Extending Euclidean constructions with dynamic geometry software

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Abstract

In order to solve cubic equations by Euclidean means, the standard ruler and compass construction tools are insufficient, as was demonstrated by Pierre Wantzel in the 19th century. However, the ancient Greek mathematicians also used another construction method, the *neusis*, which was a straightedge with two marked points. We show in this article how a neusis construction can be implemented using dynamic geometry software, and give some examples of its use.

1 Introduction

Standard Euclidean geometry, as codified by Euclid, permits of two constructions: drawing a straight line between two given points, and constructing a circle with center at one given point, and passing through another. It can be shown that the set of points constructible by these methods form the *quadratic closure* of the rationals: that is, the set of all points obtainable by any finite sequence of arithmetic operations and the taking of square roots. With the rise of Galois theory, and of field theory generally in the 19th century, it is now known that irreducible cubic equations cannot be solved by these Euclidean methods: so that the "doubling of the cube", and the "trisection of the angle" problems would need further constructions. Doubling the cube requires us to be able to solve the equation

$$x^3 - 2 = 0$$

and trisecting the angle, if it were possible, would enable us to trisect 60° (which *is* constructible), to obtain 20°. This would mean that $z = \cos 20$ would be constructible, but this satisfies

$$4z^3 - z - \frac{1}{2} = 0$$

which is again irreducible over the rationals.

The ancient Greeks indeed did have another construction: the *neusis* construction, which involves the use of a "marked ruler": a straightedge with two points on it. The use of neusis meant that angles could be trisected and cube roots taken. And in a remarkable tour-de-force of algebraic and geometric reasoning, the Renaissance mathematician François Viète showed how to construct a regular heptagon. Excellent introductions to this geometric construction method are given by Baragar [2], Martin [5], and various places online [6].

In order to explore this lovely area of mathematics, dynamic geometry software could be used, but efforts to describe neusis constructions are somewhat lacking. For example, a description of one angle trisection takes the angle α, then computes the value $\alpha/3$ numerically, and then uses the new angle to build the neusis construction. So instead of using neusis to find the angle, the construction uses the angle to construct the neusis! We show in this article how to do it the right way round.

2 A gallery of neusis constructions

We start with a few different angle trisections, the first of which is due to Archimedes. Suppose that AOB is the angle to be trisected, and both A and B are on a circle with centre O and radius 1. Extend AO outside the circle, and construct a line from B which crosses the circle at C and the line OA at D and so that $CD = 1$. (This is the where the neusis is applied). Then the angle ODB will be the trisection of AOB. We show this in figure 1 where the neusis construction is represented as a ruler with two marks on it.

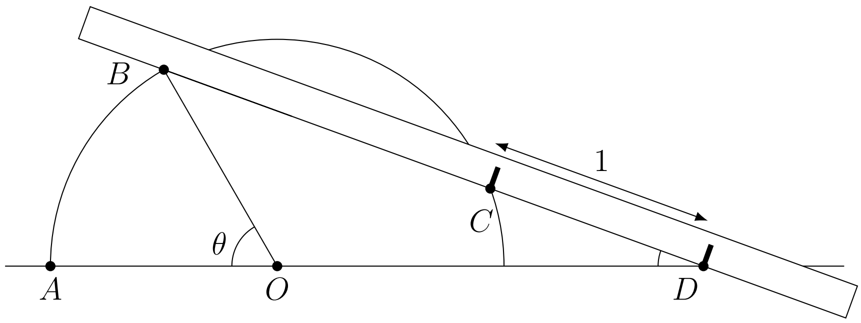

Figure 1: Archimedes' neusis trisection

The proof is remarkably easy, and involves some angle chasing. Draw the line OC and note that since $OC = CD = 1$ the triangle OCD is isosceles, with the angles $COD = CDO = \alpha$. This means that the angle $OCD = 180-2\alpha$, and thus that the angles $OCB = OBC = 2\alpha$. Thus the angle $BOC = 180 - 4\alpha$. Adding all the angles at O we have $AOB + BOC + COD = 180$, or

$$\theta + 180 - 4\alpha + \alpha = 180$$

or that

$$\theta = 3\alpha.$$

Another construction for trisection is due to Pappus of Alexandria. Let the angle to be trisected be placed in a rectangle $ABCD$, with $CAB = \theta$. Extend DC and draw a line from A to CD

crossing BC at E, meeting AD at F, and for which $EF = 2AC$. Then angles EFC is the required trisection. See figure 2.

