The Role of Generalisation in the Discovery, Proof and Visualisation of a Theorem about Cubics with the Help of CAS and DGS (*TI-Nspire* and *Cabri*)

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Abstract
Researchers know the important role of generalisation during their work because they know when it is appropriate to use it. Teachers who are not researchers, most of the time, need to convince their students that generalisation is a tool that can and must be used. But they often do not show their students examples where this process could be or is successful. Last year, while thinking about a property of cubics we had known for a long time, we wondered if it could be the particular case of a more general one. Here we give the result we got when we tried successfully to generalize the known property of polynomial of third degree defined on R by another one about polynomial of third degree defined on C replacing tangent lines to the curves of real functions by tangent planes to surfaces defined on C (considered as R^2). This paper aims to show the discovery of this brand new property and its proof using particularly the CAS (computer algebra system) of TI N'Spire. It aims also to show the power of dynamic geometry in studying and visualising such a problem. I think that the final theorem presented in this paper has neither been discovered nor proved before. We also tried (unsuccessfully) to define a group on the points of a complex cubic in order to generalize the result known on a real cubic. Nevertheless, we think it is interesting to show when and why some directions of research are not successful and especially in using the power of visualisation of DGS (dynamic geometry software), here Cabri 2 Plus.

1. The starting point: a property of tangent lines of a third degree polynomial

1.1. A property of some tangent lines of third degree polynomials (Theorem 1)

During the T3 international congress of Atlanta in 2010, we attended a session in which Jim Nakamoto from the Sir Winston Churchill Secondary School of Vancouver presented an investigation work about cubics proposed in the portfolio of the IB (International Baccalaureate). We had not heard of this result for a long time. We remind the reader of the property of cubics discovered during such an investigation and give its formal proof in Figure 2 (described here as Theorem 1).

Theorem 1: If *f* is a real function defined by $f(x) = ax^3+bx^2+cx+d$ that has three real zeros x_1, x_2 and x_3 and if *M* is the point of the curve of *f* (in a system of axis) with abscissa $\frac{x_1+x_2}{2}$, then the tangent line to the curve at point *M* is a line crossing the *x* axis at point $(x_3, 0)$ which is the intersection point of this curve and the *x* axis (as shown in Figure 1).

The following investigation was proposed by Jim Nakamoto: the student had to discover this property for a particular polynomial function *f* of third degree (having three real zeros) before generalizing experimentally this property to all polynomial functions *f* of third degree (having three real zeros). The technique for this investigation was to display the curve of such function *f* with four sliders commanding the values of *a*, *b*, *c* and *d* and to then observe the result about the tangent line as shown in Figure 1 on the left and then see evidence that this property seems to be always true when changing the values of *a*, *b*, *c* or *d* as long as the curve of *f* and the *x* axis have three intersection points. It is possible to conduct a similar experiment as shown in Figure 1 on the right where function *f* is defined by $a.(x\text{-}x_1).(x\text{-}x_2).(x\text{-}x_3)$ and where the four sliders command a, x_1, x_2 and x_3. In this case the curve of f and the x axis have always three intersection points by construction.

