as a mathematical expression, and then to use a CAS to find a limit of this expression, to solve for this expression, or to optimize this expression by differentiating and solving.

Mathematisation of a real life problem came in three steps.

1. A simplified geometric model was created, a good example being to model a kettle positioned at the focus of a parabolic solar cooker as simply a finite radius circle.
2. An appropriate parametrization was applied to the model: in the solar cooker example, the parabola is parametrized by its focal length f – treated as a variable in the problem, and the diameter of its dish d – treated as a constant.
3. A measurement made from the model and treated as an objective function of the variable parameter(s).

At this point the number of different mathematical approached which were deployed in the CAS were quite small and only a handful of CAS functions were deployed.
Having found a solution by brute force application of automated geometry modelling and CAS solution, viewing the solution back in the geometrical realm frequently yielded additional insight and indeed suggested purely geometrical solutions to the original problem. These solutions replaced the heavy algebraic machinery provided by the CAS with more elementary, but less direct geometric arguments. The technology gives the student the means to make progress with a problem while waiting for inspiration to strike. With a solution in hand, inspiration seems to strike more frequently.

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