In order to use a neusis where the marks are place 1 apart, this construction can be slightly modified as follows: Let GAC be the angle to be trisected, with both G and C on a circle of radius $1/2$ centred at A. Drop a perpendicular from C meeting AG at B. Draw CF parallel to AG. Draw a line from A through CB and DC for which the distance between the crossings is 1.

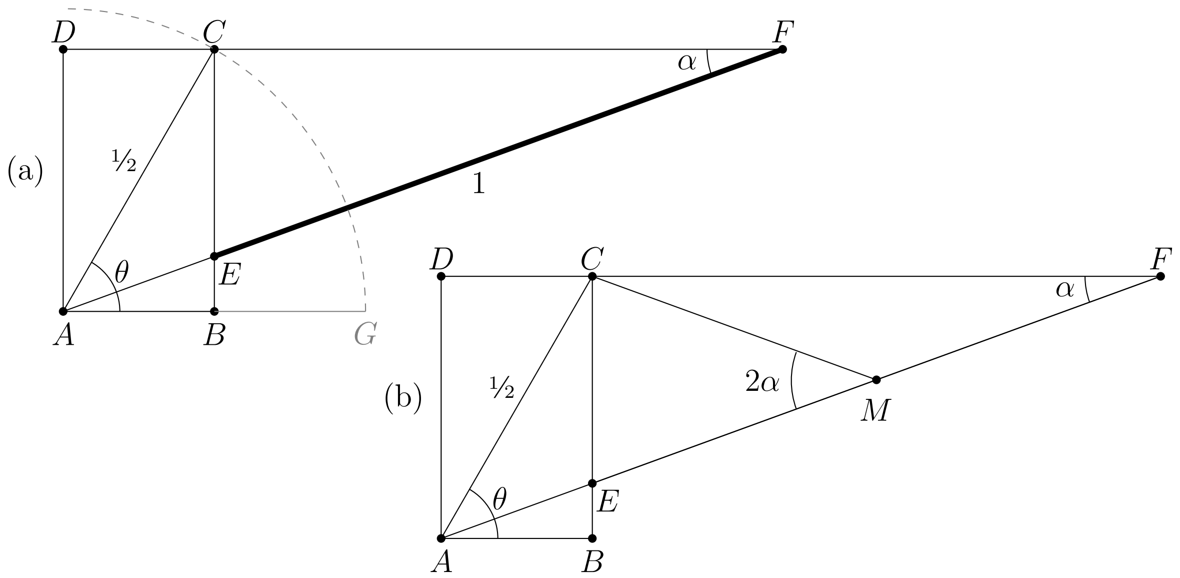

Figure 2: Pappus' (a) trisection and (b) proof

The proof is also very easy; let M be the midpoint of EF, and join CM. By the construction, all the lengths EM, MF, CM and CE are equal, so that ACM and MCF are isosceles triangles. By chasing angles we have $CMA = CAM = 2\alpha$, and $EAB = \alpha$, from which we obtain $\theta = 3\alpha$.

An elegant construction for $\sqrt[3]{2}$ is due to Nicomedes, who is a shadowy figure about whom little is known. We start with an equilateral triangle ABC, and extend the base AB. Construct BD perpendicular to CB. From C draw a line crossing BD at E and AB at F and for which $EF = 1$. Then $CE = \sqrt[3]{2}$. This construction is demonstrated in figure 3(a).

The proof of this is a little more involved than the previous, as it requires some algebra. First draw a line from C parallel to AB and which meets BD at G. Since the angle at D is 30°, the new triangle BCD is half of an equilateral triangle and so $CG = 2$. If $CE = x$, then by Pythagoras we have $BE = \sqrt{x^2 - 1}$. Let M be the midpoint of AB so that CM is perpendicular to AB, so that $MB = 1/2$ and $CM = \sqrt{3}/2$. This is shown in figure 3(b).