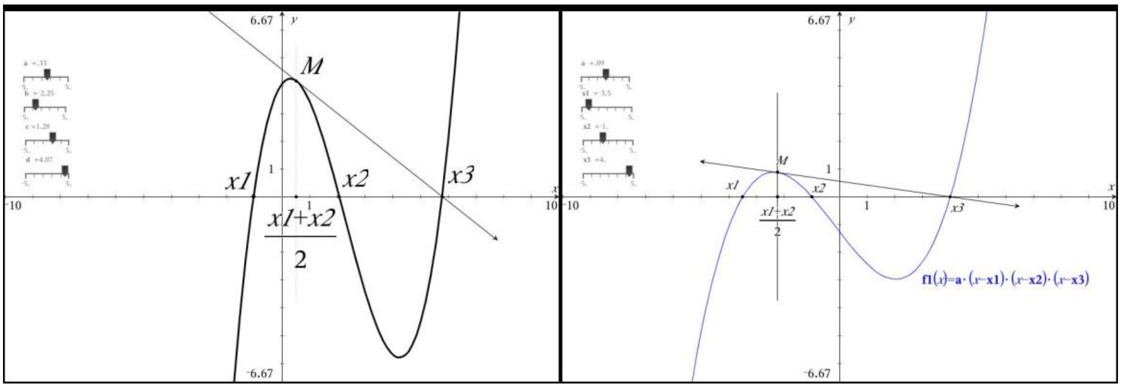

Figure 1: Illustration of Theorem 1 with *TI N'Spire*

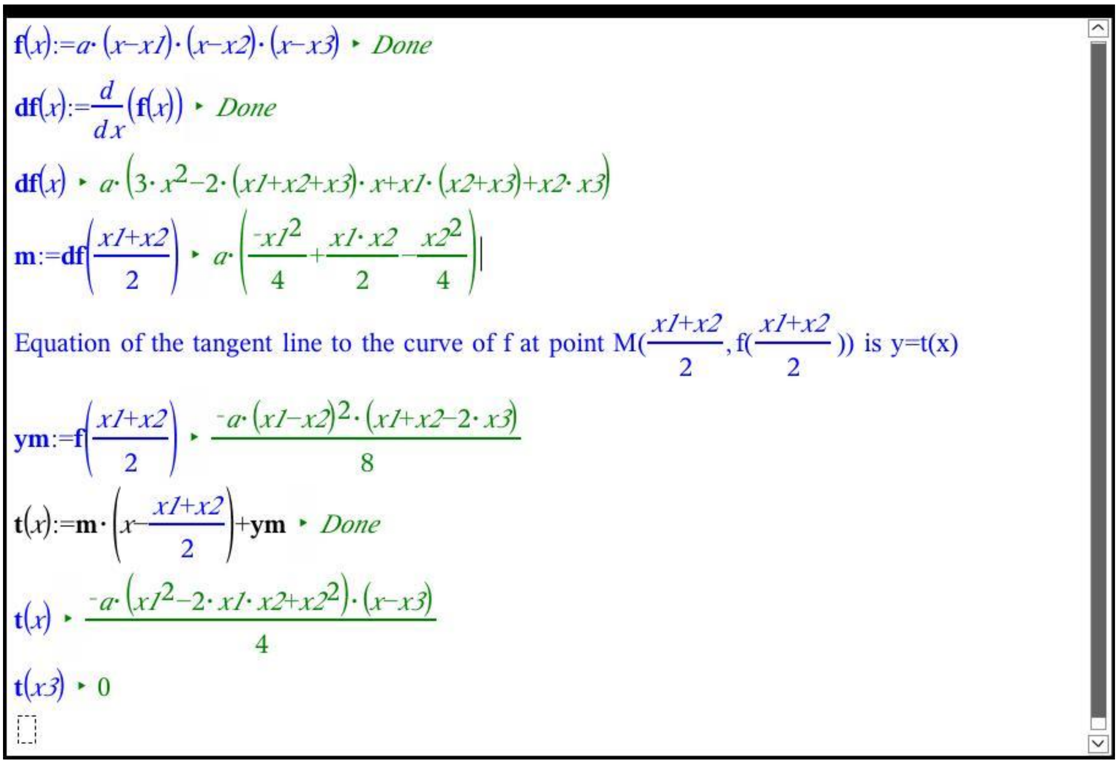

Figure 2: Note page of the formal proof of Theorem 1 with the CAS of *TI Nspire*

1.2. A first generalisation of this property from R to C (Theorem 2)

In 3D, on one hand we consider the curve of f as a curve included in the xOz plane and having $z=f(x)$ as an equation (see Figure 2 on the left) and defined on R (represented by the x axis)

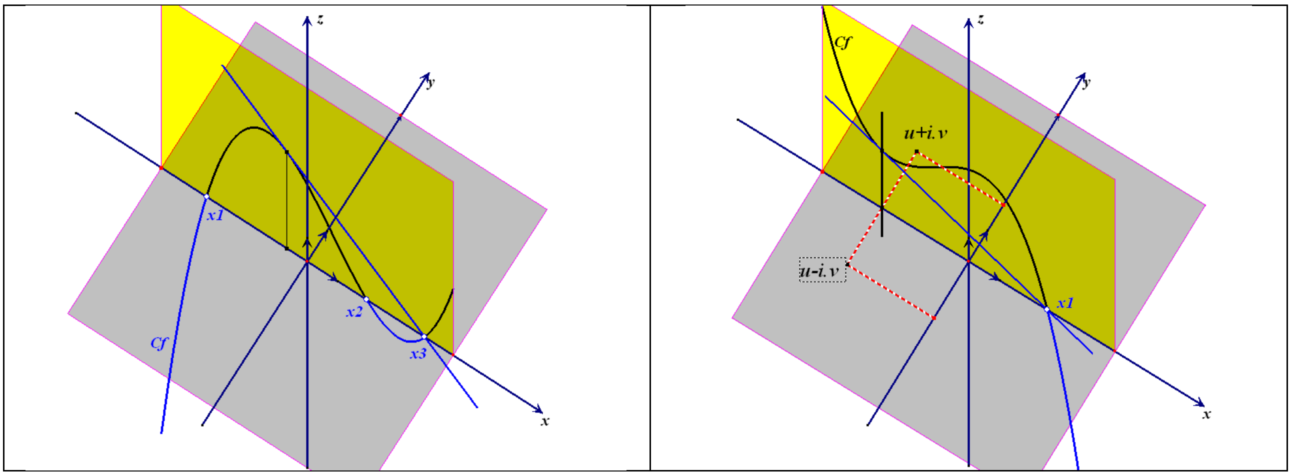

Figure 3: Representation in military perspective (*Cabri 2 Plus*)

On the other hand, we can consider this function as the restriction of a function defined on C (variable *x* being replaced by variable *x+iy*) where C is represented by the *xOy* plane (Argand plane). The function on which we focus our attention is defined by:
f: x+iy ∈ *C* ⟶ *f(x+iy)* ∈ *C* where $f(x+iy)= a.\ (x+iy)^3+b.\ (x+iy)^2+c.\ (x+iy)+d$
As the previous expression represents a complex number, the formula *z = f(x+iy)* cannot be considered as the equation of a surface, but if *f(x+iy)*= Real(*f(x+iy)*)+*i*.Imag(*f(x+iy)*), the following expressions, *z* = Real(*f(x+iy)*) and *z* = Imag(*f(x+iy)*) can be considered as the equations of two surfaces defined by two functions of two variables *x* and *y*. This remark will be used later.
For the moment, we note that when *f* admits only one real zero x_1, it admits also two other complex (and non-real) zeros x_2 and x_3 the conjugate of x_2 (so, if $x_2 = u+iv$ then $x_3 = u-iv$). These two complex zeros can be represented by two points of the *xOy* plane where each one is the symmetric point of the other with respect to the *x* axis. Therefore, $\frac{x_2+x_3}{2}$ is a real number (= *u*) belonging to the *x* axis. So it is possible to consider the point of the curve *z = f(x)* having $\frac{x_2+x_3}{2}$ as an abscissa and why not the tangent line to this curve at this point. The experiment conducted with a *TI N'Spire* suggests that this tangent line has still the same property. It passes through point (x_1, 0) of the *xOz* plane (see Figure 3 on the right). A proof from the *TI N'Spire* is given in Figure 4.
In this case: $f(x)=a.(x-x_1).(x-u-iv.((x-u+iv)$ (or its real form $a.(x-x_1)\ (x^2-2u.x+u^2+v^2)$)

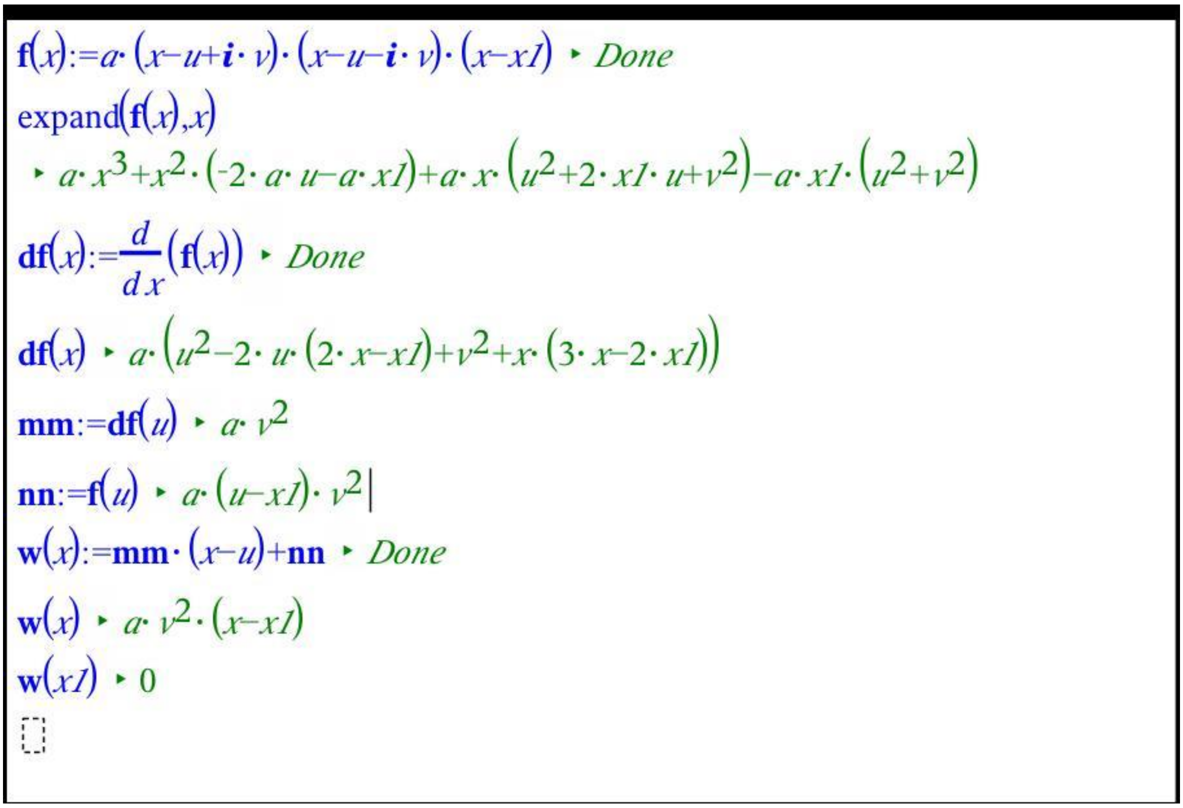

Figure 4: Formal proof of Theorem 2 with *TI-N'Spire*

1.3. Some illustrations of this theorem with surfaces

We use the CAS of *TI N'Spire*

→ to get the expressions of *Real*(*f*(*x*+*iy*)) and *Imag*(*f*(*x*+*iy*), to copy and past them in two different Cabri 2 Plus files as "expressions" of this software and to treat them to obtain the representations of the surfaces *z* = *Real*(*f*(*x*+*iy*)) and *z* = *Imag*(*f*(*x*+*iy*)) in military perspective (Figure 5 left for the first one and Figure 5 right for the second one). In reality one can notice that we have kept only the parts of these surfaces corresponding to z≥0 (the trick used was to represent *z*=(sqrt(*Real(f(x+iy))*))^2 and *z*=(sqrt(*Imag(f(x+iy))*))^2)

→ to solve the equations *Real*(*f*(*x*+*iy*)) = 0 and *Imag*(*f*(*x*+*iy*)) = 0 with respect to *y*, to copy and paste respectively the expressions we got in the two previous files, still as expressions of Cabri 2 Plus. These expressions are treated in order to obtain as loci their curves in the system of axis *xOy*. These curves are the intersection curves between the two surfaces represented previously and plane *xOy*. These curves are represented as dotted curves in Figure 5. The first intersection has three branches and the second one three branches too (one of them is the *x* axis). We can check that x_1, x_2 and x_3 belong simultaneously to these two branches because the roots of *f(x+iy)* = 0 are the common roots of *Real(f(x+iy))*=0 and *Imag(f(x+iy)*=0 (Figure 7 left: one branch in blue, the other one in red)).