By Pythagoras again, we have

$$MF = \sqrt{(1+x)^2 - 3/4}$$

and so

$$BF = \sqrt{(1+x)^2 - 3/4} - 1/2.$$

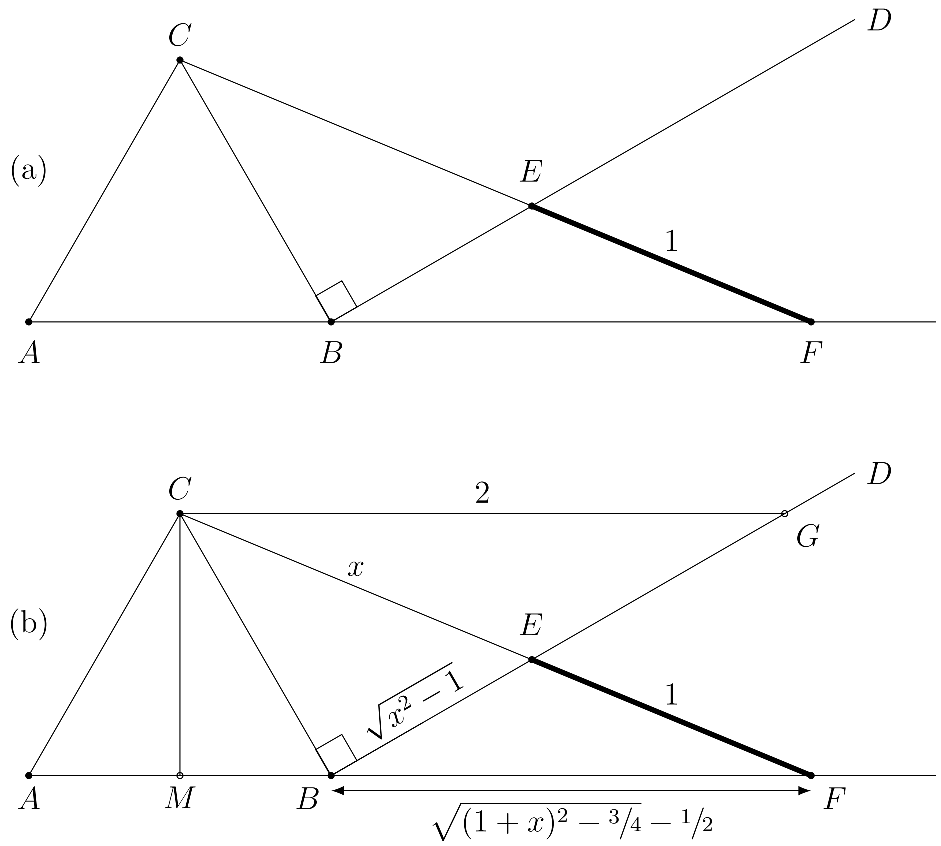

Figure 3: Nicomedes' (a) construction of $\sqrt[3]{2}$ and (b) proof

The triangles CEG and BFE are similar, so that

$$\frac{CD}{CE} = \frac{BF}{EF}$$

or that

$$2/x = \sqrt{(1+x)^2 - 3/4} - 1/2.$$

This last expression can be written as

$$2/x + 1/2 = \sqrt{(1+x)^2 - 3/4}$$

and squaring both sides produces the equation

$$4x^4 + 8x^3 - 8x - 16 = 0$$

which can be factored into

$$4(x+2)(x^3-2) = 0.$$

Since x must be positive, we have $x = \sqrt[3]{2}$.

3 Enter the conchoid

Although we have described neusis in terms of a marked straightedge, the ancient Greek mathematicians used a more general construction, which involves a curve called the *conchoid*, and which is attributed to Nicomedes. To construct a conchoid, three parameters are needed: a point P, a line L not going through P, and a distance d. From P draw a line M which crosses L at Q. Extend the line to a point R so that $QR = d$. The locus of all these points R forms a conchoid. The construction and an example of the curve is shown in figure 4.