Solutions of *Real(f(x+iy))* = 0: $y = \pm\sqrt{\frac{(u^2-2u.x+v^2+x^2)(w-x)}{2.u+w-3.x}}$

Solutions of *Imag(f(x+iy))* = 0: $y = \pm\sqrt{u^2+2.u(w-2x)+v^2-(2w-3x)x}$ and *y* = 0

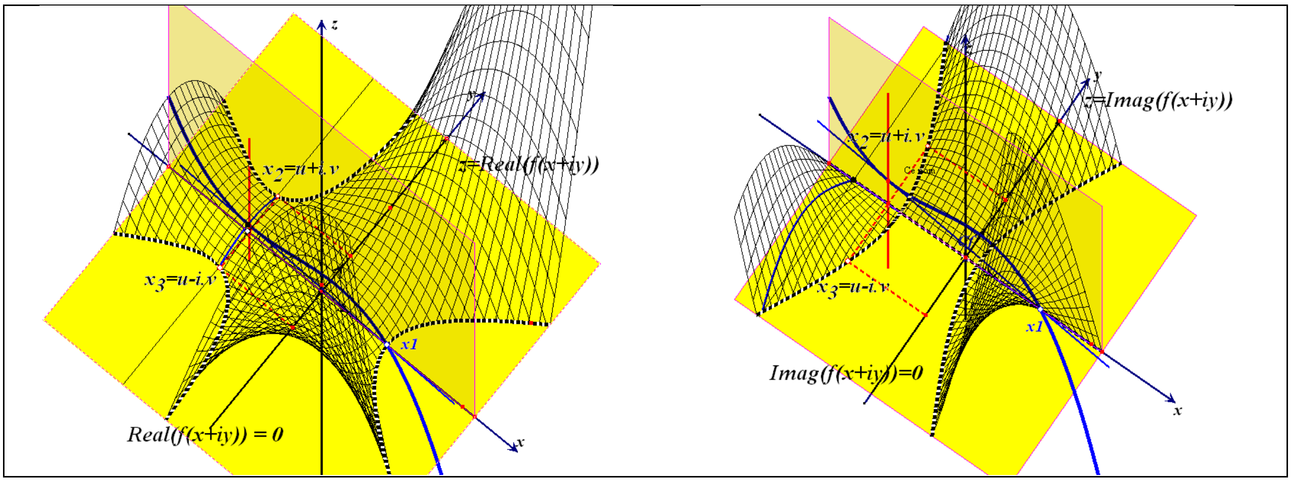

Figure 5: Various representations in military perspective with *Cabri 2 Plus* ([6])

A rather different approach can be taken with the *TI N'Spire* to get the result shown in Figure 6. But here each curve in 3D is programmed as a surface defined by a parametric equation where one of the two parameters represents the thickness of the curve.

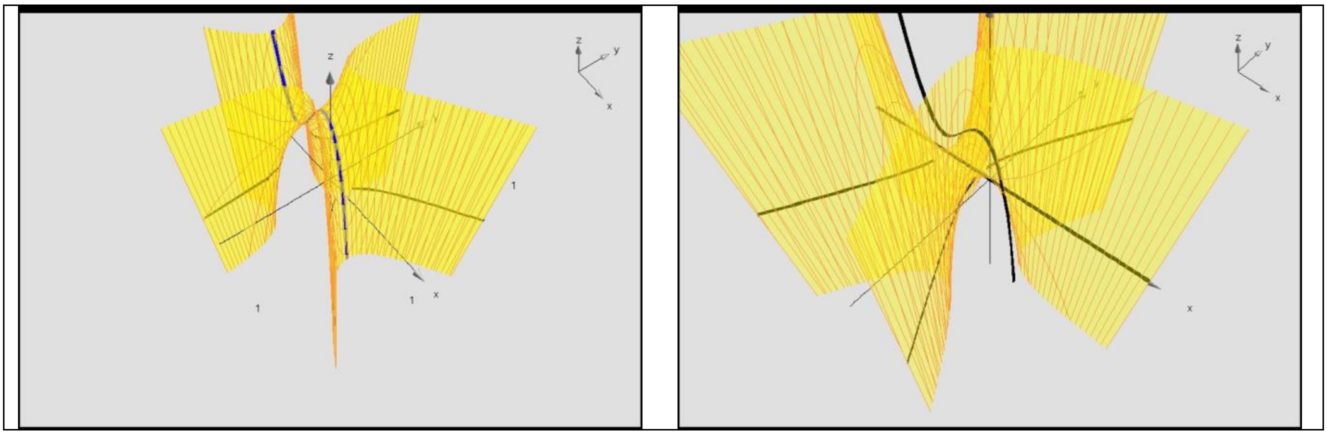

Figure 6: Various representations with *TI N'Spire*

Other calculations can be done with the same CAS to obtain an expression of *abs(f(x+iy))*. The same previous technique can be used to represent the surface *z= abs(f(x+iy))* in another *Cabri 2 Plus* file. So we can visualize the surface touching the *xOy* plane only at the three roots represented in this plane. We can notice that the curve of the real function *z=f(x)* is included in this surface when *z* is positive because in this case *abs(f(x))=Real(f(x))=f(x)* (see Figure 7 right)

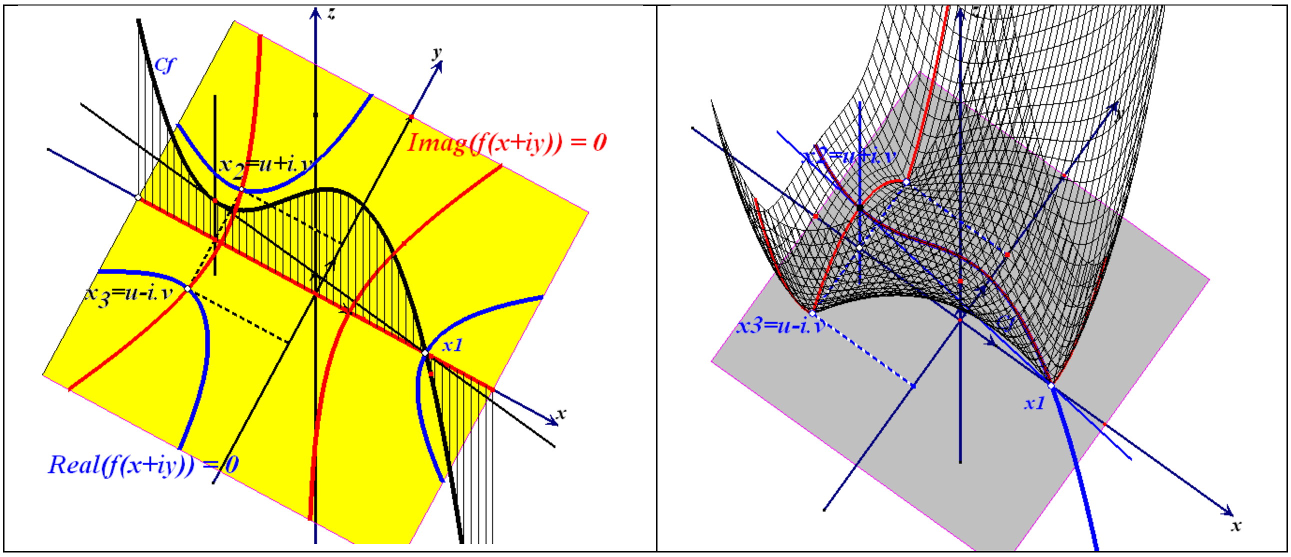

Figure 7: Various representations in military perspective with *Cabri 2 Plus* ([6])

2. Generalisation to third degree polynomials on C (Theorem 3)

2.1. Formal proof (*TI-Nspire*)

The same work can be performed with a third degree polynomial *f* having three complex roots:
Starting from *f*(z) = $a * (z - z_1) * (z - z_2) * (z - z_3)$
where $a = a_1 + i.a_2$, $z_1 = u_1 + i.v_1$, $z_2 = u_2 + i.v_2$, and $z_3 = u_3 + i.v_3$. We use *ff(x,y):=f(x+iy) to get ff1(x,y):=real (f(x+iy)) and ff2(x,y):=imag(f(x+iy)*

$$\mathbf{ff1}(x,y)$$

$$a1\cdot x^3+x^2\cdot(-3\cdot a2\cdot y-a1\cdot(u1+u2+u3)+a2\cdot(v1+v2+v3))+x\cdot(-3\cdot a1\cdot y^2+(a1\cdot(2\cdot v1+2\cdot v2+2\cdot v3)+a2\cdot(2\cdot u1+2\cdot u2+2\cdot u3))\cdot y+a1\cdot(u1\cdot(u2+u3)+u2\cdot u3+v1\cdot(-v2-v3)-v2\cdot v3)+a2\cdot(u1\cdot(-v2-v3)+u2\cdot(-v1-v3)-u3\cdot(v1+v2)))+a2\cdot y^3+(a1\cdot(u1+u2+u3)+a2\cdot(-v1-v2-v3))\cdot y^2+(a1\cdot(-u1\cdot(v2+v3)-u2\cdot(v1+v3)-u3\cdot(v1+v2))+a2\cdot(u1\cdot(-u2-u3)-u2\cdot u3+v1\cdot(v2+v3)+v2\cdot v3))\cdot y+a1\cdot(u1\cdot(v2\cdot v3-u2\cdot u3)+u2\cdot v1\cdot v3+u3\cdot v1\cdot v2)+a2\cdot(u1\cdot(u2\cdot v3+u3\cdot v2)+u2\cdot u3\cdot v1-v1\cdot v2\cdot v3)$$