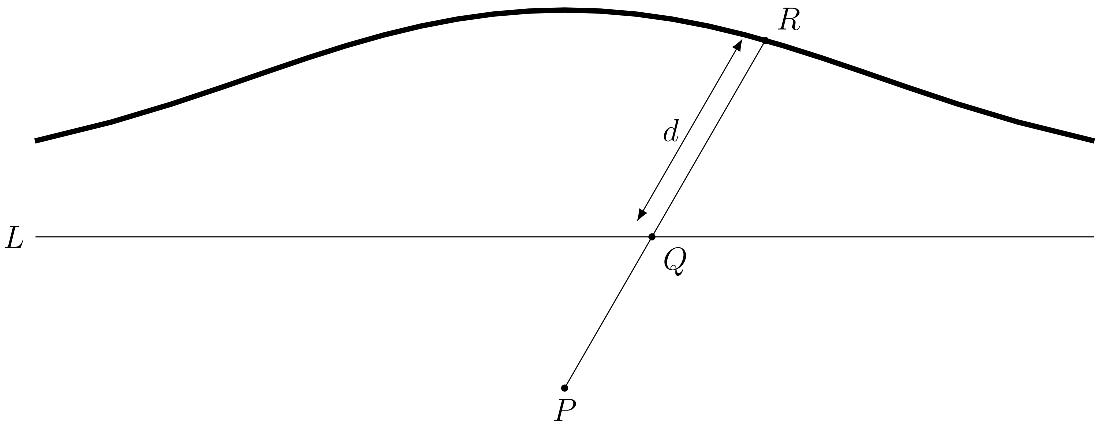

Figure 4: The conchoid

Suppose that $P = (0, 0)$, the line L is given by the equation $y = s$, and that the line M makes an angle θ with the positive x-axis. Let $PR = r$. Since $(PQ)\sin\theta = s$ we have

$$r = \frac{s}{\sin\theta} + d$$

or that

$$r = d + s\operatorname{cosec}\theta$$

as the polar equation of the conchoid. If the line is vertical, with $x = s$, then the equation would be $r = d + s\sec\theta$.

A conchoid can be used for any neusis construction involving distances between lines, such as Pappus' trisection, and Nicomedes' $\sqrt[3]{2}$. Here is how a conchoid can be used for trisection. We have our rectangle $ABCD$, with $\theta = CAB$ being the angle to be trisected. Let A be the point of the conchoid, BC the line L, and the distance $d = 1$. This is shown in figure 5. To trisect the angle $\theta = CAB$, extend DC to meet the conchoid at F. Then $\alpha = CFA$ is the trisection.

To see that this works, note that by definition of the conchoid, for any line through A, the distance between its intersection E with L and the conchoid must be constant—in this case 1. But this distance is all that is required by Pappus' construction.

4 Using dynamic geometry software

For purposes of demonstration, we shall use GeoGebra [4], which is open source and so easily available. However, the method we describe should be easily translatable to any software which

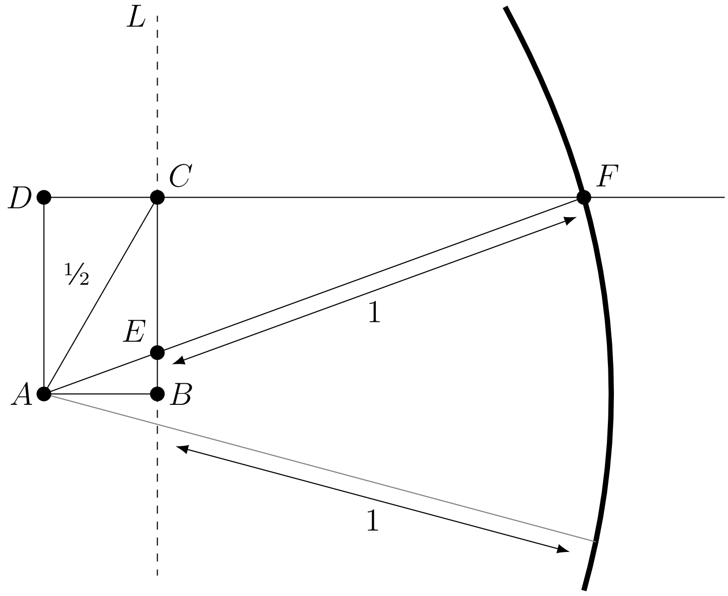

Figure 5: Using a conchoid for trisection

can draw graphs of functions in either polar or parametric form.