$$\mathbf{ff1}\left(\frac{u1+u2}{2},\frac{v1+v2}{2}\right)$$

$$a1\cdot\left(\frac{-u1^3}{8}+\frac{u1^2\cdot(u2+2\cdot u3)}{8}+u1\cdot\left(\frac{u2^2}{8}-\frac{u2\cdot u3}{2}+\frac{(v1-v2)\cdot(3\cdot v1+v2-4\cdot v3)}{8}\right)-\frac{u2^3}{8}+\frac{u2^2\cdot u3}{4}-\frac{u2\cdot(v1-v2)\cdot(v1+3\cdot v2-4\cdot v3)}{8}-\frac{u3\cdot(v1-v2)^2}{4}\right)+a2\cdot\left(u1^2\cdot\left(\frac{3\cdot v1}{8}-\frac{v2}{8}-\frac{v3}{4}\right)+u1\cdot\left(\frac{-u2\cdot(v1+v2-2\cdot v3)}{4}-\frac{u3\cdot(v1-v2)}{2}\right)+u2^2\cdot\left(\frac{-v1}{8}+\frac{3\cdot v2}{8}-\frac{v3}{4}\right)+\frac{u2\cdot u3\cdot(v1-v2)}{2}-\frac{(v1-v2)^2\cdot(v1+v2-2\cdot v3)}{8}\right)$$

Figure 8: First part of the formal proof of Theorem 3 (Note page of *TI N'Spire*)

We evaluate the gradient of surface $z = ff_1(x,y)$ which is $\left(\frac{dff_1}{dx}(x,y), \frac{dff_1}{dy}(x,y), -1\right)$ at point $(x,y,ff_1(x,y))$ and then its value at point $J\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}, ff_1\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}\right)\right)$. Then we evaluate the dot product between this gradient and vector $\overrightarrow{JM_3}$ where $M_3(u_3,v_3,0)$. As shown in Figure 9, the result obtained is 0, which means that the tangent plane to surface $z = ff_1(x,y)$ at point J passes through point $M_3(u_3,v_3,0)$.

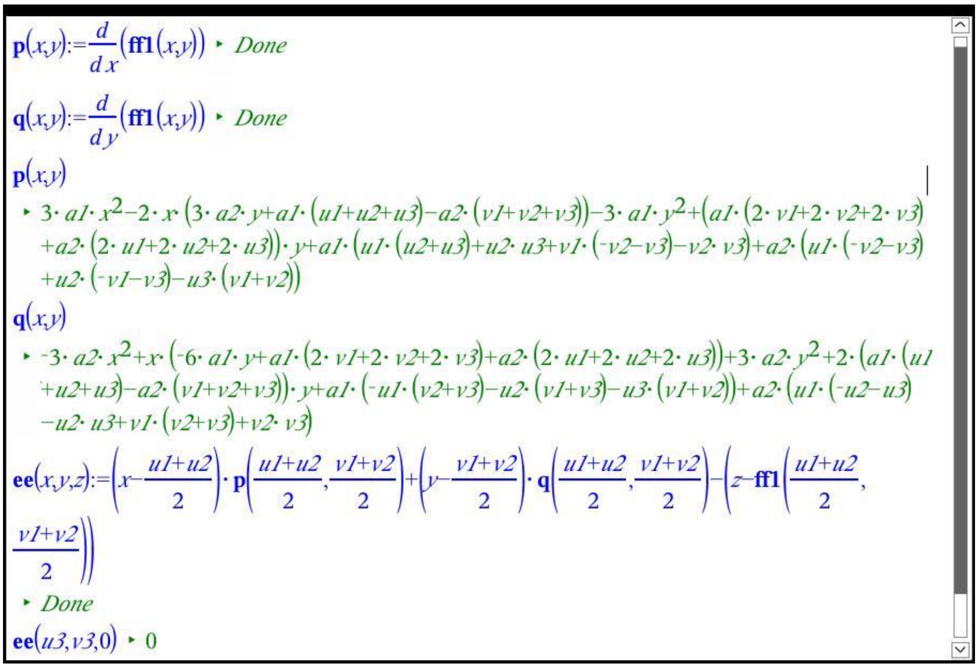

Figure 9: Second part of the formal proof of Theorem 3 (Note page of *TI N'Spire*)

The same result is obtained with the same algorithm for surface $z = ff_2(x,y)$.
It means also that the intersection curve between surface $z = ff_1(x,y)$ and the vertical plane passing through points $I\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}, 0\right)$ and point $M_3(u_3,v_3,0)$ admits a tangent line at point $J\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}, ff_1\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}\right)\right)$ passing through point $M_3(u_3,v_3,0)$ (that can be visualised in Figure 10 on the left for ff_1 and on the right for ff_2).

2.2. Visualization of this theorem (*TI N'Spire* and *Cabri 2 Plus*)

In Figure 10, we have added the parts or the same surfaces for $z \leq 0$ in using a similar trick to the one used before (we have represented the surfaces having as equations
$z = -(\text{sqrt}(-(ff1(x,y))^2$ and $z = -(\text{sqrt}(-(ff2(x,y))^2)$
So, in Figure 10 on the left, we can see the surface S_1, the points $M_1(x_1)$, $M_2(x_2)$, $M_3(x_3)$, the curve (C), the tangent line to (C) at J passing through M_3 and the vertical plane containing this curve and the tangent line. Figure 10 on the right shows the same result for S_2.

Remark 1: It is easy to prove that the two curves included in the vertical plane passing through $I\left(\frac{u_1+u_2}{2}, \frac{v_1+v_2}{2}, 0\right)$ and $M_3(u_3, v_3, 0)$ are cubics defined by a third degree polynomial defined on R[X].
We have only to use parametric equations of line (IM_3) $(x(t),y(t))$ and evaluate $ff1(x(t),y(t))$ with the CAS of *TI N'Spire* to get the result in the general case (the expression needs several lines to be

written). This remark is crucial if we want to extend to C the results about the groups associated to a cubic in R.
Remark 2: These figures (like all the other figures created and used for this paper) are in reality animated figures that can be modified by eight sliders commanding the real and imaginary parts of *a*, x_1, x_2 and x_3, a point commanding the length of the unit vector of the *z* axis and a point commanding the rotation of the *xOy* system of axis around the axis *Oz*.

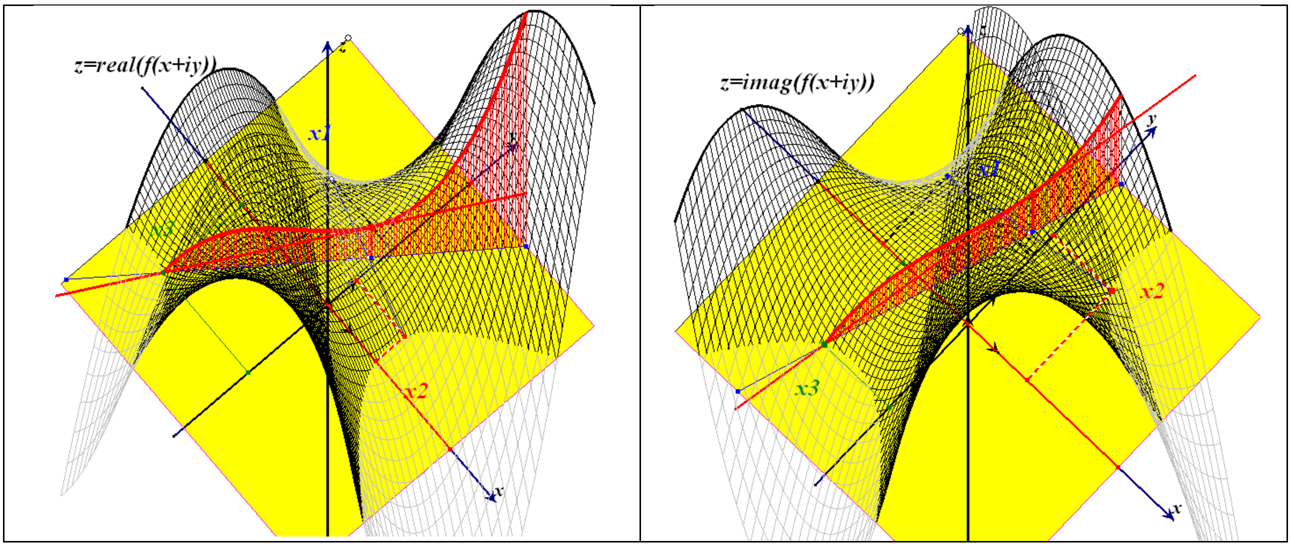

Figure 10: Visualisation of the general case in military perspective ([6])

2.3. A theorem of complex third degree polynomials

If *f* is a third degree polynomial defined on C (complex numbers), if x_1, x_2 and x_3 are its three roots representing points M_1, M_2 and M_3 in the complex plane *xOy*, if *I* is the midpoint of segment [M1M2], if J_1 (respectively J_2) is the point of the surface S_1: $z(x,y)=real(f(x+iy))$ (respectively S_2: $z(x,y)=imag(f(x+iy))$ which projection on plane *xOy* is *I*, then the tangent plane to S_1 (respectively to S_2) at point J_1 (respectively J_2) crosses the plane *xOy* at point M_3.
Remark: another way to express the conclusion is that the tangent line at point J_1 (respectively J_2) to the plane curve (C_1) (respectively C_2), which is a real cubic and the intersection between S_1 (respectively S_2) and the vertical plane passing through *I* and M_3 crosses the line (IM_3) at point M_3.
That is the generalisation in the complex plane of the initial property we presented at the beginning of the paper.

3. From the groups associated to real cubic curves to other ones associated to complex cubic surfaces

4.1. Groups of cubics and a possible extension

We know that it is possible to define an operation (+) on the points of a cubic (C) in order to get a group having as an identity a point O of this cubic. The definition is shown in Figure 11 on the left.
A possible generalisation involves defining an operation on the points of surface S_1 (respectively S_2) like the operation + on a real cubic. It is possible to define such an operation because we have noticed that two points of such a surface belong to a real vertical cubic curve. So we can define the operation + as presented on Figure 11 on the right. + is commutative, O is the identity and each point has an opposite point

4.2. Impossibility to extend such an operation to a C-cubic surface

4.2.1. Definition of a C-cubic surface: if *f* is a function belonging to C[X], the set of couple (P_1, P_2) where $P_1(x,y,real(f(x+iy)))$ and $P_2(x,y,imag(f(x+iy)))$ is called the C-cubic surface: associated to *f* .
Remark: each couple of this surface belongs to the same vertical because they have the same coordinates *x* and *y*.
4.2.2. Attempt to define (P_1, P_2) + (Q_1, Q_2): the idea is to chose (P_1+ Q_1, P_2+ Q_2). After chosing couple (O_1,O_2) as the identity, we can state on Figure 10 that points $P_1 + Q_1$ and $P_2 + Q_2$ do not belong generally to the same vertical (Figure 12). So we failed in trying to extend the concept of group of points associated to a real cubic (set of points (x, f(x)) where f is a function of R[X]) to a concept of group of couple of points associated to a complex cubic (set of couples (P_1,P_2) where f is a function of C[X])