We wish to trisect an angle θ which is given as the angle between two lines CA and GA. We can do this by using a unit circle at the origin; A is the origin, $G = (1, 0)$ and C is any chosen point on the circle. The value s of the distance of the line to the point will be the x-coordinate of C. Thus in GeoGebra, having created the circle and a point C on it, we can create the conchoid with

```
s = x(C)
Curve[s + 2cos(t), s tan(t) + 2sin(t), t, −0.7, 0.7]
```

Note that angles are measured in radians, so that 0.7 corresponds to about 40°.

Now we can create a line from C parallel to the x-axis; this could be done for example with

```
Curve[t,y(C),t,0, infinity ]
```

and then using the intersection tool to determine the point of intersection between this line and the conchoid. This produces a diagram very similar to that of figure 5. However, the beauty of using dynamic geometry software is that as C is moved around the circle, the value s which is one of the conchoid parameters also changes, and so the conchoid will be redrawn in real time.

We can easily add some measurements to our diagram, by using the point B on the x-axis:

```
B = (x(C),0)
```

and again using the intersection tool to find the intersection E between BC and AF. Since we have started with a unit circle, the distance EF will be expected to be 2, and as C is moved around the circle, this value never changes.

Implementing Archimedes' trisection method will require some adjustment, as the neusis construction doesn't use a distance beyond a line, but a distance beyond a circle. To make the construction tractable, rotate and shift the figure in diagram 1 by θ so that B is at the origin and the center of the circle is at $(1, 0)$. Figure 6 shows the setup.

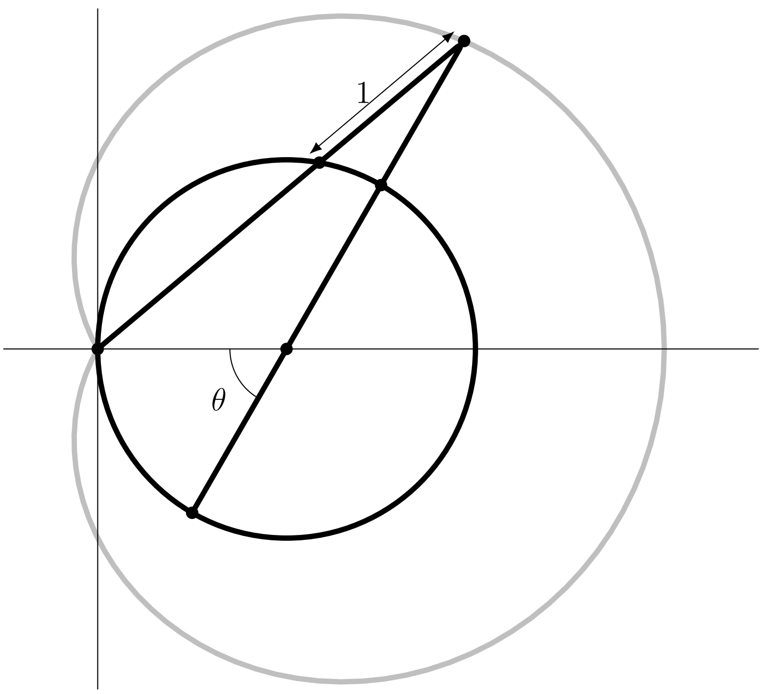

Figure 6: Setting up Archimedes' trisection for dynamic geometry

The conchoid consists of all points at a fixed distance from a given line; the curve here will consist of all points at a fixed distance beyond a circle, with all lines emanating from a position on the circle itself. Since the polar equation of a circle of radius s which passes trough the origin is $r = 2s\cos\theta$, the equation of our new curve will be $r = d + 2s\cos\theta$. Since we have $s = d = 1$, the polar equation is $r = 1 + 2\cos\theta$, and this curve is called a *limaçon* (the word comes from the French and means "small snail"). The curve is shown in gray in figure 6.

Now this is easily implemented in our DGS: start with a unit circle centred at $(1, 0)$ and let $B = (0, 0)$. Let A be any point on the curve so that the angle at $(1, 0)$ between A and B is θ. Set up the limaçon:

```
Curve[(1+2*cos(t))*cos(t),(1+2*cos(t))*sin(t),t,−2*π/3,2*π/3]
```

Now extend the line through A and $(1, 0)$ to intersect the limaçon at F. The the angle $\alpha = BFA$ will be the trisection of θ.

5 Solving cubic equations

We have seen how an ancient Greek construction, re-imagined using dynamic geometry software, can be used to create geometric constructions which can trisect angles. It is a pleasant and easy exercise to use a conchoid for Nicomedes' construction of $\sqrt[3]{2}$. We now show how to solve cubic equations.

The so called "trigonometric solution" is well known, and is based on the identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta. \tag{1}$$

We shall consider only the case of the "depressed cubic"

$$x^3 - 3ax - 2b = 0 \tag{2}$$

which is missing an x^2 term—and note that any cubic can be put into this form by a linear transformation—and writing the coefficients thus for simpler analysis later on. Even more particularly, we shall only consider equations for which $a^3 > b^2$ which ensures there are three unequal real roots. This is Cardan's *casus irreducibilis*, the form of the cubic for which purely algebraic methods of solution will always lead to the use of complex numbers.

Suppose that $x = k\cos\theta$, then by multiplying equation (1) by k^3 we have

$$4k^3\cos 3\theta - 3k^3\cos\theta - k^3\cos 3\theta = 0$$

or that

$$x^3 - \frac{3k^2}{4}x - \frac{k^3\cos 3\theta}{4} = 0.$$

Comparing coefficients with that of the depressed cubic above, we have

$$k^2 = 4a, \quad k^3\cos 3\theta = 8b$$

from which we find $k = 2\sqrt{a}$ and θ satisfies

$$\cos 3\theta = \frac{8b}{(2\sqrt{a})^3} = \frac{b}{\sqrt{a^3}}.$$

This means that given

$$\cos\phi = \frac{b}{\sqrt{a^3}}$$

then

$$x_0 = 2\sqrt{a}\cos\left(\frac{\phi}{3}\right)$$

is a solution of the cubic (2) above. The other two solutions will be

$$x_1 = 2\sqrt{a}\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) = -\sqrt{a}\cos\left(\frac{\phi}{3}\right) - \sqrt{3a}\,\sin\left(\frac{\phi}{3}\right)$$

$$x_2 = 2\sqrt{a}\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) = -\sqrt{a}\cos\left(\frac{\phi}{3}\right) + \sqrt{3a}\,\sin\left(\frac{\phi}{3}\right)$$

To implement this geometrically, start with a circle of radius $\sqrt{a}$ centred at $(0,0)$, and construct the line $x = b/a$. The intersection of this line with the circle will be at points $C_1, C_2 = (\cos\phi, \pm\sin\phi)$.

Construct the conchoid $r = 2\sqrt{a} + (b/a)\sec\theta$ and use it to trisect the angle C_1OB with $B = (b/a, 0)$ using Pappus's method. The trisection line will cross the initial circle at point E_1; and so one solution will be double its x coordinate: $2E_1[x]$. The other two solutions will be double the x coordinates of the rotations of E_1 by 120° and 240°.

In GeoGebra, this can be done by using the menu system, or by entering the following commands:

```
c = Circle [(0,0), sqrt(a)]
L = Line[(b/a,0),(b/a,1)]
C = Intersect[c,L]
r(th) = b/a*sec(th)+2*sqrt(a)
g = Curve[r(th)*cos(th), r(th)*sin*th), th, -0.7,0.7]
D = Intersect[y=x(C_1),g]
E = intersect[Line [(0,0), D],c]
s = 2*x(E)
F = Rotate[E,2*pi/3]
G = Rotate[E,4*pi/3]
t = 2*x(F)
u = 2*x(G)
```

In this sequence of commands, r(th) is the polar equation of the conchoid, and g is its implementation in Cartesian coordinates. The result will look something like figure 7.