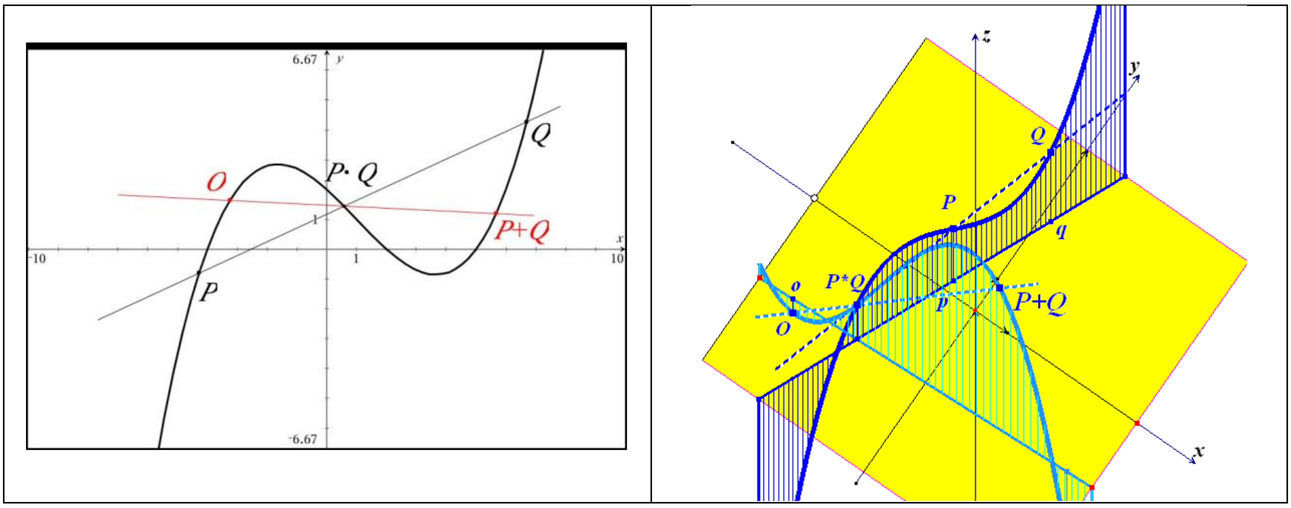

Figure 11: Definitions of a group operation on cubics

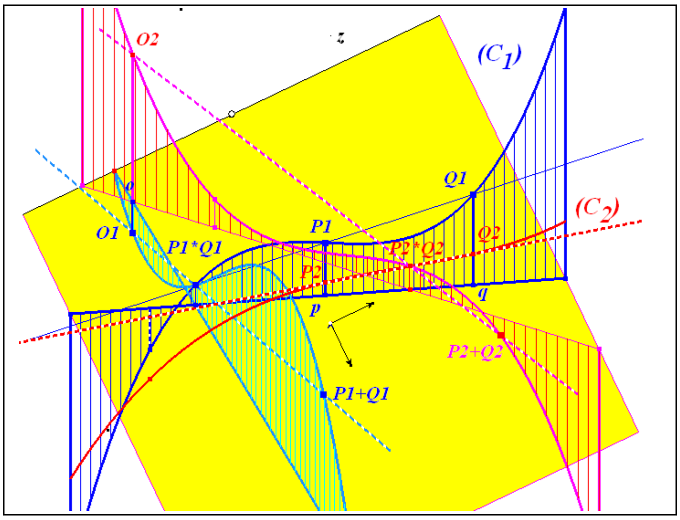

Figure 12: An attempt to define a group associated with a complex cubic

Remark: we could define an operation on the points of the surface z=real(f(x+iy) (respectively z=imag(f(x+iy)) (Figure 11 on the right) but a proof of the associativity of this operation would be difficult.

4. Conclusion

In this paper, we have illustrated how generalisation can be powerful when conducting a research work mediated by technology. DGS and CAS when connected can give confidence to a researcher who wants to check if a known result is true in a larger set than the one in which they know it. We have presented the investigations leading to the discovery of a property generalised to C from one known on R. We have opened an algebraic question about the existence of a group associated to a third degree polynomial on C[X] where the operation operates on the points of a surface instead of points of a curve. We have not solved this problem even in the restricted case of the surfaces associated to the real and imaginary part of a complex function because the proof of associativity seems very difficult and could be the centre point of another research work. We can finally notice that we have used a lot of techniques to model with *Cabri 2 Plus* most of the figures we have created for this research.

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[6] YouTube videos about military perspective
“T3_Chicago_MP” at http://youtu.be/vYStmrDNT8E
“T3_Chicago_1” at http://youtu.be/yXDhiOD1fvw
“T3_Chicago_2” at http://youtu.be/3rJZZMztbng

Software :
Cabri 2 Plus and *Cabri 3D* by Cabrilog at http://www.cabri.com
TI-Nspire by Texas Instruments at http://education.ti.com/en/us/home