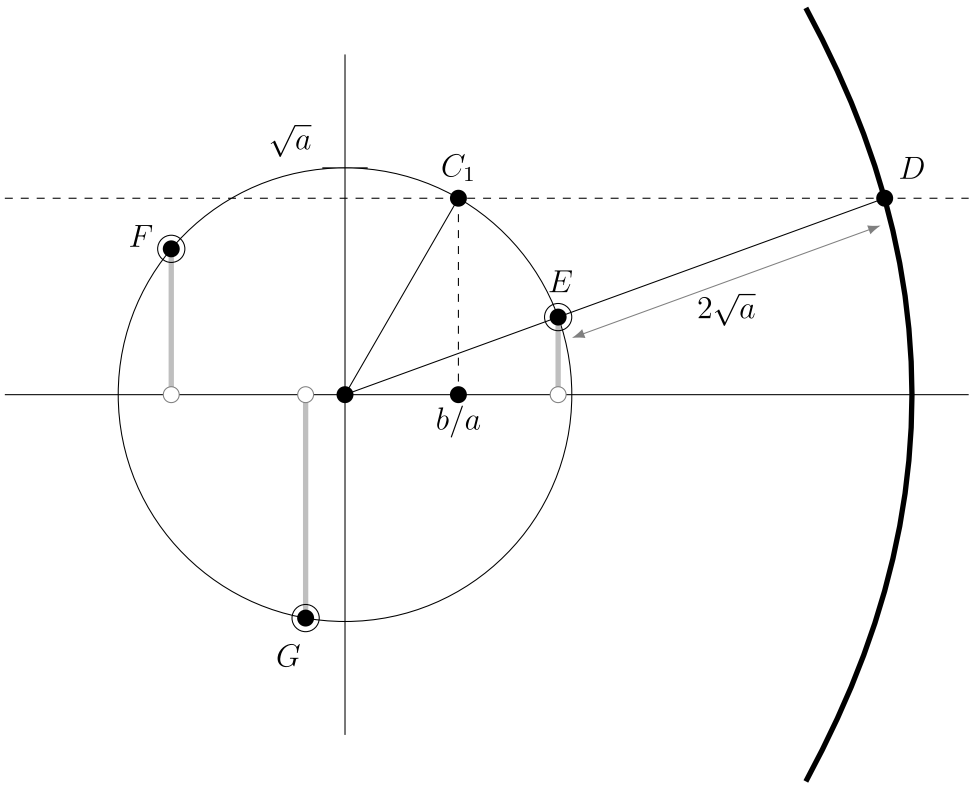

Figure 7: Solving a cubic equation using a neusis technique

The open circles on the x-axis are each half the values of the three roots. Historically, this was the most difficult of all of the three possible solutions to the cubic: one real and two imaginary (conjugates); real only, but with multiple roots; or with three distinct real roots. Cardan's initial method led him to complex numbers, which at the time were not understood. In his magisterial *Ars Magna*[3], published in 1545, he claimed that

> "Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, we obtain $25 - (-15)$. Therefore the product is 40. ...and thus far does arithmetical subtlety

go, of which this, the extreme, is, as I have said, so subtle that it is useless."

It is now known that it is not possible to avoid complex numbers in a purely algebraic treatment of the cubic equations [7], hence the use of a transcendental solution, involving circular functions, as we have done. There is nothing in our solution which is beyond ancient Greek mathematics, or in a modern setting, beyond the reach of dynamic geometry software.

6 Conclusions

We have shown how a simple technique—known and used by the ancient Greeks, but not by Euclid—can be formally and easily implemented using dynamic geometry software. This brings an elegant chapter of mathematics into the reach of modern students, teachers, or simply mathematical experimenters. Interestingly, the modern mathematical analysis of origami shows an equivalence to Euclidean geometry plus neusis [1], so the techniques of neusis are sufficient to create all possible single folds. It is quite remarkable that such a simple technique can have such far-reaching consequences!

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